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Super-replication price for asset prices having bounded increments in discrete time

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Abstract

We consider a discrete time financial model where the support of the conditional law of the risky asset is bounded. We show that, for convex option, the super-replication problem reduces to the replication one in a Cox-Ross-Rubinstein model whose parameters are the law support boundaries.

1 Introduction

We consider a discrete time financial market consisting of one risky asset S and one risk-less bond normalized to one. It is well-known that discrete time models are intrinsically incomplete and thus, as perfect replication is not always possible, the full hedging of risk goes through super-replication. The super-replication price is the minimal initial wealth needed to hedge without risk the contingent claim. It has been introduced in the binomial setup for transaction costs by Bensaid-Lesne-Pagès-Scheinkman [1], in a \mathcal{L}^2 -setup for transaction costs and short-sales constraints by Jouini-Kallal [11, 12] and in the diffusion setup for incomplete markets by El Karoui-Quenez [9]. The so called dual formulation of the super-replication price has been extensively studied and we refer to Föllmer-Kramkov [10] and the references therein. In our context, the super-replication cost of an European contingent claim H is the supremum over the risk neutral probability set of the expectation of H . Nevertheless it is well known that this dual formulation does not enable in general to effectively compute the super-replication price. Note that Cvitanić-Shreve-Soner [6], Cvitanić-Pham-Touzi [5, 4] and Patry [13] are able to prove, in various context, that for an European call option, the super-replication price is equal to the initial price of the underlying and that the hedging strategy is just the "Buy and Hold" one. In Carassus-Gobet-Temam [3], the authors consider a discrete time model and provide a closed formula in order to compute the super-replication cost of European and American style options and also the hedging strategy. In the case of European vanilla options, finding the super-replication price reduces to compute some concave envelope of the payoff function. For more general options, it involves recursive computations using again a kind of concave envelopes. The coefficients of the affine function which appears in the concave envelope give the hedging strategy.

The formula comes from the dynamic programming principle and enlighten the crucial role played by the conditional distribution of the underlying. When this distribution admits

a density with respects to the Lebesgue measure which is strictly positive over all the positive real line¹, the authors provide effective computation for the super-replication price of European and American style exotic options (including Asian, Lookback or Barrier options) and show that those price are too high to be used in practice.

Here, we focus on another class of models: the one such that the support of the conditional distribution of the underlying increments is bounded. This is of course the case for tree models. It is also true in continuous time models such that, conditionally to the information at time t , the distribution of $\frac{S_{t+1}}{S_t}$ is equivalent to the Lebesgue measure on $[d_{t+1}, u_{t+1}]$. This is in particular true if the regulator imposes some bounds on the maximal variation of the asset price in a given time interval. This for example the case in some US Stock Exchange, where the asset can not fluctuate of more of 10% in 5 minutes. We prove that, for options having convex payoff, the super-replication price is equal to the replication price in a Cox-Ross-Rubinstein model (see [7]), whose parameters are the maximum bounds of the law support. We thus generalize the result found in Scagnellato and Vargiolu [15] for a trinomial model.

The paper is organized as follows. In Section 2, we describe the financial model, give the notation of the paper and recall the algorithm found in Carassus-Gobet-Temam [3]. Then, in Section 3 we present and prove our main results.

2 The financial model and the super-replication algorithm

2.1 Notations and definitions

We consider a discrete time financial model with finite horizon T and set $\mathcal{T} = \{0, 1, \dots, T\}$. The market consists of one riskless asset with price process normalized to one and a risky asset with price process $S = \{S_t, t = 0, \dots, T\}$ which takes values in $(0, \infty)$. The stochastic price process $(S_t)_{t \in \mathcal{T}}$ is defined on a complete probability space (Ω, \mathcal{F}, P) equipped with the filtration $\mathbb{F} = \{\mathcal{F}_t, t \in \mathcal{T}\}$, where the σ -field \mathcal{F}_t is generated by the random variables S_0, S_1, \dots, S_t . We make the usual assumption that \mathcal{F}_0 is trivial and $\mathcal{F}_T = \mathcal{F}$.

Here we focus on price processes satisfying Assumption 2.1 below. Recall that the support of a generic probability measure Q in \mathbb{R} is the smallest closed set K such that $Q(K) = 1$. It is easy to see that for every $x \in K$ and for every $\varepsilon > 0$, we have $Q(B(x, \varepsilon)) > 0$, where $B(x, \varepsilon)$ is the ball of \mathbb{R} with center x and radius ε (see for example example Ex. 12.9 in Billingsley [2]).

Assumption 2.1. *For all $t \in \{0, 1, \dots, T - 1\}$, there exists real numbers u_{t+1} and d_{t+1} satisfying $d_{t+1} \neq u_{t+1}$ and $d_{t+1} \leq 1 \leq u_{t+1}$ and such that the support of the conditional law of $\frac{S_{t+1}}{S_t}$ with respect to \mathcal{F}_t is contained in $[d_{t+1}, u_{t+1}]$.*

Remark 2.2. Note that condition $d_{t+1} \leq 1 \leq u_{t+1}$ is implied by the no arbitrage assumption.

Assumption 2.1 implies that for all $t \in \{0, 1, \dots, T - 1\}$, for all $\varepsilon > 0$, $s_0, \dots, s_t \in \mathbb{R}$:

$$\mathbb{P}(S_{t+1} \in [s_t d_{t+1}, s_t u_{t+1}] \mid S_0 = s_0, \dots, S_t = s_t) = 1, \quad (2.1)$$

$$\mathbb{P}(S_{t+1} \in B(s_t d_{t+1}, \varepsilon) \mid S_0 = s_0, \dots, S_t = s_t) > 0, \quad (2.2)$$

$$\mathbb{P}(S_{t+1} \in B(s_t u_{t+1}, \varepsilon) \mid S_0 = s_0, \dots, S_t = s_t) > 0. \quad (2.3)$$

¹This case includes Black-Scholes model, general stochastic differential equations, stochastic volatility models, or models governed by Brownian motion and Poisson process, when they are observed at discrete time.

Two main kinds of price processes fulfill this assumption. The first one are processes such that conditionally to \mathcal{F}_t , the distribution of S_{t+1} is discrete and finite. Tree models are prototype of such models. The second family of models are the one such that conditionally to \mathcal{F}_t , the distribution of $\frac{S_{t+1}}{S_t}$ is equivalent to the Lebesgue measure on $[d_{t+1}, u_{t+1}]$. Of course, any combinations of both types are taken into account.

Next we define a trading portfolio by a \mathbb{R} -valued \mathbb{F} -adapted process $\phi = \{\phi_t, t = 0, \dots, T-1\}$, where ϕ_t denotes the number of risky asset held at time t . The \mathbb{R} -valued \mathbb{F} -adapted process $C = \{C_t, t \in \mathcal{T}\}$ represents the cumulative consumption process. We assume that $C_0 = 0$ and that C is non-decreasing. We also use the notation $\Delta S_t = S_t - S_{t-1}$ and $\Delta C_t = C_t - C_{t-1}$, for $t = 1, \dots, T$.

Given an initial wealth $x \in \mathbb{R}$, a trading portfolio ϕ and a cumulative consumption process C , the wealth process $X^{x, \phi, C}$ is governed by

$$\begin{aligned} X_0^{x, \phi, C} &= x \\ X_t^{x, \phi, C} &= X_{t-1}^{x, \phi, C} + \phi_{t-1} \Delta S_t - \Delta C_t, \quad \text{for } t = 1, \dots, T. \end{aligned} \quad (2.4)$$

The condition $C = 0$ means that the portfolio ϕ is self-financed. (x, ϕ, C) will be called a hedging strategy.

Following the presentation of Föllmer and Kramkov [10], we recall basic definitions related to the super-replication prices. A European contingent claim will be represented by a \mathcal{F}_T -measurable random variable H . We denote by \mathcal{A}_H^e , the set of hedging strategies for H such that $X_T^{x, \phi, C} \geq H$ \mathbb{P} -a.s. Then, $(\hat{x}, \hat{\phi}, \hat{C}) \in \mathcal{A}_H^e$ is minimal if for all $(x, \phi, C) \in \mathcal{A}_H^e$, $X_t^{x, \phi, C} \geq X_t^{\hat{x}, \hat{\phi}, \hat{C}}$ \mathbb{P} -a.s for all $t \in \mathcal{T}$. It is easy to see that \hat{x} is then the so-called super-replication cost $p^e(H)$ of H , i.e the minimal initial capital needed for hedging without risk H :

$$p^e(H) = \inf\{x \in \mathbb{R} : \exists (\phi, C) \text{ s.t. } (x, \phi, C) \in \mathcal{A}_H^e\}.$$

We now define the same notion for an American contingent claim $(H_t)_{t \in \mathcal{T}}$. \mathcal{A}_H^a will be the set of American hedging strategies such that, for all $t \in \mathcal{T}$, $X_t^{x, \phi, C} \geq H_t$ a.s. Then $(\hat{x}, \hat{\phi}, \hat{C}) \in \mathcal{A}_H^a$ is minimal if for all $(x, \phi, C) \in \mathcal{A}_H^a$, $X_t^{x, \phi, C} \geq X_t^{\hat{x}, \hat{\phi}, \hat{C}}$ a.s, for all $t \in \mathcal{T}$. Again \hat{x} is the super-replication cost $p^a(H)$ of H , i.e

$$p^a(H) = \inf\{x \in \mathbb{R} : \exists (\phi, C) \text{ s.t. } (x, \phi, C) \in \mathcal{A}_H^a\}.$$

Now, we introduce the set of equivalent martingale measure:

$$\mathcal{P} = \left\{ Q \sim P : \frac{dQ}{dP} \in L^\infty, \Delta S_t \in L^1(Q) \text{ and } E^Q[\Delta S_t | \mathcal{F}_{t-1}] = 0, 1 \leq t \leq T \text{ } P\text{-a.s.} \right\}.$$

Note that the Dalang-Morton-Willinger Theorem [8] asserts that the non-emptiness of \mathcal{P} is equivalent to the economic meaningful assumption of no-arbitrage.

2.2 Super-replication algorithm

For the reader's convenience, we now recall the Carassus-Gobet-Temam (CGT) algorithm for super-replication of derivative assets presented in [3]. We start with the European case. For a measurable function h from \mathbb{R}^{T+1} into \mathbb{R} , we define the sequence of operators

$$\Gamma_T^e h(s_0, \dots, s_T) = h(s_0, \dots, s_T) \quad (2.5)$$

$$\Gamma_t^e h(s_0, \dots, s_t) = \text{ess} \inf_{(\alpha, \beta) \in I_{\Gamma_{t+1}^e h}(s_0, \dots, s_t)} \{\alpha + \beta s_t\} \quad 0 \leq t \leq T-1 \quad (2.6)$$

where for a measurable function v from \mathbb{R}^{t+2} into \mathbb{R} we define

$$\begin{aligned} I_v(s_0, \dots, s_t) &= \\ &= \{(\alpha, \beta) \in \mathbb{R}^2 \mid \mathbb{P}(\alpha + \beta S_{t+1} \geq v(s_0, \dots, s_t, S_{t+1}) \mid S_0 = s_0, \dots, S_t = s_t) = 1\}. \end{aligned} \quad (2.7)$$

The essential infimum in (2.6) is related to the the law of the vector (S_0, \dots, S_t) , which we indicate with \mathbb{P}_t . Then, the following theorem holds.

Theorem 2.3. *Assume that $\mathcal{P} \neq \emptyset$.*

Let $H = h(S_0, \dots, S_T)$ be an European contingent claim, for some measurable function h from \mathbb{R}^{T+1} into \mathbb{R} . Assume that

$$\sup_{Q \in \mathcal{P}} E^Q [H] < \infty.$$

Then, there exists a minimal hedging strategy $(\hat{x}, \hat{\phi}, \hat{C}) \in \mathcal{A}_H^e$ and its value at time $t \leq T$ is

$$X_t^{\hat{x}, \hat{\phi}, \hat{C}} = \Gamma_t^e h(S_0, \dots, S_t) \quad \mathbb{P}_t - a.s.$$

In particular,

$$p^e(H) = \Gamma_0^e h(S_0).$$

An analogous result holds in the American case. Consider a family of measurable functions $h = (h_t)_{t \in \mathcal{T}}$ such that for $t \in \mathcal{T}$, h_t maps \mathbb{R}^{t+1} into \mathbb{R} . We define a new sequence of operators Γ^a as

$$\Gamma_T^a h(s_0, \dots, s_T) = h_T(s_0, \dots, s_T) \quad (2.8)$$

$$\begin{aligned} \Gamma_t^a h(s_0, \dots, s_t) &= \text{ess} \inf_{(\alpha, \beta) \in I_{\Gamma_{t+1}^a h(s_0, \dots, s_t)}} \{\alpha + \beta s_t\} \vee h_t(s_0, \dots, s_t), \\ &0 \leq t \leq T - 1. \end{aligned} \quad (2.9)$$

where the set I_v is still defined by (2.7). Setting $\mathcal{S}_{t,T}$ the set of all stopping w.r.t. the filtration \mathbb{F} such that $t \leq \tau \leq T$, we get that

Theorem 2.4. *Assume that $\mathcal{P} \neq \emptyset$.*

Let $H = (H_t)_{t \in \mathcal{T}}$ be an American contingent claim such that

$$\sup_{\tau \in \mathcal{S}_{0,T}, Q \in \mathcal{P}} E^Q [H_\tau] < \infty.$$

For $t \in \mathcal{T}$, we denote by h_t a measurable function from \mathbb{R}^{t+1} into $[0, \infty)$ such that $H_t = h_t(S_0, \dots, S_t)$ a.s.

Then, there exists a minimal hedging strategy $(\hat{x}, \hat{\phi}, \hat{C}) \in \mathcal{A}_H^a$ and its value at time $t \leq T$ is

$$X_t^{\hat{x}, \hat{\phi}, \hat{C}} = \Gamma_t^a h(S_0, \dots, S_t) \quad \mathbb{P}_t - a.s. \quad (2.10)$$

In particular,

$$p^a(H) = \Gamma_0^a h(S_0).$$

For both European and American option, we also get that the optimal portfolio $\hat{\phi}$ is given step by step by the optimal β from (2.6) and (2.9), see [3]

3 Main results

Now we present the result on super-replication when Assumption 2.1 is satisfied. First we prove the following lemma which shows that it is only necessary to super-replicate a convex function at the boundary of the support.

Lemma 3.1. *Let Assumption 2.1 hold true. For any convex function $v : \mathbb{R}^{t+2} \rightarrow \mathbb{R}$,*

$$I_v(s_0, \dots, s_t) = \{(\alpha, \beta) \in \mathbb{R}^2 \mid \alpha + \beta s_t x \geq v(s_0, \dots, s_t, s_t x) \text{ for } x \in \{d_{t+1}, u_{t+1}\}\}.$$

Proof. Fix some $s_0, \dots, s_t \in \mathbb{R}$. We prove the first inclusion \supseteq . By Assumption 2.1 and more precisely by its consequence (2.1), we have that for \mathbb{P} -almost every $\omega \in \Omega$ there exists $\lambda(\omega) \in [0, 1]$ such that $S_{t+1}(\omega) = \lambda(\omega)s_t d_{t+1} + (1 - \lambda(\omega))s_t u_{t+1}$. Now, let $(\alpha, \beta) \in \mathbb{R}^2$ such that $\alpha + \beta s_t x \geq v(s_0, \dots, s_t, s_t x)$ for $x \in \{d_{t+1}, u_{t+1}\}$. By convexity of v , we get that

$$\begin{aligned} v(s_0, \dots, s_t, S_{t+1}(\omega)) &\leq \lambda(\omega)v(s_0, \dots, s_t, s_t d_{t+1}) + (1 - \lambda(\omega))v(s_0, \dots, s_t, s_t u_{t+1}) \\ &\leq \alpha + \beta(\lambda(\omega)s_t d_{t+1} + (1 - \lambda(\omega))s_t u_{t+1}) = \alpha + \beta S_{t+1}(\omega). \end{aligned}$$

So we deduce that

$$\mathbb{P}(\alpha + \beta S_{t+1} \geq v(s_0, \dots, s_t, S_{t+1}) \mid S_0 = s_0, \dots, S_t = s_t) = 1.$$

For the reverse inclusion \subseteq , we argue by contradiction. Assume that for some $(\alpha, \beta) \in I_v(s_0, \dots, s_t)$, $\alpha + \beta s_t d_{t+1} < v(s_0, \dots, s_t, s_t d_{t+1})$. Then as v is a convex function on \mathbb{R}^{t+2} , it is also continuous and there exists ε such that for all $x \in B(s_t d_{t+1}, \varepsilon)$, $\alpha + \beta x < v(s_0, \dots, s_t, x)$. From (2.2), we get that

$$\mathbb{P}(\alpha + \beta S_{t+1} < v(s_0, \dots, s_t, S_{t+1}) \mid S_0 = s_0, \dots, S_t = s_t) > 0$$

The case $\alpha + \beta s_t u_{t+1} < v(s_0, \dots, s_t, s_t u_{t+1})$ works similarly. \square

Remark 3.2. Assume that $v : \mathbb{R}^{t+2} \rightarrow \mathbb{R}$ is not convex but that there exists a convex function $\hat{v} : \mathbb{R}^{t+2} \rightarrow \mathbb{R}$ such that $\hat{v} \geq v$. Then it is easy to see that

$$\{(\alpha, \beta) \in \mathbb{R}^2 \mid \alpha + \beta s_t x \geq \hat{v}(s_0, \dots, s_t, s_t x) \text{ for } x \in \{d_{t+1}, u_{t+1}\}\} \subseteq I_v(s_0, \dots, s_t),$$

but the reverse inclusion does not hold true.

We are now able to compute explicitly the operators Γ_t defined in (2.6) and (2.9). We begin by the European case.

Proposition 3.3. *Let $H = h(S_0, \dots, S_T)$ be an European contingent claim, for some convex, measurable function $h : \mathbb{R}^{T+1} \rightarrow \mathbb{R}$. Let Assumption 2.1 hold true. Then*

$$\begin{aligned} \Gamma_T^e h(s_0, \dots, s_T) &= h(s_0, \dots, s_T) \\ \Gamma_t^e h(s_0, \dots, s_t) &= \pi_{t+1} \Gamma_{t+1}^e h(s_0, \dots, s_t, s_t u_{t+1}) + \\ &\quad (1 - \pi_{t+1}) \Gamma_{t+1}^e h(s_0, \dots, s_t, s_t d_{t+1}), \quad t = 0, \dots, T-1, \end{aligned} \tag{3.11}$$

where $\pi_t := \frac{1-d_t}{u_t-d_t}$, $t = 1, \dots, T$.

Proof. The result is proved by induction. By (2.5), the result is true for T and by assumption $\Gamma_T^e h$ is convex. Now assume that $\Gamma_t^e h(s_0, \dots, s_t) = \pi_{t+1} \Gamma_{t+1}^e h(s_0, \dots, s_t, s_t u_{t+1}) + (1 - \pi_{t+1}) \Gamma_{t+1}^e h(s_0, \dots, s_t, s_t d_{t+1})$ and $\Gamma_{t+1}^e h$ is convex for some $t \in \{1, \dots, T-1\}$. First, it is straightforward that $\Gamma_t^e h$ is a convex function from \mathbb{R}^{t+1} into \mathbb{R} ($0 \leq \pi_{t+1} \leq 1$). From Lemma 3.1,

$$\begin{aligned} \Gamma_{t-1}^e h(s_0, \dots, s_{t-1}) &= \operatorname{ess} \inf_{(\alpha, \beta) \in \mathbb{R}^2 \mid \alpha + \beta s_{t-1} x \geq \Gamma_t^e h(s_0, \dots, s_{t-1}, s_{t-1} x), x \in \{d_t, u_t\}} (\alpha + \beta s_{t-1}) \\ &= \bar{\alpha} + \bar{\beta} s_{t-1} \end{aligned}$$

where $(\bar{\alpha}, \bar{\beta})$ are the unique (α, β) satisfying $\alpha + \beta s_{t-1} x = \Gamma_t^e h(s_0, \dots, s_{t-1}, s_{t-1} x)$ for $x = d_t$ and u_t , i.e.

$$\begin{cases} \bar{\alpha} = \frac{u_t \Gamma_t^e h(s_0, \dots, s_{t-1}, s_{t-1} d_t) - d_t \Gamma_t^e h(s_0, \dots, s_{t-1}, s_{t-1} u_t)}{u_t - d_t} \\ \bar{\beta} = \frac{\Gamma_t^e h(s_0, \dots, s_{t-1}, s_{t-1} u_t) - \Gamma_t^e h(s_0, \dots, s_{t-1}, s_{t-1} d_t)}{s_{t-1}(u_t - d_t)} \end{cases}$$

Thus, we obtain (3.11) for $t-1$ and the induction step follows. \square

We introduce a Cox-Ross-Rubinstein model (see [7]) and define the new risky asset S^{crr} by

$$\begin{aligned} S_0^{crr} &= S_0, \\ S_{t+1}^{crr} &= S_t^{crr} U_{t+1}^{crr}, \quad \text{for } t \in \{0, \dots, T-1\} \end{aligned}$$

where the random variables U_t^{crr} are defined on the probability space $(\Omega^{crr}, \mathcal{P}(\Omega^{crr}))$ by

$$U_t^{crr} : \Omega^{crr} := \prod_{t=1}^T \{u_t, d_t\} \longrightarrow \{u_t, d_t\}$$

We set $F^{crr} = \{\mathcal{F}_t^{crr} \mid t \in \mathcal{T}\}$ with $\mathcal{F}_0 = \{\emptyset, \Omega^{crr}\}$, $\mathcal{F}_t^{crr} = \sigma(U_1^{crr}, \dots, U_t^{crr})$, for $t \in \{1, \dots, T\}$ and we define the probability \mathbb{P}^{crr} on $\mathcal{P}(\Omega^{crr})$ by

$$\mathbb{P}^{crr}(\omega) = \prod_{t=1}^T (\mathbb{1}_{\omega_t = u_t} \pi_t + \mathbb{1}_{\omega_t = d_t} (1 - \pi_t)).$$

Proposition 3.3 can thus be rewritten as follows.

Proposition 3.4. *Under the assumptions of Proposition 3.3, for $t = 0, \dots, T$ we have*

$$\Gamma_t^e h(S_0^{crr}, \dots, S_t^{crr}) = \mathbb{E}^{crr}(h(S_0^{crr}, \dots, S_T^{crr}) \mid \mathcal{F}_t^{crr}). \quad (3.12)$$

The super-replication price of $H = h(S_0, \dots, S_T)$ is thus the replication price of $h(S_0^{crr}, \dots, S_T^{crr})$ in the Cox-Ross-Rubinstein model defined above.

Remark 3.5. Using Remark 3.2 and calling \hat{h} the smallest convex function from \mathbb{R}^{T+1} into \mathbb{R} such that $\hat{h} \geq h$, we can easily prove that

$$\Gamma_t^e h(S_0^{crr}, \dots, S_t^{crr}) \leq \mathbb{E}^{crr}(\hat{h}(S_0^{crr}, \dots, S_T^{crr}) \mid \mathcal{F}_t^{crr})$$

Remark 3.6. Now we want to see what happens when conditionally to \mathcal{F}_t , the support of the distribution of $\frac{S_{t+1}}{S_t}$ is \mathbb{R}^+ . To do that we assume that $d_1 = \dots = d_T = 0$ and $u = u_1 = \dots = u_T$ goes to ∞ . Let us fix $t = T - 1$. If $d_1 = \dots = d_T = 0$, then

$$\Gamma_{T-1}^e h(s_0, \dots, s_{T-1}) = h(s_0, \dots, s_{T-1}, 0) + \frac{h(s_0, \dots, s_{T-1}, S_{T-1}u) - h(s_0, \dots, s_{T-1}, 0)}{u}$$

If $u \rightarrow \infty$, then

$$\Gamma_{T-1}^e h(s_0, \dots, s_{T-1}) = h(s_0, \dots, s_{T-1}, 0) + \lim_{u \rightarrow \infty} \frac{h(s_0, \dots, s_{T-1}, S_{T-1}u)}{u}$$

For $h(s_0, \dots, s_T) = (s_T - K)^+$, $\Gamma_{T-1}^e h(s_0, \dots, s_{T-1}) = S_{T-1}$, while for $h(s_0, \dots, s_T) = (K - s_T)^+$, $\Gamma_{T-1}^e h(s_0, \dots, s_{T-1}) = K$. Thus, we rekind results already present in Carassus-Gobet-Temam [3].

Remark 3.7. A similar result was found in Scagnellato-Vargiolu [15] for a convex payoff h depending only of the last date in a trinomial model, where it was proved that the superreplication capital was given by the CRR price of a binomial model obtained by eliminating the “middle” branch. The proof used the characterisation of the marginals of the price process as convex combinations of two binomial models. Notice that here the result holds in wider generality, as we are only assuming that the convex hull of the support of the conditionnal law of $\frac{S_{t+1}}{S_t}$ is $[d_{t+1}, u_{t+1}]$, and it seems difficult to prove this result with the techniques of [15], especially when the law of $\frac{S_{t+1}}{S_t}$ is absolutely continuous w.r. to the Lebesgue measure.

We now turn our attention to the American case.

Proposition 3.8. *Let $H = (h_t(S_0, \dots, S_t))_{t \in \mathcal{T}}$ be an American contingent claim, where h_t are measurable and convex functions from \mathbb{R}^{t+1} into $[0, \infty)$. Let Assumption 2.1 hold true. Then*

$$\begin{aligned} \Gamma_T^a h(s_0, \dots, s_T) &= h_T(s_0, \dots, s_T), \\ \Gamma_t^a h(S_0^{crr}, \dots, S_t^{crr}) &= \mathbb{E}^{crr} (\Gamma_{t+1}^a h(S_0^{crr}, \dots, S_{t+1}^{crr}) | \mathcal{F}_t^{crr}) \vee h_t(S_0^{crr}, \dots, S_t^{crr}). \end{aligned} \tag{3.13}$$

Proof. The proof is similar to the one of Proposition 3.3 as the convexity of the operator is preserved since we consider at each time step the maximum of convex functions. \square

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