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Martingale Representations of the Lynden-Bell Estimator with Applications

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Abstract

We derive a martingale representation for the Lynden-Bell estimator F_n and show that F_n fulfills linear upper and lower bounds.

Key words: Truncated data, Lynden-Bell, martingales, linear bounds

1. Introduction And Main Results

Let X and Z be two independent random variables with unknown distribution functions (d.f.'s) F and G , respectively. Under truncation from the right we observe (X, Z) only if $X \leq Z$. Truncation typically creates some dependence between the observed X and Z . Also the distribution function of X becomes

$$F^*(x) = \mathbb{P}(X \leq x | X \leq Z) = \alpha^{-1} \int_{(-\infty, x]} (1 - G(y-)) F(dy),$$

where $\alpha = \mathbb{P}(X \leq Z)$ is unknown but assumed to be positive. Here and in the following, for any function h , we denote with

$$h^-(y) \equiv h(y-) = \lim_{z \uparrow y} h(z) \quad h^+(y) \equiv h(y+) = \lim_{z \downarrow y} h(z)$$

left and right hand limits and with

$$h\{y\} = h(y^+) - h(y^-)$$

the jump size at y . Also, for any distribution function H , we set

$$a_H = \inf\{x : H(x) > 0\} \text{ and } b_H = \inf\{x : H(x) = 1\}.$$

Given a sample (X_i, Z_i) , $1 \leq i \leq n$, of truncated replicates of (X, Z) , the goal then is to reconstruct F from the observed data. Write

$$\Lambda(x) = \int_{[x, \infty)} \frac{dF}{F}$$

for the cumulative hazard function associated with F , and set

$$C(x) = \mathbb{P}(X \leq x \leq Z | X \leq Z).$$

The function C is crucial when analyzing truncated data since, when $b_F \leq b_G$,

$$\Lambda(x) = \int_{[x, \infty)} \frac{dF^*}{C} \quad (1)$$

and F^* and C are readily estimable through

$$F_n^*(x) = n^{-1} \sum_{i=1}^n 1_{\{X_i \leq x\}} \text{ and } C_n(x) = n^{-1} \sum_{i=1}^n 1_{\{X_i \leq x \leq Z_i\}}.$$

Plugging these into (1) yields the estimator of Λ ,

$$\Lambda_n(x) = \int_{[x, \infty)} \frac{dF_n^*}{C_n} = \sum_{i=1}^n \frac{1_{\{X_i \geq x\}}}{nC_n(X_i)}.$$

The product-limit formula finally leads to the time honoured Lynden-Bell (1971) estimator of F which, if there are no ties among the X 's, equals

$$F_n(t) = \prod_{y>t} [1 + \Lambda_n\{y\}] = \prod_{X_i>t} \left[1 - \frac{1}{nC_n(X_i)} \right].$$

Stute and Wang (2008) showed how to break ties without destroying the product limit structure. Therefore, in this paper, we shall assume w.l.o.g. that there are no ties among the X 's. Note that F_n reduces to the classical

empirical d.f. when there is no truncation. In such a situation F_n^-/F^- is a martingale in reverse time for $t > a_F$.

It is the purpose of this paper to prove an analog and discuss some consequences, when the data are truncated.

A basic role in the analysis of F_n will be played by the process

$$H_n^1(t) = n^{-1} \sum_{i=1}^n 1_{\{t \leq X_i \leq Z_i\}}.$$

H_n^1 and C_n are adapted to the filtration

$$\mathcal{G}_n(t) = \sigma(\{X_i < s \leq Z_i\}, \{s \leq X_i \leq Z_i\} : t \leq s, 1 \leq i \leq n),$$

which is nondecreasing in reverse time. The process C_n is neither left- nor right-continuous. If we consider C_n^+ , the right-continuous version, we obtain a function which is predictable in reverse time. The martingale in the Doob-Meyer decomposition of H_n^1 becomes, when F and G have no jumps in common,

$$M_n(t) = H_n^1(t) - \int_{[t, \infty)} \frac{C_n(u+)}{F(u)} F(du).$$

See, e.g., Mandrekar and Thelen (1990) and Keiding and Gill (1990). The “no-common jump” condition will be assumed throughout this paper without further mentioning. Separate discontinuities will, however, be allowed.

Now, on the set $\{t : C_n(t+) > 0\}$, we obtain

$$\frac{dM_n}{C_n^+} = \frac{dH_n^1}{C_n^+} + \frac{dF}{F} = d\Lambda_n - d\Lambda,$$

upon noting that the function H_n^1 has jumps of size $-\frac{1}{n}$ at the X_i and the function C_n satisfies $C_n(X_i+) = C_n(X_i)$.

Since on the support of H_n^1 the function C_n^+ is positive we therefore obtain

$$\frac{1_{\{C_n^+ > 0\}}}{C_n^+} dM_n = d\Lambda_n - 1_{\{C_n^+ > 0\}} d\Lambda \equiv d\Lambda_n - d\hat{\Lambda}_0.$$

The hazard measure

$$\hat{\Lambda}_0(dt) = 1_{\{C_n(t+) > 0\}} \Lambda(dt)$$

is random and has distribution function

$$\hat{F}_0(t) = \prod_{s>t} [1 + \hat{\Lambda}_0\{s\}] e^{-\hat{\Lambda}_0^c(t)}.$$

Here $\hat{\Lambda}_0^c$ is the continuous part of $\hat{\Lambda}_0$. The function \hat{F}_0 will be required for a first martingale representation of F_n .

For this, since

$$\begin{aligned} \hat{\Lambda}_0(ds) &= 1_{\{C_n(s+) > 0\}} \Lambda(ds) = -1_{\{C_n(s+) > 0\}} \frac{F(ds)}{F(s)} \\ &\geq -\frac{F(ds)}{F(s)} = \Lambda(ds) \end{aligned}$$

we have

$$\begin{aligned} \hat{F}_0(t-) &= \prod_{s \geq t} [1 + \hat{\Lambda}_0\{s\}] e^{-\hat{\Lambda}_0^c(t)} \\ &= \prod_{s \geq t} [1 + 1_{\{C_n(s+) > 0\}} \Lambda\{s\}] e^{-\hat{\Lambda}_0^c(t)} \\ &= \prod_{s \geq t, C_n(s+) > 0} [1 + \Lambda\{s\}] e^{-\hat{\Lambda}_0^c(t)} \geq \prod_{s \geq t} [1 + \Lambda\{s\}] e^{-\Lambda^c(t)} = F(t-), \end{aligned} \tag{2}$$

where the second but last inequality follows from $0 \leq 1 + \Lambda\{s\} \leq 1$. Hence $a_{\hat{F}_0} \leq a_F$ and, consequently, $\hat{F}_0(t-) > 0$ for every $t > a_F$. Hence the process

$$t \rightarrow \frac{F_n(t-)}{\hat{F}_0(t-)} \tag{3}$$

is well-defined on $t > a_F$.

From Gill's lemma, see Lemma 3, we obtain

$$\frac{F_n(t-)}{\hat{F}_0(t-)} = 1 + \int_{[t, \infty)} \frac{F_n(s)}{[1 + \hat{\Lambda}_0\{s\}] \hat{F}_0(s) C_n(s+)} M_n(ds).$$

Since M_n is a martingale and the integrand is continuous from the right and hence predictable in reverse time, the process in (3) is a martingale.

When the X -data are not at risk of being truncated, \hat{F}_0^- coincides with F^- on the set $t \geq \min X_i$ so that the ratio becomes F_n^-/F^- . It is interesting to note that for censored data, i.e., for the Kaplan-Meier estimator, the indicator defining the corresponding $\hat{\Lambda}_0$ is also non-vanishing there so that \hat{F}_0^- and F^- coincide. As a conclusion we obtain the martingale property for the Kaplan-Meier estimator with the true F in the denominator. See Gill (1980) and Shorack and Wellner (1986). For truncated data, due to the non-monotonicity of C_n , a simple replacement of \hat{F}_0^- by F^- is not possible. Rather, we need to introduce

$$T = T_n = \max \{X_i : nC_n(X_i) = 1\}.$$

Note that $nC_n(X) = 1$ when X is the smallest among the X_i 's and $F_n(t)$ equals zero for $t < T$. For further analysis, the following property of T turns out to be useful.

Lemma 1. *T is a stopping time w.r.t. the filtration \mathcal{G}_n .*

We shall also need the following representation of T which connects T with $C_n(t+)$ but with t not necessarily belonging to the X -sample.

Lemma 2. *We have*

$$T = \sup \{t < \max_{1 \leq i \leq n} Z_i : C_n(t+) = 0\}. \quad (4)$$

The proofs of Lemma 1 and 2 will be postponed to the Appendix.

If we stop the reverse martingale at T , Lemma 1 may be applied to show that the process

$$Z(t) = \frac{F_n(t \vee T-)}{\hat{F}_0(t \vee T-)}$$

is also a reverse martingale. The process Z is part of the following representation of F_n^-/F^-

Theorem 1. *We have, for $t > a_F$,*

$$\frac{F_n(t-)}{F(t-)} = Z(t) \frac{1}{F(\tilde{T} \vee t-)},$$

where $\tilde{T} = \max_{1 \leq i \leq n} Z_i$.

Proof. For $t \leq T$, both sides vanish. For $T < t < \tilde{T}$, we may proceed as for (2) to get

$$\begin{aligned} \frac{F_n(t-)}{F(t-)} &= Z(t) \frac{\hat{F}_0(t-)}{F(t-)} = \frac{Z(t)}{F(t-)} \prod_{s \geq t, C_n(s+) > 0} [1 + \Lambda\{s\}] e^{-\hat{\Lambda}_0^c(t)} \\ &= \frac{Z(t)}{F(t-)} \prod_{\tilde{T} > s \geq t} [1 + \Lambda\{s\}] e^{\int_{[t, \tilde{T})} d\Lambda^c} = \frac{Z(t)}{F(t-)} \frac{F(t-)}{F(\tilde{T})} = \frac{Z(t)}{F(\tilde{T})}, \end{aligned}$$

where the third equality follows from Lemma 2. Finally, for $t \geq \tilde{T}$, $F_n(t-) = \hat{F}_0(t-) = 1$ so that the conclusion follows. \square

The case of the simple empirical d.f. may be recovered from Theorem 1 if we formally set $Z_i = \infty$. Hence $F(\tilde{T} \vee t-) = 1$ for all t and $T = \min X_i$. Since $\hat{F}_0 = F$ on $t \geq T$ we obtain the aforementioned fact that for the empirical d.f. the process F_n^-/F^- is a reverse martingale. Under truncation the additional term $F(\tilde{T} \vee t-)$ may be less than one which destroys the martingale structure. Interestingly enough the next Corollary shows that some weaker martingale structure is still obtained.

Corollary 1. F_n^-/F^- is a nonnegative reverse sub-martingale.

Proof. Fix $t < s$. Since the process $t \rightarrow F(\tilde{T} \vee t-)$ is adapted we have, by monotonicity of the denominator,

$$\begin{aligned} \mathbb{E} \left[\frac{F_n(t-)}{F(t-)} \middle| \mathcal{G}_n(s) \right] &= \mathbb{E} \left[\frac{Z(t)}{F(\tilde{T} \vee t-)} \middle| \mathcal{G}_n(s) \right] \\ &\geq \frac{\mathbb{E}[Z(t) | \mathcal{G}_n(s)]}{F(\tilde{T} \vee s-)} = \frac{Z(s)}{F(\tilde{T} \vee s-)}. \end{aligned}$$

\square

2. Linear Bounds For The Lynden-Bell Estimator

In this section we derive so-called linear bounds for F_n . Such bounds have found a lot of interest, for the classical empirical d.f. and the Kaplan-Meier estimator, see Shorack and Wellner (1986).

Theorem 2. We have, for any $\lambda > 0$,

$$\mathbb{P} \left(\sup_{t > a_F} \frac{F_n(t-)}{F(t-)} \geq \lambda \right) \leq \left(\frac{1}{\lambda \alpha} \right)^{\frac{n}{n+1}} \left[n^{\frac{1}{n+1}} + n^{-\frac{n}{n+1}} \right] \leq 2 \left(\frac{1}{\lambda \alpha} \right)^{\frac{n}{n+1}}.$$

For the classical empirical d.f., under continuity of F and for $\lambda \geq 1$, the left-hand side equals λ^{-1} . See Daniels (1945). For truncated data the factor α needs to be included to take care of truncation effects. Finally, we may write

$$\left(\frac{1}{\lambda\alpha}\right)^{\frac{n}{n+1}} = \frac{1}{\lambda\alpha}(\lambda\alpha)^{\frac{1}{n+1}}$$

so that the right side is also $O(\lambda^{-1})$ uniformly in λ for λ varying in a bounded set or tending to infinity such that $\lambda^{\frac{1}{n+1}}$ remains bounded, as $n \rightarrow \infty$. In particular, we obtain that the ratio is uniformly bounded in probability:

$$\sup_{t > a_F} \frac{F_n(t-)}{F(t-)} = O_{\mathbb{P}}(1). \quad (5)$$

Proof. For any $a > a_F$ and positive c we have from Theorem 1

$$\begin{aligned} & \mathbb{P}\left(\sup_{t > a} \frac{F_n(t-)}{F(t-)} \geq \lambda\right) \\ & \leq \mathbb{P}\left(\sup_{t > a} Z(t) \geq c\lambda\right) + \mathbb{P}(F(\tilde{T}) \leq c) \leq \frac{1}{c\lambda} + \mathbb{P}^n(F(Z_1) \leq c), \end{aligned} \quad (6)$$

where the first part of the last inequality follows from the Doob-maximal inequality for $Z(t)$ and the second is a consequence of the monotonicity of F and the independence of the Z_i 's.

To proceed, let G^* be the d.f. of the actually observed Z 's. Then

$$G^*(x) = \mathbb{P}(Z \leq x | X \leq Z) = \alpha^{-1} \mathbb{P}(Z \leq x, X \leq Z) \leq \alpha^{-1} F(x).$$

Hence we obtain

$$\mathbb{P}^n(F(Z_1) \leq c) \leq \mathbb{P}^n\left(G^*(Z_1) \leq \frac{c}{\alpha}\right) \leq \left(\frac{c}{\alpha}\right)^n.$$

In summary, the right side of (6) is bounded from above by

$$\frac{1}{c\lambda} + \alpha^{-n} c^n \equiv f(c).$$

Finally, the function $f(c)$ attains its minimum at

$$c^* = \left(\frac{\alpha^n}{\lambda n}\right)^{\frac{1}{n+1}}$$

with

$$f(c^*) = \left(\frac{1}{\lambda\alpha}\right)^{\frac{n}{n+1}} \left[n^{\frac{1}{n+1}} + n^{-\frac{n}{n+1}} \right].$$

Since the bound in (6) does not depend on a , we may let a go to a_F to complete the proof. The second bound follows from the fact that [...] equals 2 for $n = 1$ and is nonincreasing in n . \square

Next we study the ratio F/F_n . It is only defined for $t \geq T$ since the denominator vanishes for $t < T$. As it will turn out F/F_n is closely related to $(1 - G_n)/(1 - G)$ where

$$1 - G_n(t) = \prod_{Z_j \leq t} \left[1 - \frac{1}{nC_n(Z_j)} \right]$$

is the Lynden-Bell estimator for the survival function $1 - G$ of the left-truncated Z 's. Theorem 1 appropriately modified yields a representation of $(1 - G_n)/(1 - G)$ as a forward martingale, where

$$T_n^1 = \inf\{Z_j : nC_n(Z_j) = 1\}$$

plays the same role for the Z 's as T_n played for the X 's. Another important quantity in this context is the unknown probability α . Note that

$$\frac{F(t)(1 - G(t-))}{C(t)} =: \alpha(t) \quad (7)$$

equals α for all $a_F < t < b_F$. This observation led He and Yang (1998) to propose

$$\hat{\alpha}_n(t) = \frac{F_n(t)(1 - G_n(t-))}{C_n(t)} \quad (8)$$

as an estimator for α . Unfortunately, the numerator of $\hat{\alpha}_n$ vanishes everywhere if $T_n^1 < T_n$. This again holds if $T_n > \min_{1 \leq i \leq n} X_i$, i.e., if there exists a so-called hole or inner risk set among the X 's. In order to justify $\hat{\alpha}_n$ one therefore needs a careful study of the possibility of holes. In Strzalkowska-Kominiak and Stute (2008) it was shown that $T_n = \min_{1 \leq i \leq n} X_i$ with probability tending to one, as $n \rightarrow \infty$. Similarly, $T_n^1 = \max_{1 \leq i \leq n} Z_i$ with probability tending to one, as $n \rightarrow \infty$.

Hence, setting

$$\Omega_1^{(n)} = \left\{ T_n = \min_{1 \leq i \leq n} X_i \text{ and } T_n^1 = \max_{1 \leq i \leq n} Z_i \right\}$$

we get $\mathbb{P}(\Omega_1^{(n)}) \rightarrow 1$ as $n \rightarrow \infty$. Moreover, on $\Omega_1^{(n)}$, $\hat{\alpha}_n(t) = \hat{\alpha}$ is a (random) constant for $T_n < t < T_n^1$. See He and Yang (1998).

We are now in a position to formulate our next main result.

Theorem 3. *Assume $a_F \leq a_G$ and $b_F \leq b_G$. Then we have*

$$\sup_{T_n \leq t} \frac{F(t)}{F_n(t)} = O_{\mathbb{P}}(1) \text{ as } n \rightarrow \infty.$$

Moreover, $T_n \rightarrow a_F$ in probability.

Proof. In view of $\mathbb{P}(\Omega_1^{(n)}) \rightarrow 1$ it suffices to study the ratio on $\Omega_1^{(n)}$. We then have

$$\frac{F(t)}{F_n(t)} = \frac{\alpha(t)}{\hat{\alpha}(t)} \frac{C(t)}{C_n(t)} \frac{1 - G_n(t-)}{1 - G(t-)}.$$

Denote with $X_{1:n} < X_{2:n} < \dots < X_{n:n}$ the ordered X -data. Since $F_n(t) = 1$ for $t \geq X_{n:n}$ so that $F(t)/F_n(t) \leq 1$ there, it suffices to consider t 's such that $X_{i:n} \leq t < X_{i+1:n}$ for some $i = 1, \dots, n-1$. In such a situation, we have, in view of (7) and (8),

$$\frac{F(t)}{F_n(t)} \leq \frac{F(X_{i+1:n})}{F_n(X_{i:n})} = \frac{\alpha(X_{i+1:n})}{\hat{\alpha}(X_{i+1:n})} \frac{C(X_{i+1:n})}{C_n(X_{i+1:n})} \frac{1 - G_n(X_{i+1:n}-)}{1 - G(X_{i+1:n}-)} \frac{F_n(X_{i+1:n})}{F_n(X_{i:n})}.$$

The first ratio is the same for each i and converges to 1, see He and Yang (1998). The ratio C/C_n is also bounded in probability, see Stute and Wang (2008). The ratio $(1 - G_n^-)/(1 - G^-)$ is bounded according to Theorem 2, applied to the left-truncated Z 's. Finally,

$$\frac{F_n(X_{i+1:n})}{F_n(X_{i:n})} = \frac{1}{1 - \frac{1}{nC_n(X_{i+1:n})}} = \frac{nC_n(X_{i+1:n})}{nC_n(X_{i+1:n}) - 1}. \quad (9)$$

Since on $\Omega_1^{(n)}$ we have $nC_n(X_{i+1:n}) \geq 2$, the term in (9) is bounded from above by 2. That $T_n \rightarrow a_F$ in probability, is an immediate consequence of Strzalkowska-Kominiak and Stute (2008). The proof is complete. \square

Appendix

Proof of Lemma 1. We shall proof the assertion by showing that T may be approximated by a sequence $T^{(m)}$, as $m \rightarrow \infty$, where for each $m \geq 1$ the variable $T^{(m)}$ is a stopping time. For this, set

$$X_i^{(m)} = \sum_{k=0}^{\infty} k2^{-m} 1_{\{k2^{-m} \leq X_i < (k+1)2^{-m}\}}$$

and

$$T^{(m)} = \max_{1 \leq i \leq n} \left\{ X_i^{(m)} : 1_{\{X_j < X_i^{(m)} \leq Z_j\}} = 0 \text{ for } 1 \leq j \leq n \right\}.$$

Then we obtain

$$\{T^{(m)} \geq t\} = \bigcup_{i=1}^n \left\{ X_i^{(m)} \geq t : 1_{\{X_j < X_i^{(m)} \leq Z_j\}} = 0 \text{ for } 1 \leq j \leq n \right\}.$$

If t satisfies $k2^{-m} < t \leq (k+1)2^{-m}$, we have

$$\{X_i^{(m)} \geq t\} = \{X_i^{(m)} \geq (k+1)2^{-m}\} = \bigcup_{l=k+1}^{\infty} \{X_i^{(m)} = l2^{-m}\}$$

whence

$$\{X_j < X_i^{(m)} \leq Z_j\} \cap \{X_i^{(m)} \geq t\} = \bigcup_{l=k+1}^{\infty} \{X_j < l2^{-m} \leq Z_j, X_i^{(m)} = l2^{-m}\}.$$

Since by definition of \mathcal{G}_n

$$\{X_j < l2^{-m} \leq Z_j\} \in \mathcal{G}_n(l2^{-m})$$

and

$$\{X_i^{(m)} = l2^{-m}\} = \{l2^{-m} \leq X_i < (l+1)2^{-m}\} \in \mathcal{G}_n(l2^{-m}),$$

monotonicity of $\mathcal{G}_n(t)$ in reverse time yields

$$\{X_j < X_i^{(m)} \leq Z_j, X_i^{(m)} \geq t\} \in \mathcal{G}_n((k+1)2^{-m}) \in \mathcal{G}_n(t).$$

This implies that $T^{(m)}$ is a stopping time. To prove the lemma it suffices to show that $T^{(m)} \rightarrow T$ as $n \rightarrow \infty$. For this, consider any $X_i^{(m)}$ such that for each $j = 1, \dots, n$ either $X_j \geq X_i^{(m)}$ or $X_i^{(m)} > Z_j$. Since $X_i^{(m)} \leq X_i < X_i^{(m)} + 2^{-m}$, the inequality $X_i^{(m)} > Z_j$ yields $X_i > Z_j$. Furthermore, since there are no ties among the X_i 's, there exists $M_0 = M_0(n, w)$ such that for $i \neq j = 1, \dots, n$ and $m \geq M$ we have $|X_i(w) - X_j(w)| > 2^{-m}$. Hence from $X_j \geq X_i^{(m)}$ we also get $X_j \geq X_i$. Since $X_i^{(m)} \leq X_i$, we finally obtain

$$T^{(m)}(w) \leq T(w) \text{ for } m \geq M_0.$$

With similar arguments one can show that for $m \geq M_1(n, w)$

$$T^{(m)}(w) \geq T(w) - 2^{-m}.$$

This shows $T^{(m)} \rightarrow T$ pointwise and completes the proof. \square

Proof of Lemma 2. Let $t < \max Z_i$ be such that $C_n(t+) = 0$. Then $1_{\{X_j \leq t < Z_j\}} = 0$ for all $j = 1, \dots, n$. Since $t < \max Z_i$, there exists at least one $X_j > t$. For the smallest among such X_j 's we have $nC_n(X_j) = 1$. Conclude that the right-hand side of (4) is less than or equal to T . Conversely, let X_{i_0} be the maximum of the X_i 's for which $nC_n(X_i) = 1$. This set is nonempty since the smallest of the X 's always satisfies this equation. Let Z_{j_0} be the largest among the Z_j 's which is strictly smaller than X_{i_0} , if there is any. For any $Z_{j_0} \leq t < X_{i_0}$, we have $C_n(t+) = 0$. If not such Z exists, any $t < X_{i_0}$ will do the job. This shows that T is less than or equal to the right-hand side of (4). The proof is complete. \square

The following lemma is an adaptation of a result due to Gill (1980) for the survival function to the left tails of a d.f.

Lemma 3. *Let A and B be two nonincreasing, left-continuous functions satisfying*

$$A\{x\} \geq -1 \text{ and } B\{x\} > -1 \text{ for all } x \in \mathbb{R}.$$

The function

$$Z(t) = 1 - \frac{\prod_{s \geq t} (1 + A\{s\}) \exp(-A^c(t))}{\prod_{s \geq t} (1 + B\{s\}) \exp(-B^c(t))}$$

satisfies the integral equation

$$\int_{[t, \infty)} \frac{1 - Z(s^+)}{1 + B\{s\}} (B(ds) - A(ds)) = Z(t).$$

In this paper $A(t)$ and $B(t)$ are the left-continuous cumulative hazard functions of F_n and F_0 , respectively.

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