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► **To cite this version:**

Franck Assous, Marie Kray, Frédéric Nataf, Eli Turkel. Time Reversed Absorbing Conditions. 2010. hal-00511298

**HAL Id: hal-00511298**

**<https://hal.science/hal-00511298>**

Preprint submitted on 24 Aug 2010

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# Time Reversed Absorbing Conditions

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Received \*\*\*\*\*, accepted after revision +++++

Presented by xxxxx

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## Abstract

The aim of this paper is to introduce the time reversed absorbing conditions (*TRAC*) in time reversal methods. These new boundary conditions enable one to “recreate the past” without knowing the source which has emitted the signals that are back-propagated. This new method does not rely on any *a priori* knowledge of the physical properties of the inclusion. We prove an energy estimate for the resulting non-standard boundary value problem. Two applications to inverse problems are given.

*To cite this article : F. Assous, M. Kray, F. Nataf, E. Turkel, C. R. Acad. Sci. Paris, Ser. I +++++ (+++++).*

## Résumé

### Conditions aux limites absorbantes retournées temporellement.

Le but de ce papier est d’introduire les conditions absorbantes retournées temporellement (*TRAC*) dans les méthodes de retournement temporel. Elles rendent possible la “reconstruction du passé” sans connaître la source émettrice des signaux enregistrés puis rétropropagés. Cette nouvelle méthode ne nécessite pas de connaissance *a priori* des propriétés physiques de l’inclusion. Nous démontrons une estimation d’énergie pour le problème non standard obtenu. Deux applications aux problèmes inverses sont proposées.

*Pour citer cet article : F. Assous, M. Kray, F. Nataf, E. Turkel, C. R. Acad. Sci. Paris, Ser. I +++++ (+++++).*

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Version française abrégée

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*Preprint submitted to the Académie des sciences*

*29 juillet 2010*

Depuis les premiers travaux de M. Fink et al [8], le retournement temporel est un sujet très actif de recherche. L'idée principale est de tirer parti de la réversibilité des phénomènes de propagation d'ondes, tels que l'acoustique, l'élasticité ou l'électromagnétisme, dans un milieu inconnu non dissipatif. On rétropropage alors les signaux vers les sources qui les ont émis. L'expérience initiale, cf. [8], consistait à refocaliser très précisément un signal enregistré ayant traversé un ensemble de barres métalliques disposées aléatoirement. L'élément remarquable est que le signal focalise précisément après être passé à nouveau à travers les barres métalliques aléatoirement disposées, bien que l'on ne connaisse pas la position de la source. Des applications numériques sont présentées dans [7] ainsi que dans ses références. De premières analyses mathématiques se trouvent dans [2] et [5].

On peut aussi "reconstruire le passé" à partir de mesures prises sur le bord retournées temporellement. Comme il est montré expérimentalement dans [6], il est nécessaire de connaître la source émettrice afin d'éviter un phénomène de diffraction limite. La même difficulté est encore mise en avant dans [10] lors de simulations numériques de séismes par retournement temporel des sismogrammes.

La nouveauté de cet article est une méthode permettant de reconstruire le passé, sans connaître la source, en introduisant des conditions aux limites absorbantes retournées en temps (*TRAC*). Le principe de la méthode *TRAC* est le suivant : considérons une onde incidente qui arrive sur une inclusion  $D$  dans  $\mathbb{R}^3$ . L'inclusion est caractérisée par des propriétés physiques différentes de celles du milieu ambiant supposé homogène,  $c = c_0$  dans  $\mathbb{R}^3 \setminus D$ . Le champ total se décompose alors en un champ incident et un champ diffracté, soit  $u^T := u^I + u^S$ . Le problème satisfait par  $u^T$  est (1) et l'onde diffracté satisfait une condition de radiation de Sommerfeld à l'infini. Soit  $\Gamma_R$  une surface délimitant le domaine  $\Omega$  et englobant l'inclusion  $D$  et sur laquelle le champ total est enregistré jusqu'à un temps  $T_f$ . Après  $T_f$  le signal s'annule dans  $\Omega$ . Notons  $u_R^T := u^T(T_f - t, \mathbf{x})$  le champ total retourné temporellement qui satisfait également (1). Nous utiliserons des notations similaires pour les champ incident et diffracté.

Le but est de reconstruire le champ  $u_R^T$  à partir des mesures enregistrées sur  $\Gamma_R$ . Pour cela, nous introduisons un problème aux limites (*BVP*) dont  $u_R^T$  est la solution. Nous ne connaissons ni les propriétés physiques ni la position de l'inclusion  $D$ , seulement les caractéristiques physiques du milieu ambiant. Par conséquent, nous introduisons un sous-domaine  $B$  entourant  $D$  (voir Figure 1). La difficulté est de déterminer la condition aux limites à imposer sur  $\partial B$  afin que la solution du nouveau problème coïncide avec  $u_R^T$  dans  $\Omega \setminus B$ . Pour cela, nous utilisons le fait que le champ diffracté  $u^S$  vérifie (1) dans  $\Omega \setminus D$  et une condition de Sommerfeld à l'infini. Ainsi  $u^S$  satisfait une condition absorbante sur  $\partial B$  que nous approchons par (2). En retournant en temps cette condition, nous obtenons la condition absorbante retournée en temps *TRAC* (3), ce qui nous donne le *BVP* final (4) pour  $u_R^T$ .

La *TRAC* n'est plus la condition absorbante standard mais contient un terme anti-absorbant. Se pose alors la question du caractère bien posé du problème (4). Pour cela, nous démontrons une estimation d'énergie (6) pour une géométrie particulière, voir la proposition 3.1.

Une application est de localiser l'inclusion  $D$  par essais successifs. A l'instant initial,  $u^T$  vaut zéro. Donc, si  $B$  entoure  $D$ ,  $u_R^T$  doit s'annuler aussi au temps final  $T_f$  qui correspond au temps initial du problème physique. De même, si en résolvant (4)  $u_R^T$  est non nul en  $T_f$ , on en déduit que  $D$  n'est pas inclus dans  $B$ . Des résultats numériques générés avec le logiciel `FREEFEM++` [9] illustrent la méthode, voir figure 2.

En conclusion, nous avons introduit la méthode *TRAC* qui permet de "recréer le passé" à partir de signaux enregistrés rétropropagés, sans connaître la source émettrice. Deux applications en problèmes inverses sont possibles : la reconstruction de la forme de l'inclusion à partir des mesures au bord et le principe de *redatuming*, voir [4]. Des tests sur les propriétés physiques de l'inclusion ont été effectués dans [1] pour l'équation des ondes et l'équation de Helmholtz. La méthode se révèle particulièrement robuste au bruit sur les données enregistrées.

## 1. Introduction

Since the seminal paper by Fink et al. [8], time reversal is a subject of active research. The main idea is to take advantage of the reversibility of wave propagation phenomena such as occurs in acoustics, elasticity or electromagnetism in a non dissipative unknown medium to back-propagate signals to the sources that emitted them. The initial experiment, see [8], was to refocus very precisely a recorded signal after passing through a barrier consisting of randomly distributed metal rods. The remarkable feature of this experiment is the concrete possibility to focus precisely a signal after it has crossed random barriers and even without knowing its location. There have been numerous applications of this physical principle, see [7] and references therein. First mathematical analyses can be found in [2] and [5].

An interesting possibility is to “recreate the past” in a medium from time-reversed boundary measurements. As shown experimentally in [6], it is necessary to know the source that emitted the signals to overcome the diffraction limit. The same difficulty was pointed out in [10] when numerically studying the initial instants of an earthquake by sending back long period time-reversed seismograms.

In this paper, we introduce a new method that enables one to “recreate the past” without knowing the source which has emitted the signals that will be back-propagated. This is made possible by introducing time reversed absorbing conditions (*TRAC*) at the expense of removing a small region enclosing the source. This technique will have at least two applications in inverse problems : the reduction of size of the computational domain and the determination of the location and volume of an unknown inclusion from boundary measurements.

The outline of the paper is as follows. We first introduce the principle of the *TRAC* method in the time dependent domain (the time-harmonic case is also treated in [1]). We then prove a related energy estimate. Two applications to inverse problems are given.

## 2. Time Reversed Absorbing Conditions

We consider an incident wave  $u^I$  impinging on an inclusion  $D$  characterized by different physical properties from the surrounding medium which is assumed to be homogeneous,  $c = c_0$  in  $\mathbb{R}^3 \setminus D$ . The total field  $u^T$  can be decomposed into the incident and scattered fields, so  $u^T := u^I + u^S$ . We consider the problem in 3 dimensions and assume that the total field satisfies the linear wave equation

$$\frac{\partial^2 u^T}{\partial t^2} - c^2 \Delta u^T = 0 \quad \text{in } \mathbb{R}^3 \quad (1)$$

together with zero initial conditions. The scattered field  $u^S$  satisfies a radiation condition at infinity. Let  $\Gamma_R$  be a surface that defines a bounded domain  $\Omega$  and encloses the inclusion  $D$  (see Figure 1). We assume that the incident wave  $u^I$  has compact support in time and space. After a time  $T_f$  the total field  $u^T$  vanishes in  $\Omega$ . The total field  $u^T$  is recorded on  $\Gamma_R$  on the time interval  $[0, T_f]$ . Let  $u_R^T := u^T(T_f - t, \mathbf{x})$  denote the total time reversed field that satisfies the physical equation (1). Similar definitions will be used for the incident and scattered fields.

Our aim is to reconstruct the time reversed field  $u_R^T$  from the measurements on  $\Gamma_R$ . For this purpose, we derive a boundary value problem (*BVP*) whose solution is  $u_R^T$  in  $\Omega$ . We know neither the physical properties nor the exact location of the inclusion  $D$ . The only things we know are the physical properties of the surrounding medium. Therefore, we introduce  $B$  a subdomain enclosing the inclusion  $D$ . Then, we have to determine a specific boundary condition for  $u_R^T$  on the boundary  $\partial B$  so that the solution to this problem will coincide with  $u_R^T$  in the restricted domain  $\Omega \setminus B$ .

In order to derive this boundary condition, we note that  $u^I$  satisfies (1) with  $c = c_0$  the sound speed of the surrounding medium in  $\mathbb{R}^3$ , i.e. without any inclusion  $D$ , so that the scattered wave  $u^S$  also satisfies (1) but only in  $\mathbb{R}^3 \setminus D$ , with a Sommerfeld radiation condition at infinity with homogeneous initial conditions.

We look for a relation satisfied by  $u^S$  on  $\partial B$ . Absorbing boundary conditions (ABC) e.g. [3] construct accurate approximations to a perfectly absorbing boundary condition. We denote by ABC an absorbing boundary condition, that we choose to be the Bayliss-Turkel first order boundary condition. We take for  $B$  a ball of radius  $\rho$  denoted  $B_\rho$ . Then,

$$\text{ABC}(u^S) := \frac{\partial u^S}{\partial t} + c \frac{\partial u^S}{\partial r} + c \frac{u^S}{r} = 0 \text{ on } \partial B_\rho \quad (2)$$

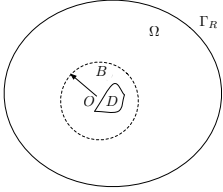
where  $r$  is the radial coordinate with the origin at the center of the ball  $B_\rho$ . Our main ingredient is to “time reverse” this relation : using  $u_R^S(t, \cdot) = u^S(T_f - t, \cdot)$ , we get

$$-\frac{\partial}{\partial t}(u_R^S) + c \frac{\partial}{\partial r}(u_R^S) + c \frac{u_R^S}{r} = 0$$

Note, that on  $\partial B_\rho$ ,  $\partial / \partial r = -\partial / \partial n$  where  $n$  is the outward normal to the restricted domain  $\Omega \setminus B_\rho$ . Multiplying by  $-1$ , we get the time reversed absorbing boundary condition TRAC:

$$\text{TRAC}(u_R^S) := \frac{\partial}{\partial t}(u_R^S(t, \cdot)) + c \frac{\partial}{\partial n}(u_R^S(t, \cdot)) - c \frac{u_R^S(t, \cdot)}{r} = 0. \quad (3)$$

Since  $u_R^T = u_R^I + u_R^S$ , we have  $\text{TRAC}(u_R^T - u_R^I) = 0$  or equivalently  $\text{TRAC}(u_R^T) = \text{TRAC}(u_R^I)$ . The time reversed problem analogous to (1) reads :



$$\begin{cases} \frac{\partial^2 u_R^T}{\partial t^2} - c^2 \Delta u_R^T = 0 & \text{in } (0, T_f) \times \Omega \setminus B_\rho \\ \text{TRAC}(u_R^T) = \text{TRAC}(u_R^I) & \text{on } \partial B_\rho \\ u_R^T(t, \mathbf{x}) = u^T(T_f - t, \mathbf{x}) & \text{on } \Gamma_R \\ \text{zero initial conditions} \end{cases} \quad (4)$$

Figure 1. Geometry – Géométrie

The TRAC is not only not the standard  $BT^1$  ABC but even has an “anti absorbing” term  $(-cu_R^T/r)$ . A natural concern arises about the well-posedness of  $BVP$  (4). Although, we have not developed a general theory, we prove an energy estimate for this problem in a special geometry. Moreover, in many computations we have never encountered stability problems, see [1] where a numerical procedure for inclusion identification is deduced from this formulation. The first two lines of Fig. 2 illustrate the principle of the  $TRAC$  method by comparing the solution to (4) to the perfect time reversed solution.

### 3. Energy estimate

Let  $g$  be a real-valued function on  $\Gamma_R$  and  $u$  satisfy the following equations (see Figure 1):

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0 & \text{in } \Omega \setminus B_\rho \\ \frac{\partial u}{\partial t} + c \left( \frac{\partial u}{\partial n} + \frac{u}{R} \right) = g & \text{on } \Gamma_R \\ \frac{\partial u}{\partial t} + c \left( \frac{\partial u}{\partial n} - \frac{u}{\rho} \right) = 0 & \text{on } \partial B_\rho \end{cases} \quad (5)$$

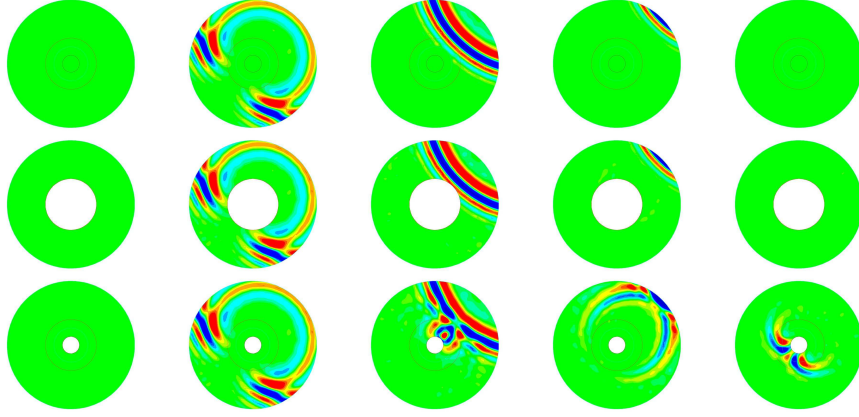


Figure 2. Time reversed solutions snapshots for a soft inclusion  $D$  : perfect time reversed (TR) solution on the upper line, TR solution for  $B_\rho$  enclosing  $D$  in the middle line and TR solution for  $B_\rho$  not enclosing  $D$  on the bottom line (final solution is not zero). The incident signal comes from top-right. – Instantanés de solutions du problème retourné temporellement pour une inclusion molle  $D$  : retournement temporel parfait sur la ligne du haut, avec  $B_\rho$  englobant  $D$  sur la ligne du milieu et avec  $B_\rho$  n’englobant pas  $D$  sur la ligne du bas (la solution finale n’est pas nulle). Le signal incident vient d’en haut à droite.

**Proposition 3.1** : *If  $\Omega$  is a ball of radius  $R$ , we have the following energy estimate written for the BVP (5) in spherical coordinates :*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left( \iiint r^2 \sin \phi \left( \frac{\partial u}{\partial t} \right)^2 + c^2 \sin \phi \left( \frac{\partial r u}{\partial r} \right)^2 + c^2 \sin \phi \left( \frac{\partial u}{\partial \phi} \right)^2 + \frac{c^2}{\sin \phi} \left( \frac{\partial u}{\partial \theta} \right)^2 \right) \\ & + \iint_{r=\rho} c \rho^2 \sin \phi \left( \frac{\partial u}{\partial t} \right)^2 + \iint_{r=R} c R^2 \sin \phi \left( \frac{\partial u}{\partial t} \right)^2 = \iint_{r=R} c R^2 \sin \phi \frac{\partial u}{\partial t} g \end{aligned} \quad (6)$$

**Proof.** The energy estimate is based on an equivalent formulation of the Laplacian in spherical coordinates :

$$\Delta = \frac{1}{r} \frac{\partial^2}{\partial r^2} (r \cdot) + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2 \sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial}{\partial \phi} \right)$$

We multiply (5) by  $\partial u / \partial t$  and integrate by parts (recall the volume element is  $r^2 \sin \phi dr d\theta d\phi$ ). We detail the computation for the term arising from the radial derivative since it is the only non classical one. We have :

$$\begin{aligned} & - \iiint \frac{c^2}{r} \frac{\partial^2}{\partial r^2} (r u) \frac{\partial u}{\partial t} r^2 \sin \phi dr d\theta d\phi \\ & = \iiint c^2 \frac{\partial(r u)}{\partial r} \frac{\partial^2(r u)}{\partial r \partial t} \sin \phi dr d\theta d\phi - \left[ \iint c^2 \frac{\partial(r u)}{\partial t} \frac{\partial(r u)}{\partial r} \sin \phi d\theta d\phi \right]_{r=\rho}^{r=R} \end{aligned}$$

We focus on the boundary term at  $r = \rho$  :

$$\iint_{r=\rho} c^2 \frac{\partial(r u)}{\partial t} \frac{\partial(r u)}{\partial r} \sin \phi d\theta d\phi = \iint_{r=\rho} c \rho \frac{\partial(r u)}{\partial t} c \left( \frac{\partial u}{\partial r} + \frac{u}{\rho} \right) \sin \phi d\theta d\phi$$

Since at  $r = \rho$ ,  $\partial / \partial r = -\partial / \partial n$  we have using (5) :

$$\iint_{r=\rho} c \rho \frac{\partial(r u)}{\partial t} c \left( \frac{\partial u}{\partial r} + \frac{u}{\rho} \right) \sin \phi d\theta d\phi = \iint_{r=\rho} c \rho \frac{\partial(r u)}{\partial t} \frac{\partial u}{\partial t} \sin \phi d\theta d\phi = \iint_{r=\rho} c \rho^2 \left( \frac{\partial u}{\partial t} \right)^2 \sin \phi d\theta d\phi$$

A similar (but different) calculus can be done on the boundary term at  $r = R$  and we find the energy estimate (6). ■

## 4. Applications & Conclusion

In addition to recreate the past, another application is to localize the inclusion  $D$  by a trial and error procedure. At the initial time  $t = 0$ , the total field  $u^T$  is zero. Thus, if  $B$  encloses the inclusion  $D$ ,  $u_R^T$  which is the time reversal of  $u^T$  is zero at the final time  $T_f$  that corresponds to the initial time of the physical problem (1). As a consequence, if after solving equation (3),  $u_R^T$  is not zero at the final time  $T_f$ , it proves that  $D$  is not a subset of  $B$ . Hence, by playing with the location and size of the subdomain  $B$  and checking the nullity of the final solution, it is possible to determine the location and volume of the inclusion  $D$ . Figure 2 illustrates this application. The computation was executed using the software FREEFEM++ [9]. A more general and detailed study can be found in [1].

We have introduced the time reversed absorbing conditions (*TRAC*). They enable one to “recreate the past” without knowing the source which has emitted the signals that are back-propagated. This is made possible at the expense of removing a small region enclosing the source. Two applications in inverse problems are possible:

- (i) the reduction of size of the computational domain by redefining the reference surface on which the receivers appear to be located, see redatuming in [4];
- (ii) the reconstruction of the shape of an unknown inclusion from boundary measurements.

We stress the fact that in contrast to many methods in inverse problems, the *TRAC* method does not rely on any *a priori* knowledge of the physical properties of the inclusion. Hard, soft and penetrable inclusions are treated in the same way. In [1], the feasibility of the method is shown for both time-dependent and harmonic equations (acoustics and electromagnetism). Moreover, the method is shown to be very robust with respect to noise on the data.

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