



**HAL**  
open science

# Cell centered Galerkin methods for diffusive problems

Daniele Antonio Di Pietro

► **To cite this version:**

Daniele Antonio Di Pietro. Cell centered Galerkin methods for diffusive problems. 2010. hal-00511125v1

**HAL Id: hal-00511125**

**<https://hal.science/hal-00511125v1>**

Preprint submitted on 23 Aug 2010 (v1), last revised 28 Apr 2011 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## CELL CENTERED GALERKIN METHODS FOR DIFFUSIVE PROBLEMS\*

DANIELE A. DI PIETRO<sup>1</sup>

**Abstract.** In this work we introduce a new class of lowest order methods for diffusive problems on general meshes with only one unknown per element. The underlying idea is to construct an incomplete piecewise affine polynomial space with sufficient approximation properties starting from values at cell centers. To do so we borrow ideas from multi-point finite volume methods, although we use them in a rather different context. The incomplete polynomial space replaces classical complete polynomial spaces in discrete formulations inspired by discontinuous Galerkin methods. Two problems are studied in this work: a heterogeneous anisotropic diffusion problem, which is used to lay the pillars of the method, and the incompressible Navier–Stokes equations, which provide a more realistic application. An exhaustive theoretical study as well as a set of numerical examples featuring different difficulties are provided.

**1991 Mathematics Subject Classification.** TO BE COMPLETED.

August 23, 2010.

## INTRODUCTION

Lowest order methods for diffusive problems on general meshes have received an increasing attention over the last few years. The interest of general meshes is multi-fold. On the one hand, allowing general polyhedral elements may ease the discretization of complex domains, and it is beneficial in the context of aggregative multigrid strategies. On the other hand, it is a mandatory requirement whenever the user cannot adapt the mesh to the needs of the numerical scheme. This is the case, e.g., in the context of computational geosciences, where the discretization of the subsoil integrates the effects of erosion and sedimentation, and is usually developed in a separate stage. In what follows we briefly recall some ideas that are instrumental to the class of methods proposed in this paper. It is a well known fact that the classical two-point finite volume (FV) method is inconsistent on non diffusion-orthogonal meshes. In the context of reservoir simulation, a successful attempt to adapt FV methods to general meshes and full diffusion tensors has been independently proposed in the 90s by Aavatsmark, Barkve, Bøe, and Mannseth [1, 2] and by Edwards and Rogers [25, 26]. These methods are usually referred to as *multi-point* since the main idea is to express consistent numerical fluxes using a larger stencil than in the classical two-point FV method. In this paper we borrow some ideas from the recent multi-point scheme of Aavatsmark, Eigestad, Mallison, and Nordbotten [3], where the authors propose a compact construction to derive a consistent multi-point flux approximation honoring the heterogeneity of the diffusion tensor. A convergence analysis of the *L-construction* proposed therein has been recently performed by Agélas,

---

*Keywords and phrases:* Cell centered Galerkin, finite volumes, discontinuous Galerkin, heterogeneous anisotropic diffusion, incompressible Navier–Stokes equations

\* This work has been partially supported by the VFSitCom ANR project; see <http://ens.math.univ-montp2.fr/droniou/vfsitcom>

<sup>1</sup> IFP, 1 & 4 avenue de Bois Préau, 92582 Rueil-Malmaison Cedex

Di Pietro, and Droniou [4]. The main drawback of multi-point FV methods is related to the difficulty in finding easily computable stability conditions. A possible remedy to the lack of stability in multi-point methods has been proposed independently by Brezzi, Lipnikov, Shashkov, and Simoncini [10, 11] (Mimetic Finite Difference methods, MFD) and by Droniou, Eymard, Gallouët, and Herbin [23, 30] (Mixed/Hybrid Finite Volume methods, MHFV). Both classes of schemes are inspired from the variational form of the problem as in finite element (FE) methods rather than from local balances as in FV methods. The unconditional stability of both MFD and MHFV methods results from stabilization terms incorporated in the discrete bilinear form. Also, in both cases auxiliary face unknowns are added, which can be in some cases related to the enforcement of a flux conservation constraint as in the mixed FE method. A way to eliminate face unknowns has been proposed in [30], where it is suggested that a barycentric interpolator can be used to express face values in terms of a few neighboring cell unknowns. The main drawback of this construction is that it does not respect the heterogeneity of the diffusion tensor; moreover, the resulting method has a larger stencil compared to multi-point FV methods, which results in denser matrices as well as in the increase of data exchange in parallel implementations. The algebraic analogy between generalized versions of the MFD and MHFV methods has been recently pointed out by Droniou, Eymard, Gallouët, and Herbin [24].

In this work we propose a different philosophical approach to variational lowest-order methods which is inspired by discontinuous Galerkin (dG) methods rather than mixed FEs. The motivation is twofold. A first objective is to devise a suitable framework for a multi-physics platform based on lowest-order methods. In this respect, the main requirement is to dispose of a method easily adapted to a variety of diffusive problems including, e.g., pure diffusion, diffusion-advection-reaction, linear and nonlinear elasticity as well as incompressible flows. A second important point is related to the robustness in the context of heterogeneous anisotropic diffusion, which is crucial in the field of computational geosciences. Cell centered Galerkin (ccG) methods have been introduced in [18, 19] with application to a homogeneous diffusion problem. The main idea of ccG methods is to build an incomplete space of piecewise affine functions related to a given set of degrees of freedom (DOFs) on the mesh  $\mathcal{T}_h$ . In this work we consider, in particular, ccG methods with cell unknowns only, *i.e.*, where the algebraic space of DOFs given by

$$\mathbb{V}_h \stackrel{\text{def}}{=} \mathbb{R}^{\mathcal{T}_h}.$$

The unknown in each mesh element is interpreted as the value of the discrete function at a given point (the cell center); a piecewise affine representation is then obtained by reconstructing a constant gradient in each mesh element. In the present work, the elementary gradient is obtained in two steps: first, trace values are reconstructed from cell center values by means of the L-construction, then Green's formula is used to infer a local value for the gradient. The piecewise affine function inside each element is then chosen as the unique function with prescribed value at cell center and gradient equal to the reconstructed gradient. Formally, this procedure amounts to introducing an injective linear operator  $\mathfrak{R}_h^{\text{ccg}} : \mathbb{V}_h \rightarrow \mathbb{P}_d^1(\mathcal{T}_h)$  and defining the discrete space

$$V_h^{\text{ccg}} \stackrel{\text{def}}{=} \mathfrak{R}_h^{\text{ccg}}(\mathbb{V}_h) \subset \mathbb{P}_d^1(\mathcal{T}_h).$$

The space  $V_h^{\text{ccg}}$  is then used as a test/trial space in a suitable nonconforming FE setting. In particular, since the functions in  $V_h^{\text{ccg}}$  are discontinuous across mesh interfaces, the discrete setting largely borrows from dG methods. In this work we consider two applications to problems naturally set in  $H_0^1(\Omega)$ : a homogeneous anisotropic scalar diffusion problem and the incompressible Navier–Stokes equations. The scalar diffusion problem offers a simplified context to outline the main ideas of the method. The convergence analysis is here discussed in detail. In particular, we show that both classical dG arguments relying on error estimates [7] and compactness arguments inspired by [29, 30] apply. For the latter, an important remark is that the functional analytic results independently derived by Di Pietro and Ern [20] and Buffa and Ortner [13] hold *a fortiori* since  $V_h^{\text{ccg}} \subset \mathbb{P}_d^1(\mathcal{T}_h)$ . Moreover, in both cases, the main technical issue is related to the approximation properties of the  $V_h^{\text{ccg}}$  space. The application to the incompressible Navier–Stokes equations provides a nonlinear case study to illustrate how a ccG method can be derived from an existing dG method.

The material is organized as follows: in §1 we discuss the discrete setting, provide a careful description of admissible mesh sequences in arbitrary space dimension, introduce the L-construction, and define the ccG space

used throughout the rest of this work. Some functional analytical results are also recalled; in §2 we show an application of the ccG method to a heterogeneous anisotropic diffusion problem, derive basic error estimates and carefully study convergence rates. The main sources of inspiration are here the work of Arnold [6] on the weak enforcement of boundary and interface conditions and the paper of Di Pietro, Ern, and Guermond [22] on the robust handling of heterogeneous anisotropic diffusion tensors. To infer convergence rates we study the approximation properties of the space  $V_h^{\text{ccg}}$  with respect to the energy norm naturally associated to the discrete problem. In particular, we distinguish between the heterogeneous case, where optimal energy estimates are obtained for functions belonging to the the space introduced and analyzed in [4], and the homogeneous case, where weaker regularity assumptions are sufficient and  $L^2$  error estimates can also be derived; in §3 we discuss the application of ccG methods to the incompressible Navier–Stokes equations inspired on the dG scheme of [20]. In this case, an  $H^1$ -stability result for ccG spaces is crucial to infer the stability of the velocity-pressure coupling. The analysis closely follows the guidelines of [20]; finally, in §4 we present numerical results to assess the theoretical study for both problems at hand.

## 1. THE CCG SPACE

### 1.1. Discrete setting

Closely following [21, Chapter 1], we introduce the concept of *admissible mesh sequence* of a bounded connected polyhedral domain  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 1$ . Let  $\mathcal{H} \subset \mathbb{R}_+^*$  denote a countable set having 0 as its unique accumulation point. For all  $h \in \mathcal{H}$  we denote by  $\mathcal{T}_h$  a finite collection of disjoint open polyhedra  $\mathcal{T}_h = \{T\}$  forming a partition of  $\Omega$  such that  $h = \max_{T \in \mathcal{T}_h} h_T$ , with  $h_T$  denoting the diameter of the element  $T \in \mathcal{T}_h$ . Mesh nodes are collected in the set  $\mathcal{N}_h$ . We say that a hyperplanar closed subset  $F$  of  $\bar{\Omega}$  is a mesh face if it has positive  $(d-1)$ -dimensional measure and if either there exist  $T_1, T_2 \in \mathcal{T}_h$  such that  $F \subset \partial T_1 \cap \partial T_2$  (and  $F$  is called an *interface*) or there exist  $T \in \mathcal{T}_h$  such that  $F \subset \partial T \cap \partial \Omega$  (and  $F$  is called a *boundary face*). Interfaces are collected in the set  $\mathcal{F}_h^i$ , boundary faces in  $\mathcal{F}_h^b$  and we let  $\mathcal{F}_h \stackrel{\text{def}}{=} \mathcal{F}_h^i \cup \mathcal{F}_h^b$ . Moreover, we set, for all  $T \in \mathcal{T}_h$ ,

$$\mathcal{F}_T \stackrel{\text{def}}{=} \{F \in \mathcal{F}_h \mid F \subset \partial T\}. \quad (1)$$

Similarly, for all  $F \in \mathcal{F}_h$ , we define

$$\mathcal{T}_F \stackrel{\text{def}}{=} \{T \in \mathcal{T}_h \mid F \subset \partial T\}.$$

The set  $\mathcal{T}_F$  consists of exactly two mesh elements if  $F \in \mathcal{F}_h^i$  and of one if  $F \in \mathcal{F}_h^b$ . For all mesh nodes  $P \in \mathcal{N}_h$ ,  $\mathcal{F}_P$  denotes the set of mesh faces sharing  $P$ , *i.e.*

$$\mathcal{F}_P \stackrel{\text{def}}{=} \{F \in \mathcal{F}_h \mid P \in F\}. \quad (2)$$

For every interface  $F \in \mathcal{F}_h^i$  we introduce an arbitrary but fixed ordering of the elements in  $\mathcal{T}_F$  and let  $\mathbf{n}_F = \mathbf{n}_{T_1, F} = -\mathbf{n}_{T_2, F}$ , where  $\mathbf{n}_{T_i, F}$ ,  $i \in \{1, 2\}$ , denotes the unit normal to  $F$  pointing out  $T_i \in \mathcal{T}_F$ . For all  $T \in \mathcal{T}_h$ , we also introduce the symbol  $\mathbf{n}_T$  to denote the vector field such that  $\mathbf{n}_T|_F = \mathbf{n}_{T, F}$  for all  $F \in \mathcal{F}_T$ . On a boundary face  $F \in \mathcal{F}_h^b$  we let  $\mathbf{n}_F$  denote the unit normal pointing out of  $\Omega$ . The barycenter of a face  $F \in \mathcal{F}_h$  is denoted by  $\bar{\mathbf{x}}_F \stackrel{\text{def}}{=} \int_F \mathbf{x} / |F|_{d-1}$ .

**Definition 1** (Jumps and weighted averages). *Let  $v$  be a scalar-valued function defined on  $\Omega$ , and assume that  $v$  is smooth enough to admit on all  $F \in \mathcal{F}_h$  a possibly two-valued trace. To any interface  $F \in \mathcal{F}_h^i$  with  $F \subset \partial T_1 \cap \partial T_2$  we assign two non-negative real numbers  $\omega_{T_1, F}$  and  $\omega_{T_2, F}$  such that*

$$\omega_{T_1, F} + \omega_{T_2, F} = 1.$$

*Then, if  $F \in \mathcal{F}_h^i$  with  $F = \partial T_1 \cap \partial T_2$  the jump and weighted average of  $v$  at  $F$  are respectively defined for a.e.  $\mathbf{x} \in F$  as*

$$\llbracket v \rrbracket_F(\mathbf{x}) \stackrel{\text{def}}{=} v|_{T_1} - v|_{T_2}, \quad \{v\}_{\omega, F}(\mathbf{x}) \stackrel{\text{def}}{=} \omega_{T_1, F} v|_{T_1}(\mathbf{x}) + \omega_{T_2, F} v|_{T_2}(\mathbf{x}),$$

while, if  $F \in \mathcal{F}_h^b$  with  $F = \partial T \cap \partial \Omega$ , we set  $\{v\}_{\omega, F}(\mathbf{x}) = \llbracket v \rrbracket_F(\mathbf{x}) = v|_T(\mathbf{x})$ .

When  $v$  is vector-valued, the jump and average operators act component-wise. Whenever no confusion can arise, the subscript  $F$  and the variable  $\mathbf{x}$  are omitted, and we simply write  $\{v\}_\omega$  and  $\llbracket v \rrbracket$ . Moreover, in the classical case  $\omega_{T_1, F} = \omega_{T_2, F} = 1/2$  we also omit the subscript  $\omega$  and write  $\{v\}$  instead of  $\{v\}_\omega$ . The use of weighted averages in dG methods has been pointed out and used in various contexts, e.g., by Stenberg [36] and by Heinrich and Pietsch [32].

**Definition 2** (Matching simplicial submesh). *Let  $\mathcal{T}_h$  be a general mesh. We say that  $\mathfrak{S}_h$  is a matching simplicial submesh of  $\mathcal{T}_h$  if (i)  $\mathfrak{S}_h$  is a matching simplicial mesh, (ii) for all  $T' \in \mathfrak{S}_h$ , there is only one  $T \in \mathcal{T}_h$  such that  $T' \subset T$ , (iii) for all  $F' \in \mathfrak{F}_h$ , the set collecting the mesh faces of  $\mathfrak{S}_h$ , there is only one  $F \in \mathcal{F}_h$  such that  $F' \subset F$ .*

**Definition 3** (Shape- and contact-regularity). *We say that the mesh sequence  $(\mathcal{T}_h)_{h \in \mathcal{H}}$  is shape- and contact-regular if for all  $h \in \mathcal{H}$ ,  $\mathcal{T}_h$  admits a matching simplicial submesh  $\mathfrak{S}_h$  such that (i) the mesh sequence  $(\mathfrak{S}_h)_{h \in \mathcal{H}}$  is shape-regular in the usual sense of Ciarlet [16], meaning that there is a parameter  $\varrho_1 > 0$  independent of  $h$  such that for all  $T' \in \mathfrak{S}_h$ ,*

$$\varrho_1 h_{T'} \leq \delta_{T'},$$

where  $h_{T'}$  is the diameter of  $T'$  and  $\delta_{T'}$  the radius of the largest ball inscribed in  $T'$ , (ii) there is a parameter  $\varrho_2 > 0$  independent of  $h$  such that for all  $T \in \mathcal{T}_h$  and for all  $T' \in \mathfrak{S}_h$ ,

$$\varrho_2 h_T \leq h_{T'}.$$

**Lemma 4** (Bounds on geometric quantities). *Let  $(\mathcal{T}_h)_{h \in \mathcal{H}}$  be a shape- and contact-regular mesh sequence. Then, for all  $h \in \mathcal{H}$  and all  $T \in \mathcal{T}_h$ , (i) the number of mesh faces belonging to the boundary of an element is uniformly bounded in  $h$ ,*

$$N_\partial \stackrel{\text{def}}{=} \sup_{h \in \mathcal{H}, T \in \mathcal{T}_h} \text{card}(\mathcal{F}_T) < +\infty;$$

(ii) for all  $F \in \mathcal{F}_T$ ,

$$h_F \geq \varrho_1 \varrho_2 h_T. \quad (3)$$

*Proof.* See [21, Lemmata 1.33–1.34]. □

For all  $h \in \mathcal{H}$  and  $k \geq 0$ , we introduce the broken polynomial spaces of total degree  $\leq k$

$$\mathbb{P}_d^k(\mathcal{T}_h) \stackrel{\text{def}}{=} \{v \in L^2(\Omega) \mid v|_T \in \mathbb{P}_d^k(T)\},$$

with  $\mathbb{P}_d^k(T)$  given by the restriction to  $T$  of the functions in  $\mathbb{P}_d^k$ . Broken polynomial spaces are a special instance of broken Sobolev spaces: For all  $k \geq 1$ ,  $H^k(\mathcal{T}_h) \stackrel{\text{def}}{=} \{v \in L^2(\Omega) \mid v|_T \in H^k(T), \forall T \in \mathcal{T}_h\}$ . The shape- and contact-regularity of the mesh sequence  $(\mathcal{T}_h)_{h \in \mathcal{H}}$  are essential to infer the following results; see, e.g., [21, §1.4.3].

**Lemma 5** (Trace inequalities). *Let  $(\mathcal{T}_h)_{h \in \mathcal{H}}$  be a shape- and contact-regular mesh sequence. Then, for all  $h \in \mathcal{H}$ , all  $T \in \mathcal{T}_h$ , and all  $F \in \mathcal{F}_T$ ,*

$$\forall v_h \in \mathbb{P}_d^k(\mathcal{T}_h), \quad \|v_h\|_{L^2(F)} \leq C_{\text{tr}} h_F^{-1/2} \|v_h\|_{L^2(T)}, \quad (4)$$

$$\forall v \in H^1(\mathcal{T}_h), \quad \|v\|_{L^2(F)} \leq C_{\text{tr}, c} \left( h_T^{-1} \|v\|_{L^2(T)}^2 + h_T |v|_{H^1(T)}^2 \right)^{1/2}, \quad (5)$$

with  $C_{\text{tr}}$  and  $C_{\text{tr}, c}$  independent of the meshsize  $h$ .

The following property is necessary to obtain optimal estimates of the convergence rate.

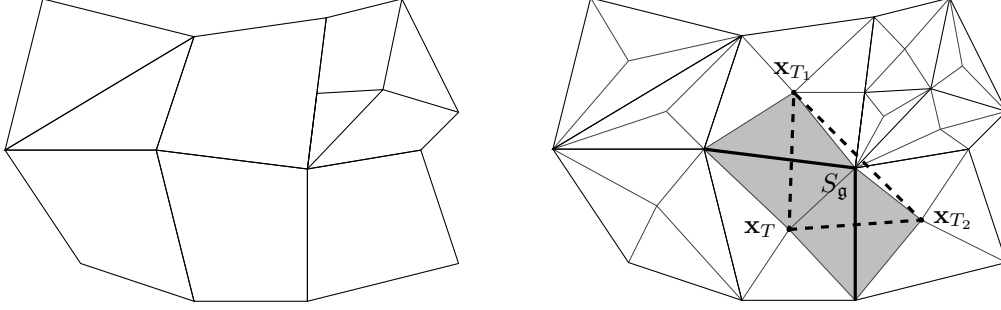


FIGURE 1. Mesh  $\mathcal{T}_h$  (left panel), submesh  $\mathcal{S}_h$  (right panel, thin lines), and simplex  $S_g$  (right panel, dashed) for the L-group  $\mathbf{g}$  in bold line. The patch  $\mathcal{P}_g$  is shaded.

**Definition 6** (Optimal polynomial approximation). *We say that the mesh sequence  $(\mathcal{T}_h)_{h \in \mathcal{H}}$  has optimal polynomial approximation properties if for all  $h \in \mathcal{H}$ , all  $T \in \mathcal{T}_h$ , and all polynomial degrees  $k$ , there is a linear interpolation operator  $\mathcal{I}_T^k : L^2(T) \rightarrow \mathbb{P}_d^k(T)$  such that for all  $s \in \{0, \dots, k+1\}$  and all  $v \in H^s(T)$ , there holds*

$$|v - \mathcal{I}_T^k v|_{H^m(T)} \leq C_{\text{app}} h_T^{s-m} |v|_{H^s(T)} \quad \forall m \in \{0, \dots, s\},$$

with  $C_{\text{app}}$  independent of  $T$  and  $h$ .

A sufficient condition to prove optimal approximation properties is, e.g., that the mesh is *finitely shaped*, i.e., that there exists a finite set of reference polyhedra such that every mesh element is the image of a reference polyhedron via an affine bijective mapping. This assumption yields a uniformly bounded Deny–Lions constant; see, e.g., [27, Theorem 1.103]. Other, more general, sufficient conditions can be devised. It is a simple matter to prove that the  $L^2$ -orthogonal projection has optimal approximation properties on mesh sequences with optimal approximation properties. In order to obtain a piecewise affine representation, it is useful to associate the unique degree of freedom of each element to a point with suitable properties.

**Definition 7** (Cell centers). *Let  $(\mathcal{T}_h)_{h \in \mathcal{H}}$  be a shape- and contact-regular mesh sequence. We say that  $(\mathcal{T}_h)_{h \in \mathcal{H}}$  admits a set of cell centers if (i) for every  $T \in \mathcal{T}_h$  there exists a point  $\mathbf{x}_T$  such that  $T$  is star-shaped with respect to  $\mathbf{x}_T$  (the cell-center) and (ii) there exists  $\varrho_3 > 0$  such that for all  $h \in \mathcal{H}$ , all  $T \in \mathcal{T}_h$ , and all  $F \in \mathcal{F}_T$ ,*

$$d_{T,F} \stackrel{\text{def}}{=} \text{dist}(\mathbf{x}_T, F) \geq \varrho_3 h_T. \quad (6)$$

**Definition 8** (Admissible mesh sequence). *We say that the mesh sequence  $(\mathcal{T}_h)_{h \in \mathcal{H}}$  is admissible if it is shape- and contact-regular, it has optimal polynomial approximation properties, and there exists a set of cell centers.*

Admissible mesh sequences include general polyhedral discretizations with possibly nonconforming interfaces; see Figure 1 for an example. For all  $h \in \mathcal{H}$  we can define a pyramidal submesh of  $\mathcal{T}_h$  as follows:

$$\mathcal{S}_h \stackrel{\text{def}}{=} \{\mathcal{P}_{T,F}\}_{T \in \mathcal{T}_h, F \in \mathcal{F}_T}, \quad (7)$$

where, for all  $T \in \mathcal{T}_h$  and all  $F \in \mathcal{F}_T$ ,  $\mathcal{P}_{T,F}$  denotes the open pyramid of apex  $\mathbf{x}_T$  and base  $F$ , i.e.,

$$\mathcal{P}_{T,F} \stackrel{\text{def}}{=} \{\mathbf{x} \in T \mid \exists \mathbf{y} \in F \setminus \partial F, \exists \theta \in (0, 1) \mid \mathbf{x} = \theta \mathbf{y} + (1 - \theta) \mathbf{x}_T\}.$$

The pyramids  $\{\mathcal{P}_{T,F}\}_{T \in \mathcal{T}_h, F \in \mathcal{F}_T}$  are non-degenerated owing to assumption (6). Since faces are planar, for all  $T \in \mathcal{T}_h$  and all  $F \in \mathcal{F}_T$  there holds

$$|\mathcal{P}_{T,F}|_d = \frac{|F|_{d-1} d_{T,F}}{d}, \quad (8)$$

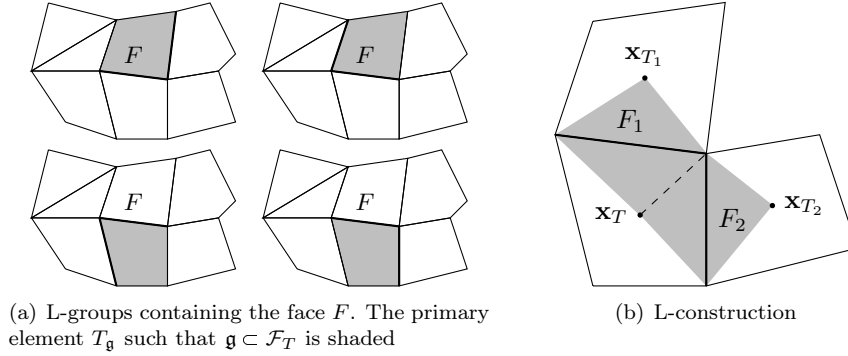


FIGURE 2. L-construction

and, for all  $T \in \mathcal{T}_h$ ,

$$\sum_{F \in \mathcal{F}_T} |\mathcal{P}_{T,F}|_d = \sum_{F \in \mathcal{F}_T} \frac{|F|_{d-1} d_{T,F}}{d} = |T|_d. \quad (9)$$

## 1.2. The L-construction

In this section we briefly recall the L-construction originally introduced by Aavatsmark, Eigestad, Mallison, and Nordbotten [3] and analyzed by Agélas, Di Pietro, and Droniou [4]. This construction is a fundamental ingredient in the definition of the ccG spaces presented in this work. Let  $\boldsymbol{\kappa} \in [L^\infty(\Omega)]^{d,d}$  denote a symmetric, uniformly elliptic tensor field such that the spectrum of  $\boldsymbol{\kappa}(\mathbf{x})$  lies in  $[\underline{\lambda}, \bar{\lambda}]$  for a.e.  $\mathbf{x} \in \Omega$ . For the sake of simplicity we assume henceforth that

$$\forall h \in \mathcal{H}, \quad \boldsymbol{\kappa} \in [\mathbb{P}_d^0(\mathcal{T}_h)]^{d,d},$$

and denote by  $\boldsymbol{\kappa}^{1/2}$  the piecewise symmetric, uniformly elliptic tensor field such that  $\boldsymbol{\kappa}^{1/2}(\mathbf{x})\boldsymbol{\kappa}^{1/2}(\mathbf{x}) = \boldsymbol{\kappa}(\mathbf{x})$  for a.e.  $\mathbf{x} \in \Omega$ . The piecewise regular case can be handled with minor modifications. For all  $F \in \mathcal{F}_h$  and all  $T \in \mathcal{T}_F$  we denote the diffusion in the normal direction by

$$\lambda_{T,F} \stackrel{\text{def}}{=} \boldsymbol{\kappa}|_T \mathbf{n}_F \cdot \mathbf{n}_F.$$

The key idea of the L-construction is to use  $d$  cell and boundary face values (provided, in this case, by a homogeneous boundary condition) to express a continuous piecewise affine function with continuous diffusive fluxes. The values are selected using  $d$  neighboring faces belonging to a cell and sharing a common vertex. More precisely, we define the set of *L-groups* as follows:

$$\mathcal{G} \stackrel{\text{def}}{=} \{\mathbf{g} \subset \mathcal{F}_T \cap \mathcal{F}_P, T \in \mathcal{T}_h, P \in \mathcal{N}_h \mid \text{card}(\mathbf{g}) = d\},$$

with  $\mathcal{F}_T$  and  $\mathcal{F}_P$  given by (1) and (2) respectively. For each  $\mathbf{g} \in \mathcal{G}$  we select a *primary element*  $T_{\mathbf{g}}$  such that  $\mathbf{g} \subset \mathcal{F}_{T_{\mathbf{g}}}$ ; see Figure 2(a). Such an element may not be unique as non-convex elements are allowed. We let, for the sake of brevity,

$$\mathbf{g}^i \stackrel{\text{def}}{=} \mathbf{g} \cap \mathcal{F}_h^i, \quad \mathbf{g}^b \stackrel{\text{def}}{=} \mathbf{g} \cap \mathcal{F}_h^b.$$

It is also useful to introduce a symbol for the set of cells concurring to the L-construction as well as for the union of the pyramids based on the group faces (see Figure 2): For all  $\mathbf{g} \in \mathcal{G}$ ,

$$\mathcal{T}_{\mathbf{g}} \stackrel{\text{def}}{=} \{T \in \mathcal{T}_h \mid T \in \mathcal{T}_F, F \in \mathbf{g}\}, \quad \mathcal{P}_{\mathbf{g}} \stackrel{\text{def}}{=} \bigcup_{F \in \mathbf{g}, T \in \mathcal{T}_F} \mathcal{P}_{T,F}. \quad (10)$$

The patch  $\mathcal{P}_{\mathbf{g}}$  is shaded in Figures 1 and 2(b). Let now  $\mathbf{g} \in \mathcal{G}$ . In what follows, for any  $D \subset \mathbb{R}$  of codimension  $l$  we denote by  $\langle \varphi \rangle_D$  the average  $\langle \varphi \rangle_D \stackrel{\text{def}}{=} \int_D \varphi / |D|_{d-l}$ .

For all  $\mathbf{v}_h \in \mathbb{V}_h$  we construct the function  $\xi_{\mathbf{v}_h}^{\mathbf{g}}$  piecewise affine on the family of pyramids  $\{\mathcal{P}_{T,F}\}_{F \in \mathbf{g}, T \in \mathcal{T}_F}$  such that:

- (i)  $\xi_{\mathbf{v}_h}^{\mathbf{g}}(\mathbf{x}_T) = v_T$  for all  $T \in \mathcal{T}_{\mathbf{g}}$  and  $\langle \xi_{\mathbf{v}_h}^{\mathbf{g}} \rangle_F = \xi_{\mathbf{v}_h}^{\mathbf{g}}(\bar{\mathbf{x}}_F) = 0$  for all  $F \in \mathbf{g}^b$ ;
- (ii)  $\xi_{\mathbf{v}_h}^{\mathbf{g}}$  is continuous across every interface in the group: For all  $F \in \mathbf{g}^i$  such that  $F \subset \partial T_1 \cap \partial T_2$ ,

$$\forall \mathbf{x} \in F, \quad \xi_{\mathbf{v}_h}^{\mathbf{g}}|_{T_1}(\mathbf{x}) = \xi_{\mathbf{v}_h}^{\mathbf{g}}|_{T_2}(\mathbf{x});$$

- (iii)  $\xi_{\mathbf{v}_h}^{\mathbf{g}}$  has continuous diffusive flux across every interface in the group: For all  $F \in \mathbf{g}^i$  such that  $F \subset \partial T_1 \cap \partial T_2$ ,

$$(\boldsymbol{\kappa} \xi_{\mathbf{v}_h}^{\mathbf{g}})|_{T_1} \cdot \mathbf{n}_F = (\boldsymbol{\kappa} \xi_{\mathbf{v}_h}^{\mathbf{g}})|_{T_2} \cdot \mathbf{n}_F.$$

(The quantities in both sides are constant since  $\xi_{\mathbf{v}_h}^{\mathbf{g}}$  is piecewise affine and the face  $F$  is planar).

The following result is instrumental to derive approximation properties for the ccG space.

**Proposition 9** (L-construction). *For all  $\mathbf{v}_h \in \mathbb{V}_h$  and all  $\mathbf{g} \in \mathcal{G}$ , there holds*

$$\mathbf{A}_{\mathbf{g}}(\nabla \xi_{\mathbf{v}_h}^{\mathbf{g}})|_{T_{\mathbf{g}}} = \mathbf{b}_{\mathbf{g}}(\mathbf{v}_h), \quad (11)$$

where the matrix  $\mathbf{A}_{\mathbf{g}} \in \mathbb{R}^{d,d}$  and the linear application  $\mathbf{b}_{\mathbf{g}} : \mathbb{V}_h \rightarrow \mathbb{R}^d$  are defined row-wise by

$$\mathbf{A}_{\mathbf{g}} \stackrel{\text{def}}{=} \begin{bmatrix} \left( \frac{\lambda_{T,F}}{d_{T,F}}(\mathbf{x}_T - \mathbf{x}_{T_{\mathbf{g}}}) + \boldsymbol{\kappa}_{T_{\mathbf{g}}} \mathbf{n}_{T_{\mathbf{g}},F} + \boldsymbol{\kappa}_{T,F} \mathbf{n}_{T,F} \right)_{\mathbf{g}^i \ni F \subset T_{\mathbf{g}} \cap T}^t \\ \left( \frac{\lambda_{T_{\mathbf{g}},F}}{d_{T_{\mathbf{g}},F}}(\bar{\mathbf{x}}_F - \mathbf{x}_{T_{\mathbf{g}}}) \right)_{F \in \mathbf{g}^b}^t \end{bmatrix}, \quad \mathbf{b}_{\mathbf{g}}(\mathbf{v}_h) = \begin{bmatrix} \left( \frac{\lambda_{T,F}}{d_{T,F}}(v_T - v_{T_{\mathbf{g}}}) \right)_{\mathbf{g}^i \ni F \subset T_{\mathbf{g}} \cap T} \\ \left( -\frac{\lambda_{T_{\mathbf{g}},F}}{d_{T_{\mathbf{g}},F}} v_{T_{\mathbf{g}}} \right)_{F \in \mathbf{g}^b} \end{bmatrix}. \quad (12)$$

*Proof.* See [4, Lemma 3.1]. □

In order to express  $\xi_{\mathbf{v}_h}^{\mathbf{g}}$  terms of the values  $\{v_T\}_{T \in \mathcal{T}_{\mathbf{g}}}$ , the matrix  $\mathbf{A}_{\mathbf{g}}$  must be invertible. Simple sufficient invertibility conditions are discussed in what follows. In practice, however, the invertibility can be checked directly, and backup strategies can be devised; see Remark 11.

### 1.3. The ccG space

In this section we introduce a ccG space based on the L-construction and on a local gradient inspired by Green's formula. For a face  $F \in \mathcal{F}_h^i$ , let  $\mathcal{G}_F$  denote the set of L-groups containing  $F$ ,

$$\mathcal{G}_F \stackrel{\text{def}}{=} \{\mathbf{g} \in \mathcal{G} \mid F \in \mathbf{g}\}.$$

**Assumption 10** (Existence of an L-group leading to an invertible matrix for each interface). *We assume that, for all  $F \in \mathcal{F}_h^i$ ,  $\mathcal{G}_F$  is non-empty and there exists  $\mathbf{g} \in \mathcal{G}_F$  such that the matrix  $\mathbf{A}_{\mathbf{g}}$  defined by (12) is invertible.*

For the sake of simplicity, Assumption 10 holds tacitly from this point on. Should this not be true, backup strategies can be envisaged, as discussed in Remark 11. The discrete gradient is obtained as follows:

- (i) For every  $F \in \mathcal{F}_h^i$  we select a *unique* L-group  $\mathbf{g}_F$  yielding an invertible matrix and, for all  $\mathbf{v}_h \in \mathbb{V}_h$ , we denote by  $\xi_{\mathbf{v}_h}^{\mathbf{g}_F}$  the piecewise affine function on  $\{\mathcal{P}_{T,F}\}_{T \in \mathcal{T}_F}$  obtained from the L-construction. Whenever more than one such group is present, we select  $\mathbf{g} \in \mathcal{G}_F$  for which  $\|\mathbf{A}_{\mathbf{g}}^{-1}\|_2$  is minimal (this ensures the best approximation properties; see Lemma 20 and Assumption 21). For convenience of notation, for all boundary faces  $F \subset \partial T \cap \partial \Omega$  we introduce the affine function  $\xi_{\mathbf{v}_h}^{\mathbf{g}_F}$  on  $\mathcal{P}_{T,F}$  such that  $\xi_{\mathbf{v}_h}^{\mathbf{g}_F}|_F \equiv 0$  on  $F$  and  $\xi_{\mathbf{v}_h}^{\mathbf{g}_F}(\mathbf{x}_T) = v_T$ . Such a function is well-defined since faces are planar.



- (ii) We then define the *trace reconstruction operator*  $\mathfrak{T}_h : \mathbb{V}_h \rightarrow \mathbb{R}^{\mathcal{F}_h}$  which realizes the mapping  $\mathbb{V}_h \ni \mathbf{v}_h \mapsto \mathfrak{T}_h(\mathbf{v}_h) = (v_F)_{F \in \mathcal{F}_h}$  with  $(v_F)_{F \in \mathcal{F}_h} \in \mathbb{R}^{\mathcal{F}_h}$  and, for all  $F \in \mathcal{F}_h$ ,

$$v_F = \langle \xi_{\mathbf{v}_h}^{\mathfrak{g}F} \rangle_F = \xi_{\mathbf{v}_h}^{\mathfrak{g}F}(\bar{\mathbf{x}}_F). \quad (13)$$

- (iii) The *gradient reconstruction operator*  $\mathfrak{G}_h : \mathbb{V}_h \rightarrow [\mathbb{P}_d^0(\mathcal{T}_h)]^d$  is defined following Droniou and Eymard [23] as the application  $\mathbb{V}_h \ni \mathbf{v}_h \mapsto \mathfrak{G}_h(\mathbf{v}_h) \in [\mathbb{P}_d^0(\mathcal{T}_h)]^d$  with

$$\forall T \in \mathcal{T}_h, \quad \mathfrak{G}_h(\mathbf{v}_h)|_T = \frac{1}{|T|^d} \sum_{F \in \mathcal{F}_T} |F|_{d-1} (v_F - v_T) \mathbf{n}_{T,F}. \quad (14)$$

where we have set  $(v_F)_{F \in \mathcal{F}_h} = \mathfrak{T}_h(\mathbf{v}_h)$ .

The expression (14) is inspired by Green's formula. As such,  $v_F$  has to be related to the average on the face  $F$ . For affine functions, the average coincides with the barycentric value, thereby justifying the choice in (13). The linear reconstruction operator  $\mathfrak{R}_h^{\text{ccg}} : \mathbb{V}_h \rightarrow \mathbb{P}_d^1(\mathcal{T}_h)$  leading to the ccG space is finally defined as the mapping  $\mathbb{V}_h \ni \mathbf{v}_h \mapsto v_h \in \mathbb{P}_d^1(\mathcal{T}_h)$  with

$$\forall T \in \mathcal{T}_h, \forall \mathbf{x} \in T, \quad v_h|_T(\mathbf{x}) = v_T + \mathfrak{G}_h(\mathbf{v}_h)|_T \cdot (\mathbf{x} - \mathbf{x}_T).$$

In the above formula,  $v_T$  is interpreted as the point value in  $\mathbf{x}_T$ . The incomplete polynomial space  $V_h^{\text{ccg}}$  upon which the ccG method relies is the image of the algebraic space of degrees of freedom  $\mathbb{V}_h$  through  $\mathfrak{R}_h^{\text{ccg}}$ ,

$$V_h^{\text{ccg}} \stackrel{\text{def}}{=} \mathfrak{R}_h^{\text{ccg}}(\mathbb{V}_h) \subset \mathbb{P}_d^1(\mathcal{T}_h).$$

Proceeding by contradiction, it is a simple matter to prove that  $\mathfrak{R}_h^{\text{ccg}}$  is injective from  $\mathbb{V}_h$  to  $\mathbb{P}_d^1$  and, hence, bijective from  $\mathbb{V}_h$  to  $V_h^{\text{ccg}}$ . It is also useful to observe that the discrete space  $V_h^{\text{ccg}}$  is contained in the larger broken Sobolev space  $H^1(\mathcal{T}_h)$ .

*Remark 11* (Backup strategies). Numerical evidence [3, 4] shows that Assumption 10 is true in most circumstances. In the presence of highly heterogeneous diffusion tensors or for extremely deformed meshes, it may occur, however, that no L-group yielding an invertible matrix can be found for some interfaces. In this case, several backup strategies are possible, and the choice can be mostly guided by the implementation at hand. We mention two possibilities, although many more are possible. A first strategy relying on the tight link of ccG methods to dG methods consists in using a full  $\mathbb{P}_d^1$  basis on the mesh elements which have at least one face for which no invertible L-group exists. This results in a local increase of the number of unknowns and yields a completely robust method. Whenever exactly one unknown per cell is allowed, the L-construction can be replaced by the barycentric interpolator of [30, §2.2]. The method can then be constructed provided the  $d$  points involved in the barycentric interpolation form a non-degenerate simplex (an extremely mild mesh regularity assumption). In this case, however, a local loss of precision may be observed as the barycentric interpolator does not honor the heterogeneity of  $\kappa$ . We emphasize that no backup strategy whatsoever was required in the numerical examples of §4.

*Remark 12* (Polynomial order adaptivity). In the spirit of the previous remark, the polynomial degree can be adapted in ccG methods by using full polynomial spaces inside selected elements. This is naturally handled whenever the discrete formulation relying on the ccG space is inspired by a dG method, as is the case for all the examples provided in this work.

#### 1.4. Discrete functional analysis

This section collects some discrete functional analysis results that will be used in the rest of the paper. The material is mainly adapted from [20, §6], to which we refer for further details. We state, in particular, the

Sobolev embeddings for broken polynomial spaces in the Hilbertian case and the discrete counterpart of the Rellich–Kondrachov theorem. To this end, we introduce the following norm on  $H^1(\mathcal{T}_h)$ :

$$\|v\|^2 \stackrel{\text{def}}{=} \|\nabla_h v\|_{[L^2(\Omega)]^d}^2 + |v|_J^2, \quad |v|_J^2 \stackrel{\text{def}}{=} \sum_{F \in \mathcal{F}_h} \frac{1}{h_F} \|\llbracket v \rrbracket\|_{L^2(F)}^2, \quad (15)$$

where  $\nabla_h$  denotes the broken gradient on  $\mathcal{T}_h$ , *i.e.*, for all  $v \in H^1(\mathcal{T}_h)$ ,  $\nabla_h v|_T = \nabla v$  for all  $T \in \mathcal{T}_h$ .

**Theorem 13** (Discrete Sobolev embeddings, Hilbertian case). *For all  $q$  such that (i)  $1 \leq q < +\infty$  if  $d \leq 2$ , (ii)  $1 \leq q \leq \frac{2d}{d-2}$  if  $d > 2$ , there is  $\sigma_q$  independent of  $h$  such that*

$$\forall v_h \in \mathbb{P}_d^k(\mathcal{T}_h), \quad \|v_h\|_{L^q(\Omega)} \leq \sigma_q \|v_h\|. \quad (16)$$

The constant  $\sigma_q$  additionally depends on  $k$ ,  $|\Omega|_d$  and on the mesh regularity parameters.

The discrete Poincaré inequality is obtained as a special case for  $q = 2$ . For a proof in the more general non-Hilbertian case we refer to [20]. Sequences in  $(V_h^{\text{ccg}})_{h \in \mathcal{H}}$  uniformly bounded in the  $\|\cdot\|$ -norm possess an important compactness property. Following the idea of Brezzi, Manzini, Marini, Pietra, and Russo [12] modified according to [5, §3.1], we introduce for all  $F \in \mathcal{F}_h$  the local lifting operator  $\mathbf{r}_{\omega, F} : L^2(F) \rightarrow [\mathbb{P}_d^0(\mathcal{T}_h)]^d$  which maps every  $\varphi \in L^2(F)$  to  $\mathbf{r}_{\omega, F}(\varphi) \in [\mathbb{P}_d^0(\mathcal{T}_h)]^d$  solution to:

$$\int_{\Omega} \mathbf{r}_{\omega, F}(\varphi) \cdot \boldsymbol{\tau}_h = \int_F \varphi \{\boldsymbol{\tau}_h\}_{\omega} \cdot \mathbf{n}_F, \quad \forall \boldsymbol{\tau}_h \in [\mathbb{P}_d^0(\mathcal{T}_h)]^d. \quad (17)$$

For further use we also introduce the global lifting  $\mathbf{R}_{\omega, h}(\varphi) \stackrel{\text{def}}{=} \sum_{F \in \mathcal{F}_h} \mathbf{r}_{\omega, F}(\varphi)$ . If  $\omega_{T_1, F} = \omega_{T_2, F} = 1/2$  for all  $F \in \mathcal{F}_h^i$  with  $F \subset \partial T_1 \cap \partial T_2$  we simply write  $\mathbf{r}_F$  and  $\mathbf{R}_h$ . The lifting operators can be used to define a corrected discrete gradient accounting for the jumps across mesh interfaces and on  $\partial\Omega$ . More precisely, we introduce the linear operator  $\mathbf{G}_{\omega, h} : H^1(\mathcal{T}_h) \rightarrow [L^2(\Omega)]^d$  defined as follows: For all  $v \in H^1(\mathcal{T}_h)$ ,

$$\mathbf{G}_{\omega, h}(v) \stackrel{\text{def}}{=} \nabla_h v - \mathbf{R}_{\omega, h}(\llbracket v \rrbracket). \quad (18)$$

As before, if  $\omega_{T_1, F} = \omega_{T_2, F} = 1/2$  for all  $F \in \mathcal{F}_h^i$  with  $F \subset \partial T_1 \cap \partial T_2$  we omit the subscript  $\omega$  and write  $\mathbf{G}_h$ .

**Lemma 14** (Discrete Rellich–Kondrachov). *Let  $(v_h)_{h \in \mathcal{H}}$  be a sequence in  $(\mathbb{P}_d^k(\mathcal{T}_h))_{h \in \mathcal{H}}$ ,  $k > 0$ , uniformly bounded in the  $\|\cdot\|$ -norm. Then, there exists a function  $v \in H_0^1(\Omega)$  such that as  $h \rightarrow 0$ , up to a subsequence  $v_h \rightarrow v$  strongly in  $L^2(\Omega)$ .*

*Proof.* See [21, Theorem 6.3]. □

**Lemma 15** (Weak asymptotic consistency of  $\mathbf{G}_{\omega, h}$  for sequences of discrete functions). *Let  $(v_h)_{h \in \mathcal{H}}$  be a sequence in  $(\mathbb{P}_d^k(\mathcal{T}_h))_{h \in \mathcal{H}}$ ,  $k > 0$ , uniformly bounded in the  $\|\cdot\|$ -norm. Then, as  $h \rightarrow 0$ ,  $\mathbf{G}_{\omega, h}(v_h) \rightharpoonup \nabla v$  weakly in  $[L^2(\Omega)]^d$ , where  $v \in H_0^1$  is the limit provided by Theorem 14.*

*Proof.* Denote by  $\pi_h$  the  $L^2$ -orthogonal projection onto  $[\mathbb{P}_d^1(\mathcal{T}_h)]^d$ . To prove the weak convergence of  $\mathbf{G}_{\omega, h}(v_h)$  to  $\nabla v$ , let  $\Phi \in [C_0^\infty(\Omega)]^d$ , set  $\Phi_h \stackrel{\text{def}}{=} \pi_h \Phi$ , and observe that

$$\int_{\Omega} \mathbf{G}_{\omega, h}(v_h) \cdot \Phi = - \int_{\Omega} v_h \nabla \cdot \Phi + \sum_{T \in \mathcal{T}_h} \int_{\partial T} \Phi \cdot \mathbf{n}_T v_h - \sum_{F \in \mathcal{F}_h} \int_{\Omega} \mathbf{r}_{\omega, F}(\llbracket v_h \rrbracket) \cdot \Phi_h = - \int_{\Omega} v_h \nabla \cdot \Phi + \sum_{F \in \mathcal{F}_h} \int_F \llbracket v_h \rrbracket \{\Phi - \Phi_h\}_{\omega} \cdot \mathbf{n}_F,$$

where we have used the definition of the  $L^2$ -orthogonal projection, the fact that  $\{\Phi\}_{\omega} = \Phi$  on every  $F \in \mathcal{F}_h$ , and (17). Denote by  $\mathfrak{T}_1$  and  $\mathfrak{T}_2$  the addends in the right-hand side. Clearly,  $\mathfrak{T}_1 \rightarrow \int_{\Omega} v \nabla \cdot \Phi$ . For the second term, the Cauchy–Schwarz inequality yields  $\mathfrak{T}_2 \leq |v_h|_J \times \left( \sum_{F \in \mathcal{F}_h} h_F \int_F |\{\Phi - \Phi_h\}_{\omega}|^2 \right)^{1/2}$ , which tends to zero

owing to the approximation properties of the  $L^2$ -orthogonal projection for the smooth function  $\Phi$  together with the fact that  $|v_h|_J$  is uniformly bounded by assumption.  $\square$

It is important to observe that, since  $V_h^{\text{ccg}} \subset \mathbb{P}_d^1(\mathcal{T}_h)$ , both Theorems 13 and 14 and Lemma 15 hold *a fortiori* for the sequence of ccG spaces  $(V_h^{\text{ccg}})_{h \in \mathcal{H}}$ .

## 2. PURE DIFFUSION

### 2.1. The discrete problem

The space  $V_h^{\text{ccg}}$  is used in this section to discretize the heterogeneous diffusion problem

$$\begin{aligned} -\nabla \cdot (\boldsymbol{\kappa} \nabla u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

with source term  $f \in L^2(\Omega)$ . The weak form of this problem is the following:

$$\text{Find } u \in V \text{ s.t. } a(u, v) = \int_{\Omega} f v \text{ for all } v \in V, \quad (19)$$

with  $V = H_0^1(\Omega)$  and  $a(u, v) \stackrel{\text{def}}{=} \int_{\Omega} \boldsymbol{\kappa} \nabla u \cdot \nabla v$ . The functions in  $V_h^{\text{ccg}}$  are possibly discontinuous across interfaces, and  $V_h^{\text{ccg}}$  is therefore not  $V$ -conforming. In order to devise a suitable discretization, we take inspiration from the work of Arnold [6] on the weak enforcement of potential continuity across interfaces. In particular, we consider the modification proposed by Di Pietro, Ern, and Guermond [22] in the context of degenerate diffusion-advection-reaction problems to attain robustness with respect to the diffusion tensor  $\boldsymbol{\kappa}$ . In [22] robustness is achieved by relating the weights introduced in Definition 1 to the diffusion on both sides of an interface. The idea of diffusion-dependent weights can be traced back to the work of Burman and Zunino [15] on mortaring techniques for a singularly perturbed diffusion-advection equation. For all  $F \in \mathcal{F}_h^i$  such that  $F \subset \partial T_1 \cap \partial T_2$ , we let

$$\omega_{T_1, F} \stackrel{\text{def}}{=} \frac{\lambda_{T_2, F}}{\lambda_{T_1, F} + \lambda_{T_2, F}}, \quad \omega_{T_2, F} \stackrel{\text{def}}{=} \frac{\lambda_{T_1, F}}{\lambda_{T_1, F} + \lambda_{T_2, F}}.$$

The analysis in the spirit of Céa's Lemma requires to extend the discrete bilinear form to a continuous space containing the exact solution. In order to guarantee that boundary terms remain well-defined when doing so, it is useful to introduce a space  $V_{\dagger}$  featuring additional local regularity with respect to  $V$ .

**Definition 16** (Space  $V_{\dagger}$ ). *We let  $V_{\dagger}$  denote the subspace of  $V$  spanned by functions  $v$  such that, for all  $T \in \mathcal{T}_h$ ,  $\nabla v \cdot \mathbf{n}_T|_T \in L^2(\partial T)$ .*

The space containing both the discrete and the continuous solution is defined as

$$V_{\dagger h} \stackrel{\text{def}}{=} V_h^{\text{ccg}} + V_{\dagger}.$$

We are now ready to define the bilinear form  $a_h \in \mathcal{L}(V_{\dagger h} \times V_{\dagger h}, \mathbb{R})$ ,

$$a_h(v, w) \stackrel{\text{def}}{=} \int_{\Omega} \boldsymbol{\kappa} \nabla_h v \cdot \nabla_h w - \sum_{F \in \mathcal{F}_h} \int_F \{ \boldsymbol{\kappa} \nabla_h v \}_{\omega} \cdot \mathbf{n}_F \llbracket w \rrbracket - \sum_{F \in \mathcal{F}_h} \int_F \llbracket v \rrbracket \{ \boldsymbol{\kappa} \nabla_h w \}_{\omega} \cdot \mathbf{n}_F + \sum_{F \in \mathcal{F}_h} \eta \frac{\gamma_F}{h_F} \int_F \llbracket v \rrbracket \llbracket w \rrbracket, \quad (20)$$

where  $\eta > 0$  denotes a user-dependent penalty parameter while  $\gamma_F$  is such that

$$\gamma_F \stackrel{\text{def}}{=} \begin{cases} \frac{\lambda_{T_1, F} \lambda_{T_2, F}}{\lambda_{T_1, F} + \lambda_{T_2, F}} & \text{if } F \in \mathcal{F}_h^i, F \subset \partial T_1 \cap \partial T_2, \\ \lambda_{T, F} & \text{if } F \in \mathcal{F}_h^b, F \subset \partial T \cap \partial\Omega. \end{cases}$$

The discrete problem reads

$$\text{Find } u_h \in V_h^{\text{ccg}} \text{ s.t. } a_h(u_h, v_h) = \int_{\Omega} f v_h \text{ for all } v_h \in V_h^{\text{ccg}}. \quad (21)$$

## 2.2. Basic error estimate

We introduce the following data dependent norms on  $V_{\dagger h}$ :

$$\|v\|_{\kappa}^2 \stackrel{\text{def}}{=} \|\kappa^{1/2} \nabla_h v\|_{[L^2(\Omega)]^d}^2 + |v|_{\text{J}, \kappa}^2, \quad \|v\|_{\kappa, \dagger}^2 \stackrel{\text{def}}{=} \|v\|_{\kappa}^2 + \sum_{T \in \mathcal{T}_h} h_T \|\kappa^{1/2} \nabla v \cdot \mathbf{n}_T\|_{L^2(\partial T)}^2, \quad (22)$$

where the jump seminorm is given by

$$|v|_{\text{J}, \kappa}^2 \stackrel{\text{def}}{=} \sum_{F \in \mathcal{F}_h} \frac{\gamma_F}{h_F} \|\llbracket v \rrbracket\|_{L^2(F)}^2.$$

**Lemma 17** (Properties of the bilinear form  $a_h$ ). *The bilinear form  $a_h$  enjoys the following properties:*

(i) Consistency. Assume  $u \in V_{\dagger}$ . Then, for all  $v_h \in V_h^{\text{ccg}}$ ,

$$a_h(u, v_h) = \int_{\Omega} f v_h;$$

(ii) Coercivity. For all  $\eta > \underline{\eta} = C_{\text{tr}}^2 N_{\delta}$  there holds

$$\forall v_h \in V_h^{\text{ccg}}, \quad a_h(v_h, v_h) \geq C_{\text{sta}} \|v_h\|_{\kappa}^2,$$

with  $C_{\text{sta}} \stackrel{\text{def}}{=} (\eta - C_{\text{tr}}^2 N_{\delta}) \{\max(1/2, \eta + C_{\text{tr}}^2 N_{\delta})\}^{-1}$  independent of both  $\kappa$  and  $h$ ;

(iii) Boundedness. There is  $C_{\text{bnd}}$  independent of the meshsize  $h$  and of the diffusion coefficient  $\kappa$  such that

$$\forall (v, w_h) \in V_{\dagger h} \times V_h^{\text{ccg}}, \quad a_h(v, w_h) \leq C_{\text{bnd}} \|v\|_{\kappa, \dagger} \|w_h\|_{\kappa}.$$

*Proof.* We preliminarily note the following bound resulting from the Cauchy–Schwarz inequality:

$$\forall (v, w_h) \in V_{\dagger h} \times V_h^{\text{ccg}}, \quad \left| \sum_{F \in \mathcal{F}_h} \int_F \{\kappa \nabla_h v\}_{\omega} \cdot \mathbf{n}_F \llbracket w_h \rrbracket \right| \leq \left( \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} h_F \|\kappa^{1/2} \nabla v|_{T \cdot \mathbf{n}_F}\|_{L^2(F)}^2 \right)^{1/2} |w_h|_{\text{J}, \kappa}. \quad (23)$$

*Consistency.* Plugging the exact solution  $u$  into the first argument of  $a_h$  and integrating by parts we obtain, for all  $v_h \in V_h^{\text{ccg}}$ ,

$$a_h(u, v_h) = - \int_{\Omega} \nabla \cdot (\kappa \nabla u) v_h + \sum_{F \in \mathcal{F}_h} \int_F \llbracket \kappa \nabla_h u \rrbracket \cdot \mathbf{n}_F \{v_h\}_{\overline{\omega}} - \sum_{F \in \mathcal{F}_h} \int_F \llbracket u \rrbracket \{\kappa \nabla_h v_h\}_{\omega} \cdot \mathbf{n}_F + \sum_{F \in \mathcal{F}_h} \eta \frac{\gamma_F}{h_F} \int_F \llbracket u \rrbracket \llbracket v_h \rrbracket,$$

where  $\{v_h\}_{\overline{\omega}} \stackrel{\text{def}}{=} \omega_{T_2, F} v_h|_{T_1} + \omega_{T_1, F} v_h|_{T_2}$ . The conclusion follows using the fact that  $-\nabla \cdot (\kappa \nabla u) = f$  for a.e.  $x \in \Omega$  for the first term and the fact that both  $u$  and its diffusive flux are continuous across interfaces and  $u$  vanishes on  $\partial\Omega$  to infer that the remaining terms are zero.

*Coercivity.* It is inferred from the bound (23) together with the discrete trace inequality (4) that

$$\forall v_h \in V_h^{\text{ccg}}, \quad \left| \sum_{F \in \mathcal{F}_h} \int_F \{\kappa \nabla_h v_h\}_{\omega} \cdot \mathbf{n}_F \llbracket v_h \rrbracket \right| \leq C_{\text{tr}} N_{\delta}^{1/2} \|\kappa^{1/2} \nabla_h v_h\|_{[L^2(\Omega)]^d} |v_h|_{\text{J}, \kappa}.$$

Using the inequality  $2ab \leq \epsilon a^2 + 1/\epsilon b^2$  valid for any  $\epsilon > 0$  together with the above bound, we obtain

$$\begin{aligned} a_h(v_h, v_h) &= \|\boldsymbol{\kappa}^{1/2} \nabla_h v_h\|_{[L^2(\Omega)]^d}^2 - 2 \sum_{F \in \mathcal{F}_h} \int_F \{\boldsymbol{\kappa} \nabla_h v_h\}_{\omega} \cdot \mathbf{n}_F \llbracket v_h \rrbracket + \eta |v_h|_{J, \boldsymbol{\kappa}}^2 \\ &\geq (1 - C_{\text{tr}}^2 N_\epsilon \epsilon) \|\boldsymbol{\kappa}^{1/2} \nabla_h v_h\|_{[L^2(\Omega)]^d}^2 + (\eta - 1/\epsilon) |v_h|_{J, \boldsymbol{\kappa}}^2. \end{aligned}$$

The desired result follows by properly selecting  $\epsilon$ .

*Boundedness.* Let  $(v, w_h) \in V_{\dagger h} \times V_h^{\text{ccg}}$  and let  $\mathfrak{T}_1, \dots, \mathfrak{T}_4$  denote the addends in the expression of  $a_h(v, w_h)$  obtained from (20). Using the Cauchy–Schwarz inequality it is readily inferred that  $|\mathfrak{T}_1 + \mathfrak{T}_4| \leq (1 + \eta) \|v\|_{\boldsymbol{\kappa}} \|w_h\|_{\boldsymbol{\kappa}}$ . Moreover, owing to the bound (23),  $|\mathfrak{T}_2| \leq \|v\|_{\boldsymbol{\kappa}, \dagger} |w_h|_{J, \boldsymbol{\kappa}} \leq \|v\|_{\boldsymbol{\kappa}, \dagger} \|w_h\|_{\boldsymbol{\kappa}}$ . Finally,  $|\mathfrak{T}_3| \leq C_{\text{tr}} N_\epsilon^{1/2} |v|_{J, \boldsymbol{\kappa}} \|\boldsymbol{\kappa}^{1/2} \nabla_h w_h\|_{[L^2(\Omega)]^d}$ , and the conclusion follows.  $\square$

**Theorem 18** ( $\|\cdot\|_{\boldsymbol{\kappa}}$ -error estimate). *Assume  $u \in V_{\dagger}$ . There holds*

$$\|u - u_h\|_{\boldsymbol{\kappa}} \leq \left(1 + \frac{C_{\text{bnd}}}{C_{\text{sta}}}\right) \inf_{v_h \in V_h^{\text{ccg}}} \|u - v_h\|_{\boldsymbol{\kappa}, \dagger}.$$

*Proof.* By the triangular inequality, for all  $v_h \in V_h^{\text{ccg}}$  there holds  $\|u - u_h\|_{\boldsymbol{\kappa}} \leq \|u - v_h\|_{\boldsymbol{\kappa}} + \|v_h - u_h\|_{\boldsymbol{\kappa}}$ . Using the coercivity, consistency and boundedness of  $a_h$  it is inferred

$$C_{\text{sta}} \|u_h - v_h\|_{\boldsymbol{\kappa}}^2 \leq a_h(u_h - v_h, u_h - v_h) = a_h(u - v_h, u_h - v_h) \leq C_{\text{bnd}} \|u - v_h\|_{\boldsymbol{\kappa}, \dagger} \|u_h - v_h\|_{\boldsymbol{\kappa}},$$

hence  $\|u_h - v_h\|_{\boldsymbol{\kappa}} \leq C_{\text{bnd}} C_{\text{sta}}^{-1} \|u - v_h\|_{\boldsymbol{\kappa}, \dagger}$ . The conclusion follows observing that  $\|u - v_h\|_{\boldsymbol{\kappa}} \leq \|u - v_h\|_{\boldsymbol{\kappa}, \dagger}$ .  $\square$

An important remark is that the error estimate in Theorem 18 is robust in that the multiplicative constant in the right-hand side does not depend on  $\boldsymbol{\kappa}$ . To infer a convergence rate from Theorem 18 we have to study the approximation properties of the discrete space  $V_h^{\text{ccg}}$  and further bound the right-hand side.

### 2.3. Convergence rate

#### 2.3.1. The heterogeneous case

We first consider the heterogeneous case and focus on exact solutions exhibiting further local regularity. For  $k \geq 0$  define the spaces of piecewise regular functions

$$C^k(\mathcal{T}_h) \stackrel{\text{def}}{=} \{v \in L^2(\Omega) \mid v|_T \in C^k(\overline{T}), \forall T \in \mathcal{T}_h\},$$

classically equipped with the norm

$$\|v\|_{C^k(\mathcal{T}_h)} \stackrel{\text{def}}{=} \max_{T \in \mathcal{T}_h} \max_{0 \leq l \leq k, 1 \leq i \leq d} \|\partial_i^l v\|_{C^0(\overline{T})},$$

and  $\|w\|_{C^0(\overline{T})} \stackrel{\text{def}}{=} \max_{x \in \overline{T}} |w(x)|$  for all  $w \in C^0(\overline{T})$ .

**Theorem 19** (Test space). *Let  $\mathcal{Q}_{\mathcal{T}_h, \boldsymbol{\kappa}}$  be the space of functions  $\varphi : \overline{\Omega} \rightarrow \mathbb{R}$  enjoying the following properties:*

- (i) Global and local regularity. *The function  $\varphi$  belongs to  $C_0(\overline{\Omega}) \cap C^2(\mathcal{T}_h)$ ;*
- (ii) Continuity of the tangential derivatives at interfaces. *For all  $F \in \mathcal{F}_h^i$  such that  $F \subset \partial T_1 \cap \partial T_2$ , all vector  $\mathbf{t}$  parallel to  $F$ , and all  $\mathbf{x} \in F$ ,*

$$\nabla \varphi|_{T_1}(\mathbf{x}) \cdot \mathbf{t} = \nabla \varphi|_{T_2}(\mathbf{x}) \cdot \mathbf{t};$$

- (iii) Continuity of the diffusive flux at interfaces. *For all  $F \in \mathcal{F}_h^i$  such that  $F \subset \partial T_1 \cap \partial T_2$ , and all  $\mathbf{x} \in F$ ,*

$$(\boldsymbol{\kappa} \nabla \varphi)|_{T_1}(\mathbf{x}) \cdot \mathbf{n}_F = (\boldsymbol{\kappa} \nabla \varphi)|_{T_2}(\mathbf{x}) \cdot \mathbf{n}_F.$$

Then,  $\mathcal{Q}_{\mathcal{T}_h, \kappa}$  is dense in  $H_0^1(\Omega)$ .

An important remark is that  $\mathcal{Q}_{\mathcal{T}_h, \kappa}$  is a subspace of  $V_{\dagger}$ , and so the assumptions required to prove the error estimate in Theorem 18 are met by solutions that are in  $\mathcal{Q}_{\mathcal{T}_h, \kappa}$ . In what follows we denote by  $\mathcal{I}_h^{\text{ccg}} : C^0(\mathcal{T}_h) \rightarrow \mathbb{V}_h$  the interpolator at cell centers which maps every function  $v \in C^0(\mathcal{T}_h)$  onto the vector

$$\mathcal{I}_h^{\text{ccg}}(v) = (v(\mathbf{x}_T))_{T \in \mathcal{T}_h} \in \mathbb{V}_h.$$

**Lemma 20** (Consistency of the gradient reconstruction). *For all  $h \in \mathcal{H}$ , all  $v \in \mathcal{Q}_{\mathcal{T}_h, \kappa}$ , and all  $F \in \mathcal{F}_h$ ,*

$$|\nabla \xi_{\mathbf{v}_h}^{\mathbf{g}_F} - \nabla v(\mathbf{x}_{T_{\mathbf{g}_F}})| \leq C (1 + \|\mathbf{A}_{\mathbf{g}_F}^{-1}\|_2) h_{T_{\mathbf{g}_F}},$$

where  $\mathbf{v}_h = \mathcal{I}_h^{\text{ccg}}(v)$  and  $C$  depends on  $\kappa$ , on the mesh regularity parameters, and on  $\|v\|_{C^2(\mathcal{T}_h)}$  but not on  $h$ .

*Proof.* Let  $F \in \mathcal{F}_h^i$ . It is proved in [4, Lemma 3.3] that  $|\nabla \xi_{\mathbf{v}_h}^{\mathbf{g}_F} - \nabla v(\mathbf{x}_{T_{\mathbf{g}_F}})| \leq C (1 + \|\mathbf{A}_{\mathbf{g}_F}^{-1}\|_2) \max_{T \in \mathcal{T}_{\mathbf{g}_F}} h_T$ . To bound the term in the right-hand side, use the shape- and contact-regularity of the mesh to conclude that, for all  $T \in \mathcal{T}_{\mathbf{g}_F} \setminus \{T_{\mathbf{g}_F}\}$ ,  $\varrho_1 \varrho_2 h_T \leq h_F \leq h_{T_{\mathbf{g}_F}}$ . Hence,  $\max_{T \in \mathcal{T}_{\mathbf{g}_F}} h_T \leq (\varrho_1 \varrho_2)^{-1} h_{T_{\mathbf{g}_F}}$ , which yields the desired result. A similar argument can be used for boundary faces.  $\square$

In order to estimate the convergence rate of the method (21) for solutions in  $\mathcal{Q}_{\mathcal{T}_h, \kappa}$ , we introduce the following.

**Assumption 21** (Uniform bound on  $\|\mathbf{A}_{\mathbf{g}_F}^{-1}\|_2$ ). *We assume that there exists  $\Lambda < +\infty$  independent of the meshsize  $h$  uniformly bounding the set  $\{\|\mathbf{A}_{\mathbf{g}_F}^{-1}\|_2\}_{h \in \mathcal{H}, F \in \mathcal{F}_h^i}$  from above.*

**Lemma 22** (Consistency of the trace reconstruction). *Let  $v \in \mathcal{Q}_{\mathcal{T}_h, \kappa}$  and set  $\mathbf{v}_h \stackrel{\text{def}}{=} \mathcal{I}_h^{\text{ccg}}(v)$  and  $(v_F)_{F \in \mathcal{F}_h} = \mathfrak{T}_h(\mathbf{v}_h)$ . Under Assumption 21 there holds for all  $h \in \mathcal{H}$ , all  $v \in \mathcal{Q}_{\mathcal{T}_h, \kappa}$ , and all  $F \in \mathcal{F}_h$ ,*

$$|v_F - v(\bar{\mathbf{x}}_F)| \leq C h_{T_{\mathbf{g}_F}}^2,$$

where  $C$  depends on  $\kappa$ , on the mesh regularity parameters, on  $\Lambda$ , and on  $\|v\|_{C^2(\mathcal{T}_h)}$ .

*Proof.* The assertion is trivially verified for  $F \in \mathcal{F}_h^b$ . Let now  $F \in \mathcal{F}_h^i$ . Using the Taylor expansion of  $v$  about  $\mathbf{x}_{T_{\mathbf{g}_F}}$  together with the fact that  $T_{\mathbf{g}_F}$  is star-shaped with respect to  $\mathbf{x}_{T_{\mathbf{g}_F}}$ , we conclude that there exists  $\mathbf{y}(\bar{\mathbf{x}}_F) \in [\mathbf{x}_{T_{\mathbf{g}_F}}, \mathbf{x}]$  such that

$$v(\bar{\mathbf{x}}_F) = v(\mathbf{x}_{T_{\mathbf{g}_F}}) + \nabla v(\mathbf{x}_{T_{\mathbf{g}_F}}) \cdot (\bar{\mathbf{x}}_F - \mathbf{x}_{T_{\mathbf{g}_F}}) + \frac{1}{2} (\bar{\mathbf{x}}_F - \mathbf{x}_{T_{\mathbf{g}_F}})^t \mathbf{H}v(\mathbf{y}(\bar{\mathbf{x}}_F)) (\bar{\mathbf{x}}_F - \mathbf{x}_{T_{\mathbf{g}_F}}),$$

where  $\mathbf{H}v(\mathbf{y}(\bar{\mathbf{x}}_F))$  denotes the Hessian of  $v$  evaluated at  $\mathbf{y}(\bar{\mathbf{x}}_F)$ . Similarly, letting  $\mathbf{v}_h = \mathcal{I}_h^{\text{ccg}}(v)$  and  $(v_F)_{F \in \mathcal{F}_h} \stackrel{\text{def}}{=} \mathfrak{T}_h(\mathbf{v}_h) \in \mathbb{R}^{\mathcal{F}_h}$ ,

$$v_F = v(\mathbf{x}_{T_{\mathbf{g}_F}}) + \nabla \xi_{\mathbf{v}_h}^{\mathbf{g}_F}|_{T_{\mathbf{g}_F}} \cdot (\bar{\mathbf{x}}_F - \mathbf{x}_{T_{\mathbf{g}_F}}).$$

Hence,  $|v_F - v(\bar{\mathbf{x}}_F)| \leq h_T |\nabla \xi_{\mathbf{v}_h}^{\mathbf{g}_F}|_{T_{\mathbf{g}_F}} - \nabla v(\bar{\mathbf{x}}_F)| + \frac{h_T^2}{2} \|v\|_{C^2(\bar{T})}$ . The conclusion follows from Lemma 20.  $\square$

For further use we introduce the following augmented version of the  $\|\cdot\|$ -norm on  $V_{\dagger}$ :

$$\|v\|_{\dagger}^2 \stackrel{\text{def}}{=} \|v\|^2 + \sum_{T \in \mathcal{T}_h} h_T \|\nabla v|_{T \cdot \mathbf{n}_T}\|_{L^2(\partial T)}^2. \quad (24)$$

**Theorem 23** (Approximation of functions in  $\mathcal{Q}_{\mathcal{T}_h, \kappa}$ ). *Let  $v \in \mathcal{Q}_{\mathcal{T}_h, \kappa}$  and set  $v_h \stackrel{\text{def}}{=} (\mathfrak{R}_h^{\text{ccg}} \circ \mathcal{I}_h^{\text{ccg}})(v) \in V_h^{\text{ccg}}$ . Then, under Assumption 21 there holds*

$$\|v - v_h\|_{\kappa, \dagger} \leq \bar{\lambda}^{-1/2} \|v - v_h\|_{\dagger} \leq C_v h, \quad (25)$$

with  $C_v$  depending on  $\kappa$ , on the mesh regularity parameters, on  $\Lambda$ , and on  $\|v\|_{C^2(\mathcal{T}_h)}$ .

*Proof.* The proof closely follows that of [5, Lemma 3.10]. Since it is clear that  $\|w\|_{\kappa, \dagger} \leq \bar{\lambda}^{1/2} \|w\|_{\dagger}$  for all  $w \in H^1(\mathcal{T}_h)$ , we only show the second inequality in (25). In the rest of the proof we abbreviate  $a \lesssim b$  the inequality  $a \leq Cb$  where  $C$  can depend on  $\kappa$ , on the mesh regularity parameters, on  $\Lambda$ , and on  $\|v\|_{C^2(\mathcal{T}_h)}$ . Let  $v \in \mathcal{Q}_{\mathcal{T}_h, \kappa}$  and set

$$\mathbf{v}_h \stackrel{\text{def}}{=} \mathcal{I}_h^{\text{ccg}}(v) \in \mathbb{V}_h, \quad v_h \stackrel{\text{def}}{=} \mathfrak{R}_h^{\text{ccg}}(\mathbf{v}_h) \in V_h^{\text{ccg}}, \quad (v_F)_{F \in \mathcal{F}_h} = \mathfrak{T}_h(\mathbf{v}_h) \in \mathbb{R}^{\mathcal{F}_h}.$$

(i) *Estimate of  $\|\nabla v - \nabla_h v_h\|_{[L^2(\Omega)]^d}$ .* For all  $T \in \mathcal{T}_h$  and all  $\mathbf{x} \in T$  there holds

$$\begin{aligned} \nabla_h v_h|_T - \nabla v(\mathbf{x}) &= \left\{ \nabla v(\mathbf{x}_T) - \nabla v(\mathbf{x}) \right\} + \left\{ \frac{1}{|T|^d} \sum_{F \in \mathcal{F}_T} |F|_{d-1} (v_F - v(\bar{\mathbf{x}}_F)) \mathbf{n}_{T,F} \right\} \\ &\quad + \left\{ \frac{1}{|T|^d} \sum_{F \in \mathcal{F}_T} |F|_{d-1} (v(\bar{\mathbf{x}}_F) - v(\mathbf{x}_T)) \mathbf{n}_{T,F} - \nabla v(\mathbf{x}_T) \right\} \stackrel{\text{def}}{=} \mathfrak{T}_1 + \mathfrak{T}_2 + \mathfrak{T}_3. \end{aligned}$$

It is clear that  $|\mathfrak{T}_1| \leq \|v\|_{C^2(\bar{T})} h_T$ , hence  $\sum_{T \in \mathcal{T}_h} \|\mathfrak{T}_1\|_{L^2(T)}^2 \lesssim |\Omega|_d h^2$ . Using Lemma 22 together with the geometric relation (9) and mesh regularity it is inferred that

$$|\mathfrak{T}_2| \leq \sum_{F \in \mathcal{F}_T} \frac{|F|_{d-1} d_{T,F} |v_F - v(\bar{\mathbf{x}}_F)|}{|T|^d d_{T,F}} \lesssim d \frac{h_{T_{\mathfrak{g}F}}^2}{d_{T,F}} \lesssim h_{T_{\mathfrak{g}F}}.$$

As a result,  $\sum_{T \in \mathcal{T}_h} \|\mathfrak{T}_2\|_{L^2(T)}^2 \lesssim |\Omega|_d h^2$ . To handle the last term, we use the magic formula [5, eq. (33)]: For all  $T \in \mathcal{T}_h$  and all  $\mathbf{x} \in T$ ,

$$\frac{1}{|T|^d} \sum_{F \in \mathcal{F}_T} |F|_{d-1} (\bar{\mathbf{x}}_F - \mathbf{x})_i (\mathbf{n}_{T,F})_j = \delta_{ij}, \quad (26)$$

where  $\delta_{ij}$  is Kronecker's symbol. By virtue of (26), there holds

$$\mathfrak{T}_3 = \frac{1}{|T|^d} \sum_{F \in \mathcal{F}_T} |F|_{d-1} [v(\bar{\mathbf{x}}_F) - v(\mathbf{x}_T) - \nabla v(\mathbf{x}_T) \cdot (\bar{\mathbf{x}}_F - \mathbf{x}_T)] \mathbf{n}_{T,F} \leq \sum_{F \in \mathcal{F}_T} \frac{|F|_{d-1} d_{T,F}}{|T|^d} \|v\|_{C^2(\bar{T})} h_T.$$

Proceeding as for  $\mathfrak{T}_2$ , we easily infer that  $\sum_{T \in \mathcal{T}_h} \|\mathfrak{T}_3\|_{L^2(T)}^2 \lesssim |\Omega|_d h^2$ , thereby concluding that

$$\|\nabla v - \nabla_h v_h\|_{[L^2(\Omega)]^d} \lesssim h.$$

(ii) *Estimate of the remaining terms in  $\|\cdot\|_{\dagger}$ .* By the continuous trace inequality (5),

$$|v - v_h|_J^2 \lesssim \sum_{F \in \mathcal{F}_h} h_F^{-1} \sum_{T \in \mathcal{T}_F} \|(v - v_h)|_T\|_{L^2(F)}^2 \leq C_{\text{tr},c}^2 \sum_{F \in \mathcal{F}_h} h_F^{-1} \sum_{T \in \mathcal{T}_F} \left\{ h_T^{-1} \|v - v_h\|_{L^2(T)}^2 + h_T \|v - v_h\|_{H^1(T)}^2 \right\}.$$

On the other hand, owing to the previous point, for all  $T \in \mathcal{T}_h$  and all  $\mathbf{x} \in T$ ,  $|v_T - v_h(\mathbf{x})| + h_T |\nabla_h v_h|_T - \nabla v(\mathbf{x})| \lesssim h^2$ . It follows easily that  $|v - v_h|_J \lesssim h$ . The term  $\sum_{T \in \mathcal{T}_h} h_T \|\nabla(v - v_h)|_T \cdot \mathbf{n}_T\|_{L^2(\partial T)}$  can be handled in a similar way, thereby concluding the proof.  $\square$

**Corollary 24** (Convergence rate, heterogeneous case). *Assume  $u \in \mathcal{Q}_{\mathcal{T}_h, \kappa}$ . Then, under Assumption 21, there holds*

$$\|u - u_h\|_{\kappa} \leq Ch$$

with  $C = C_u \bar{\lambda}^{-1/2} \left(1 + \frac{C_{\text{bnd}}}{C_{\text{sta}}}\right)$  and  $C_u$  results from Theorem 23.

### 2.3.2. The homogeneous isotropic case

We assume throughout this section that

$$\boldsymbol{\kappa} = \mathbf{1}_{d,d}, \quad d \in \{2, 3\}. \quad (27)$$

The more general case  $\boldsymbol{\kappa} = \nu \mathbf{1}_d$ ,  $\nu > 0$ , can be handled with minor modifications. When (27) holds, convergence rates can be estimated with milder regularity assumptions on the exact solution, and  $L^2$  error estimates for the method (21) can be obtained by the Aubin–Nietsche trick [8, 35]. The key point is here to show how optimal convergence rates can be obtained avoiding pointwise estimations, thereby removing the need for the strong local regularity assumption  $u \in C^2(\mathcal{T}_h)$  used all along the previous section. To this end, we need to introduce further mild assumptions on the mesh family. For all faces we define a patch of pyramids on which the piecewise affine functions  $\{\xi_{\mathbf{v}_h}^{\mathbf{g}_F}\}_{F \in \mathcal{F}_h}$  are required to exhibit approximation properties. More precisely, for all  $F \in \mathcal{F}_h$  let

$$\mathcal{P}_F \stackrel{\text{def}}{=} \begin{cases} \mathcal{P}_{\mathbf{g}_F} & \text{if } F \in \mathcal{F}_h^i, \\ \mathcal{P}_{T,F} & \text{if } F \subset \partial T \cap \partial \Omega, \end{cases}$$

with  $\mathcal{P}_{\mathbf{g}_F}$  defined by (10); see Figure 1.

**Assumption 25** (Approximation property for L-constructions). *We assume that the L-constructions are such that, for all  $0 \leq l \leq 1$  with  $l > d/2 - 1$ , there exists  $C$  independent of the meshsize  $h$  such that, for all  $0 \leq m \leq l + 1$ , and all  $F \in \mathcal{F}_h$*

$$\forall v \in H^{l+1}(\mathcal{P}_F), \quad |v - \xi_{\mathbf{v}_h}^{\mathbf{g}_F}|_{H^m(\mathcal{P}_F)} \leq Ch_{\mathcal{P}_F}^{l+1-m} |v|_{H^{l+1}(\mathcal{P}_F)},$$

with  $\mathbf{v}_h = \mathcal{I}_h^{\text{ccg}}(v)$ . Moreover, there exists  $\varrho_4$  independent of the meshsize  $h$  such that, for all  $F \in \mathcal{F}_h$  and all  $T \in \mathcal{T}_F$ ,  $h_{\mathcal{P}_F} \leq \varrho_4 h_T$ .

Some comments are of order. It is worth giving a hint of how Assumption 25 could be proved. We start by observing that Assumption 10 has a straightforward interpretation in the homogeneous case, as it amounts to requiring that, for all  $F \in \mathcal{F}_h^i$ , there exists at least a group  $\mathbf{g} \in \mathcal{G}_F$  such that the cell centers of the elements in  $\mathcal{T}_{\mathbf{g}}$  and the barycenters of the faces  $F \in \mathbf{g}^b$  form a non-degenerate simplex  $S_{\mathbf{g}}$ ; see Figure 1. Assumption 21 is then essentially a shape-regularity requirement on the family of simplices  $\{S_{\mathbf{g}_F}\}_{h \in \mathcal{H}, F \in \mathcal{F}_h^i}$ . A second important remark is that, for all  $v$  sufficiently regular, the functions  $\xi_{\mathbf{v}_h}^{\mathbf{g}_F}$  with  $\mathbf{v}_h = \mathcal{I}_h^{\text{ccg}}(v)$  coincides with the Lagrange interpolator on  $S_{\mathbf{g}}$ . Assumption 25 can then be proved by classical FE techniques by using, e.g., the affine map onto the reference simplex and estimating the norms on the ball circumscribed to  $\mathcal{P}_F$ . Using the circumscribed ball allows to have a uniform bound on the Deny–Lions constant appearing when proceeding as in [27, Theorem 1.103]. Clearly, this reasoning breaks down close to the boundary of the domain  $\Omega$ , as the ball may no longer be contained in  $\Omega$ . As a result, we can interpret Assumption 25 as a requirement that the patches associated to the L-construction near the boundary of  $\Omega$  be “not too far” from the simplex formed by the cell centers. To conclude, we point out that the assumption  $l > d/2 - 1$  is necessary to ensure that point values are well-defined inside the elements. This is why the space dimension has been restricted to  $\{2, 3\}$  in (27).

**Theorem 26** (Approximation of functions in  $V \cap H^2(\Omega)$ ). *Under Assumption 25, there holds*

$$\forall v \in V \cap H^2(\Omega), \quad \|v - v_h\|_{\dagger} \leq Ch \|v\|_{H^2(\Omega)},$$

with  $v_h \stackrel{\text{def}}{=} (\mathfrak{R}_h^{\text{ccg}} \circ \mathcal{I}_h^{\text{ccg}})(v) \in V_h^{\text{ccg}}$  and  $C$  depending on the mesh regularity parameters but not on  $h$ .

*Proof.* Let  $v \in V \cap H^2(\Omega)$ , and set  $\mathbf{v}_h \stackrel{\text{def}}{=} \mathcal{I}_h^{\text{ccg}}(v) \in \mathbb{V}_h$  and  $(v_F)_{F \in \mathcal{F}_h} = \mathfrak{T}_h(\mathbf{v}_h) \in \mathbb{R}^{\mathcal{F}_h}$ . In the rest of the proof we abbreviate  $a \lesssim b$  the inequality  $a \leq Cb$  where  $C$  can depend on the mesh regularity parameters but not on the meshsize  $h$ .



(i) *Estimate of  $\|\nabla v - \nabla_h v_h\|_{[L^2(\Omega)]^d}$ .* The quantity to estimate is decomposed as follows:

$$\begin{aligned} \|\nabla v - \nabla_h v_h\|_{[L^2(\Omega)]^d}^2 &= \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \int_{\mathcal{P}_{T,F}} |\nabla v - \nabla \xi_{\mathbf{v}_h}^{\mathbf{g}_F} + \nabla \xi_{\mathbf{v}_h}^{\mathbf{g}_F} - \nabla v_h|^2 \\ &\lesssim \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \int_{\mathcal{P}_{T,F}} |\nabla v - \nabla \xi_{\mathbf{v}_h}^{\mathbf{g}_F}|^2 + \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \int_{\mathcal{P}_{T,F}} |\nabla \xi_{\mathbf{v}_h}^{\mathbf{g}_F} - \nabla v_h|^2 \stackrel{\text{def}}{=} \mathfrak{I}_1 + \mathfrak{I}_2. \end{aligned}$$

For the first term it suffices to use Assumption 25 to infer

$$\mathfrak{I}_1 = \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} |v - \xi_{\mathbf{v}_h}^{\mathbf{g}_F}|_{H^1(\mathcal{P}_{T,F})}^2 \lesssim h^2 |v|_{H^2(\Omega)}^2.$$

To estimate the second term, preliminarily observe that, for all  $T \in \mathcal{T}_h$ ,

$$\nabla_h v_h|_T = \sum_{F' \in \mathcal{F}_T} \frac{|F'|_{d-1}}{|T|^d} \left( \xi_{\mathbf{v}_h}^{\mathbf{g}_{F'}}|_{\mathcal{P}_{T,F'}}(\bar{\mathbf{x}}_{F'}) - v_T \right) \mathbf{n}_{T,F'} = \sum_{F' \in \mathcal{F}_T} \frac{|F'|_{d-1}}{|T|^d} \nabla \xi_{\mathbf{v}_h}^{\mathbf{g}_{F'}}|_{\mathcal{P}_{T,F'}} \cdot (\bar{\mathbf{x}}_{F'} - \mathbf{x}_T) \mathbf{n}_{T,F'},$$

where we have used the linearity of  $\xi_{\mathbf{v}_h}^{\mathbf{g}_{F'}}$  to infer that  $\xi_{\mathbf{v}_h}^{\mathbf{g}_{F'}}|_{\mathcal{P}_{T,F'}}(\bar{\mathbf{x}}_{F'}) = v_T + \nabla \xi_{\mathbf{v}_h}^{\mathbf{g}_{F'}}|_{\mathcal{P}_{T,F'}} \cdot (\bar{\mathbf{x}}_{F'} - \mathbf{x}_T)$ . Using formula (26), we obtain

$$\langle \nabla v \rangle_T - \nabla_h v_h|_T = \sum_{F' \in \mathcal{F}_T} \frac{|F'|_{d-1}}{|T|^d} \left( \langle \nabla v \rangle_T - \nabla \xi_{\mathbf{v}_h}^{\mathbf{g}_{F'}}|_{\mathcal{P}_{T,F'}} \right) \cdot (\bar{\mathbf{x}}_{F'} - \mathbf{x}_T) \mathbf{n}_{T,F'}.$$

Exploiting the above remark, the second term is then decomposed as follows:

$$\begin{aligned} \mathfrak{I}_2 &\lesssim \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} |\mathcal{P}_{T,F}|^d |\nabla \xi_{\mathbf{v}_h}^{\mathbf{g}_F}|_{\mathcal{P}_{T,F}} - \langle \nabla v \rangle_{\mathcal{P}_{T,F}}|^2 + \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} |\mathcal{P}_{T,F}|^d |\langle \nabla v \rangle_{\mathcal{P}_{T,F}} - \langle \nabla v \rangle_T|^2 \\ &+ \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} |\mathcal{P}_{T,F}|^d \left| \sum_{F' \in \mathcal{F}_T} \frac{|F'|_{d-1}}{|T|^d} \left( \langle \nabla v \rangle_{\mathcal{P}_{T,F'}} - \nabla \xi_{\mathbf{v}_h}^{\mathbf{g}_{F'}}|_{\mathcal{P}_{T,F'}} \right) \cdot (\bar{\mathbf{x}}_{F'} - \mathbf{x}_T) \mathbf{n}_{T,F'} \right|^2 \\ &+ \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} |\mathcal{P}_{T,F}|^d \left| \sum_{F' \in \mathcal{F}_T} \frac{|F'|_{d-1}}{|T|^d} \left( \langle \nabla v \rangle_T - \langle \nabla v \rangle_{\mathcal{P}_{T,F'}} \right) \cdot (\bar{\mathbf{x}}_{F'} - \mathbf{x}_T) \mathbf{n}_{T,F'} \right|^2 \stackrel{\text{def}}{=} \mathfrak{I}_{2,1} + \mathfrak{I}_{2,2} + \mathfrak{I}_{2,3} + \mathfrak{I}_{2,4}. \end{aligned}$$

The Cauchy–Schwarz inequality yields

$$\nabla \xi_{\mathbf{v}_h}^{\mathbf{g}_F}|_{\mathcal{P}_{T,F}} - \langle \nabla v \rangle_{\mathcal{P}_{T,F}} = \frac{1}{|\mathcal{P}_{T,F}|^d} \int_{\mathcal{P}_{T,F}} (\nabla \xi_{\mathbf{v}_h}^{\mathbf{g}_F} - \nabla v) \leq \frac{1}{|\mathcal{P}_{T,F}|_d^{1/2}} |\xi_{\mathbf{v}_h}^{\mathbf{g}_F} - v|_{H^1(\mathcal{P}_{T,F})},$$

whence, by Assumption 25,

$$\mathfrak{I}_{2,1} \leq \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} |\xi_{\mathbf{v}_h}^{\mathbf{g}_F} - v|_{H^1(\mathcal{P}_{T,F})}^2 \lesssim h^2 |v|_{H^2(\Omega)}^2.$$

The term  $\mathfrak{I}_{2,2}$  can be estimated in a similar fashion using the approximation properties of the  $L^2$  projection of  $\nabla v$  onto  $[\mathbb{P}_d^0(\mathcal{T}_h)]^d$  to conclude. To estimate  $\mathfrak{I}_{2,3}$ , observe preliminarily that, for all  $T \in \mathcal{T}_h$  and all  $F' \in \mathcal{F}_T$ , equation (8) yields

$$|F'|_{d-1} \left( \langle \nabla v \rangle_{\mathcal{P}_{T,F'}} - \nabla \xi_{\mathbf{v}_h}^{\mathbf{g}_{F'}}|_{\mathcal{P}_{T,F'}} \right) = \frac{d}{d_{T,F'}} \int_{\mathcal{P}_{T,F'}} (\nabla v - \nabla \xi_{\mathbf{v}_h}^{\mathbf{g}_{F'}}).$$

The Cauchy–Schwarz inequality together with (6) yield

$$\begin{aligned} \left| \sum_{F' \in \mathcal{F}_T} \frac{|F'|^{d-1}}{|T|^d} \left( \langle \nabla v \rangle_{\mathcal{P}_{T,F'}} - \nabla \xi_{\mathbf{v}_h}^{\mathbf{g}_{F'}}|_{\mathcal{P}_{T,F'}} \right) \cdot (\bar{\mathbf{x}}_{F'} - \mathbf{x}_T) \right|^2 &\leq \frac{d^2 h_T^2}{d_{T,F'}^2 |T|^2} \left| \sum_{F' \in \mathcal{F}_T} \int_{\mathcal{P}_{T,F'}} (\nabla v - \nabla \xi_{\mathbf{v}_h}^{\mathbf{g}_{F'}}) \right|^2 \\ &\leq \frac{d^2 h_T^2}{d_{T,F'}^2 |T|^2} \left\{ \sum_{F' \in \mathcal{F}_T} |\mathcal{P}_{T,F'}|_d \right\} \times \left\{ \sum_{F' \in \mathcal{F}_T} |v - \xi_{\mathbf{v}_h}^{\mathbf{g}_{F'}}|_{H^1(\mathcal{P}_{T,F'})}^2 \right\} = \frac{d^2}{|T|^d \varrho_3^2} \sum_{F' \in \mathcal{F}_T} |v - \xi_{\mathbf{v}_h}^{\mathbf{g}_{F'}}|_{H^1(\mathcal{P}_{T,F'})}^2. \end{aligned}$$

We therefore have

$$\begin{aligned} \mathfrak{T}_{2,3} &\leq \frac{d^2}{\varrho_3^2} \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \left\{ \frac{|\mathcal{P}_{T,F}|_d}{|T|^d} \sum_{F' \in \mathcal{F}_T} |v - \xi_{\mathbf{v}_h}^{\mathbf{g}_{F'}}|_{H^1(\mathcal{P}_{T,F'})}^2 \right\} \\ &\leq \frac{d^2}{\varrho_3^2} \sum_{T \in \mathcal{T}_h} \left\{ \sum_{F \in \mathcal{F}_T} \frac{|\mathcal{P}_{T,F}|_d}{|T|^d} \right\} \times \left\{ \sum_{F' \in \mathcal{F}_T} |v - \xi_{\mathbf{v}_h}^{\mathbf{g}_{F'}}|_{H^1(\mathcal{P}_{T,F'})}^2 \right\} \leq \frac{d^2}{\varrho_3^2} \sum_{T \in \mathcal{T}_h} \sum_{F' \in \mathcal{F}_T} |v - \xi_{\mathbf{v}_h}^{\mathbf{g}_{F'}}|_{H^1(\mathcal{P}_{T,F'})}^2, \end{aligned}$$

and, by Assumption 25, we infer that  $\mathfrak{T}_{2,3} \lesssim h^2 \|v\|_{H^2(\Omega)}^2$ . Proceeding in a similar way, one can prove that  $\mathfrak{T}_{2,4} \lesssim h^2 \|v\|_{H^{l+1}(\Omega)}^2$ . In conclusion,

$$\|\nabla v - \nabla_h v_h\|_{[L^2(\Omega)]^d} \lesssim h \|v\|_{H^2(\Omega)}.$$

To conclude, we observe that the fact that  $v$  vanishes on the boundary has been tacitly used to estimate the error associated to the functions  $\xi_{\mathbf{v}_h}^{\mathbf{g}_F}$  on boundary faces  $F \in \mathcal{F}_h^b$ .

(ii) *Estimate of the remaining terms in  $\|\cdot\|_{\dagger}$ .* We start by estimating  $\|v - v_h\|_{L^2(\Omega)}$ . Let  $\pi_h^1$  denote the  $L^2$ -projection on to  $\mathbb{P}_d^1(\mathcal{S}_h)$  with  $\mathcal{S}_h$  defined by (7). An application of the triangular inequality yields

$$\|v - v_h\|_{L^2(\Omega)} \leq \|v - \pi_h^1 v\|_{L^2(\Omega)} + \|\pi_h^1 v - v_h\|_{L^2(\Omega)} \stackrel{\text{def}}{=} \mathfrak{T}_1 + \mathfrak{T}_2.$$

By the approximation properties of the pyramidal submesh  $\mathcal{S}_h$  (which follow from the shape- and contact-regularity of the mesh  $\mathcal{T}_h$  as well as from the assumption that cell centers are uniformly away from the cell boundaries), there holds  $\mathfrak{T}_1 \lesssim h^2 \|v\|_{H^2(\Omega)}$ . To proceed, we rewrite the second term as follows:

$$\mathfrak{T}_2^2 = \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \|\pi_h^1 v - v_h\|_{L^2(\mathcal{P}_{T,F})}^2.$$

Since, for all  $T \in \mathcal{T}_h$ , all  $F \in \mathcal{F}_T$ , and all  $\mathbf{x} \in \mathcal{P}_{T,F}$ ,  $\pi_h^1 v|_{\mathcal{P}_{T,F}}(\mathbf{x}) = \langle v \rangle_{\mathcal{P}_{T,F}} + \nabla \pi_h^1 v|_{\mathcal{P}_{T,F}} \cdot (\mathbf{x} - \mathbf{x}_T)$ , there holds,

$$\begin{aligned} \|\pi_h^1 v - v_h\|_{L^2(\mathcal{P}_{T,F})} &\leq \|\langle v \rangle_{\mathcal{P}_{T,F}} - \langle \xi_{\mathbf{v}_h}^{\mathbf{g}_F} \rangle_{\mathcal{P}_{T,F}}\|_{L^2(\mathcal{P}_{T,F})} + \|\nabla(\pi_h^1 v - \xi_{\mathbf{v}_h}^{\mathbf{g}_F}) \cdot (\mathbf{x} - \mathbf{x}_T)\|_{L^2(\mathcal{P}_{T,F})} \\ &\leq \left\{ \int_{\mathcal{P}_{T,F}} \left( \int_{\mathcal{P}_{T,F}} (v - \xi_{\mathbf{v}_h}^{\mathbf{g}_F}) \right)^2 \right\}^{1/2} + h_T |\pi_h^1 v - \xi_{\mathbf{v}_h}^{\mathbf{g}_F}|_{H^1(\mathcal{P}_{T,F})} \\ &\leq \|v - \xi_{\mathbf{v}_h}^{\mathbf{g}_F}\|_{L^2(\mathcal{P}_{T,F})} + h_T |\pi_h^1 v - \xi_{\mathbf{v}_h}^{\mathbf{g}_F}|_{H^1(\mathcal{P}_{T,F})}. \end{aligned}$$

Assumption 25 then yields  $\mathfrak{T}_2 \lesssim h^2 \|v\|_{H^2(\Omega)}$ . We therefore conclude that

$$\|v - v_h\|_{L^2(\Omega)} \lesssim h^2 \|v\|_{H^2(\Omega)}. \quad (28)$$

To bound  $|v - v_h|_J$  and  $\sum_{T \in \mathcal{T}_h} h_T \|\nabla v|_{T \cdot \mathbf{n}_F}\|_{L^2(\partial T)}$  use the continuous trace inequality (5) as in point (ii) of Theorem 23 together with (28).  $\square$

**Lemma 27** (Convergence rate, homogeneous case). *Let  $u \in V \cap H^2(\Omega)$ . Then, under Assumption 25,*

$$\|u - u_h\| \leq Ch \|u\|_{H^2(\Omega)}, \quad (29)$$

with  $C$  independent of the meshsize  $h$ . Moreover, if elliptic regularity holds,

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^2 \|u\|_{H^2(\Omega)}. \quad (30)$$

*Proof.* (i) *Energy estimate* (29). Use the error estimate of Theorem 18 together with the assumption (27) and Theorem 26 to conclude that  $\|u - u_h\| \leq C \left(1 + \frac{C_{\text{bnd}}}{C_{\text{sta}}}\right) \|v\|_{H^2(\Omega)}$ .

(ii)  *$L^2$ -error estimate* (30). We only give a sketch of the proof and refer, e.g., to [7] or [21, §4.1.3] for further details. Preliminarily remark that the bilinear form  $a_h$  is coercive in  $V_h^{\text{ccg}} \times V_h^{\text{ccg}}$  with respect to the augmented norm  $\|\cdot\|_{\dagger}$ , i.e., there exists  $C'_{\text{sta}}$  such that, for all  $\eta > \underline{\eta}$  and all  $v_h \in V_h^{\text{ccg}}$ ,  $a_h(v_h, v_h) \geq C'_{\text{sta}} \|v_h\|_{\dagger}^2$  (indeed, the  $\|\cdot\|_{\dagger}$ -norm and the  $\|\cdot\|$ -norm are uniformly equivalent on  $V_h^{\text{ccg}}$ ). Also, there exists  $C'_{\text{bnd}}$  independent of the meshsize  $h$  such that, for all  $w, v \in V_{\dagger h}$ ,  $a_h \leq C'_{\text{bnd}} \|w\|_{\dagger} \|v\|_{\dagger}$ . Hence, proceeding as in Theorem 18 and using Theorem 26 we conclude that  $\|u - u_h\|_{\dagger} \leq Ch \|v\|_{H^2(\Omega)}$  with  $C$  independent of the meshsize  $h$ . Consider now the auxiliary problem

$$\text{Find } \chi \in V \text{ such that } a(\chi, v) = \int_{\Omega} (u - u_h)v \text{ for all } v \in V.$$

By the elliptic regularity assumption, there exists  $C_{\text{ell}}$  such that  $\|\chi\|_{H^2(\Omega)} \leq C_{\text{ell}} \|u - u_h\|_{L^2(\Omega)}$ . Moreover, owing to the symmetry and the consistency of  $a_h$ ,  $a_h(u - u_h, \chi) = -\int_{\Omega} \Delta \chi (u - u_h)$  and, for  $\chi_h = (\mathfrak{R}_h^{\text{ccg}} \circ \mathcal{T}_h^{\text{ccg}})(\chi) \in V_h^{\text{ccg}}$ ,  $a_h(u - u_h, \chi_h) = 0$ . As a result,

$$\|u - u_h\|_{L^2(\Omega)}^2 = a_h(u - u_h, \chi - \chi_h) \lesssim \|u - u_h\|_{\dagger} \|\chi - \chi_h\|_{\dagger} \lesssim \|u - u_h\|_{\dagger} h \|\chi\|_{H^2(\mathcal{T}_h)} \lesssim \|u - u_h\|_{\dagger} h \|u - u_h\|_{L^2(\Omega)},$$

with  $\lesssim$  indicating inequalities up to a multiplicative constant independent of the meshsize  $h$ . To conclude the proof, use the fact that  $\|u - u_h\|_{\dagger} \leq Ch \|v\|_{H^2(\Omega)}$ .  $\square$

#### 2.4. Convergence to minimal regularity solutions

We investigate the convergence of the method (21) to minimal regularity solutions, i.e., solutions that barely sit in  $H_0^1(\Omega)$ . Throughout this section we restore the original assumptions on the diffusion coefficient  $\kappa$ , and we consider an arbitrary space dimension, i.e.

$$\kappa \in [\mathbb{P}_d^0(\mathcal{T}_h)]^{d,d}, \quad d \geq 1.$$

The analysis follows the ideas of Eymard, Gallouët, and Herbin [29, 30] originally developed in the context of FV methods and recently transposed to dG methods by Di Pietro and Ern [20]. An important remark is that the bilinear form  $a_h$  admits the following equivalent form on  $V_h^{\text{ccg}} \times V_h^{\text{ccg}}$ :

$$a_h(u_h, v_h) = \int_{\Omega} \kappa \mathbf{G}_{\omega, h}(u_h) \cdot \mathbf{G}_{\omega, h}(v_h) + j_h(u_h, v_h), \quad (31)$$

with  $j_h(u_h, v_h) \stackrel{\text{def}}{=} \int_{\Omega} \kappa \mathbf{R}_{\omega, h}(\llbracket u_h \rrbracket) \cdot \mathbf{R}_{\omega, h}(\llbracket v_h \rrbracket) + \sum_{F \in \mathcal{F}_h} \eta \gamma_F h_F^{-1} \int_F \llbracket u_h \rrbracket \llbracket v_h \rrbracket$  and discrete gradients defined by (18). When extended to  $V_{\dagger h} \times V_h^{\text{ccg}}$ , this alternative form is no longer consistent in the sense of point (i) in Lemma 17; see [20, Remark 3.3] and [21, §5.2.1]. However,  $a_h$  retains a different form of consistency which suffices to infer the convergence of the method when  $u$  only exhibits the minimal regularity.

**Definition 28** (Asymptotic consistency). *We say that the bilinear form  $a_h$  is asymptotically consistent with the exact bilinear form  $a$  on  $(V_h^{\text{ccg}})_{h \in \mathcal{H}}$  if, for any sequence  $(v_h)_{h \in \mathcal{H}}$  in  $(V_h^{\text{ccg}})_{h \in \mathcal{H}}$  uniformly bounded in the*

$\|\cdot\|$ -norm, and for all  $\varphi \in \mathcal{Q}_{T_h, \kappa}$  with  $\varphi_h \stackrel{\text{def}}{=} (\mathfrak{R}_h^{\text{ccg}} \circ \mathcal{I}_h^{\text{ccg}})(\varphi) \in V_h^{\text{ccg}}$ ,

$$\lim_{h \rightarrow 0} a_h(v_h, \varphi_h) = a(v, \varphi) = \int_{\Omega} \kappa \nabla v \cdot \nabla \varphi,$$

and  $v \in H_0^1(\Omega)$  results from Theorem 14.

To prove the convergence of the method, we then proceed as follows: (i) using the coercivity of  $a_h$  we prove a uniform bound for the  $\|\cdot\|$ -norm of the discrete solutions  $(u_h)_{h \in \mathcal{H}}$ ; (ii) by virtue of Theorem 14, we infer the existence of  $\bar{u} \in V = H_0^1(\Omega)$  limit of  $(u_h)_{h \in \mathcal{H}}$ ; (iii) using the asymptotic consistency of  $a_h$  together with the fact that  $\mathcal{Q}_{T_h, \kappa}$  is dense in  $V$ , we conclude that  $\bar{u} = u$  (and, by the uniqueness of  $u$ , that the convergence property extends to the whole sequence); (iv) using the above result, we prove the strong convergence of the sequence  $(\mathbf{G}_{\omega, h}(u_h))_{h \in \mathcal{H}}$  to  $\nabla u$  in  $[L^2(\Omega)]^d$  as  $h \rightarrow 0$ . An important intermediate result to prove the asymptotic consistency of  $a_h$  is the consistency of the discrete gradient  $\mathbf{G}_{\omega, h}$  defined by (18) for the interpolates of functions in  $\mathcal{Q}_{T_h, \kappa}$ . We first prove the following.

**Lemma 29** (Bound on global lifting). *For all  $v \in V_{\dagger}$ , there holds*

$$\|\kappa^{1/2} \mathbf{R}_{\omega, h}(\llbracket v \rrbracket)\|_{[L^2(\Omega)]^d}^2 \leq C_{\text{tr}} N_{\rho}^{1/2} |v|_{J, \kappa}^2.$$

*Proof.* By definition,

$$\|\kappa^{1/2} \mathbf{R}_{\omega, h}(\llbracket v \rrbracket)\|_{[L^2(\Omega)]^d}^2 = \sum_{F \in \mathcal{F}_h} \int_{\Omega} \kappa \mathbf{R}_{\omega, h}(\llbracket v \rrbracket) \cdot \mathbf{r}_{\omega, F}(\llbracket v \rrbracket) = \sum_{F \in \mathcal{F}_h} \int_F \{\kappa \mathbf{R}_{\omega, h}(\llbracket v_h \rrbracket)\}_{\omega} \cdot \mathbf{n}_F \llbracket v \rrbracket.$$

For brevity of notation, for all  $F \in \mathcal{F}_h^i$  with  $F = \partial T_1 \cap \partial T_2$ , let  $\omega_i = \omega_{T_i, F}$ ,  $\lambda_{T_i, F} = \lambda_i$ ,  $\kappa_i = \kappa|_{T_i}$ , and  $\mathbf{a}_i = \kappa_i^{1/2} \mathbf{R}_{\omega, h}(\llbracket v \rrbracket)|_{T_i}$ ,  $i \in \{1, 2\}$ . The Cauchy–Schwarz inequality yields

$$\begin{aligned} \int_F \{\kappa \mathbf{R}_{\omega, h}(\llbracket v \rrbracket)\}_{\omega} \cdot \mathbf{n}_F \llbracket v \rrbracket &= \int_F (\omega_1 \kappa_1^{1/2} \mathbf{n}_{T_1} \cdot \mathbf{a}_1 + \omega_2 \kappa_2^{1/2} \mathbf{n}_{T_2} \cdot \mathbf{a}_2) \llbracket v \rrbracket \\ &\leq \left\{ \frac{1}{2} h_F \left( \|\mathbf{a}_1\|_{[L^2(F)]^d}^2 + \|\mathbf{a}_2\|_{[L^2(F)]^d}^2 \right) \right\}^{1/2} \times \left\{ 2(\omega_1^2 \lambda_1 + \omega_2^2 \lambda_2) \frac{1}{h_F} \|\llbracket v \rrbracket\|_{L^2(F)}^2 \right\}^{1/2}, \end{aligned}$$

and since  $2(\omega_1^2 \lambda_1 + \omega_2^2 \lambda_2) = \gamma_F$ , it is inferred that

$$\int_F \{\kappa \mathbf{R}_{\omega, h}(\llbracket v \rrbracket)\}_{\omega} \cdot \mathbf{n}_F \llbracket v \rrbracket \leq \left\{ \frac{1}{2} h_F \left( \|\mathbf{a}_1\|_{[L^2(F)]^d}^2 + \|\mathbf{a}_2\|_{[L^2(F)]^d}^2 \right) \right\}^{1/2} \times \left\{ \frac{\gamma_F}{h_F} \|\llbracket v \rrbracket\|_{L^2(F)}^2 \right\}^{1/2}.$$

Moreover, for all  $F \in \mathcal{F}_h^b$  with  $F = \partial T \cap \partial \Omega$ ,

$$\int_F \{\kappa \mathbf{R}_{\omega, h}(\llbracket v \rrbracket)\}_{\omega} \cdot \mathbf{n}_F \llbracket v \rrbracket \leq h_F^{1/2} \|(\kappa^{1/2} \mathbf{R}_{\omega, h}(\llbracket v \rrbracket))|_T \cdot \mathbf{n}_F\|_{L^2(F)} \times \left\{ \frac{\gamma_F}{h_F} \|\llbracket v \rrbracket\|_{L^2(F)}^2 h_F \sum_{T \in \mathcal{T}_F} \|\kappa^{1/2} \mathbf{R}_{\omega, h}(\llbracket v \rrbracket)|_T \right\}^{1/2}.$$

Summing over mesh faces, and using the Cauchy–Schwarz inequality we obtain

$$\|\kappa^{1/2} \mathbf{R}_{\omega, h}(\llbracket v \rrbracket)\|_{[L^2(\Omega)]^d}^2 \leq \sum_{F \in \mathcal{F}_h} \left\{ h_F \sum_{T \in \mathcal{T}_F} \|(\kappa^{1/2} \mathbf{R}_{\omega, h}(\llbracket v \rrbracket))|_T\|_{L^2(F)}^2 \right\}^{1/2} |v|_{J, \kappa} \leq C_{\text{tr}} N_{\rho}^{1/2} \|\kappa^{1/2} \mathbf{R}_{\omega, h}(\llbracket v \rrbracket)\|_{[L^2(\Omega)]^d} |v|_{J, \kappa}.$$

This concludes the proof.  $\square$

**Lemma 30** (Strong convergence of  $\mathbf{G}_{\omega,h}$  for smooth functions). *Let  $\varphi \in \mathcal{Q}_{\mathcal{T}_h,\kappa}$  and set  $\varphi_h \stackrel{\text{def}}{=} (\mathfrak{R}_h^{\text{ccg}} \circ \mathcal{I}_h^{\text{ccg}})(\varphi) \in V_h^{\text{ccg}}$ . Under Assumption 21, there holds*

$$\mathbf{G}_{\omega,h}(\varphi_h) \rightarrow \nabla\varphi \quad \text{strongly in } [L^2(\Omega)]^d$$

*Proof.* The triangular inequality yields

$$\|\mathbf{G}_{\omega,h}(\varphi_h) - \nabla\varphi\|_{[L^2(\Omega)]^d} \leq \|\nabla_h\varphi_h - \nabla\varphi\|_{[L^2(\Omega)]^d} + \|\mathbf{R}_{\omega,h}([\varphi_h])\|_{[L^2(\Omega)]^d} \stackrel{\text{def}}{=} \mathfrak{T}_1 + \mathfrak{T}_2.$$

Using Theorem 23 it is readily inferred that  $\mathfrak{T}_1 \rightarrow 0$  as  $h \rightarrow 0$ . For the second term, use Lemma 29 together with the fact that  $[\![\varphi]\!]_F(\mathbf{x}) = 0$  for all  $F \in \mathcal{F}_h$  and all  $\mathbf{x} \in F$  to infer

$$\Delta^{1/2} \|\mathbf{R}_{\omega,h}([\varphi_h])\|_{[L^2(\Omega)]^d} \leq \|\kappa \mathbf{R}_{\omega,h}([\varphi_h])\|_{[L^2(\Omega)]^d} \leq C_{\text{tr}} N_\partial^{1/2} |\varphi_h|_{J,\kappa} \leq C_{\text{tr}} N_\partial^{1/2} |\varphi_h - \varphi|_{J,\kappa},$$

and the right-hand side tends to zero as  $h \rightarrow 0$  again by virtue of Theorem 23. This concludes the proof.  $\square$

We are now ready to prove the following.

**Lemma 31** (Asymptotic consistency of the bilinear form  $a_h$ ). *Under Assumption 21, the bilinear form  $a_h$  is asymptotically consistent with the exact bilinear form  $a$  on  $(V_h^{\text{ccg}})_{h \in \mathcal{H}}$ .*

*Proof.* Let  $(v_h)_{h \in \mathcal{H}}$  be a sequence in  $(V_h^{\text{ccg}})_{h \in \mathcal{H}}$  bounded in the  $\|\cdot\|$ -norm and let  $\varphi \in \mathcal{Q}_{\mathcal{T}_h,\kappa}$ . For all  $h \in \mathcal{H}$ , we set  $\varphi_h \stackrel{\text{def}}{=} (\mathfrak{R}_h^{\text{ccg}} \circ \mathcal{I}_h^{\text{ccg}})(\varphi) \in V_h^{\text{ccg}}$ . By Theorem 23, it is clear that  $\|\varphi - \varphi_h\|_{\kappa} \rightarrow 0$  as  $h \rightarrow 0$ . Observe that

$$a_h(v_h, \varphi_h) = \int_{\Omega} \kappa \mathbf{G}_{\omega,h}(v_h) \cdot \mathbf{G}_{\omega,h}(\varphi_h) + j_h(v_h, \varphi_h) \stackrel{\text{def}}{=} \mathfrak{T}_1 + \mathfrak{T}_2.$$

Clearly, as  $h \rightarrow 0$ ,  $\mathfrak{T}_1 \rightarrow \int_{\Omega} \kappa \nabla v \cdot \nabla \varphi$  owing to the weak convergence of  $\mathbf{G}_{\omega,h}(v_h)$  to  $\nabla v$  stated in Lemma 15 and to the strong convergence of  $\mathbf{G}_{\omega,h}(\varphi_h)$  to  $\nabla \varphi$  proved in Lemma 30. Furthermore, using the Cauchy–Schwarz inequality together with Lemma 29 and the fact that, for all  $w \in H^1(\mathcal{T}_h)$ ,  $|w|_{J,\kappa} \leq \bar{\lambda}^{1/2} |w|_J$ , it is inferred that

$$|\mathfrak{T}_2| = |j_h(v_h, \varphi_h)| \leq (C_{\text{tr}}^2 N_\partial + \eta) |v_h|_{J,\kappa} |\varphi_h|_{J,\kappa} \leq (C_{\text{tr}}^2 N_\partial + \eta) \bar{\lambda}^{1/2} |v_h|_J |\varphi_h|_{J,\kappa}$$

Since  $|v_h|_J$  is bounded by assumption, and since  $|\varphi_h|_{J,\kappa} = |\varphi_h - \varphi|_{J,\kappa}$  tends to zero as  $h \rightarrow 0$ , it is inferred that  $\mathfrak{T}_2 \rightarrow 0$ . The proof is complete.  $\square$

*Remark 32* (Weakening Assumption 21). To prove the asymptotic consistency of  $a_h$ , and hence the convergence to minimal regularity solutions, we only need that

$$\forall \varphi \in \mathcal{Q}_{\mathcal{T}_h,\kappa}, \quad \|\varphi - \varphi_h\| \rightarrow 0 \text{ as } h \rightarrow 0, \quad (32)$$

with  $\varphi_h = (\mathfrak{R}_h^{\text{ccg}} \circ \mathcal{I}_h^{\text{ccg}})(\varphi) \in V_h^{\text{ccg}}$ . Property (32) holds, e.g., if  $\max_{F \in \mathcal{F}_h} \|\mathbf{A}_{\mathbf{g}F}^{-1}\|_2 \leq Ch^{-\epsilon}$  with  $0 \leq \epsilon < 1$  and  $C$  independent of the meshsize  $h$ . Whenever the solution exhibits sufficient regularity, however, one may wish to have  $\epsilon = 0$  to attain optimal convergence rates. For the sake of simplicity, Assumption 21 is required in the statements of Lemma 31 and Theorem 33, although (32) could have been used instead.

**Theorem 33** (Convergence to minimal regularity solutions). *Let  $(u_h)_{h \in \mathcal{H}}$  be the sequence of approximate solutions generated by solving the discrete problems (21). Then, under Assumption 21, as  $h \rightarrow 0$ , (i)  $u_h \rightarrow u$  strongly in  $L^2(\Omega)$ , (ii)  $\nabla_h u_h \rightarrow \nabla u$  strongly in  $[L^2(\Omega)]^d$ , (iii)  $|u_h|_J \rightarrow 0$ , with  $u \in V$  unique solution to (19).*

*Proof.* We follow the four steps outlined above.

(i) *A priori estimate.* Owing to Lemma 17ii and to the discrete Poincaré inequality obtained from (16) with  $q = 2$ ,

$$C_{\text{sta}}\Delta\|u_h\|^2 \leq C_{\text{sta}}\|u_h\|_{\kappa}^2 \leq a(u_h, u_h) = \int_{\Omega} f u_h \leq \|f\|_{L^2(\Omega)} \|u_h\|_{L^2(\Omega)} \leq \sigma_2 \|f\|_{L^2(\Omega)} \|u_h\|,$$

hence  $\|u_h\| \leq \sigma_2 (C_{\eta}\Delta)^{-1} \|f\|_{L^2(\Omega)}$ .

(ii) *Compactness.* Owing to Theorem 14 together with Lemma 15, there exists  $\bar{u} \in V = H_0^1(\Omega)$  such that, as  $h \rightarrow 0$ , up to a subsequence,  $u_h \rightarrow \bar{u}$  strongly in  $L^2(\Omega)$  and  $\mathbf{G}_{\omega,h}(u_h) \rightharpoonup \nabla \bar{u}$  weakly in  $[L^2(\Omega)]^d$ .

(iii) *Identification of the limit.* Owing to the asymptotic consistency of  $a_h$  proved in Lemma 31, for all  $\varphi \in \mathcal{Q}_{T_h, \kappa}$  with  $\varphi_h \stackrel{\text{def}}{=} (\mathfrak{N}_h^{\text{ccg}} \circ \mathcal{I}_h^{\text{ccg}})(\varphi) \in V_h^{\text{ccg}}$ ,

$$\int_{\Omega} f \varphi \leftarrow \int_{\Omega} f \varphi_h = a_h(u_h, \varphi_h) \rightarrow \int_{\Omega} \kappa \nabla \bar{u} \cdot \nabla \varphi,$$

*i.e.*,  $\bar{u}$  solves problem (19) by the density of  $\mathcal{Q}_{T_h, \kappa}$  in  $V$  stated in Theorem 19 and, hence,  $\bar{u} = u$ . Moreover, since the solution  $u$  to problem (19) is unique, the whole sequence converges (prove by contradiction).

(iv) *Strong convergence of the gradient and of the jumps.* Lemma 29 and (31) yield

$$\forall v_h \in V_h^{\text{ccg}}, \quad a_h(v_h, v_h) \geq \|\kappa^{1/2} \mathbf{G}_{\omega,h}(v_h)\|_{[L^2(\Omega)]^d}^2 + (\eta - C_{\text{tr}}^2 N_{\hat{\rho}}) |v_h|_{J, \kappa}^2 \quad (33)$$

Moreover, from the weak convergence of  $\mathbf{G}_{\omega,h}(u_h)$  to  $\nabla u$ , we readily infer the weak convergence of  $\kappa^{1/2} \mathbf{G}_{\omega,h}(u_h)$  to  $\kappa^{1/2} \nabla u$ . Owing to (33) and to weak convergence,

$$\liminf_{h \rightarrow 0} a_h(u_h, u_h) \geq \liminf_{h \rightarrow 0} \|\kappa^{1/2} \mathbf{G}_{\omega,h}(u_h)\|_{[L^2(\Omega)]^d}^2 \geq \|\kappa^{1/2} \nabla u\|_{[L^2(\Omega)]^d}^2.$$

Furthermore, still owing to (33),

$$\limsup_{h \rightarrow 0} \|\kappa^{1/2} \mathbf{G}_{\omega,h}(u_h)\|_{[L^2(\Omega)]^d}^2 \leq \limsup_{h \rightarrow 0} a_h(u_h, u_h) = \limsup_{h \rightarrow 0} \int_{\Omega} f u_h = \int_{\Omega} f u = \|\kappa^{1/2} \nabla u\|_{[L^2(\Omega)]^d}^2.$$

This classically proves the strong convergence of  $\kappa^{1/2} \mathbf{G}_{\omega,h}(u_h)$  to  $\kappa^{1/2} \nabla u$  in  $[L^2(\Omega)]^d$  and, hence, the strong convergence of  $\mathbf{G}_{\omega,h}(u_h)$  to  $\nabla u$  in  $[L^2(\Omega)]^d$ . Note that  $a_h(u_h, u_h) \rightarrow \|\kappa^{1/2} \nabla u\|_{[L^2(\Omega)]^d}^2$  also. Owing to (33),

$$(\eta - C_{\text{tr}}^2 N_{\hat{\rho}}) |u_h|_{J, \kappa}^2 \leq a_h(u_h, u_h) - \|\kappa^{1/2} \mathbf{G}_{\omega,h}(u_h)\|_{[L^2(\Omega)]^d}^2,$$

and, since  $\eta > C_{\text{tr}}^2 N_{\hat{\rho}}$  and the right-hand side tends to zero,  $|u_h|_{J, \kappa} \rightarrow 0$ . To infer that  $|u_h|_J \rightarrow 0$ , simply observe that  $|u_h|_J \leq \underline{\Delta}^{-1/2} |u_h|_{J, \kappa}$  and that the right-hand side tends to zero.  $\square$

*Remark 34* (Rough forcing terms). A possible way to handle forcing terms  $f$  in  $H^{-1}(\Omega)$  consists in replacing the test function by an interpolate in  $H_0^1(\Omega)$  in the second member. For the sake of simplicity, assume that  $T_h$  is conforming (if this is not the case,  $\mathfrak{S}_h$  can be used instead) and let  $\mathcal{I}_{\text{Os}}$  denote the Oswald interpolator discussed, e.g., by Burman and Ern [14]. It can be proved that there exists  $C$  independent of the meshsize  $h$  such that, for all  $v_h \in V_h^{\text{ccg}}$ ,  $\|\mathcal{I}_{\text{Os}} v_h\|_{H^1(\Omega)} \leq C_{\text{sta}, \text{Os}} \|v_h\|$ . We consider the following modification of the method (21):

$$\text{Find } u_h \in V_h^{\text{ccg}} \text{ s.t. } a_h(u_h, v_h) = \langle f, \mathcal{I}_{\text{Os}} v_h \rangle_{-1, 1}.$$

The *a priori* estimate for the discrete solutions on the admissible mesh family  $(T_h)_{h \in \mathcal{H}}$  is obtained as follows:

$$C_{\text{sta}}\Delta\|u_h\|^2 \leq a_h(u_h, u_h) = \langle f, \mathcal{I}_{\text{Os}} u_h \rangle_{-1, 1} \leq \|f\|_{H^{-1}(\Omega)} \|\mathcal{I}_{\text{Os}} u_h\|_{H^1(\Omega)} \leq C_{\text{sta}, \text{Os}} \|f\|_{H^{-1}(\Omega)} \|u_h\|,$$

hence  $\|u_h\| \leq C_{\text{sta}, \text{Os}} / C_{\text{sta}} \|f\|_{H^{-1}(\Omega)}$ . The convergence to minimal regularity solutions can then be proved as in Theorem 33.

### 3. STEADY INCOMPRESSIBLE NAVIER–STOKES EQUATIONS

#### 3.1. The continuous setting

In the second part of this work we corroborate the claim that ccG methods are easily extended to problems for which a dG scheme can be devised. Our focus is on the steady incompressible Navier–Stokes (INS) equations for  $d \in \{2, 3\}$ ,

$$\begin{aligned} -\nu \Delta u_i + \partial_j (u_i u_j) + \partial_i p &= f_i & \text{in } \Omega, \quad i \in \{1, \dots, d\}, \\ \partial_i u_i &= 0 & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \\ \langle p \rangle_\Omega &= 0, \end{aligned} \tag{34}$$

where the positive real  $\nu$  denotes the kinematic viscosity and  $f \in [L^2(\Omega)]^d$ . In (34) and throughout this section Einstein’s convention on repeated indices is adopted. The natural spaces for the weak formulation of (34) are

$$U \stackrel{\text{def}}{=} [H_0^1(\Omega)]^d, \quad P \stackrel{\text{def}}{=} L_0^2(\Omega), \quad X \stackrel{\text{def}}{=} U \times P,$$

where we have set  $L_0^2(\Omega) \stackrel{\text{def}}{=} \{v \in L^2(\Omega) \mid \langle v \rangle_\Omega = 0\}$ . We define the linear forms  $a \in \mathcal{L}(U \times U, \mathbb{R})$  and  $b \in \mathcal{L}(U \times P, \mathbb{R})$ , and the trilinear form  $t \in \mathcal{L}(U \times U \times U, \mathbb{R})$  such that

$$a(u, v) \stackrel{\text{def}}{=} \int_\Omega \nu \nabla u_i \cdot \nabla v_i, \quad b(v, q) \stackrel{\text{def}}{=} - \int_\Omega q \nabla \cdot v, \quad t(w, u, v) \stackrel{\text{def}}{=} \int_\Omega (w \cdot \nabla u_i) v_i - \frac{1}{2} \int_\Omega (\nabla \cdot w)(u \cdot v).$$

The trilinear form  $t$  includes Temam’s device [37] to control the kinetic energy balance as this is needed in what follows for the asymptotic consistency of its discrete counterpart. The weak formulation of system (34) is:

$$\text{Find } (u, p) \in X \text{ s.t. } c((u, p), (v, q)) + t(u, u, v) = \int_\Omega f \cdot v \text{ for all } (v, q) \in X, \tag{35}$$

with bilinear form  $c \in \mathcal{L}(X \times X, \mathbb{R})$  such that  $c((u, p), (v, q)) = a(u, v) + b(v, p) - b(u, q)$ .

#### 3.2. The discrete setting

We seek a discretization of (35) based on the following discrete spaces:

$$U_h \stackrel{\text{def}}{=} [V_h^{\text{ccg}}]^d, \quad P_h \stackrel{\text{def}}{=} \mathbb{P}_d^0(\mathcal{T}_h)/\mathbb{R}, \quad X_h \stackrel{\text{def}}{=} U_h \times P_h.$$

The main difficulties in the approximation of the INS equations lie in the discretization of the velocity-pressure coupling and of the convective term. In our case, the velocity-pressure coupling is stabilized by penalizing the pressure jumps across interfaces with a weight proportional to the meshsize; see, e.g., [17]. As regards the convective term, we use the non-dissipative trilinear form recently proposed by Di Pietro and Ern [20], which has proved suitable to convection-dominated regimes; see also Botti and Di Pietro [9] for the application to a dG discretization of the advection step in the context of a pressure-correction time-integration scheme. As the convergence analysis is similar as for the dG method of [20], the proofs of the results that hold *a fortiori* are sometimes omitted to leave room to specific issues related to the ccG method.

##### 3.2.1. Velocity-pressure coupling

The velocity-pressure coupling is based on the bilinear form  $b_h \in \mathcal{L}(U_h \times P_h, \mathbb{R})$  such that

$$b_h(v_h, q_h) \stackrel{\text{def}}{=} - \int_\Omega q_h \nabla \cdot v_h + \sum_{F \in \mathcal{F}_h} \int_F \llbracket v_h \rrbracket \cdot \mathbf{n}_F \{q_h\} = - \sum_{F \in \mathcal{F}_h^i} \int_F \{v_h\} \cdot \mathbf{n}_F \llbracket q_h \rrbracket. \tag{36}$$

A useful equivalent form for  $b_h$  can be inferred introducing the discrete divergence operator  $D_h : [H^1(\mathcal{T}_h)]^d \rightarrow \mathbb{P}_d^0(\mathcal{T}_h)$  and the discrete gradient operator  $\tilde{\mathbf{G}}_h : H^1(\mathcal{T}_h) \rightarrow [\mathbb{P}_d^0(\mathcal{T}_h)]^d$  defined as follows: For all  $v \in [H^1(\mathcal{T}_h)]^d$  and all  $w \in H^1(\mathcal{T}_h)$ ,

$$D_h(v) \stackrel{\text{def}}{=} \mathbf{G}_h(v_i) \cdot \mathbf{e}_i, \quad \tilde{\mathbf{G}}_h(w) \stackrel{\text{def}}{=} \nabla_h w - \sum_{F \in \mathcal{F}_h^i} \mathbf{r}_F(\llbracket w \rrbracket).$$

The subscript  $\omega$  has been omitted from discrete gradients and lifting operators since  $\kappa = \nu \mathbf{1}_{d,d}$  (as the kinematic viscosity is homogeneous and isotropic) implies  $\omega_{T_1, F} = \omega_{T_2, F} = 1/2$  for all  $F \in \mathcal{F}_h^i$  with  $F \subset \partial T_1 \cap \partial T_2$ . The discrete divergence  $D_h$  is defined as the trace of  $\mathbf{G}_h$  applied to a vector function, whereas  $\tilde{\mathbf{G}}_h$  only differs from  $\mathbf{G}_h$  in that boundary faces are not included in the summation in the right-hand side. It follows from (36) that

$$\forall (v_h, q_h) \in X_h, \quad b_h(v_h, q_h) = \int_{\Omega} v_h \tilde{\mathbf{G}}_h(q_h) = - \int_{\Omega} q_h D_h(v_h). \quad (37)$$

We let, for all vector functions  $v_h \in U_h$ ,

$$\|v_h\|_{\mathbf{J}}^2 \stackrel{\text{def}}{=} \sum_{i=1}^d \|v_{h,i}\|_{\mathbf{J}}^2, \quad |v_h|_{\mathbf{J}}^2 \stackrel{\text{def}}{=} \sum_{i=1}^d |v_{h,i}|_{\mathbf{J}}^2.$$

As the discrete operator associated to the discrete bilinear form  $b_h$  is not surjective, pressure stabilization must be introduced. To this end, we define the bilinear form  $s_h \in \mathcal{L}(P_h \times P_h, \mathbb{R})$  and the associated seminorm  $|\cdot|_p$  such that

$$s_h(p_h, q_h) \stackrel{\text{def}}{=} \sum_{F \in \mathcal{F}_h^i} h_F \int_F \llbracket p_h \rrbracket \llbracket q_h \rrbracket, \quad |q_h|_p \stackrel{\text{def}}{=} \sum_{F \in \mathcal{F}_h^i} h_F \|\llbracket q_h \rrbracket\|_{L^2(F)}^2.$$

We are now ready to state the main result of this section.

**Lemma 35** (Stability of the velocity-pressure coupling). *Under Assumption 21, there exists  $\beta > 0$  independent of the meshsize  $h$  such that*

$$\forall q_h \in P_h, \quad \beta \|q_h\|_{L^2(\Omega)} \leq \sup_{w_h \in U_h \setminus \{0\}} \frac{b_h(w_h, q_h)}{\|w_h\|} + |q_h|_p.$$

*Proof.* In the proof we abbreviate  $a \lesssim b$  the inequality  $a \leq Cb$  where  $C$  can depend on the mesh regularity parameters and on  $\Omega$  but not on the meshsize  $h$ . Owing to the surjectivity of the divergence operator from  $U$  to  $P$ , there exists  $C_{\Omega} > 0$  uniquely depending on the domain  $\Omega$  such that, for all  $q \in L_0^2(\Omega)$ , there exists  $v \in [H_0^1(\Omega)]^d$  such that

$$\nabla \cdot v = q, \quad C_{\Omega} \|v\|_{[H^1(\Omega)]^d} \leq \|q\|_{L^2(\Omega)}. \quad (38)$$

Let now  $q = q_h$ , denote by  $v$  the element of  $[H_0^1(\Omega)]^d$  satisfying (38) and set  $\mathbf{v}_h \stackrel{\text{def}}{=} (\langle v \rangle_T)_{T \in \mathcal{T}_h} \in \mathbb{V}_h$  and  $v_h \stackrel{\text{def}}{=} \mathfrak{R}_h^{\text{ccg}}(\mathbf{v}_h) \in V_h^{\text{ccg}}$ . Then,

$$\begin{aligned} C_{\Omega} \|v\|_{[H^1(\Omega)]^d} \|q_h\|_{L^2(\Omega)} &\leq \|q_h\|_{L^2(\Omega)}^2 = \int_{\Omega} q_h \nabla \cdot v = \sum_{F \in \mathcal{F}_h^i} \int_F \llbracket q_h \rrbracket \{v\} \cdot \mathbf{n}_F = -b_h(v_h, q_h) + \sum_{F \in \mathcal{F}_h^i} \int_F \llbracket q_h \rrbracket \{v - v_h\} \cdot \mathbf{n}_F \\ &\leq \left\{ \sup_{w_h \in U_h \setminus \{0\}} \frac{b_h(w_h, q_h)}{\|w_h\|} \right\} \|v_h\| + |q_h|_p \times \left\{ \sum_{F \in \mathcal{F}_h^i} h_F^{-1} \int_F |v - v_h|^2 \right\}. \end{aligned}$$

It follows from Lemma 36 that  $\|v_h\| \lesssim \|v\|_{[H^1(\Omega)]^d}$ . Moreover, using the continuous trace inequality (5) together with the approximation properties of the mesh sequence  $(\mathcal{T}_h)_{h \in \mathcal{H}}$ , it is readily proved that  $\left\{ \sum_{F \in \mathcal{F}_h^i} h_F^{-1} \int_F |v - v_h|^2 \right\} \lesssim \|v\|_{H^1(\Omega)}$ . The result follows.  $\square$



The following lemma establishes the  $H^1$ -stability property for functions in  $V_h^{\text{ccg}}$  used in the proof of Lemma 35. Observe that, unlike elsewhere, the degrees of freedom are here interpreted as average values over the cells, since the regularity of the function  $v$  is in general insufficient for point values to be defined inside elements.

**Lemma 36** ( $H^1$ -stability). *Under Assumption 21, for all  $v \in H_0^1(\Omega)$  there holds*

$$\|v_h\| \leq C \|v\|_{H^1(\Omega)},$$

where  $\mathbf{v}_h = (\langle v \rangle_T)_{T \in \mathcal{T}_h} \in \mathbb{V}_h$ ,  $v_h = \mathfrak{R}_h^{\text{ccg}}(\mathbf{v}_h) \in V_h^{\text{ccg}}$  and  $C$  is independent of the meshsize  $h$ .

*Proof.* In the proof we abbreviate  $a \lesssim b$  the inequality  $a \leq Cb$  where  $C$  can depend on the mesh regularity parameters and on  $\Omega$  but not on the meshsize  $h$ . By definition,

$$\|v_h\|^2 = \|\nabla_h v_h\|_{[L^2(\Omega)]^d}^2 + |v_h|_J^2 \stackrel{\text{def}}{=} \mathfrak{T}_1 + \mathfrak{T}_2.$$

The first term can be bounded as follows:

$$\mathfrak{T}_1 \lesssim \sum_{T \in \mathcal{T}_h} |T|_d \left| \frac{1}{|T|_d} \sum_{F \in \mathcal{F}_T} |F|_{d-1} (\langle v \rangle_T - \langle v \rangle_F) \mathbf{n}_{T,F} \right|^2 + \sum_{T \in \mathcal{T}_h} |T|_d \left| \frac{1}{|T|_d} \sum_{F \in \mathcal{F}_T} |F|_{d-1} (\langle v \rangle_F - \langle \xi_{\mathbf{v}_h}^{\mathfrak{g}F} \rangle_F) \mathbf{n}_{T,F} \right|^2.$$

Denote by  $\mathfrak{T}_{1,1}$  and  $\mathfrak{T}_{1,2}$  the addends in the right-hand side. Using the Cauchy–Schwarz inequality together with formula (9)

$$\mathfrak{T}_{1,1} \leq \sum_{T \in \mathcal{T}_h} \left\{ \left( \sum_{F \in \mathcal{F}_T} \frac{|F|_{d-1}}{|T|_d} d_{T,F} \right) \times \left( \sum_{F \in \mathcal{F}_T} d_{T,F}^{-1} \|\langle v \rangle_T - v\|_{L^2(F)}^2 \right) \right\} \leq dC^2 \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \frac{h_T}{d_{T,F}} \|v\|_{H^1(T)}^2 \lesssim \|v\|_{H^1(\Omega)}^2,$$

where we have used the classical estimate  $\|\langle v \rangle_T - v\|_{L^2(F)} = \|\pi_h^0 v - v\|_{L^2(F)} \leq Ch_T^{1/2} |v|_{H^1(T)}$ . As for the term  $\mathfrak{T}_{1,2}$ , repeated applications of the Cauchy–Schwarz inequality together with (8) yield

$$\begin{aligned} \mathfrak{T}_{1,2} &\leq \sum_{T \in \mathcal{T}_h} \frac{1}{|T|_d} \left\{ \sum_{F \in \mathcal{F}_T} \int_F (v - \xi_{\mathbf{v}_h}^{\mathfrak{g}F}) \mathbf{n}_{T,F} \right\}^2 \leq \sum_{T \in \mathcal{T}_h} \frac{1}{|T|_d} \left\{ \sum_{F \in \mathcal{F}_T} |F|_{d-1}^{1/2} \|v - \xi_{\mathbf{v}_h}^{\mathfrak{g}F}\|_{L^2(F)} \right\}^2 \\ &\leq \sum_{T \in \mathcal{T}_h} \left\{ \sum_{F \in \mathcal{F}_T} \frac{|F|_{d-1} d_{T,F}}{|T|_d} \right\} \times \left\{ \frac{1}{d_{T,F}} \sum_{F \in \mathcal{F}_T} \|v - \xi_{\mathbf{v}_h}^{\mathfrak{g}F}\|_{L^2(F)}^2 \right\} \leq d \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \frac{1}{d_{T,F}} \|v - \xi_{\mathbf{v}_h}^{\mathfrak{g}F}\|_{L^2(F)}^2. \end{aligned}$$

Since  $\boldsymbol{\kappa} = \nu \mathbf{1}_d$ , for all  $F \in \mathcal{F}_h^i$ ,  $\xi_{\mathbf{v}_h}^{\mathfrak{g}F}(\bar{\mathbf{x}}_F)$  can be expressed as a linear combination of the values  $\{\langle v \rangle_T\}_{T \in \mathcal{T}_g}$  with coefficients  $\{\tau_T^F\}_{T \in \mathcal{T}_g}$  such that  $0 \leq \tau_T^F \leq 1$ . Hence, for all  $F \in \mathcal{F}_h^i$ , using the triangular inequality, the continuous trace inequality (5), and the approximation properties of the  $L^2$ -projector onto  $\mathbb{P}_d^0(\mathcal{T}_h)$  we infer

$$\begin{aligned} \|v - \xi_{\mathbf{v}_h}^{\mathfrak{g}F}\|_{L^2(F)} &\leq \sum_{T \in \mathcal{T}_g} \tau_T^F \|v - \langle v \rangle_T\|_{L^2(F)} \leq C_{\text{tr},c} \sum_{T \in \mathcal{T}_g} \tau_T^F \left( h_T^{-1} \|v - \langle v \rangle_T\|_{L^2(T)}^2 + h_T |v|_{H^1(T)}^2 \right)^{1/2} \\ &\lesssim \max_{T \in \mathcal{T}_g} |\tau_T^F| h_T^{1/2} \sum_{T \in \mathcal{T}_g} \|v\|_{H^1(T)}, \end{aligned} \tag{39}$$

and  $\max_{T \in \mathcal{T}_g} |\tau_T^F|$  is uniformly bounded owing to Assumption 21. Moreover, for all  $F \in \mathcal{F}_h^b$ ,  $\|v - \xi_{\mathbf{v}_h}^{\mathfrak{g}F}\|_{L^2(F)} = 0$ . We conclude that  $\mathfrak{T}_{1,2} \lesssim \|v\|_{H^1(\Omega)}$ . To bound  $\mathfrak{T}_2$  observe that since,  $v$  is continuous across interfaces and it vanishes on  $\partial\Omega$ ,  $|v_h|_J = |v_h - v|_J$ . The conclusion follows from (3), (6), and (39).  $\square$

### 3.2.2. A non-dissipative convective trilinear form

The discrete convective trilinear form  $t_h \in \mathcal{L}(U_h \times U_h \times U_h, \mathbb{R})$  is given by

$$t_h(w_h, u_h, v_h) \stackrel{\text{def}}{=} \int_{\Omega} (w_h \cdot \nabla_h u_{h,i}) v_{h,i} - \sum_{F \in \mathcal{F}_h^i} \int_F \{w_h\} \cdot \mathbf{n}_F \llbracket u_h \rrbracket \cdot \{v_h\} + \frac{1}{2} \int_{\Omega} (\nabla_h \cdot w_h)(u_h \cdot v_h) - \frac{1}{2} \sum_{F \in \mathcal{F}_h} \int_F \llbracket w_h \rrbracket \cdot \mathbf{n}_F \{u_h \cdot v_h\}.$$

The following lemma collects some important results. The proof essentially follows [20, Proposition 5.2]. In particular, the Sobolev embedding 16 for  $q = 4$  is required, which limits the space dimension to  $d \in \{2, 3\}$ .

**Lemma 37** (Properties of the trilinear form  $t_h$ ). *For all  $h \in \mathcal{H}$ , let  $\mathcal{U}_h \stackrel{\text{def}}{=} [\mathbb{P}_d^k(\mathcal{T}_h)]^d$  for some  $k > 0$ . The trilinear form  $t_h$  defined by (3.2.2) enjoys the following properties:*

- (i) Non-dissipativity. *For all  $w_h, v_h \in \mathcal{U}_h$ ,  $t_h(w_h, v_h, v_h) = 0$ .*
- (ii) Boundedness. *There is  $C_{\text{bnd},t}$  independent of the meshsize  $h$  such that, for all  $w_h, u_h, v_h \in \mathcal{U}_h$ ,  $t_h(w_h, u_h, v_h) \leq C_{\text{bnd},t} \|w_h\| \|u_h\| \|v_h\|$ .*
- (iii) Asymptotic consistency for smooth functions. *Let  $(v_h)_{h \in \mathcal{H}}$  be a sequence in  $(\mathcal{U}_h)_{h \in \mathcal{H}}$  bounded in the  $\|\cdot\|$ -norm. Then, for all  $\Phi \in [C_0^\infty(\Omega)]^d$ , as  $h \rightarrow 0$ , up to a subsequence,  $t_h(v_h, v_h, \Phi_h) \rightarrow t(v, v, \Phi)$ , where  $\Phi_h \stackrel{\text{def}}{=} (\mathfrak{R}_h^{\text{ccg}} \circ \mathcal{I}_h^{\text{ccg}})(\Phi) \in U_h$  and  $v \in U$  is the limit provided by Theorem 14.*
- (iv) Asymptotic consistency for test functions. *In the setting of point (iii), further suppose that  $\mathbf{G}_h(v_h) \rightarrow \nabla v$  strongly in  $[L^2(\Omega)]^d$  and that  $|v_h|_J \rightarrow 0$ . Let  $(w_h)_{h \in \mathcal{H}}$  be another sequence in  $(\mathcal{U}_h)_{h \in \mathcal{H}}$  bounded in the  $\|\cdot\|$ -norm. Then, as  $h \rightarrow 0$ , up to a subsequence,  $t_h(v_h, v_h, w_h) \rightarrow t(v, v, w)$ , with both  $v, w \in U$  resulting from Theorem 14.*

The discrete problem reads

$$\text{Find } (u_h, p_h) \in X_h \text{ s.t. } c_h((u_h, p_h), (v_h, q_h)) + t_h(u_h, u_h, v_h) = \int_{\Omega} f \cdot v_h \text{ for all } (v_h, q_h) \in X_h, \quad (40)$$

with bilinear form  $c_h \in \mathcal{L}(X_h \times X_h, \mathbb{R})$  such that  $c_h \stackrel{\text{def}}{=} a_h(u_{h,i}, v_{h,i}) + b_h(v_h, p_h) - b_h(u_h, q_h) + s_h(p_h, q_h)$ , where we have set  $\kappa = \nu \mathbf{1}_{d,d}$  in the expression of  $a_h$  and a sum over the index  $i$  is understood in the first term.

### 3.3. Convergence

We study the convergence of the method (40) in the spirit of §2.4. As the ccG space  $V_h^{\text{ccg}}$  is a subspace of  $\mathbb{P}_d^1(\mathcal{T}_h)$ , some of the results presented in [20, §5] in the context of dG methods hold *a fortiori*. In such cases, the details of the proofs are omitted in order to restrict the focus to the peculiarities of the proposed ccG method. Also, since the diffusion coefficient is homogeneous, the standard test space  $C_0^\infty(\Omega)$  can replace  $\mathcal{Q}_{\mathcal{T}_h, \kappa}$  in the convergence proof. The following lemmata contain results that are instrumental to the analysis.

**Lemma 38** (Properties of  $D_h$ ). *The discrete divergence  $D_h$  enjoys the following properties:*

- (i) Consistency for smooth functions. *Let  $\Phi \in [C_0^\infty(\Omega)]^d$  and set  $\Phi_h \stackrel{\text{def}}{=} (\mathfrak{R}_h^{\text{ccg}} \circ \mathcal{I}_h^{\text{ccg}})(\Phi) \in U_h$ . Then, under Assumption 21, as  $h \rightarrow 0$ ,  $D_h(\Phi_h) \rightarrow \nabla \cdot \Phi$  strongly in  $L^2(\Omega)$ .*
- (ii) Weak asymptotic consistency for test functions. *Let  $(v_h)_{h \in \mathcal{H}}$  be a sequence in  $U_h$ . Then, as  $h \rightarrow 0$ , up to a subsequence,  $D_h(v_h) \rightarrow \nabla \cdot v$  weakly in  $L^2(\Omega)$ , where  $v \in [H_0^1(\Omega)]^d$  is the limit resulting from Theorem 14.*

*Proof.* Point (i) is a direct consequence of Theorem 23 together with the fact that  $\kappa = \nu \mathbf{1}_d$  implies  $C_0^\infty(\Omega) \subset \mathcal{Q}_{\mathcal{T}_h, \kappa}$ , whereas point (ii) immediately follows from Lemma 15.  $\square$

**Lemma 39** (Weak asymptotic consistency of  $\tilde{\mathbf{G}}_h$ ). *Let  $(v_h)_{h \in \mathcal{H}}$  be a sequence in  $(\mathbb{P}_d^k(\mathcal{T}_h))_{h \in \mathcal{H}}$ ,  $k > 0$ , uniformly bounded in the  $\|\cdot\|$ -norm. Then, as  $h \rightarrow 0$ , up to a subsequence,  $\tilde{\mathbf{G}}_h(v_h) \rightarrow \nabla v$  weakly in  $[L^2(\Omega)]^d$ , where  $v \in H_0^1(\Omega)$  is the limit resulting from Theorem 14.*

*Proof.* See [20, §2.3].  $\square$

**Lemma 40** (Existence of a solution to (40)). *There exists  $(u_h, p_h) \in X_h$  solution to (40).*

*Proof.* The proof is based on a topological degree argument and it follows [20, Proposition 5.1]. The use of a topological degree argument to assert the existence of a discrete solution can be traced back to Eymard, Herbin, and Latché [31].  $\square$

The first step in the convergence proof is to derive a uniform *a priori* estimate on a suitable norm of the discrete solution. This estimate is used to infer a compactness property for the sequence of discrete solutions. To this end, we introduce the following norm on  $X_h$ :

$$\| (v_h, q_h) \|_{\text{ns}}^2 \stackrel{\text{def}}{=} \| v_h \|^2 + \| q_h \|_{L^2(\Omega)}^2 + |q_h|_p^2.$$

**Lemma 41** (*A priori estimate*). *There exists  $\gamma > 0$  independent of the meshsize  $h$  such that*

$$\gamma \| (u_h, p_h) \| \leq \sigma_2 \| f \|_{[L^2(\Omega)]^d} + C_{\text{bnd},t} (\nu C_{\text{sta}})^{-1} (\sigma_2 \| f \|_{[L^2(\Omega)]^d})^2. \quad (41)$$

where  $C_{\text{bnd},t}$  results from Lemma 37ii,  $C_{\text{sta}}$  is the coercivity constant of  $a_h$ , and  $\sigma_2$  results from Theorem 13.

*Proof.* The proof proceeds along the lines of [20, Lemma 5.1]. The details are omitted for the sake of brevity.  $\square$

**Theorem 42** (Convergence). *Let  $((u_h, p_h))_{h \in \mathcal{H}}$  be a sequence of approximate solutions generated by solving the discrete problems (40) on the admissible mesh sequence  $(\mathcal{T}_h)_{h \in \mathcal{H}}$ . Then, under Assumption 21, as  $h \rightarrow 0$ , up to a subsequence, (i)  $u_h \rightarrow u$  strongly in  $[L^2(\Omega)]^d$ , (ii)  $\nabla_h u_h \rightarrow \nabla u$  strongly in  $[L^2(\Omega)]^{d,d}$ , (iii)  $|u_h|_J \rightarrow 0$ , (iv)  $p_h \rightarrow p$  strongly in  $L^2(\Omega)$ , (v)  $|p_h|_p \rightarrow 0$ , with  $(u, p) \in X$  solution to (35). If the continuous solution  $(u, p)$  is unique, the convergence property extends to the whole sequence.*

*Proof.* (i) *Compactness.* Owing to the *a priori* estimate (41), by Theorem 14 together with Lemma 15 there exists  $(\bar{u}, \bar{p}) \in X$  such that, up to a subsequence,  $u_h \rightarrow \bar{u}$  strongly in  $[L^2(\Omega)]^d$ ,  $\mathbf{G}_h(u_{h,i}) \rightharpoonup \nabla \bar{u}_i$  weakly in  $[L^2(\Omega)]^d$  for  $i \in \{1, \dots, d\}$ , and  $p_h \rightharpoonup \bar{p}$  weakly in  $L^2(\Omega)$  (as the sequence  $(p_h)_{h \in \mathcal{H}}$  is uniformly bounded in the  $L^2$ -norm). For the sake of conciseness, subsequences are not renumbered in what follows.

(ii) *Identification of the limit and convergence of a subsequence.* Let  $\Phi \in [C_0^\infty(\Omega)]^d$  and set  $\Phi_h \stackrel{\text{def}}{=} (\mathfrak{R}_h^{\text{ccg}} \circ \mathcal{I}_h^{\text{ccg}})(\Phi) \in U_h$ . Testing against  $(\Phi_h, 0)$  yields

$$a_h(u_h, \Phi_h) + b_h(\Phi_h, p_h) + t_h(u_h, u_h, \Phi_h) = \int_{\Omega} f \cdot \Phi_h.$$

Clearly, as  $h \rightarrow 0$ , the right-hand side tends to  $\int_{\Omega} f \cdot \Phi$ . Furthermore, by virtue of Lemma 31, the first term in the left-hand side converges to  $a(u, \Phi) = \int_{\Omega} \nu \nabla u_i \cdot \nabla \Phi_i$ . Using (37), the second term can be written as  $b_h(\Phi_h, p_h) = -\int_{\Omega} p_h \mathbf{D}_h(\Phi_h)$ . Owing to the weak convergence of  $(p_h)_{h \in \mathcal{H}}$  to  $\bar{p}$  in  $L^2(\Omega)$  and to the strong convergence of  $(\mathbf{D}_h(\Phi_h))_{h \in \mathcal{H}}$  to  $\nabla \cdot \Phi$  in  $L^2(\Omega)$  stated in Lemma 38, this term converges to  $b(\Phi, \bar{p}) = -\int_{\Omega} \bar{p} \nabla \cdot \Phi$ . Finally  $t_h(u_h, u_h, \Phi_h)$  tends to  $t(\bar{u}, \bar{u}, \Phi)$  owing to Lemma 37iii. As a result,

$$a(\bar{u}, \Phi) + b(\Phi, \bar{p}) + t(\bar{u}, \bar{u}, \Phi) = \int_{\Omega} f \cdot \Phi.$$

Let now  $\varphi \in C_0^\infty(\Omega)/\mathbb{R}$  and set  $\varphi_h \stackrel{\text{def}}{=} \pi_h^0 \varphi$ , where  $\pi_h^0$  denotes the  $L^2$ -orthogonal projection onto  $\mathbb{P}_d^0(\mathcal{T}_h)$ . Testing against  $(0, \varphi_h)$  yields

$$-b_h(u_h, \varphi_h) + s_h(p_h, \varphi_h) = 0.$$

Clearly,  $-b_h(u_h, \varphi_h) = \int_{\Omega} \varphi_h \mathbf{D}_h(u_h)$  tends to  $\int_{\Omega} \varphi \nabla \cdot \bar{u}$  as  $h \rightarrow 0$  since  $(\mathbf{D}_h(u_h))_{h \in \mathcal{H}}$  weakly converges to  $\nabla \cdot \bar{u}$  in  $L^2(\Omega)$  and  $(\varphi_h)_{h \in \mathcal{H}}$  strongly converges to  $\varphi$  in  $L^2(\Omega)$ . Furthermore, using the *a priori* estimate (41),

$|s_h(p_h, \varphi_h)| \leq |p_h|_p |\varphi_h|_p \leq C |\varphi_h|_p$  with  $C$  independent of the meshsize  $h$  and this upper bound tends to zero. Hence,

$$\int_{\Omega} \varphi \nabla \cdot \bar{u} = 0.$$

By density of  $[C_0^\infty(\Omega)]^d \times (C_0^\infty(\Omega)/\mathbb{R})$  in  $X$ , this shows that  $(\bar{u}, \bar{p}) = (u, p)$  solution to (35).

(iii) *Strong convergence of the velocity gradient and of the velocity and pressure jumps.* Owing to the non-dissipativity of  $t_h$ ,

$$\int_{\Omega} f \cdot u_h = c_h((u_h, p_h), (u_h, p_h)) = a_h(u_h, u_h) + s_h(p_h, p_h) \geq a_h(u_h, u_h) \geq \nu \sum_{i=1}^d \|\mathbf{G}_h(u_{h,i})\|_{[L^2(\Omega)]^d}^2.$$

Thus,  $\nu \limsup_{h \rightarrow 0} \sum_{i=1}^d \|\mathbf{G}_h(u_{h,i})\|_{[L^2(\Omega)]^d}^2 \leq \limsup_{h \rightarrow 0} \int_{\Omega} f \cdot u_h = \int_{\Omega} f \cdot u = \nu \|\nabla u\|_{[L^2(\Omega)]^d}^2$ . Proceeding as in point (iv) of Theorem 33, it is then inferred that  $\mathbf{G}_h(u_{h,i}) \rightarrow \nabla u_i$  in  $[L^2(\Omega)]^d$  for all  $i \in \{1, \dots, d\}$  and that  $|u_h|_J \rightarrow 0$ . Finally, since  $|p_h|_p^2 = b_h(u_h, p_h) = \int_{\Omega} f \cdot u_h - a_h(u_h, u_h)$ , we conclude that  $|p_h|_p \rightarrow 0$ .

(v) *Strong convergence of the pressure.* Let  $v \in [H_0^1(\Omega)]^d$  be such that  $\nabla \cdot v = p_h$  with  $\|v\|_{[H^1(\Omega)]^d} \leq C_{\Omega} \|p_h\|_{L^2(\Omega)}$ , and set  $v_h \stackrel{\text{def}}{=} \mathfrak{R}_h^{\text{ccg}}(\mathbf{v}_h) \in V_h^{\text{ccg}}$  with  $\mathbf{v}_h = (\langle v \rangle_T)_{T \in \mathcal{T}_h} \in \mathbb{V}_h$ . In the rest of the proof we abbreviate  $a \lesssim b$  the inequality  $a \leq Cb$  with  $C$  independent of the meshsize  $h$ . Proceeding as in the proof of Lemma 35 yields

$$\|p_h\|_{L^2(\Omega)}^2 \lesssim |p_h|_p \|p_h\|_{L^2(\Omega)} - b_h(v_h, p_h) \lesssim |p_h|_p \|p_h\|_{L^2(\Omega)} + a_h(u_h, v_h) + t_h(u_h, u_h, v_h) - \int_{\Omega} f \cdot v_h.$$

Let  $\mathfrak{T}_i$ ,  $i \in \{1, \dots, 4\}$  denote the terms in the right-hand side. Since  $|p_h|_p$  tends to zero and  $\|p_h\|_{L^2(\Omega)}$  is bounded,  $\mathfrak{T}_1$  converges to zero. Furthermore, since the sequence  $(v_h)_{h \in \mathcal{H}}$  is bounded in the  $\|\cdot\|$ -norm because  $\|v_h\| \lesssim \|v\|_{[H^1(\Omega)]^d} \lesssim \|p_h\|_{L^2(\Omega)}$  there is  $\bar{v} \in [H_0^1(\Omega)]^d$  such that, up to a subsequence,  $v_h \rightarrow \bar{v}$  strongly in  $[L^2(\Omega)]^d$  and  $\mathbf{G}_h(v_{h,i}) \rightarrow \nabla \bar{v}_i$  weakly in  $[L^2(\Omega)]^d$  for  $i \in \{1, \dots, d\}$ . Owing to the uniqueness of the limit in the distribution sense, it is inferred that  $\nabla \cdot \bar{v} = p$ . There holds

$$\mathfrak{T}_2 = a_h(u_h, v_h) = \int_{\Omega} \nu \mathbf{G}_h(u_{h,i}) \cdot \mathbf{G}_h(v_{h,i}) + s_h(u_h, v_h) = \mathfrak{T}_{2,1} + \mathfrak{T}_{2,2}.$$

Owing to the strong convergence of  $(\mathbf{G}_h(u_{h,i}))_{h \in \mathcal{H}}$  in  $[L^2(\Omega)]^d$  proved in the previous point and to the weak convergence of  $(\mathbf{G}_h(v_{h,i}))_{h \in \mathcal{H}}$  in  $[L^2(\Omega)]^d$ , it is inferred that  $\mathfrak{T}_{2,1} \rightarrow \int_{\Omega} \nabla u_i \cdot \nabla \bar{v}_i$ . Moreover,  $|\mathfrak{T}_{2,2}| \lesssim |u_h|_J |v_h|_J$ , which converges to zero. Owing to Lemma 37iv,  $\mathfrak{T}_3 \rightarrow t(u, u, \bar{v})$ . Finally, since  $\mathfrak{T}_4 \rightarrow \int_{\Omega} f \cdot \bar{v}$ , the strong convergence of the pressure in  $L^2$  classically follows from

$$\limsup_{h \rightarrow 0} \|p_h\|_{L^2(\Omega)}^2 \leq \int_{\Omega} \nabla u_i \cdot \nabla \bar{v}_i + t(u, u, \bar{v}) - \int_{\Omega} f \cdot \bar{v} = \int_{\Omega} p \nabla \cdot \bar{v} = \|p\|_{L^2(\Omega)}^2. \quad \square$$

## 4. NUMERICAL EXAMPLES

### 4.1. Pure diffusion

**Anisotropy.** To assess the robustness of the method with respect to the anisotropy of the diffusion tensor we consider the following exact solution to problem (19) in  $d = 2$ :

$$u = \sin(\pi x) \sin(\pi y), \quad \boldsymbol{\kappa} = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}, \quad f = (1 + \epsilon)\pi^2 \sin(\pi x) \sin(\pi y),$$

with anisotropy ratio  $\epsilon = 10^{-3}$ . The discrete problem is solved on the Kershaw mesh sequence 4.2 of the FVCA5 benchmark [33], and the results are listed in Table 1. Besides the errors in the  $L^2$ - and energy-norms,

TABLE 1. Anisotropic test case

$\text{card}(\mathcal{T}_h)$	$\ u - u_h\ _{L^2(\Omega)}$	order	$\ u - u_h\ $	order	$\max_{F \in \mathcal{F}_h^i} \ \mathbf{A}_{\mathfrak{g}F}^{-1}\ _\infty$	cg+AMG it.
9801	1.2396e-02	–	5.1296e-02	–	1.0028e+03	41
17424	6.8589e-03	2.06	3.3572e-02	1.47	1.0018e+03	49
27225	3.9340e-03	2.49	2.3897e-02	1.52	1.0013e+03	55
39204	2.5485e-03	2.38	1.8058e-02	1.54	1.0009e+03	62

TABLE 2. Heterogeneous test case

$\text{card}(\mathcal{T}_h)$	$\ u - u_h\ _{L^2(\Omega)}$	order	$\ u - u_h\ $	order	$\max_{F \in \mathcal{F}_h^i} \ \mathbf{A}_{\mathfrak{g}F}^{-1}\ _\infty$	cg+AMG it.
224	7.3209e-03	–	1.1526e-01	–	1.2800e+02	7
896	1.9172e-03	1.93	5.6440e-02	1.03	1.2800e+02	8
3584	4.8802e-04	1.97	2.7925e-02	1.02	1.2800e+02	9
14336	1.2330e-04	1.98	1.3891e-02	1.01	1.2800e+02	10

Table 1 lists the maximum  $\|\cdot\|_\infty$ -norm of the matrices defined by (12) as well as the number of conjugate gradient iterations with AMG preconditioner required to solve the linear system. An inspection of column 6 shows that Assumption 21 is satisfied as the  $\|\cdot\|_\infty$ -norm of the matrices involved in the local construction remains almost constant when refining the mesh. Here and in what follows, we estimate the order of convergence as

$$\text{order} = d \ln(e_1/e_2) / \ln(\text{card}(\mathcal{T}_{h_2})/\text{card}(\mathcal{T}_{h_1})),$$

where  $e_1$  and  $e_2$  denote, respectively, the errors committed on  $\mathcal{T}_{h_1}$  and  $\mathcal{T}_{h_2}$ ,  $h_1, h_2 \in \mathcal{H}$ .

**Heterogeneity.** To assess the robustness with respect to the heterogeneity of  $\kappa$ , consider the following pseudo 2d exact solution to (19) on the unit square domain  $\Omega = (0, 1)^2$ :

$$u = \begin{cases} -\frac{1}{2}x^2 + \frac{3+\epsilon}{4(1+\epsilon)}x & \text{if } x \leq \frac{1}{2}, \\ -\frac{1}{2\epsilon}x^2 + \frac{3+\epsilon}{4\epsilon(1+\epsilon)}x + \frac{\epsilon-1}{4\epsilon(1+\epsilon)} & \text{if } x > \frac{1}{2}. \end{cases} \quad \kappa = \begin{cases} \mathbf{1}_2 & \text{if } x < \frac{1}{2}, \\ \epsilon \mathbf{1}_2 & \text{if } x > \frac{1}{2}, \end{cases} \quad f = 1. \quad (42)$$

The parameter  $\epsilon$  represents here the heterogeneity ratio, and it has been taken equal to  $10^{-2}$ . The numerical results are collected in Table 2, and symbols have the same meaning as in the previous case.

#### 4.2. Steady incompressible Navier–Stokes equations

**The Kovasznay problem.** To verify the asymptotic convergence properties of the method (40), we consider Kovasznay’s analytical solution of the INS equations [34] on the square domain  $\Omega = (-0.5, 1.5) \times (0, 2)$ ,

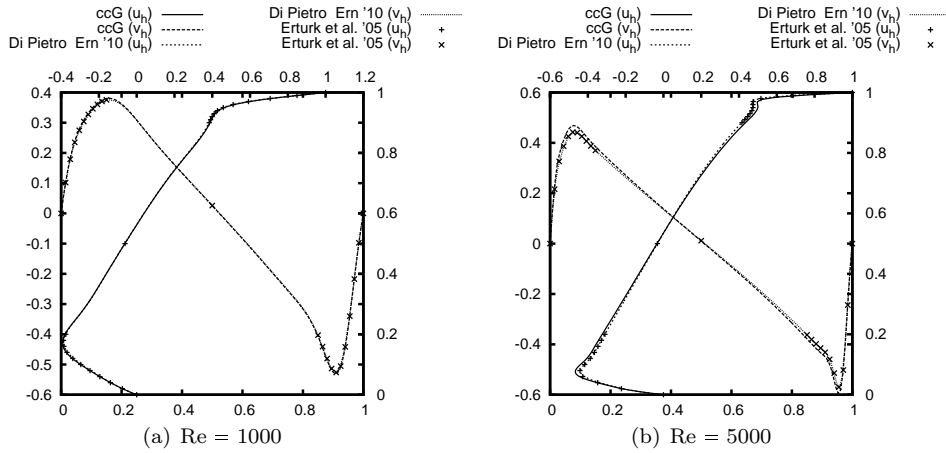
$$u_1 = 1 - e^{\pi x_2} \cos(2\pi x_2), \quad u_2 = -1/2 e^{\pi x_1} \sin(2\pi x_2), \quad p = -1/2 e^{\pi x_1} \cos(2\pi x_2) - \bar{p},$$

where  $\bar{p} = \langle -\frac{1}{2} e^{\pi x_1} \cos(2\pi x_2) \rangle_\Omega \approx -0.920735694$  ensures the zero mean constraint for the pressure,  $\nu = 3\pi$ , and  $f = 0$ . The example is run on a family of uniformly refined triangular meshes with mesh sizes ranging from 0.5 down to 0.03125. According to Table 1, the errors  $\|u - u_h\|_{\text{sto}}$  and  $\|p - p_h\|_{L^2(\Omega)}$  converge to first order, while second order is attained for  $\|u - u_h\|_{[L^2(\Omega)]^d}$ . The results are collected in Table 4.2.

**The lid-driven cavity problem.** To assess the behavior of the method in more complex situations we consider the classical lid-driven cavity problem. Despite its simple geometry, at large Reynolds numbers this problem presents complex flow patterns with counter-rotating vortices of significantly different scale. The domain is here the unit square with imposed horizontal velocity on the upper side and zero velocity on the others. In Figure 3 we provide the values of the velocity components on the centerlines of the domain. For the sake of completeness,

TABLE 3. Convergence results for the Kovasznay problem

$\text{card}(\mathcal{T}_h)$	$\ u - u_h\ _{[L^2(\Omega)]^d}$	order	$\ p - p_h\ _{L^2(\Omega)}$	order	$\ u - u_h\ _{\text{sto}}$	order
224	1.6539e-01	—	2.5536e-01	—	4.7777e-01	—
896	4.3732e-02	1.92	1.0737e-01	1.25	2.1759e-01	1.13
3584	1.1847e-02	1.88	3.9802e-02	1.43	1.0763e-01	1.02
14336	3.1620e-03	1.91	1.7385e-02	1.19	5.5182e-02	0.96

FIGURE 3. Lid-driven cavity problem, comparison of centerline velocity values (ccG = present work, Di Pietro Ern '10 = ref. [20], Erturk *et al.* '05 = ref. [28]).

we compare against the method of [20] with piecewise linear approximations of the velocity and the pressure. In both cases a uniform  $128 \times 128$  Cartesian orthogonal mesh was used. The reference data of Erturk, Corke, and Gökçöl [28] are also included. The proposed method shows essentially the same accuracy as the dG method of [20] at  $\text{Re} = 1000$ . To observe more sizable differences, we also present the results for  $\text{Re} = 5000$  on the same mesh. In this case, where a slight loss of accuracy can be observed towards the boundaries of the domain.

## REFERENCES

- [1] I. Aavatsmark, T. Barkve, Ø. Bøe, and T. Mannseth. Discretization on unstructured grids for inhomogeneous, anisotropic media, Part I: Derivation of the methods. *SIAM J. Sci. Comput.*, 19(5):1700–1716, 1998.
- [2] I. Aavatsmark, T. Barkve, Ø. Bøe, and T. Mannseth. Discretization on unstructured grids for inhomogeneous, anisotropic media, Part II: Discussion and numerical results. *SIAM J. Sci. Comput.*, 19(5):1717–1736, 1998.
- [3] I. Aavatsmark, G. T. Eigestad, B. T. Mallison, and J. M. Nordbotten. A compact multipoint flux approximation method with improved robustness. *Numer. Methods Partial Differ. Eq.*, 24:1329–1360, 2008.
- [4] L. Agélas, D. A. Di Pietro, and J. Droniou. The G method for heterogeneous anisotropic diffusion on general meshes. *M2AN Math. Model. Numer. Anal.*, 44(4):597–625, 2010. DOI: 10.1051/m2an/2010021.
- [5] L. Agélas, D. A. Di Pietro, R. Eymard, and R. Masson. An abstract analysis framework for nonconforming approximations of diffusion problems on general meshes. *IJFV*, 7(1):1–29, 2010.
- [6] D. N. Arnold. An interior penalty finite element method with discontinuous elements. *SIAM J. Numer. Anal.*, 19:742–760, 1982.
- [7] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39(5):1749–1779, 2002.
- [8] J.-P. Aubin. *Analyse fonctionnelle appliquée*. Presses Universitaires de France, Paris, 1987.
- [9] L. Botti and D. A. Di Pietro. A pressure-correction scheme for convection-dominated incompressible flows with discontinuous velocity and continuous pressure. Submitted. Preprint available at <http://hal.archives-ouvertes.fr/hal-00458293/en/>, 2010.

- [10] F. Brezzi, K. Lipnikov, and M. Shashkov. Convergence of mimetic finite difference methods for diffusion problems on polyhedral meshes. *SIAM J. Numer. Anal.*, 45:1872–1896, 2005.
- [11] F. Brezzi, K. Lipnikov, and V. Simoncini. A family of mimetic finite difference methods on polygonal and polyhedral meshes. *M3AS*, 15:1533–1553, 2005.
- [12] F. Brezzi, G. Manzini, L. D. Marini, P. Pietra, and A. Russo. Discontinuous Galerkin approximations for elliptic problems. *Numer. Methods Partial Differential Equations*, 16(4):365–378, 2000.
- [13] A. Buffa and C. Ortner. Compact embeddings of broken Sobolev spaces and applications. *IMA J. Numer. Anal.*, 4(29):827–855, 2009.
- [14] E. Burman and A. Ern. Continuous interior penalty  $hp$ -finite element methods for advection and advection-diffusion equations. *Math. Comp.*, 76(259):1119–1140, 2007.
- [15] E. Burman and P. Zunino. A domain decomposition method for partial differential equations with non-negative form based on interior penalties. *SIAM J. Numer. Anal.*, 44:1612–1638, 2006.
- [16] P. G. Ciarlet. *The finite element method for elliptic problems*, volume 40 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002. Reprint of the 1978 original [North-Holland, Amsterdam; MR0520174 (58 #25001)].
- [17] D. A. Di Pietro. Analysis of a discontinuous Galerkin approximation of the Stokes problem based on an artificial compressibility flux. *Int. J. Numer. Methods Fluids*, 55:793–813, 2007.
- [18] D. A. Di Pietro. Cell centered Galerkin methods. *C. R. Acad. Sci. Paris, Ser. I*, 348:31–34, 2010. DOI: 10.1016/j.crma.2009.11.012.
- [19] D. A. Di Pietro. A compact cell-centered Galerkin method with subgrid stabilization. Submitted. Preprint available at <http://hal.archives-ouvertes.fr/hal-00476222/en/>, April 2010.
- [20] D. A. Di Pietro and A. Ern. Discrete functional analysis tools for discontinuous Galerkin methods with application to the incompressible Navier–Stokes equations. *Math. Comp.*, 79:1303–1330, 2010. DOI: 10.1090/S0025-5718-10-02333-1.
- [21] D. A. Di Pietro and A. Ern. *Mathematical aspects of discontinuous Galerkin methods*. Mathematics & Applications. Springer-Verlag, Berlin, 2010. In press.
- [22] D. A. Di Pietro, A. Ern, and J.-L. Guermond. Discontinuous Galerkin methods for anisotropic semi-definite diffusion with advection. *SIAM J. Numer. Anal.*, 46(2):805–831, 2008.
- [23] J. Droniou and R. Eymard. A mixed finite volume scheme for anisotropic diffusion problems on any grid. *Num. Math.*, 105(1):35–71, 2006.
- [24] J. Droniou, R. Eymard, T. Gallouët, and R. Herbin. A unified approach to mimetic finite difference, hybrid finite volume and mixed finite volume methods. *M3AS, Math. Models Methods Appl. Sci.*, 20(2):265–295, 2010.
- [25] M. G. Edwards and C. F. Rogers. A flux continuous scheme for the full tensor pressure equation. In *Prov. of the 4th European Conf. on the Mathematics of Oil Recovery*, volume D, Røros, Norway, 1994.
- [26] M. G. Edwards and C. F. Rogers. Finite volume discretization with imposed flux continuity for the general tensor pressure equation. *Comput. Geosci.*, 2:259–290, 1998.
- [27] A. Ern and J.-L. Guermond. *Theory and Practice of Finite Elements*, volume 159 of *Applied Mathematical Sciences*. Springer-Verlag, New York, NY, 2004.
- [28] E. Erturk, T. C. Corke, and C. Gökçöl. Numerical solutions of 2-D steady incompressible driven cavity flow at high Reynolds numbers. *Int. J. Num. Meth. Fluids*, 48:747–774, 2005.
- [29] R. Eymard, Th. Gallouët, and R. Herbin. *The Finite Volume Method*. Ph. Charlet and J.L. Lions eds, North Holland, 2000.
- [30] R. Eymard, Th. Gallouët, and R. Herbin. Discretization of heterogeneous and anisotropic diffusion problems on general nonconforming meshes SUSHI: a scheme using stabilization and hybrid interfaces. *IMA J. Numer. Anal.*, June 2009. DOI: 10.1093/imanum/drn084.
- [31] R. Eymard, R. Herbin, and J.-C. Latché. Convergence analysis of a collocated finite volume scheme for the incompressible Navier-Stokes equations on general 2D or 3D meshes. *SIAM J. Numer. Anal.*, 45(1):1–36, 2007.
- [32] B. Heinrich and K. Pietsch. Nitsche type mortaring for some elliptic problem with corner singularities. *Computing*, 68(3):217–238, 2002.
- [33] R. Herbin and F. Hubert. Benchmark on discretization schemes for anisotropic diffusion problems on general grids. In R. Eymard and J.-M. Hérard, editors, *Finite Volumes for Complex Applications V*, pages 659–692. John Wiley & Sons, 2008.
- [34] L. S. G. Kovaszny. Laminar flow behind a two-dimensional grid. *Proc. Camb. Philos. Soc.*, 44:58–62, 1948.
- [35] J. Nitsche. On Dirichlet problems using subspaces with nearly zero boundary conditions. In *The mathematical foundations of the finite element method with applications to partial differential equations (Proc. Sympos., Univ. Maryland, Baltimore, Md., 1972)*, pages 603–627. Academic Press, New York, 1972.
- [36] R. Stenberg. Mortaring by a method of J.A. Nitsche. In Idelsohn S.R., Oñate E., and Dvorkin E.N., editors, *Computational Mechanics: New trends and applications*, pages 1–6, Barcelona, Spain, 1998. Centro Internacional de Métodos Numéricos en Ingeniería.
- [37] R. Temam. *Navier-Stokes Equations*, volume 2 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, revised edition, 1979. Theory and numerical analysis, With an appendix by F. Thomasset.