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# Cross-Spectral Properties of a Spatial Point-Lattice

## Process

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### Abstract

This paper shows how spectral analysis can be used to study a hybrid process involving a spatial point process and a lattice process. Asymptotic distributions of spectral statistics for such processes are derived. *Keywords: Cross spectrum, Cross periodogram, hybrid process, lattice process, spatial point process.*

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# 1 Introduction

In studying a spatial point process one might be interested in investigating whether the structure of the process is related to some quantitative variable that also exhibits spatial structure. For example, one might be interested in studying the relationship between the location of trees in a forest, a spatial point process, and a quantitative variable such as altitude or depth of soil. Often the quantitative variable will have been measured at positions on a grid, or *lattice*, either because a systematic sampling scheme has been used, or because the recording device produces a digital image. Few papers have attempted to analyze datasets that involve both a spatial point process and a lattice process. Those analyses that have been conducted were restricted to the spatial domain (see Augustin et al., 1996; Dessard and Goulard, 1998). In this paper, a frequency domain approach, based on two-dimensional cross-spectral analysis, will be used. This approach is analogous to the analysis carried out by Mugglestone and Renshaw (1996) to investigate properties of a bivariate spatial point process. Furthermore, it is an extension of the cross-spectral analysis used for one-dimensional hybrid processes: a one-dimensional hybrid process is a process with two components where one component is a one-dimensional point process and the other is a time series (see Rigas, 1983; Brillinger, 1994). Using spectral analysis, one can determine the nature and scale of association, if any, between the two processes. Specifically, one can establish whether a spatial lead-lag effect exists between the two processes and its direction.

In Section 2 we introduce the notion of a spatial point-lattice process and provide some definitions of spectral statistics and assumptions related to this process. In Section 3 we derive the asymptotic distributions of these spectral statistics. A test for association between

the two components of such hybrid processes is also derived. We close with a discussion and conclusions in Section 4.

## 2 Definitions

In this section, we will define the types of processes that we are studying and their theoretical spectra.

### 2.1 Theoretical Processes and Spectra

A spatial point-lattice process is a hybrid process consisting of a spatial point process,  $X$ , and a lattice process,  $Y$ , where  $Y$  is quantitative measurement made at grid points and both processes are observed within a study region  $\Omega \subseteq \mathbb{R}^2$ . In order to study a spatial point-lattice process, we need to explore the relationship between its components.

Let the hybrid process be denoted by  $Z(\mathbf{a}) = \{N_X(\mathbf{a}), Y_{\mathbf{a}}\}$ , where  $\mathbf{a} \in \Omega$ ,  $N_X(A)$  is the number of events in a given region  $A \subseteq \Omega$ , and  $Y_{\mathbf{a}}$  is the quantitative measurement made at  $\mathbf{a}$ . Henceforth,  $N_X(A)$  will be written as  $N(A)$  and  $N(\Omega)$  as  $N$ . The spatial point-lattice process satisfies the following assumptions. First, the hybrid process is strictly stationary, that is the process  $Z(\mathbf{a} + \mathbf{c})$  has the same probability distribution as  $Z(\mathbf{a})$  for any  $\mathbf{a}, \mathbf{c} \in \Omega$ . Second, the point process is orderly, that is, the probability of more than one event at a particular location is negligible. Third, the hybrid process satisfies the strong mixing condition, that is, values of the process that are well separated in space become independent, see Appendix A; this is generally a reasonable assumption in practical problems. Finally, the lattice process is mean corrected.

In what follows, we define parameters associated with the hybrid process in both the spatial and frequency domains. By virtue of the stationarity assumption and the fact that the lattice process is mean corrected, the *cross-covariance function* of a point-lattice process is defined as

$$\gamma_{NY}(\mathbf{c}) = \lim_{|d\mathbf{a}| \rightarrow 0} \left\{ \frac{\mathbb{E}[dN(\mathbf{a})Y_{\mathbf{a}+\mathbf{c}}]}{|d\mathbf{a}|} \right\} \text{ for } \mathbf{a}, \mathbf{c} \in \mathbb{R}^2. \quad (1)$$

In equation (1),  $d\mathbf{a}$  represents a small neighbourhood of  $\mathbf{a}$ ,  $|d\mathbf{a}|$  is the area of this neighbourhood and  $dN(\mathbf{a}) = N(\mathbf{a} + d\mathbf{a}) - N(\mathbf{a})$  is the number of events within this neighbourhood.

The cross-spectral density function (or *cross-spectrum*),  $f_{NY}$ , for a stationary spatial point-lattice process is defined as the Fourier transform of the cross-covariance function. Thus, at a given frequency,  $\boldsymbol{\omega} \in \mathbb{R}^2$ ,  $f_{NY}(\boldsymbol{\omega}) = (2\pi)^{-2} \int \gamma_{NY}(\mathbf{c}) \exp(-i\boldsymbol{\omega}\mathbf{c}^\top) d\mathbf{c}$ . In general,  $\gamma_{NY}(\mathbf{c}) \neq \gamma_{NY}(-\mathbf{c})$ , so  $f_{NY}$  will usually be a complex number. By analogy with bivariate time series analysis,  $f_{NY}$  can be decomposed into its real and imaginary parts as follows, see for example (Chatfield, 2003, chapter 8)

$$f_{NY}(\boldsymbol{\omega}) = (2\pi)^{-2} \int \gamma_{NY}(\mathbf{c}) \cos(\boldsymbol{\omega}\mathbf{c}^\top) d\mathbf{c} - i(2\pi)^{-2} \int \gamma_{NY}(\mathbf{c}) \sin(\boldsymbol{\omega}\mathbf{c}^\top) d\mathbf{c} \equiv c_{NY}(\boldsymbol{\omega}) - iq_{NY}(\boldsymbol{\omega}). \quad (2)$$

The function  $c_{NY}(\boldsymbol{\omega})$  is known as the *co-spectrum* and  $q_{NY}(\boldsymbol{\omega})$  is known as the *quadrature spectrum*. Alternatively,  $f_{NY}$  can be represented in polar form; hence, from (2) we have

$$f_{NY}(\boldsymbol{\omega}) = \sqrt{c_{NY}^2(\boldsymbol{\omega}) + q_{NY}^2(\boldsymbol{\omega})} \exp\{i \tan^{-1}(-q_{NY}(\boldsymbol{\omega})/c_{NY}(\boldsymbol{\omega}))\} \equiv \alpha_{NY}(\boldsymbol{\omega}) \exp\{i\phi_{NY}(\boldsymbol{\omega})\},$$

where  $\alpha_{NY}$  is known as the *amplitude spectrum*, and  $\phi_{NY}$  is known as the *phase spectrum*.

In addition to the above spectra, the *squared coherency spectrum* is defined as

$$v_{NY}(\omega) \equiv \frac{c_{NY}^2(\omega) + q_{NY}^2(\omega)}{f_{NN}(\omega)f_{YY}(\omega)} = \frac{\alpha_{NY}^2(\omega)}{f_{NN}(\omega)f_{YY}(\omega)}, \quad (3)$$

where  $f_{NN}(\omega)$  is the Fourier transform of the complete auto-covariance function of the point process (see Bartlett, 1964; Mugglestone and Renshaw, 1996) and  $f_{YY}(\omega)$  is the Fourier transform of the auto-covariance function of the lattice process (see Priestley, 1996). Using the Cauchy-Schwartz inequality, it can be shown that the squared coherency spectrum satisfies the inequality  $0 \leq v_{NY}(\omega) \leq 1$ .

Here, we briefly explain what each of the above spectra measures. The co-spectrum,  $c_{NY}$ , represents the covariance between the coefficients of the in-phase components of the two patterns. The quadrature spectrum,  $q_{NY}$ , represents the covariance between the coefficients of the out-of-phase components. The amplitude spectrum measures the relative value of the power at the frequency  $\omega$  in the components  $N$  and  $Y$ . The phase spectrum represents the mean value of the phase shift between the components  $N$  and  $Y$  at frequency  $\omega$ , in the sense described by Priestley (1996), assuming that the phase and amplitude of the spectral representations for each process are independent random variables. The squared coherency spectrum measures the square of the linear correlation between the components of the point-lattice process at frequency  $\omega$ . The closer the square root of this value to unity the stronger the relationship between the two processes at frequency  $\omega$ .

In practice when studying a spatial point-lattice process the study region,  $\Omega$ , is required to be rectangular and to coincide with where the lattice process is observed. Let  $\Omega = [0, \ell_1] \times [0, \ell_2]$  where  $\ell_1$  and  $\ell_2$  represent the length and breadth of the study region.

The empirical version of the cross-spectrum is called the *cross-periodogram statistic* and it is defined by  $F_{NY}(\boldsymbol{\omega}) = (2\pi)^{-2}|\Omega|^{-1}F_N(\boldsymbol{\omega})\overline{F_Y(\boldsymbol{\omega})}$ , for  $\boldsymbol{\omega} \in \mathbb{R}^2$ , where  $|\Omega| = \ell_1 \times \ell_2$ ,  $F_N(\boldsymbol{\omega}) = \int_{\Omega} \exp(-i\boldsymbol{\omega}\mathbf{a}^\top) dN(\mathbf{a})$ , and  $F_Y(\boldsymbol{\omega}) = \int_{\Omega} \exp(-i\boldsymbol{\omega}\mathbf{a}^\top) Y_{\mathbf{a}} d\mathbf{a}$  where  $F$  is the *finite Fourier transform* (see Rigas, 1983; Brillinger, 1994). Furthermore, we can decompose the cross-periodogram in the same ways as the cross-spectrum.

In order to avoid bias near  $\boldsymbol{\omega} = 0$ , we use the mean-corrected periodograms, namely,  $\check{F}_N(\boldsymbol{\omega}) = F_N(\boldsymbol{\omega}) - \mathbb{E}[F_N(\boldsymbol{\omega})] = F_N(\boldsymbol{\omega}) - \lambda_N \Delta(\boldsymbol{\omega})$ , and  $\check{F}_Y(\boldsymbol{\omega}) = F_Y(\boldsymbol{\omega}) - \mathbb{E}[F_Y(\boldsymbol{\omega})] = F_Y(\boldsymbol{\omega}) - \mu_Y \Delta(\boldsymbol{\omega}) = F_Y(\boldsymbol{\omega})$ , where  $\Delta(\boldsymbol{\omega}) = \int_{\Omega} \exp\{-i\boldsymbol{\omega}\mathbf{a}^\top\} d\mathbf{a}$ ,  $\lambda_N = \lim_{|d\mathbf{a}| \rightarrow 0} \left\{ \frac{\mathbb{E}(dN_X(\mathbf{a}))}{|d\mathbf{a}|} \right\}$  is the *first-order intensity* function of the point pattern as defined in Diggle (2003), and  $\mu_Y$  is the mean of the lattice process which is assumed to be zero.

### 3 Distributional Properties

In this section, we provide the asymptotic distributional properties of the above cross-periodogram statistics; proofs are outlined in Appendix B. In what follows,  $Z$  denotes a hybrid process defined as in Section 2 and satisfying the assumptions mentioned there.

**Theorem 1** *For any two non-zero frequencies  $\boldsymbol{\omega}_1$  and  $\boldsymbol{\omega}_2 \in \mathbb{R}^2$ , such that  $\boldsymbol{\omega}_1 \pm \boldsymbol{\omega}_2 \neq 0$  and as  $\ell_1, \ell_2 \rightarrow \infty$ ,  $\check{F}_Z(\boldsymbol{\omega}_j)$ , where  $\check{F}_Z^\top(\boldsymbol{\omega}_j) = [\check{F}_N(\boldsymbol{\omega}_j), \check{F}_Y(\boldsymbol{\omega}_j)]$  for  $j = 1, 2$  are asymptotically independent and distributed as  $N_2^{\mathbb{C}}(\mathbf{0}, (2\pi)^2|\Omega|f_{ZZ}(\boldsymbol{\omega}_j))$ , where  $N_2^{\mathbb{C}}$  denotes the bivariate complex normal distribution (see Kotz, 1997),*

$$f_{ZZ}(\boldsymbol{\omega}_j) = \begin{bmatrix} f_{NN}(\boldsymbol{\omega}_j) & f_{NY}(\boldsymbol{\omega}_j) \\ f_{YN}(\boldsymbol{\omega}_j) & f_{YY}(\boldsymbol{\omega}_j) \end{bmatrix}$$



and  $f_{YN}$  is the Fourier transform of the cross-covariance function  $\gamma_{YN}$ , which is defined in a similar manner to that of  $\gamma_{NY}$ . Furthermore,  $f_{YN}$  satisfies the property  $f_{YN}(\boldsymbol{\omega}) = f_{NY}(-\boldsymbol{\omega})$  since the hybrid process is assumed to be stationary. In addition,  $f_{NN}(\boldsymbol{\omega})$  and  $f_{YY}(\boldsymbol{\omega})$  are as defined above.

**Remark 1** If  $\boldsymbol{\omega} = \mathbf{0}$  then  $\check{F}_Z(\boldsymbol{\omega})$  is asymptotically  $N_2(\mathbf{0}, (2\pi)^2|\Omega|f_{ZZ}(\mathbf{0}))$  and  $\check{F}_Z(\mathbf{0})$  is independent of  $\check{F}_Z(\boldsymbol{\omega})$  for non-zero  $\boldsymbol{\omega}$ .

**Theorem 2** Consider frequencies  $\boldsymbol{\omega}_j \in \mathbb{R}^2$  for  $j = 1, \dots, n$  such that  $\boldsymbol{\omega}_j \rightarrow \boldsymbol{\omega}$  as  $\ell_1, \ell_2 \rightarrow \infty$  then  $\check{F}_Z(\boldsymbol{\omega}_j)$  are asymptotically independent variates distributed as  $N_2^{\mathbb{C}}(\mathbf{0}, (2\pi)^2|\Omega|f_{ZZ}(\boldsymbol{\omega}))$  for  $\boldsymbol{\omega} \neq \mathbf{0}$  and as  $N_2(\mathbf{0}, (2\pi)^2|\Omega|f_{ZZ}(\mathbf{0}))$  for  $\boldsymbol{\omega} = \mathbf{0}$ .

**Corollary 1** As a consequence of Theorem 2 and the definition of the (complex) Wishart distribution introduced by Goodman (1963), it can be shown that for the periodogram  $\check{\mathbf{F}}_{ZZ} \equiv (2\pi)^{-2}|\Omega|^{-1}\check{F}_Z\check{F}_Z^{\overline{\top}}$ , and at a frequency  $\boldsymbol{\omega}$ , we have asymptotically that  $\check{\mathbf{F}}_{ZZ}(\boldsymbol{\omega}) \sim W_2^{\mathbb{C}}(1, f_{ZZ}(\boldsymbol{\omega}))$  if  $\boldsymbol{\omega} \neq \mathbf{0}$ , and  $W_2(1, f_{ZZ}(\mathbf{0}))$  if  $\boldsymbol{\omega} = \mathbf{0}$ . Here,  $W_2^{\mathbb{C}}$  denotes the complex Wishart distribution of dimension two and one degree of freedom;  $W_2$  denotes the Wishart distribution of dimension two and one degree of freedom.

Therefore, the periodogram ordinates are distributed as independent (complex) Wishart variates with one degree of freedom and covariance matrix  $f_{ZZ}$ . Being a Wishart distribution with just one degree of freedom implies that  $\check{\mathbf{F}}_{ZZ}(\boldsymbol{\omega})$  cannot be considered a reasonable estimate (see Brillinger, 1981). However, under the assumptions described in Section 2 the spectral density function is a smooth function of frequency (see Brillinger, 1970). Therefore, a *reasonable* estimate at a particular frequency might be constructed by averaging over

frequencies that are close together in the periodogram. Thus, one can consider the uniformly smoothed periodogram,  $\mathbb{F}_{ZZ}(\boldsymbol{\omega}) \equiv \frac{1}{m} \sum_{k=1}^m \check{\mathbb{F}}_{ZZ}(\boldsymbol{\omega}_k)$ , where  $\boldsymbol{\omega}_k$  for  $k = 1, \dots, m$  are frequencies close to  $\boldsymbol{\omega}$ .

**Corollary 2** *Under the conditions of Theorem 2 and using the results of Corollary 1 and the properties of the (complex) Wishart distribution, the following holds asymptotically:  $\mathbb{F}_{ZZ}(\boldsymbol{\omega}) \sim m^{-1}W_2^{\mathbb{C}}(m, f_{ZZ}(\boldsymbol{\omega}))$  if  $\boldsymbol{\omega} \neq \mathbf{0}$ , and  $m^{-1}W_2(m, f_{ZZ}(\mathbf{0}))$  if  $\boldsymbol{\omega} = \mathbf{0}$ .*

Having established the asymptotic joint distribution of  $\check{\mathbb{F}}_{ZZ}(\boldsymbol{\omega})$ , we proceed to give the asymptotic properties of the cross-periodogram statistic  $\check{F}_{NY}(\boldsymbol{\omega}) = (2\pi)^{-2} |\Omega|^{-1} \check{F}_N(\boldsymbol{\omega}) \overline{\check{F}_Y(\boldsymbol{\omega})}$ .

**Theorem 3** *It can be shown that asymptotically  $\check{F}_{NY}(\boldsymbol{\omega})$  has the following properties:*

$$\lim_{\ell_1, \ell_2 \rightarrow \infty} \mathbb{E} \left\{ \check{F}_{NY}(\boldsymbol{\omega}) \right\} = f_{NY}(\boldsymbol{\omega}) \quad \text{for } \boldsymbol{\omega} \neq \mathbf{0}, \text{ and} \quad (4)$$

$$\lim_{\ell_1, \ell_2 \rightarrow \infty} \text{Cov} \left\{ \check{F}_{NY}(\boldsymbol{\omega}_1), \check{F}_{NY}(\boldsymbol{\omega}_2) \right\} = \delta(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) f_{NN}(\boldsymbol{\omega}_1) f_{YY}(-\boldsymbol{\omega}_1) + \delta(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2) [f_{NY}(\boldsymbol{\omega}_1)]^2 \quad (5)$$

for  $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \neq \mathbf{0}$ , where  $\delta$  is the Kronecker delta; that is  $\delta(\alpha)$  is equal to one if  $\alpha = 0$  and it is equal to zero otherwise.

Thus asymptotically the cross-periodogram is an unbiased estimator of the cross-spectral density function. Using equations (4) and (5), we derive the asymptotic distributional properties of the covariance matrix of  $\check{F}_{NN}$ ,  $\check{F}_{YY}$ ,  $\check{c}_{NY}$ , and  $\check{q}_{NY}$ , where  $\check{c}_{NY} = (\overline{\check{F}_{NY}} + \check{F}_{NY})/2$  and  $\check{q}_{NY} = (\overline{\check{F}_{NY}} - \check{F}_{NY})/2i$  are the co- and quadrature periodograms, respectively. The covariance matrix of  $\check{F}_{NN}$ ,  $\check{F}_{YY}$ ,  $\check{c}_{NY}$ , and  $\check{q}_{NY}$ , respectively, is given by

$$\Sigma_{zz} = \begin{bmatrix} f_{NN}^2 & |f_{NY}|^2 & f_{NN}c_{NY} & f_{NN}q_{NY} \\ & f_{YY}^2 & f_{YY}c_{NY} & f_{YY}q_{NY} \\ & & (f_{NN}f_{YY} + c_{NY}^2 - q_{NY}^2)/2 & c_{NY}q_{NY} \\ & & & (f_{NN}f_{YY} + q_{NY}^2 - c_{NY}^2)/2 \end{bmatrix}.$$

However, if  $\mathbb{F}$  is used instead of  $\check{\mathbb{F}}$  then  $\Sigma_{zz}$  should be divided by  $m$ . Only the upper triangle of the covariance matrix is reported; the lower triangle is derived by symmetry. The matrix  $\Sigma_{zz}$  is similar to that reported in Priestley (1996, Chapter 9) for any two components of a one-dimensional multivariate time series.

The asymptotic results for amplitude, phase and squared coherency spectra are derived by expanding these functions around their means using Taylor series expansions, and retaining the first two or three terms. Taylor expansions are needed since the amplitude, phase and squared coherency spectra are non-linear functions of  $\check{c}_{NY}$ ,  $\check{q}_{NY}$ ,  $\check{F}_{NN}$ , and  $\check{F}_{YY}$ . In fact, using just the first two terms of the Taylor expansion for functions of several variables gives  $\mathbb{E}\{\check{\alpha}_{NY}\} = \alpha_{NY}$ ,  $\mathbb{E}\{\check{\phi}_{NY}\} = \phi_{NY}$ ,  $\mathbb{E}\{\check{v}_{NY}\} = v_{NY}$ ,  $\text{Var}(\check{\alpha}_{NY}) \sim \frac{\alpha_{NY}^2}{2m} \left\{ \frac{1}{v_{NY}} + 1 \right\}$ ,  $\text{Var}(\check{\phi}_{NY}) \sim \frac{1}{2m} \left\{ \frac{1}{v_{NY}} - 1 \right\}$ ,  $\text{Cov}(\check{\phi}_{NY}, \check{\alpha}_{NY}) \sim 0$ , and  $\text{Var}(\check{v}_{NY}) \sim \frac{2}{m} v_{NY} (1 - v_{NY})^2$ . Based on the above distributional properties, one can construct confidence intervals for the different auto- and cross-spectra. Note that under independence the expressions for the variance estimates of the phase and squared coherency spectra are invalid (see Brillinger, 1981). However, one can derive appropriate distributions in this case as reported in Priestley (1996) and Brillinger (1981).

A test of whether the squared coherency is zero is now derived; it is an adaptation of a test for the one-dimensional multivariate case discussed by Priestley (1996, Chapter 9).

Under the null hypothesis that the coherency is zero,  $\frac{(m-1)\check{v}_{NY}}{1-\check{v}_{NY}}$  is distributed as  $F_{2,2(m-1)}$ ,

where  $F_{s,t}$  is the F-distribution with  $s$  and  $t$  degrees of freedom and, as before,  $m$  is the number of ordinates used in smoothing the periodogram at a particular frequency.

This test can be used as a test of independence between the point and lattice processes under the assumption that the hybrid process is a spatially random point-lattice process in which the two components are independent. The independence of the two components implies that the cross-covariance function is equal to zero for all lags, which in turn implies that the cross-spectral density function is identically equal to zero for all frequencies. Thus, the co-, quadrature, amplitude, and coherency spectra are all equal to zero. It might seem that  $\phi_{NY}(\omega)$  is indeterminate but it can be shown (see Jenkins and Donald, 1968, chapter 8) that it is uniformly distributed over the range  $(-\pi/2, \pi/2)$ . Therefore, any deviations from zero in any of the cross-spectra except for the phase spectrum indicate that the two components are correlated.

## 4 Discussion and Conclusions

In this paper we have derived asymptotic distributions of the spectral density functions of the hybrid process consisting of a spatial point process and a lattice process. These distributions were analogous to results from studies of one-dimensional processes. For non-zero frequencies, we found that the asymptotic distribution of the cross-spectral matrix followed a complex Wishart distribution of dimension two with one degree of freedom. For zero frequencies, the distribution followed a Wishart distribution of dimension two with one degree of freedom. In addition, the cross-periodogram statistic was shown to be an unbiased estimator of the cross-spectral statistic and cross-spectral estimates at different frequencies were shown to be

asymptotically independent of each other. Furthermore, a test for zero coherency, based on a statistic which followed an F-distribution, was introduced.

With the widespread use of geographical information systems, global positioning systems and digital imaging, recording spatial point-lattice processes is becoming increasingly common across a wide range of disciplines, including ecology, engineering, epidemiology and geography. The results presented in this paper provide a framework for a spectral approach to the analysis of such data. This framework holds potential for assessing various forms of spatial dependence between the components of the spatial point-lattice process, including directional dependence. This contrasts with other forms of analysis that have been used for such datasets, for example where dependence between the different components has been assessed via a regression approach treating the lattice component as an explanatory variable in the model used for occurrence of events (see Augustin et al., 1996). Using spectral analysis allows exploring the relationship between the two processes without having to treat one of the processes as an outcome. In addition, for processes of a periodic nature and where a linear shift exists between the two processes, spectral analysis is known to be a powerful tool to detect such patterns.

## Appendix A: Definitions

In this section, we provide the definitions of some functions that are not covered elsewhere in the article and are needed for the proofs of the results that are presented in Section 3.

The  $k^{\text{th}}$ -order cumulant function is defined as

$$\text{Cum}\{dZ_{j_1}(\mathbf{u}_1 + \mathbf{t}), \dots, dZ_{j_{k-1}}(\mathbf{u}_{k-1} + \mathbf{t}), dZ_{j_k}(\mathbf{t})\} = \gamma_{j_1 \dots j_k}(\mathbf{u}_1, \dots, \mathbf{u}_{k-1}) d\mathbf{u}_1 \dots d\mathbf{u}_{k-1} dt$$

$j_1, \dots, j_k = (Y \text{ or } N)$ ;  $k = 1, 2, \dots$ , and  $\mathbf{u}_1, \dots, \mathbf{u}_{k-1}, \mathbf{t} \in \mathbb{R}^2$ . Furthermore,  $dZ$  is defined as  $dZ_j(\mathbf{u}) = dN(\mathbf{u})$  if  $j = N$  and  $Y_{\mathbf{u}}d\mathbf{u}$  if  $j = Y$  (see Brillinger, 1970).

The hybrid process is assumed to possess moments of all orders and to satisfy

$$\int \cdots \int \sum_{s=1}^{k-1} |\mathbf{u}_s| |\gamma_{j_1 \dots j_k}(\mathbf{u}_1, \dots, \mathbf{u}_{k-1})| d\mathbf{u}_1 \cdots d\mathbf{u}_{k-1} < \infty, \quad \text{where} \quad (\text{A.1})$$

$|\mathbf{u}| = \sqrt{\mathbf{u}\mathbf{u}^\top} = \sqrt{u^2 + v^2}$  and  $\mathbf{u} = (u, v)$ . Equation (A.1) is a form of *mixing* condition. The mixing condition implies that the process of increments  $\{N(\Delta), Y(\Delta)\}$ , where  $N(\Delta) = \int_{\Delta} dN(\mathbf{a})$ ,  $Y(\Delta) = \int_{\Delta} Y_{\mathbf{a}}d\mathbf{a}$  and  $\Delta$  is a subregion of  $\mathbb{R}^2$ , has the property that values of the process that are well separated in space become independent. This is generally a reasonable assumption in practical problems.

The  $k^{\text{th}}$ -order *cumulant spectral* density function is defined as

$$f_{j_1 \dots j_k}(\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_{k-1}) = (2\pi)^{-2(k-1)} \int \cdots \int \exp\left(-i \sum_{s=1}^{k-1} \boldsymbol{\omega}_s \mathbf{u}_s^\top\right) \times \gamma_{j_1 \dots j_k}(\mathbf{u}_1, \dots, \mathbf{u}_{k-1}) d\mathbf{u}_1 \cdots d\mathbf{u}_{k-1}$$

for  $\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_k \in \mathbb{R}^2$ , where  $\sum_{s=1}^k \boldsymbol{\omega}_s = \mathbf{0}$ , and the other terms are defined as above.

## Appendix B: Proofs

Here we outline proofs for the results presented in Section 3. Before proceeding, we present a result that is needed for the proofs.

Let  $A = \text{Cum}\{F_{j_1}(\boldsymbol{\omega}_1), \dots, F_{j_k}(\boldsymbol{\omega}_k)\}$ , and  $L = \max(\ell_1, \ell_2)$  then as  $\ell_1, \ell_2 \rightarrow \infty$  we have

$$A = (2\pi)^{2(k-1)} \Delta \left( \sum_{s=1}^k \boldsymbol{\omega}_s \right) f_{j_1 \dots j_k}(\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_{k-1}) + O(L). \quad (\text{A.2})$$

**Proof.**  $A = \int_{\Omega} \dots \int_{\Omega} \exp \left( -i \sum_{s=1}^k \boldsymbol{\omega}_s \mathbf{a}_s^{\top} \right) \times \text{Cum}\{dZ_{j_1}(\mathbf{a}_1), \dots, dZ_{j_k}(\mathbf{a}_k)\} = \int_0^{\ell_2} \dots \int_0^{\ell_2} \exp \left( -i \sum_{s=1}^k \omega_{2_s} a_{2_s} \right) B da_{2_1} \dots da_{2_k}$ , where  
 $B = \int_{-\ell_1}^{\ell_1} \dots \int_{-M_s}^{\ell_1 - m_s} \dots \int_{m_1}^{M_1} \exp \left( -i \sum_{s=1}^{k-1} \omega_{1_s} u_s - i u \sum_{s=1}^k \omega_{1_s} \right) \gamma_{j_1 \dots j_k}(u_1, \dots, u_{k-1}) du \dots du_s \dots du_{k-1}$ ,  
 $u_s = a_{1_s} - a_{1_k}, v_s = a_{2_s} - a_{2_k}, M_s = \ell_1 - \max(0, u_s, \dots, u_{k-1})$ , and  $m_s = -\min(0, u_s, \dots, u_{k-1})$   
for  $s = 1, \dots, k-1$ ;  $a_{1_k} = u$  and  $a_{2_k} = v$ . Therefore,

$$A = \int_{\Omega_k} \dots \int_{\Omega_2} D \exp \left( -i \sum_{s=1}^{k-1} \boldsymbol{\omega}_s \mathbf{u}_s^{\top} \right) \gamma_{j_1 \dots j_k}(\mathbf{u}_1, \dots, \mathbf{u}_{k-1}) d\mathbf{u}_1 \dots d\mathbf{u}_{k-1}$$
, where  
 $D = \int_{n_1}^{N_1} \int_{m_1}^{M_1} \exp \left\{ -i \sum_{s=1}^k (\omega_{1_s} u + \omega_{2_s} v) \right\} du dv$  with  $\mathbf{u}_s = (u_s, v_s), d\mathbf{u}_s = du_s dv_s, N_s = \ell_2 - \max(0, v_s, \dots, v_{k-1}), n_s = -\min(0, v_s, \dots, v_{k-1}), \Omega_s = [-M_s, \ell_1 - m_s] \times [-N_s, \ell_2 - n_s]$ , for  
 $s = 1, \dots, k-1$ . Using the definition of  $\Delta(\boldsymbol{\omega})$ , one can show that  
 $\left| D - \Delta \left( \sum_{s=1}^k \boldsymbol{\omega}_s \right) \right| \leq 2(\ell_2 + \ell_1) \sum_{s=1}^{k-1} |\mathbf{u}_s| \leq 4L \sum_{s=1}^{k-1} |\mathbf{u}_s|$ . Hence, as  $\ell_1, \ell_2 \rightarrow \infty$ , equation (A.2) holds. ■

Using equation (A.2) and the fact that cumulants of order greater than two vanish for normally distributed variables (see Kendall and Stuart, 1963, chapter 5), one can easily proceed to prove Theorems 1 and 2.

### Proof of equation (4) in Theorem 3

The proof of equation (4) in Theorem 3 is as follows. Consider

$$\mathbb{E} \left[ \check{F}_{NY}(\boldsymbol{\omega}) \right] = (2\pi)^{-2} |\Omega|^{-1} \mathbb{E} \left[ \check{F}_N(\boldsymbol{\omega}) \overline{\check{F}_Y(\boldsymbol{\omega})} \right] = (2\pi)^{-2} |\Omega|^{-1} \text{Cum}\{F_N(\boldsymbol{\omega}), F_Y(-\boldsymbol{\omega})\}, \text{ then}$$

using equation (A.2)

$\mathbb{E} \left[ \check{F}_{NY}(\boldsymbol{\omega}) \right] = (2\pi)^{-2} |\Omega|^{-1} [(2\pi)^2 \Delta(\boldsymbol{\omega} - \boldsymbol{\omega}) f_{NY}(\boldsymbol{\omega}) + O(L)]$  as  $\ell_1, \ell_2 \rightarrow \infty$ . However,  $\Delta(\mathbf{0}) = \ell_1 \ell_2 = |\Omega|$ , and  $O(L)/|\Omega| \rightarrow 0$  as  $\ell_1, \ell_2 \rightarrow \infty$ . Therefore,  $\mathbb{E} \left[ \check{F}_{NY}(\boldsymbol{\omega}) \right] = f_{NY}(\boldsymbol{\omega})$  which proves equation (4).

### Proof of equation (5) in Theorem 3

The proof of equation (5) in Theorem 3 is as follows. Consider  $\mathcal{F} = \text{Cov} \left\{ \check{F}_{NY}(\boldsymbol{\omega}_1), \check{F}_{NY}(\boldsymbol{\omega}_2) \right\} = (2\pi)^{-4} |\Omega|^{-2} \times \text{Cum} \left\{ \check{F}_N(\boldsymbol{\omega}_1) \check{F}_Y(-\boldsymbol{\omega}_1), \check{F}_N(-\boldsymbol{\omega}_2) \check{F}_Y(\boldsymbol{\omega}_2) \right\}$ . Using the properties of cumulants,  $\mathcal{F}$  can be decomposed as  $(2\pi)^{-4} |\Omega|^{-2} \{ (2\pi)^6 |\Omega| f_{NYNY}(\boldsymbol{\omega}_1, -\boldsymbol{\omega}_1, -\boldsymbol{\omega}_2) + (2\pi)^4 |\Delta(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2)|^2 f_{NN}(\boldsymbol{\omega}_1) f_{YY}(-\boldsymbol{\omega}_1) + (2\pi)^4 |\Delta(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2)|^2 f_{NY}(\boldsymbol{\omega}_1) f_{YN}(-\boldsymbol{\omega}_1) + O(L^3) \}$ . Now  $O(|\Omega|) = O(L^2)$  so that as  $\ell_1, \ell_2 \rightarrow \infty$  we prove equation (5) for  $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \neq \mathbf{0}$ . Note that  $|\Omega|^{-2} |\Delta(\mathbf{a})|^2 \rightarrow \delta(\mathbf{a})$  as  $\ell_1, \ell_2 \rightarrow \infty$ , where  $\delta$  is the Kronecker delta.

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