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ON A DISCRETE VERSION OF THE LAUGESSEN-MORPURGO CONJECTURE

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Abstract

In this paper we use coupling arguments to prove a discrete 1-dimensional version of the Laugesen-Morpurgo conjecture. As an application, we derive a probabilistic proof of the 1-dimensional Laugesen-Morpurgo conjecture (for the 1-dimensional reflecting Brownian motion).

Key words: reflecting random walk, reflecting Brownian motion, transition density

2000 MSC: 60J10, 60J65, 60J35.

1. Introduction

The Laugesen-Morpurgo conjecture appeared, as we learned from Rodrigo Bañuelos, in connection with their work (see [Laugesen and Morpurgo \(1998\)](#)) on conformal extremals of the Riemann zeta function of eigenvalues. The conjecture states the diagonal element of the Neumann heat kernel of the Laplacian in the unit ball $\mathbb{U} = \{x \in \mathbb{R}^2 : |x| < 1\}$ in \mathbb{R}^2 is a radially increasing function, that is

$$p(t, x, x) < p(t, y, y), \quad t \geq 0, \quad (1)$$

for all $x, y \in U$ with $0 \leq |x| < |y| \leq 1$, where $p(t, x, y)$ denotes the heat kernel for the Laplacian with Neumann boundary conditions (or, equivalently, the transition density for the Brownian motion with normal reflection on the boundary) in the unit disk U . The conjecture extends naturally to the Neumann heat kernel of the Laplacian in the unit ball $\mathbb{B} = \{x \in \mathbb{R}^d : \|x\| < 1\}$ in \mathbb{R}^d , $d \geq 1$.

The probabilistic interpretation of the conjecture is that a reflecting Brownian motion starting closer to the boundary is more likely to return to its starting

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position (after t units of time), than a reflecting Brownian motion starting further away from the boundary (after the same t units of time).

The physical interpretation is that introducing an atom of heat in a circular room with thermally insulated boundary, the closer this point to the boundary, the warmer we feel at this point, after any fixed number of units of time.

The intuition for this phenomenon is that starting closer to the boundary, the reflecting Brownian motion has a better chance of returning to this point since it will get more “push” directed towards this point, compared to a Brownian motion starting further away from the boundary. However, this is just an intuitive argument, and it is quite difficult to arrange it in a proof (despite the fact that it is known for some time), the Laugesen-Morpurgo conjecture is still open at the present moment in its full generality (see [Pascu and Pascu \(2007\)](#) and [Pascu and Pascu \(2008\)](#) for some partial results towards the resolution of the conjecture).

Recently, [Bañuelos, Kulczycki and Siudeja \(2008\)](#) proved the following result related to the Laugesen-Morpurgo conjecture:

Theorem 1. *The diagonal element $p_{\mathbb{B}}(t, x, x)$ of the transition probabilities for the d -dimensional Bessel processes on $(0; 1]$, reflected at 1, is an increasing function of $x \in (0, 1]$ for $d > 2$ and this is false for $d = 2$.*

Remark 2. Since the norm of a d -dimensional Brownian motion is a Bessel process of order d , the above result is equivalent to the monotonicity with respect to $x \in (0, 1)$ (for any $t > 0$ arbitrarily fixed) of the integral mean

$$\int_0^{2\pi} p(t, x, xe^{i\theta}) d\theta$$

of the transition probabilities of the d -dimensional reflecting Brownian motion in the unit ball in \mathbb{R}^d .

In this paper we will prove a discrete 1-dimensional version of the Laugesen-Morpurgo conjecture, as follows: if X_n is a simple random walk on $\{-s, \dots, s\}$ with reflecting barriers at $\pm s$, then for any $n \in \mathbb{N}$ arbitrarily fixed, $P^i(X_n = i)$ is a strictly increasing function of $|i|$, that is:

$$P^i(X_n = i) \leq P^j(X_n = j), \quad (2)$$

for any $i, j \in \{-s + 1, \dots, s - 1\}$ with $|i| < |j|$ and any $n \in \mathbb{N}$.

Trying to prove (2) by combinatorial methods (by explicit computation of the probabilities $P^i(X_n = i)$ for various i, n and s) is rather difficult, requiring the computation of the elements on the diagonal of a n^{th} power of a certain $s \times s$ matrix.

It is interesting to note that the inequality (2) does not hold for $j = s$, as it

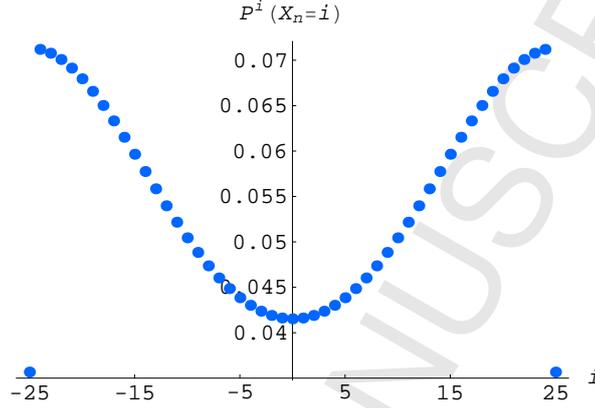


Figure 1: The graph of the probabilities $P^i(X_n=i)$, $i = -s, \dots, s$ for $s = 25$ and $n = 500$.

can be seen from the Figure 1 above. The reason for this is that

$$\begin{aligned}
 P^s(X_n = s) &= P^s(X_1 = s-1) P^{s-1}(X_{n-1} = s) \\
 &= P^{s-1}(X_{n-1} = s) \\
 &= P^{s-1}(X_{n-2} = s-1) P^{s-1}(X_1 = s) \\
 &= \frac{1}{2} P^{s-1}(X_{n-2} = s-1)
 \end{aligned}$$

which is smaller than $P^{s-1}(X_n = s-1)$, since

$$\begin{aligned}
 P^{s-1}(X_n = s-1) &= P^{s-1}(X_2 = s-1) P^{s-1}(X_{n-2} = s-1) + \\
 &\quad + P^{s-1}(X_2 = s-3) P^{s-3}(X_{n-2} = s-1) \\
 &= \frac{3}{4} P^{s-1}(X_{n-2} = s-1) + \frac{1}{4} P^{s-3}(X_{n-2} = s-1) \\
 &> \frac{1}{2} P^{s-1}(X_{n-2} = s-1).
 \end{aligned}$$

Also note that when n is odd, (2) is trivial, since in this case $P^i(X_n = i) = 0$ for any $i \in \{-s, \dots, s\}$.

Coupling methods are powerful tools which can be used to prove certain inequalities for the associated processes (see for example [Pascu \(2002\)](#) for a partial resolution of the Hot Spots conjecture of J. Rauch using coupling arguments). Our proof of the discrete version of Laugesen-Morpurgo conjecture for the 1-dimensional random walk on $\{-s, \dots, s\}$ with reflecting barriers at s and $-s$ uses synchronous and mirror couplings of (reflecting) simple random walks, introduced in Section 3.

As an application of our main result (Theorem 15), we also derive a proof of the Laugesen-Morpurgo conjecture for the 1-dimensional Brownian motion, that

is, we show that the inequality (1) holds for the transition density $p(t, x, y)$ of the 1-dimensional reflecting Brownian motion on the interval $(-1, 1)$ (see Corollary 21 and Remark 22).

2. Preliminaries

Definition 3. (See, e.g. [Iosifescu \(1980\)](#)). Let $(X_n)_{n \in \mathbb{N}}$ be a homogeneous Markov chain with state space S , transition matrix P and initial distribution p . Consider a decomposition $S = S_1 \cup \dots \cup S_q$ of the state space in pairwise disjoint sets. For simplicity, we denote the sets S_1, \dots, S_q by $\hat{1}, \dots, \hat{q}$, respectively. Define a new sequence of random variable $(Y_n)_{n \in \mathbb{N}}$ by

$$Y_n = \hat{k} \text{ if and only if } X_n \in S_k,$$

for any $k \in \{1, \dots, q\}$ and for any $n \in \mathbb{N}$. If $(Y_n)_{n \in \mathbb{N}}$ is a homogeneous Markov chain under P_p , for any choice of p , with transition probabilities independent of p , then $(X_n)_{n \in \mathbb{N}}$ is said to be groupable with respect to the partition $S = S_1 \cup \dots \cup S_q$ and $(Y_n)_{n \in \mathbb{N}}$ is called a grouped Markov chain.

Remark 4. In the context above, we will call $(Y_n)_{n \in \mathbb{N}}$ the corresponding grouped Markov chain of the chain $(X_n)_{n \in \mathbb{N}}$ with respect to the given partition of the state space.

For $A \subseteq S$, $A \neq \emptyset$, and $i \in S$, set $P_{i,A} = \sum_{j \in A} P_{ij}$. The following result gives a necessary and sufficient condition for a Markov chain to be groupable:

Proposition 5. A necessary and sufficient condition for a Markov chain to be groupable with respect to a partition $S = S_1 \cup \dots \cup S_q$ is that the probabilities P_{i,S_l} have the same value $\hat{P}_{\hat{k},\hat{l}}$ for all states $i \in S_k$, for any pair of subsets S_k, S_l , $1 \leq k, l \leq q$. Further, the transition matrix of the grouped Markov chain is given by

$$\hat{P} = (\hat{P}_{\hat{k},\hat{l}})_{1 \leq k, l \leq q}.$$

Proof. For a proof, see for example [Iosifescu \(1980\)](#), pp. 166 – 167. \square

Let $S = \{-s, -s+1, \dots, s-1, s\}$, where $s \in \mathbb{N} - \{0\}$. Define new states $s^+ = s^- = s$, $(-s)^+ = (-s)^- = -s$ and let i^\pm , $i \in \{-s+1, \dots, s-1\}$ be distinct, such that

$$S^+ \cap S^- := \{i^+ \mid i \in S\} \cap \{i^- \mid i \in S\} = \{-s, s\}.$$

Setting $S^\pm = S^+ \cup S^-$ and $S_i = \{i^+, i^-\}$, $i \in S$,

$$S^\pm = S_{-s} \cup S_{-s+1} \cup \dots \cup S_{s-1} \cup S_s \quad (3)$$

is a decomposition of S^\pm in disjoint sets.

By a finite cyclic random walk on S^\pm (or simply a random walk on S^\pm), we understand a random walk $(X_n^\pm)_{n \in \mathbb{N}}$ with state space S^\pm and transition matrix $P^\pm = (P_{ij}^\pm)_{i,j \in S^\pm}$ given by

$$P_{i^+, (i+1)^+}^\pm = P_{i^+, (i-1)^+}^\pm = P_{i^-, (i+1)^-}^\pm = P_{i^-, (i-1)^-}^\pm = \frac{1}{2}, \quad (4)$$

for $i \in \{-s+1, \dots, s-1\}$, and

$$P_{-s, (-s+1)^+}^\pm = P_{-s, (-s+1)^-}^\pm = P_{s, (s-1)^+}^\pm = P_{s, (s-1)^-}^\pm = \frac{1}{2}. \quad (5)$$

Given the random walk $(X_n^\pm)_{n \in \mathbb{N}}$ on S^\pm , we define a new sequence of random variable $(X_n)_{n \in \mathbb{N}}$ with state space S by setting

$$X_n = i \text{ if and only if } X_n^\pm \in \{i^+, i^-\}, \quad (6)$$

where $i \in S$ and $n \in \mathbb{N}$.

Remark 6. Since the condition of Proposition 5 is fulfilled, it follows that $(X_n^\pm)_{n \in \mathbb{N}}$ is groupable with respect to the partition (3) and that $(X_n)_{n \in \mathbb{N}}$ is the corresponding grouped Markov chain with respect to the partition (3), having the transition probability matrix P given by

$$P_{i, i-1} = P_{i, i+1} = \frac{1}{2}, \quad (7)$$

for $i \in \{-s+1, \dots, s-1\}$, and

$$P_{-s, -s+1} = P_{s, s-1} = 1. \quad (8)$$

Remark 7. Defining the projection function $pr : S^\pm \rightarrow S$ by

$$pr(i^+) = pr(i^-) = i,$$

for $i \in \{-s+1, \dots, s-1\}$, and

$$pr(-s) = -s \text{ and } pr(s) = s,$$

it can be seen that

$$X_n = pr(X_n^\pm), \quad n \in \mathbb{N}.$$

Remark 8. From (7) and (8), it can be seen that $(X_n)_{n \in \mathbb{N}}$ is a random walk on $S = \{-s, \dots, s\}$, with reflecting barriers at $-s$ and s . We shall refer to it as the reflecting random walk on S corresponding to the random walk $(X_n^\pm)_{n \in \mathbb{N}}$.

For an arbitrary fixed starting point $X_0^\pm = x \in S^\pm$, we denote by \mathbf{P}^x the probability measure associated with the random walk $(X_n^\pm)_{n \in \mathbb{N}}$ and by $\mathbf{P}^{pr(x)}$ the probability measure associated with the corresponding reflecting random walk $(X_n)_{n \in \mathbb{N}}$.

The next result gives a relation between the transition probabilities of a random walk on S^\pm and those of the corresponding reflecting random walk on S , as follows:

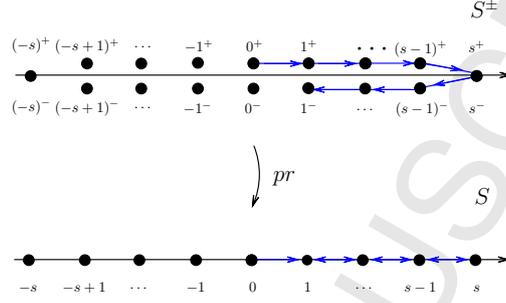


Figure 2: The projection of a random walk on S^\pm onto a reflecting random walk on S .

Proposition 9. For any $i \in S - \{-s, s\}$ and $n \in \mathbb{N}$ we have

$$\mathbf{P}^i(X_n = i) = \mathbf{P}^{i^+}(X_n^\pm = i^+) + \mathbf{P}^{i^-}(X_n^\pm = i^-),$$

where $(X_n^\pm)_{n \in \mathbb{N}}$ is a random walk on S^\pm and $(X_n)_{n \in \mathbb{N}}$ is the corresponding reflecting random walk on S .

Proof. First note that for $i \in S - \{-s, s\}$ we have $i^+ \neq i^-$. The equality follows now from the fact that $X_n = pr(X_n^\pm)$, and therefore $X_n = i$ if and only if $X_n^\pm = i^+$ or $X_n^\pm = i^-$, for any $i \in S - \{-s, s\}$ and $n \in \mathbb{N}$. \square

Remark 10. Alternately, letting $U_{4s} = \{\exp(\frac{ik\pi}{2s}) : k \in \{0, 1, \dots, 4s-1\}\}$ denote the vertices of a regular polygon with $4s$ sides and defining the bijection $f : S^\pm \rightarrow U_{4s}$ by

$$f(k^+) = \exp\left(\frac{(s-k)i\pi}{2s}\right), \quad k \in S,$$

and

$$f(k^-) = \exp\left(\frac{(3s+k)i\pi}{2s}\right), \quad k \in S,$$

we can view a random walk on $S^\pm = \{-s, \dots, s\}$ as a rotationally invariant random walk on the vertices of the polygon U_{4s} (see Figure 3).

3. Main results

In this section we will introduce two couplings of random walks on S^\pm : translation and mirror couplings, respectively.

To construct the translation coupling with starting points $(x, y) \in S^\pm \times S^\pm$ (without loss of generality we may assume that $pr(x) < pr(y)$), we consider a random walk $(X_n^\pm)_{n \in \mathbb{N}}$ on S^\pm with starting point $x \in S^\pm$, and we define a new

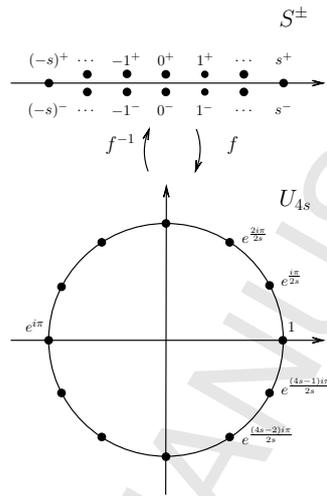


Figure 3: The bijective correspondence between random walks on S^\pm and U_{4s} .

sequence of random variables $(Y_n^\pm)_{n \in \mathbb{N}}$ as follows. The idea is that by Remark 10, we can view (X_n^\pm) and (Y_n^\pm) as random walks on U_{4s} , and thus we will define the random variable Y_n^\pm as a translation of X_n^\pm on the unit circle by $(pr(y) - pr(x)) \frac{\pi}{2s}$ units (or equivalently, a rotation of X_n^\pm with respect to the origin in the clockwise direction of angle $(pr(y) - pr(x)) \frac{\pi}{2s}$).

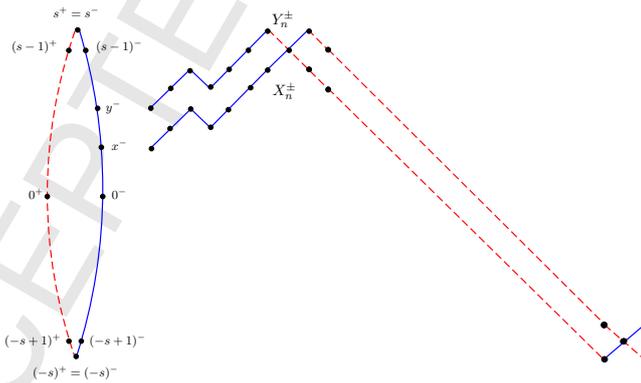


Figure 4: Sample paths of random walks on S^\pm coupled by translation.

In order to define Y_n^\pm formally, for $a \in \mathbb{Z}$ we introduce the translation function $tr_a : S^\pm \rightarrow S^\pm$ on S^\pm by

$$tr_a(t) = f^{-1} \left(f(t) \exp \left(-i \frac{a\pi}{2s} \right) \right), \quad t \in S^\pm, \quad (9)$$

where f is the bijection defined in Remark 10, and we set

$$Y_n^\pm = tr_{pr(y)-pr(x)}(X_n^\pm), \quad n \in \mathbb{N}.$$

We call the pair $(X_n^\pm, Y_n^\pm)_{n \in \mathbb{N}}$ constructed above a translation coupling of random walks on S^\pm with starting points (x, y) .

Remark 11. The fact that the sequence $(Y_n^\pm)_{n \in \mathbb{N}}$ constructed above is a random walk on S^\pm with starting point $Y_0^\pm = y$ and transition probabilities given by (4) and (5), follows from

$$\begin{aligned} & \mathbf{P}^y(Y_{n+1}^\pm = (j+1)^+ | Y_n^\pm = j^+) \\ &= \mathbf{P}^x(X_{n+1}^\pm = tr_{-(pr(y)-pr(x))}((j+1)^+) | X_n^\pm = tr_{-(pr(y)-pr(x))}(j^+)) \\ &= \frac{1}{2}, \end{aligned}$$

for any $j \in S - \{s\}$ and $n \in \mathbb{N}$,

$$\begin{aligned} & \mathbf{P}^y(Y_{n+1}^\pm = (s-1)^+ | Y_n^\pm = s^+) \\ &= \mathbf{P}^x(X_{n+1}^\pm = tr_{-(pr(y)-pr(x))}((s-1)^+) | X_n^\pm = tr_{-(pr(y)-pr(x))}(s^+)) \\ &= \frac{1}{2}, \end{aligned}$$

for any $n \in \mathbb{N}$, and similarly for the other transition probabilities.

To construct the mirror coupling with starting points $(x, y) \in S^- \times S^-$ chosen such that $pr(x) + pr(y)$ is an even number (without loss of generality we may assume that $pr(x) < pr(y)$), we consider a random walk $(X_n^\pm)_{n \in \mathbb{N}}$ on S^\pm with starting point $x \in S^-$, and we define a new sequence of random variables $(Y_n^\pm)_{n \in \mathbb{N}}$ as follows. The idea is that by Remark 10, we can view (X_n^\pm) as a random walk on U_{4s} , and we define the random variable Y_n^\pm as the symmetric point of X_n^\pm with respect to the line passing through the origin and the point $f \left(\left(\frac{pr(x)+pr(y)}{2} \right)^- \right) = \exp \left(\left(3s + \frac{pr(x)+pr(y)}{2} \right) \frac{i\pi}{2s} \right)$, for $n \leq \tau$, where

$$\tau = \inf \left\{ n \geq 0 : X_n^\pm = \left(\frac{pr(x) + pr(y)}{2} \right)^- \text{ or } X_n^\pm = \left(-\frac{pr(x) + pr(y)}{2} \right)^+ \right\}$$

is the coupling time, and we let $Y_n^\pm = X_n^\pm$ for $n \geq \tau$ (see Figure 5).

In order to define Y_n^\pm formally, for $a \in S^\pm$ we introduce the symmetry function $sym_a : S^\pm \rightarrow S^\pm$ on S^\pm (symmetry with respect to the line passing through the origin and the point $f(a)$), by

$$sym_a(t) = f^{-1} \left(\overline{f(t)} f^2(a) \right), \quad t \in S^\pm, \quad (10)$$

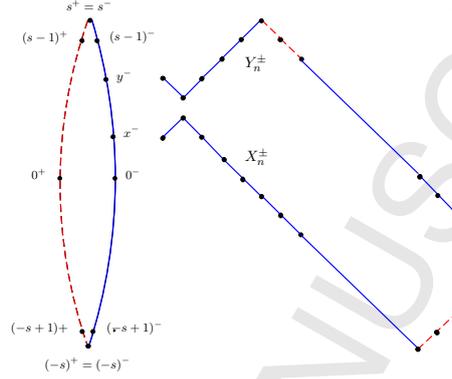


Figure 5: Sample paths of random walks on S^\pm coupled by mirror coupling.

and we set

$$Y_n^\pm = \begin{cases} \text{sym}_{\left(\frac{pr(x)+pr(y)}{2}\right)^-} (X_n^\pm) & , \quad n \leq \tau \\ X_n^\pm & , \quad n > \tau \end{cases}$$

We call the pair $(X_n^\pm, Y_n^\pm)_{n \in \mathbb{N}}$ constructed above a mirror coupling of random walks on S^\pm with starting points (x, y) .

Remark 12. The fact that the sequence $(Y_n^\pm)_{n \in \mathbb{N}}$ constructed above is a random walk on S^\pm with starting point $Y_0^\pm = y$ and transition probabilities given by (4) and (5), follows from

$$\begin{aligned} & \mathbf{P}^y(Y_{n+1}^\pm = (j+1)^\mp | Y_n^\pm = j^\mp) \\ &= \mathbf{P}^x(X_{n+1}^\pm = \text{sym}_{\left(\frac{pr(x)+pr(y)}{2}\right)^-} (j+1)^\mp | X_n^\pm = \text{sym}_{\left(\frac{pr(x)+pr(y)}{2}\right)^-} (j^\mp)) \\ &= \frac{1}{2}, \end{aligned}$$

for any $j \in S - \{s\}$ and $n < \tau$, and

$$\begin{aligned} & \mathbf{P}^y(Y_{n+1}^\pm = (s-1)^\mp | Y_n^\pm = s^\mp) \\ &= \mathbf{P}^x(X_{n+1}^\pm = \text{sym}_{\left(\frac{pr(x)+pr(y)}{2}\right)^-} (s-1)^\mp | X_n^\pm = \text{sym}_{\left(\frac{pr(x)+pr(y)}{2}\right)^-} (s^\mp)) \\ &= \frac{1}{2}, \end{aligned}$$

for any $n < \tau$, and similarly for the other transition probabilities.

Proposition 13. For any $i, j \in S$ with $0 \leq i \leq j$ and any $n \in \mathbb{N}$ we have

$$\mathbf{P}^{i^+}(X_n^\pm = i^+) + \mathbf{P}^{i^+}(X_n^\pm = i^-) = \mathbf{P}^{j^+}(X_n^\pm = j^+) + \mathbf{P}^{j^+}(X_n^\pm = (2i-j)^-), \quad (11)$$

where X_n^\pm is a random walk on S^\pm .

Proof. For arbitrarily fixed $i, j \in S$ with $0 \leq i \leq j$ let (X_n^\pm, Y_n^\pm) be a translation coupling of random walks on S^\pm with starting points (i^+, j^+) , that is

$$Y_n^\pm = tr_{j-i}(X_n^\pm), \quad n \in \mathbb{N}.$$

Since $0 \leq j-i \leq s$, from (9) it can be seen that $tr_{j-i}(t) = j^+$ if and only if $t = i^+$, and similarly, $tr_{j-i}(t) = (2i-j)^-$ if and only if $t = i^-$, and therefore we obtain

$$\mathbf{P}^{i^+}(X_n^\pm = i^+) = \mathbf{P}^{j^+}(Y_n^\pm = j^+)$$

and

$$\mathbf{P}^{i^+}(X_n^\pm = i^-) = \mathbf{P}^{j^+}(Y_n^\pm = (2i-j)^-),$$

for any $n \in \mathbb{N}$, which shows that

$$\mathbf{P}^{i^+}(X_n^\pm = i^+) + \mathbf{P}^{i^+}(X_n^\pm = i^-) = \mathbf{P}^{j^+}(Y_n^\pm = j^+) + \mathbf{P}^{j^+}(Y_n^\pm = (2i-j)^-),$$

and since Y_n^\pm is a random walk with starting point j^+ , we obtain equivalently (11), as needed. \square

Proposition 14. For any $i, j \in S$ with $0 \leq i < j$ such that $i+j$ is an even number and any $n \in \mathbb{N}$, we have

$$\mathbf{P}^{i^-}(X_n^\pm = j^+) < \mathbf{P}^{j^-}(X_n^\pm = j^+),$$

where X_n^\pm is a random walk on S^\pm .

Proof. For arbitrarily fixed $i, j \in S$ with $0 \leq i < j$ such that $i+j$ is an even number let (X_n^\pm, Y_n^\pm) be a mirror coupling of random walks on S^\pm with starting points (i^-, j^-) , that is

$$Y_n^\pm = \begin{cases} \text{sym}_{\left(\frac{i+j}{2}\right)^-} X_n^\pm, & n \leq \tau \\ X_n^\pm, & n > \tau \end{cases}.$$

Under the hypothesis of the theorem it can be seen that we cannot have $X_n^\pm = j^+$ unless $n \geq \tau$, that is, unless $X_n^\pm = Y_n^\pm$. We obtain therefore

$$\begin{aligned} \mathbf{P}^{i^-}(X_n^\pm = j^+) &= \mathbf{P}^{i^-}(X_n^\pm = j^+, n \geq \tau) \\ &= \mathbf{P}^{j^-}(Y_n^\pm = j^+, n \geq \tau) \\ &< \mathbf{P}^{j^-}(Y_n^\pm = j^+). \end{aligned}$$

Since Y_n^\pm is a random walk with starting point j^- , we have

$$\mathbf{P}^{j^-}(Y_n^\pm = j^+) = \mathbf{P}^{j^-}(X_n^\pm = j^+),$$

concluding the proof. \square

We are now ready to prove the main result of this section, as follows:

Theorem 15. For any $s \in \mathbb{N} - \{0\}$ and $n \in \mathbb{N}$, $P^i(X_n = i)$ is a strictly increasing function of $i \in \{0, \dots, s-1\}$, that is, for any $i, j \in \{0, \dots, s-1\}$, with $i < j$ and any $n \in \mathbb{N}$, we have

$$\mathbf{P}^i(X_n = i) < \mathbf{P}^j(X_n = j),$$

where X_n is a reflecting random walk on S with reflecting barriers at $\pm s$.

Proof. Let X_n^\pm be a random walk on S^\pm and let $X_n = pr(X_n^\pm)$. By Proposition 9, for $i \in S - \{-s, s\}$ and $n \in \mathbb{N}$ we have

$$\mathbf{P}^i(X_n = i) = \mathbf{P}^{i^+}(X_n^\pm = i^+) + \mathbf{P}^{i^-}(X_n^\pm = i^-).$$

Consider first the case $i \in \{1, \dots, s-2\}$. Using Proposition 13 with j replaced by $i+1$, we have

$$\begin{aligned} \mathbf{P}^i(X_n = i) &= \mathbf{P}^{i^+}(X_n^\pm = i^+) + \mathbf{P}^{i^-}(X_n^\pm = i^-) \\ &= \mathbf{P}^{(i+1)^+}(X_n^\pm = (i+1)^+) + \mathbf{P}^{(i+1)^-}(X_n^\pm = (i-1)^-) \\ &< \mathbf{P}^{(i+1)^+}(X_n^\pm = (i+1)^+) + \mathbf{P}^{(i+1)^+}(X_n^\pm = (i+1)^-), \end{aligned}$$

for any $n \in \mathbb{N}$, where the last inequality follows from the symmetry of the transition matrix of a random walk on S^\pm and Proposition 14 with j replaced by $i+1$ and i replaced by $i-1$. Using again Proposition 9 we obtain

$$\mathbf{P}^i(X_n = i) < \mathbf{P}^{i+1}(X_n = i+1),$$

for any $n \in \mathbb{N}$ and $i \in \{1, \dots, s-2\}$.

To conclude the proof, we need to show that the previous inequality also holds for $i = 0$. In this case, using an argument similar to the one in the proofs of Proposition 13 and Proposition 14 it can be shown that we have

$$\begin{aligned} \mathbf{P}^{0^+}(X_n^\pm = 0^+) + \mathbf{P}^{0^-}(X_n^\pm = 0^-) &= \mathbf{P}^{1^+}(X_n^\pm = 1^+) + \mathbf{P}^{1^-}(X_n^\pm = (-1)^-) \\ &< \mathbf{P}^{1^+}(X_n^\pm = 1^+) + \mathbf{P}^{1^+}(X_n^\pm = 1^-), \end{aligned}$$

which by Proposition 9 shows that

$$\mathbf{P}^0(X_n = 0) < \mathbf{P}^1(X_n = 1),$$

for any $n \in \mathbb{N}$, concluding the proof. \square

4. Extensions and applications

In this section we will extend the main result obtained in Theorem 15. First, we need the following:

Proposition 16. For any $i, k \in S$ with $|i+k| < s$ and $n \in \mathbb{N}$ we have

$$\mathbf{P}^i(X_n = i+k) = \mathbf{P}^{i^+}(X_n^\pm = (i+k)^+) + \mathbf{P}^{i^-}(X_n^\pm = (i+k)^-),$$

where $(X_n^\pm)_{n \in \mathbb{N}}$ is a random walk on S^\pm and $(X_n)_{n \in \mathbb{N}}$ is the corresponding reflecting random walk on S .

Proof. The equality follows from the fact that $X_n = pr(X_n^\pm)$, and therefore $X_n = i+k$ if and only if $X_n^\pm = (i+k)^+$ or $X_n^\pm = (i+k)^-$, for any $i, k \in S$ with $|i+k| < s$ and $n \in \mathbb{N}$. \square

Using a proof similar to the one in Proposition 13, we can establish the following:

Proposition 17. For any $s, n \in \mathbb{N} - \{0\}$ and $i, j, k \in S$ with $|i+k| < s$, $|j+k| < s$ and $0 \leq i \leq j \leq 2i+k+s$ we have

$$\begin{aligned} & \mathbf{P}^{i^+}(X_n^\pm = (i+k)^+) + \mathbf{P}^{i^+}(X_n^\pm = (i+k)^-) \\ &= \mathbf{P}^{j^+}(X_n^\pm = (j+k)^+) + \mathbf{P}^{j^+}(X_n^\pm = (2i-j+k)^-) \end{aligned}$$

where X_n^\pm is a random walk on S^\pm .

We can generalize the result in Theorem 15 as follows:

Theorem 18. For any $s, n \in \mathbb{N} - \{0\}$, $i, j \in \{0, \dots, s-1\}$ with $i < j$ and any $k \in S$ with $|i+k| < s$, $|j+k| < s$ and $k > -\min\{i+j, 2i-j+s\}$, we have

$$\mathbf{P}^i(X_n = i+k) < \mathbf{P}^j(X_n = j+k),$$

where X_n is a reflecting random walk on $S = \{-s, \dots, s\}$ with reflecting barriers at $\pm s$.

Proof. Let $i, j \in \{0, \dots, s-1\}$ with $i < j$ be arbitrarily fixed. Consider X_n^\pm a random walk on S^\pm and let $X_n = pr(X_n^\pm)$. By Proposition 16 we have

$$\mathbf{P}^i(X_n = i+k) = \mathbf{P}^{i^+}(X_n^\pm = (i+k)^+) + \mathbf{P}^{i^+}(X_n^\pm = (i+k)^-),$$

for all $n \in \mathbb{N}$ and $k \in S$ with $|i+k| < s$. By Proposition 17, for all $n \in \mathbb{N} - \{0\}$ and $k \in S$ with $|i+k| < s$, $|j+k| < s$ and $j < 2i+k+s$ we have

$$\begin{aligned} & \mathbf{P}^{i^+}(X_n^\pm = (i+k)^+) + \mathbf{P}^{i^+}(X_n^\pm = (i+k)^-) \\ &= \mathbf{P}^{j^+}(X_n^\pm = (j+k)^+) + \mathbf{P}^{j^+}(X_n^\pm = (2i-j+k)^-). \end{aligned}$$

Using the fact that the transition matrix of a random walk on S^\pm is (by definition) symmetric and an argument similar to the one used in the proof of Proposition 14, for all $k \in S$ with $k > -i-j$, we get

$$\mathbf{P}^{j^+}(X_n^\pm = (2i-j+k)^-) < \mathbf{P}^{j^+}(X_n^\pm = (j+k)^-),$$

and combining with the above equality, we obtain

$$\begin{aligned} \mathbf{P}^i(X_n = (i+k)) &< \mathbf{P}^{j^+}(X_n^\pm = (j+k)^+) + \mathbf{P}^{j^+}(X_n^\pm = (j+k)^-) \\ &= \mathbf{P}^j(X_n = (j+k)), \end{aligned}$$

by Proposition 16, and the claim follows. \square

From the previous theorem we obtain the following:

Corollary 19. For any $s, n \in \mathbb{N} - \{0\}$, $i, j \in \{0, \dots, s-1\}$ with $i < j$ and any $k \in \{1, \dots, s\}$ with $k \leq \min\{s-j, i+j\}$, we have

$$\mathbf{P}^i(|X_n - i| < k) < \mathbf{P}^j(|X_n - j| < k),$$

where X_n is a reflecting random walk on $S = \{-s, \dots, s\}$ with reflecting barriers at $\pm s$.

Proof. The proof follows from the previous theorem by summing over k . \square

Using the fact that the reflecting Brownian motion can be approximated by random walks (see for example [Burdzy and Chen \(2008\)](#)), we obtain the following

Corollary 20. For any $x, y, \varepsilon \in (0, 1)$ with $x < y$ and $\varepsilon < \min\{1-y, x+y\}$, we have

$$\mathbf{P}^x(B_t \in (x-\varepsilon, x+\varepsilon)) \leq \mathbf{P}^y(B_t \in (y-\varepsilon, y+\varepsilon)), \quad t > 0,$$

where B_t is a 1-dimensional reflecting Brownian motion on $[-1, 1]$.

Proof. In the case when both x and y are dyadic rationals in $[0, 1]$, the proof follows from the previous corollary by using the fact that a reflecting Brownian motion B_t on $[-1, 1]$ starting at $B_0 = \frac{k}{2^l}$ ($k, l \in \mathbb{Z}$, $l > 0$) can be approximated by a reflecting random walk $(X_m^n)_{m \geq 0}$, more precisely it can be shown (see for example [Burdzy and Chen \(2008\)](#)) that we have

$$X_{[2^{2n}t]}^n \xrightarrow{n \rightarrow \infty} B_t, \quad t \geq 0,$$

where $(X_m^n)_{m \geq 0}$ is a (simple) reflecting random walk on

$$\left\{ -1, -\frac{2^n-1}{2^n}, -\frac{2^n-2}{2^n}, \dots, \frac{2^n-2}{2^n}, \frac{2^n-1}{2^n}, 1 \right\}$$

with reflecting barriers at ± 1 and starting at $X_0^n = \frac{k}{2^l}$.

The general case follows by approximating x and y by dyadic rationals and using the previous part of the proof. \square

As a corollary, we obtain the proof of the Laugesen-Morpurgo conjecture in the 1-dimensional case, as follows:

Corollary 21. For any $x, y \in (0, 1)$ with $x < y$, we have

$$p(t, x, x) < p(t, y, y), \quad t > 0,$$

where $p(t, x, y)$ denotes the transition probabilities of the 1-dimensional reflecting Brownian motion on $[-1, 1]$.

Proof. The fact that $p(t, x, x)$ is increasing in $x \in (0, 1)$ follows from the previous corollary, using the fact that by the continuity of $p(t, x, y)$ in the second variable we have

$$p(t, x, x) = \lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} p(t, x, y) dy = \lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} P^x(|B_t - x| < \varepsilon).$$

To show that $p(t, x, x)$ is in fact strictly increasing, note that since $p(t, x, x)$ is a real analytic function of $x \in (-1, 1)$ for any $t > 0$ arbitrarily fixed (it is the diagonal of the heat kernel of an operator with real analytic coefficients), it cannot be constant on a nonempty open subset of $[-1, 1]$ unless it is identically constant on the entire interval $[-1, 1]$.

It can be shown (see for example [Bañuelos, Kulczycki and Siudeja \(2008\)](#)) that for any $t > 0$ arbitrarily fixed we have

$$p(t, x, x) + \tilde{p}(t, x, x) = c, \quad x \in (-1, 1),$$

where c is a constant depending on $t > 0$ and $\tilde{p}(t, x, y)$ denotes the transition density of Brownian motion on $(-1, 1)$ killed on hitting the boundary of the interval.

If $p(t, x, x)$ were constant in $x \in (-1, 1)$ for an arbitrarily fixed $t > 0$, then $\tilde{p}(t, x, x)$ would also be constant in $x \in (-1, 1)$. However, this leads to a contradiction, since

$$\lim_{x \nearrow 1} \tilde{p}(t, x, x) = 0 < \tilde{p}(t, 0, 0).$$

This, together with the fact that $p(t, x, x)$ is increasing in $x \in (0, 1)$ for any $t > 0$ arbitrarily fixed, shows that $p(t, x, x)$ is in fact strictly increasing in $x \in (0, 1)$, concluding the proof. \square

Remark 22. The fact that the Laugesen-Morpurgo conjecture holds in the 1-dimensional case can also be proved by other methods (see for example the Remark 5.4 in [Bañuelos, Kulczycki and Siudeja \(2008\)](#) for an analytic proof, or [Pascu and Pascu \(2008\)](#) for a probabilistic proof). We presented it here in order to show that the same result can be obtained by probabilistic techniques, as a corollary of the results presented above.

It is possible that the idea used in proving the 1-dimensional version of the Laugesen-Morpurgo conjecture in the present paper (viewing the reflecting random walk as an un-reflected random walk on a different state space and proving the desired monotonicity in this new setting) can also be extended to higher-dimensional cases, but aside from particular cases (parallepipeds in \mathbb{R}^d , $d \geq 1$) we were unable to implement it.

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