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A sequential estimation procedure for the parameter of an exponential distribution under asymmetric loss function

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Abstract

A problem of Bayesian sequential estimating an unknown parameter of an exponential distribution is considered. It is supposed that the loss associated with the error of estimation is asymmetric (LINEX) and the cost of observing the process is a linear function of time and the number of observations. A Bayes sequential procedure for estimating the unknown parameter is presented.

Key words: Bayesian estimation, Bayes sequential procedure, LINEX loss function, optimal stopping

1. Introduction

This paper deals with the problem of estimating sequentially the parameter ϑ of the exponential distribution defined by the density

$$f(x; \vartheta) = \vartheta \exp(-\vartheta x) \mathbf{1}_{(0, \infty)}(x). \quad (1)$$

The prior knowledge about the parameter ϑ is that it has a gamma $\mathcal{G}(\alpha, \beta)$ distribution with the density function $\pi_{\alpha, \beta}$ of the form

$$\pi_{\alpha, \beta}(\vartheta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \vartheta^{\alpha-1} \exp(-\beta\vartheta) \mathbf{1}_{(0, \infty)}(\vartheta),$$

where $\alpha, \beta > 0$ are known parameters.

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We want to estimate the parameter ϑ on the basis of at most n independent observations X_1, \dots, X_n from the distribution given by (1). If the estimate is determined at time t , the values of X_1, \dots, X_n not exceeding t are exactly known, whereas the other X_i are only known to be larger than t . The estimator $\hat{\vartheta}_t$ of the parameter ϑ used if the observation process is stopped at time t must be \mathcal{F}_t -measurable, where $\mathcal{F}_t := \sigma\{(X_1 \wedge t, \mathbf{1}_{(0,t]}(X_1)), \dots, (X_n \wedge t, \mathbf{1}_{(0,t]}(X_n))\}$, for $t \geq 0$.

If observation is stopped at time t , the loss incurred is defined by

$$L_t(\vartheta, \hat{\vartheta}_t) = L(\vartheta, \hat{\vartheta}_t) + c_A N(t) + c_T t,$$

where $L(\vartheta, \hat{\vartheta}_t)$ denotes the loss associated with the error of estimation, when ϑ is the true value of the parameter and $\hat{\vartheta}_t$ is the chosen \mathcal{F}_t -measurable estimate, $N(t) := \sum_{i=1}^n \mathbf{1}_{(0,t]}(X_i)$, c_A , c_T are known nonnegative constants. We assume the following asymmetric linear exponential (LINEX) loss function

$$L(\vartheta, \hat{\vartheta}_t) = b\{\exp[a(\hat{\vartheta}_t - \vartheta)] - a(\hat{\vartheta}_t - \vartheta) - 1\}, \quad (2)$$

where $a \neq 0$ is a shape parameter and $b > 0$ is a factor of proportionality.

The purpose of this article is to determine the optimal Bayes stopping time and the corresponding Bayes sequential estimator of the unknown parameter ϑ .

The problem under consideration (with the estimation loss function $\vartheta^{-r}(\hat{\vartheta}_t - \vartheta)^2$ for some $r \geq 0$) has been treated in the paper of Chen and Wardrop (1980), where so-called infinitesimal look-ahead (ILA) stopping rule is seen to be optimal in certain cases and studied asymptotically. In the case $r = 0$ (the quadratic loss function) the ILA rule is not optimal for some values n and α . In this case Stadje (1990) gave an optimal solution in rather explicit form by using the "free boundary method". In the model considered in this paper, the ILA stopping time turns out also to be not optimal for some values n and α , namely for $n - \alpha > 1$. Using an analogous method as in the paper of Stadje (1990), we present an optimal solution to the problem stated.

2. The results

Let $\vartheta, X_1, \dots, X_n$ be defined on the probability space $(\Omega, \mathcal{F}, P_{\alpha, \beta})$. Denote by $E_{\alpha, \beta}(\cdot)$ the expectation with respect to $P_{\alpha, \beta}$. Our task is to find a stopping time σ relative to $(\mathcal{F}_t)_{t \geq 0}$ and a family of estimators $(\hat{\vartheta}_t)_{t \geq 0}$ such that the risk

$$E_{\alpha, \beta}(L_\sigma(\vartheta, \hat{\vartheta}_\sigma)) = E_{\alpha, \beta}\left\{b\{\exp[a(\hat{\vartheta}_\sigma - \vartheta)] - a(\hat{\vartheta}_\sigma - \vartheta) - 1\} + c_A N(\sigma) + c_T \sigma\right\}$$

is minimized.

Denote $U(t) := \sum_{i=1}^n (X_i \wedge t)$.

Lemma 1 *The conditional distribution of ϑ given \mathcal{F}_t is the gamma distribution $\mathcal{G}(N(t) + \alpha, U(t) + \beta)$.*

Let us assume that $a > -\beta$. The following lemma gives the form of the Bayes estimator of the parameter ϑ with respect to a priori gamma distribution $\mathcal{G}(\alpha, \beta)$ under the LINEX loss function.

Lemma 2 Under the loss function given by (2) and for any stopping time t the estimator

$$\hat{\vartheta}_t = \frac{N(t) + \alpha}{a} \ln \left(1 + \frac{a}{U(t) + \beta} \right) \quad (3)$$

is Bayes with respect to a priori gamma $\mathcal{G}(\alpha, \beta)$ distribution and its posterior risk is of the form

$$\bar{R}(\pi_{\alpha, \beta}, \hat{\vartheta}_t) = b[N(t) + \alpha] \left[\frac{a}{U(t) + \beta} - \ln \left(1 + \frac{a}{U(t) + \beta} \right) \right]. \quad (4)$$

Proof: The form (3) of the Bayes estimator we obtain by using the general formula given in Zellner (1986). A straightforward calculation shows that the posterior risk $\bar{R}(\pi_{\alpha, \beta}, \hat{\vartheta}_t)$ is of the form (4). \square

Lemma 2 implies that the sequential procedure can be identified with the stopping time. Thus we have to find a stopping time σ relative to $(\mathcal{F}_t)_{t \geq 0}$ which minimizes the expectation

$$\mathbb{E}_{\alpha, \beta} \left[\bar{R}(\pi_{\alpha, \beta}, \hat{\vartheta}_\tau) + c_A N(\tau) + c_T \tau \right]$$

over all such stopping times τ (relative to $(\mathcal{F}_t)_{t \geq 0}$), where $\bar{R}(\pi_{\alpha, \beta}, \hat{\vartheta}_t)$ is given by (4).

Denote

$$g(j, x) := b(j + \alpha) \left[\frac{a}{x + \beta} - \ln \left(1 + \frac{a}{x + \beta} \right) \right] + c_A j, \quad (5)$$

for $(j, x) \in E := \{0, 1, \dots, n\} \times [0, \infty)$.

Lemma 3 The process $(N(t), U(t))$, $t \geq 0$, with values in E is a stationary Markov one. It is also strong Markov. If $\phi : E \rightarrow \mathbb{R}$ is a given function such that $\phi(j, \cdot) \in C^1[0, \infty)$ and $\phi(j, \cdot)$ and $\phi'(j, \cdot)$ are bounded for $j = 0, 1, \dots, n$, then ϕ belongs to the domain $D_{\mathcal{I}}$ of the weak infinitesimal generator \mathcal{I} of the process $(N(t), U(t))$, and we have

$$\mathcal{I}\phi(j, x) = [\phi(j+1, x) - \phi(j, x)](n-j)(\alpha+j)(\beta+x)^{-1} + (n-j)\phi'(j, x),$$

where $\phi(n+1, x) := \phi(n, x)$ and $\phi'(j, x) := \frac{d}{dx}\phi(j, x)$. \square

For the proof of this lemma see Stadje (1990).

Theorem 1 The stopping time σ defined by

$$\sigma = \inf\{t \geq 0 : \exists j \in \{0, \dots, n\}, N(t) = j, U(t) \geq \gamma_j\}$$

is optimal, the boundaries being determined by $\gamma_j = 0$, $j = 0, \dots, n-1$, when $\psi(j, x) > 0$ for all $x \in [0, \infty)$ or γ_j is a unique root of the equation $\psi(j, x) = 0$, and $\gamma_n = 0$, where

$$\begin{aligned} \psi(j, x) := & \phi(j+1, x) - g(j+1, x) + b \ln \left(\frac{\beta+x}{\beta+x+a} \right) \\ & + \frac{ab}{\beta+x+a} + \frac{c_T(\beta+x)}{(n-j)(\alpha+j)} + c_A, \end{aligned} \quad (6)$$

$g(j, x)$ is given by (5),

$$\phi(n, x) := g(n, x), \quad (7)$$

$$\phi(j, x) := (\beta+x)^{\alpha+j} \left[g_j(x) + \int_x^{\gamma_j} \frac{\alpha+j}{(\beta+t)^{\alpha+j+1}} \phi(j+1, t) dt \right], \quad (8)$$

for $j = 0, \dots, n-1$, where

$$\begin{aligned} g_j(x) := & \frac{c_T}{(n-j)(\alpha+j-1)} \left[(\beta+x)^{-\alpha-j+1} - (\beta+\gamma_j)^{-\alpha-j+1} \right] + \frac{c_A j}{(\beta+\gamma_j)^{\alpha+j}} \\ & + \frac{b(\alpha+j)}{(\beta+\gamma_j)^{\alpha+j}} \left[\frac{a}{\beta+\gamma_j} + \ln \left(\frac{\beta+\gamma_j}{\beta+\gamma_j+a} \right) \right], \end{aligned} \quad (9)$$

when $\alpha+j \neq 1$, and

$$\begin{aligned} g_j(x) := & -\frac{c_T}{n-j} \ln(\beta+x) + \frac{ab}{(\beta+\gamma_j)^2} + \frac{b}{\beta+\gamma_j} \ln \frac{\beta+\gamma_j}{\beta+\gamma_j+a} \\ & + \frac{c_A j}{(\beta+\gamma_j)^{\alpha+1}} + \frac{c_T}{n-j} \ln(\beta+\gamma_j), \end{aligned} \quad (10)$$

when $\alpha+j = 1$.

Proof: By Dynkin's identity (Dynkin (1965)) it can be shown that if $\phi : E \rightarrow \mathbb{R}$ satisfies the following conditions

- (i) $\phi \in D_{\mathcal{I}}$,
- (ii) $\phi(j, x) \leq g(j, x)$ and $\mathcal{I}\phi(j, x) \geq -c_T$, $(j, x) \in E$,
- (iii) $\phi(j, x) < g(j, x) \Rightarrow \mathcal{I}\phi(j, x) = -c_T$, $(j, x) \in E$,

then the stopping time

$$\sigma = \inf\{t \geq 0 : \phi(N(t), U(t)) = g(N(t), U(t))\} \quad (11)$$

satisfies $E_{\alpha, \beta}[g(N(\sigma), U(\sigma)) + c_T \sigma] = \inf_t E_{\alpha, \beta}[g(N(t), U(t)) + c_T t] = \phi(0, 0)$, provided

$$E_{\alpha, \beta}(\sigma) < \infty. \quad (12)$$

For the stopping time σ , given by (11), condition (12) is satisfied, because

$$E_{\alpha,\beta}(\sigma) \leq E_{\alpha,\beta}(\max(X_1, \dots, X_n)) < \infty.$$

The form of the "payoff function"

$$g(j, x) + c_T t = b(j + \alpha) \left[\frac{a}{x + \beta} - \ln \left(1 + \frac{a}{x + \beta} \right) \right] + c_A j + c_T t$$

suggests that one stops at time t iff $U(t) = x$ is large. This idea leads to the consideration of stopping times of the following form $\sigma = \inf\{t \geq 0 : \exists j \in \{0, \dots, n\}; N(t) = j, U(t) \geq \gamma_j\}$, where $\gamma_n = 0$ and $\gamma_0, \dots, \gamma_{n-1}$ are constants. Thus we should look for the constants $\gamma_0, \dots, \gamma_{n-1}$ and the function $\phi : E \rightarrow \mathbb{R}$ satisfying condition (i) and having the following properties

$$\phi(n, x) = g(n, x) = b(n + \alpha) \left[\frac{a}{x + \beta} - \ln \left(1 + \frac{a}{x + \beta} \right) \right] + c_A n, \quad (13)$$

$$\phi(j, x) = g(j, x), \quad \text{when } x \geq \gamma_j, \quad (14)$$

$$\mathcal{I}\phi(j, x) \geq -c_T, \quad \text{when } x \geq \gamma_j, \quad (15)$$

$$\phi(j, x) \leq g(j, x), \quad \text{when } x \leq \gamma_j, \quad (16)$$

$$\begin{aligned} \mathcal{I}\phi(j, x) &= [\phi(j + 1, x) - \phi(j, x)](n - j)(\alpha + j)(\beta + x)^{-1} + (n - j)\phi'(j, x) \\ &= -c_T, \quad \text{when } x \leq \gamma_j. \end{aligned} \quad (17)$$

Condition (17) is a recursive system of differential equations for $\phi(n - 1, \cdot), \dots, \phi(0, \cdot)$, and (13) gives $\phi(n, \cdot) = g(n, \cdot)$. To treat this system in a convenient way it turns out that if one sets

$$\phi(j, x) = f_j(x)[g_j(x) + \int_x^{\gamma_j} h_j(t)\phi(j + 1, t)dt],$$

condition (17) gives equations for f_j , g_j and h_j which can be easily solved. Namely, for $j = 0, \dots, n - 1$,

$$f_j(x) = (\beta + x)^{\alpha + j},$$

g_j is given by formulas (9), (10), and

$$h_j(x) = \frac{\alpha + j}{(\beta + x)^{\alpha + j + 1}}.$$

We shall now determine the constants $\gamma_0, \dots, \gamma_{n-1}$ such that $\phi \in D_{\mathcal{I}}$ and conditions (14)–(16) hold. For the function ψ defined by (6) it is easily checked that

$$\psi(j, x) = \phi(j + 1, x) - g(j, x) - ab \left[\frac{1}{\beta + x} - \frac{1}{\beta + x + a} \right] + \frac{c_T(\beta + x)}{(n - j)(\alpha + j)},$$

and for $x \leq \gamma_j$

$$\int_x^{\gamma_j} (\beta + t)^{-\alpha - j - 1} \psi(j, t) dt = [\phi(j, x) - g(j, x)] / (\alpha + j)(\beta + x)^{\alpha + j}. \quad (18)$$

Further, when $x \geq \gamma_j$,

$$\mathcal{I}\phi(j, x) = -c_T + \psi(j, x)(n - j)(\alpha + j)(\beta + x)^{-1}. \quad (19)$$

In an analogous way as in the paper of Stadjé (1990), it can be shown that for certain constants γ_j

$$x < \gamma_j \Rightarrow \psi(j, x) \leq 0 \quad (20)$$

and

$$x \geq \gamma_j \Rightarrow \psi(j, x) \geq 0. \quad (21)$$

It follows from (18) and (20) that $\phi(j, x) \leq g(j, x)$ for $x \in [0, \gamma_j]$. If $\gamma_j > 0$, it follows from (18) that the lefthand derivative of $\phi(j, x)$ at $x = \gamma_j$ is equal to its righthand derivative $g'(j, x)$. Thus it is clear that $\phi(j, \cdot) \in C^1[0, \infty)$ and that $\phi(j, \cdot)$ and $\phi'(j, \cdot)$ are bounded. Thus we have $\phi \in D_{\mathcal{I}}$. Equation (19) and implication (21) yield $\mathcal{I}\phi \geq -c_T$. Thus the function ϕ given by (7) and (8) satisfies conditions (13)–(17). This completes the proof of the theorem. \square

Corollary 1 *If $n - \alpha \leq 1$, the constants $\gamma_0, \dots, \gamma_n$ satisfy $0 = \gamma_n \leq \gamma_{n-1} \leq \dots \leq \gamma_0$.*

If $n - \alpha > 1$, we have $0 = \gamma_n \leq \gamma_{n-1} \leq \dots \leq \gamma_{i+1} \geq \gamma_i \geq \dots \geq \gamma_0$, where $i = [(n - \alpha - 1)/2]$.

Proof: The theorem can be proved in an analogous way as Theorem 2 in the paper of Stadjé (1990).

3. Description of the optimal stopping time and a numerical example

The optimal stopping time derived in this paper can be described as follows. Suppose first that one has observed the n units up to time t and no failure has occurred. Then one

stops at t iff $t \geq \gamma_0/n$. Suppose next that up to time t exactly one failure has happened and one has not stopped yet. Then one stops at t if $X_{(1)} + (n-1)t \geq \gamma_1$. In general, if up to the current time t exactly j units have failed, i.e., $t \in [X_{(j)}, X_{(j+1)})$, where $j \in \{0, 1, \dots, n-1\}$, one stops iff

$$U(t) = X_{(1)} + \dots + X_{(j)} + (n-j)t \geq \gamma_j.$$

Finally let us look at an explicit example. Consider the case $n = 2$, $a = b = 1$, $\alpha = \beta = 1$, $c_A = c_T = 0.1$. We shall compute γ_0 and γ_1 , ($\gamma_2 = 0$ by definition). First we have to find γ_1 which is the unique solution to the equation

$$\psi(1, x) = 0, \tag{22}$$

or $\gamma_1 = 0$, when equation (22) has no solution. Function $\psi(1, x)$ is of the form

$$\psi(1, x) = \phi(2, x) - g(2, x) + \ln\left(\frac{x+1}{x+2}\right) + \frac{x+1}{20} + \frac{1}{x+2} + \frac{1}{10},$$

where

$$\phi(2, x) = g(2, x) = \frac{3}{x+1} + \ln\left(\frac{x+1}{x+2}\right) + \frac{1}{5}.$$

Thus

$$\psi(1, x) = \ln\left(\frac{x+1}{x+2}\right) + \frac{x+1}{20} + \frac{1}{x+2} + \frac{1}{10},$$

and $\gamma_1 = 0.0163$. The boundary γ_0 is the unique solution to the equation

$$\psi(0, x) = 0, \tag{23}$$

or $\gamma_1 = 0$, when equation (23) has no solution. Function $\psi(0, x)$ is of the form

$$\psi(0, x) = \phi(1, x) - g(1, x) + \ln\left(\frac{x+1}{x+2}\right) + \frac{x+1}{20} + \frac{1}{x+2} + \frac{1}{10},$$

where

$$\phi(1, x) = \frac{1}{(x+1)^2} \left[g_1(x) + \int_x^{\gamma_1} \frac{6}{(t+1)^4} + \frac{1}{25(t+1)^3} dt \int_x^{\gamma_1} \frac{6}{(t+1)^3} \ln\left(\frac{t+1}{t+2}\right) dt \right],$$

where

$$g_1(x) = \frac{1}{10(x+1)} + \frac{2}{(\gamma_1+1)^3} + \frac{2}{(\gamma_1+1)^2} \ln\left(\frac{\gamma_1+1}{\gamma_1+2}\right) + \frac{1}{10(\gamma_1+1)^2} - \frac{1}{10(\gamma_1+1)},$$

and

$$g(1, x) = \frac{2}{x+1} + 2 \ln \left(\frac{x+1}{x+2} \right) + \frac{1}{10}.$$

Equation (23) has the solution $\gamma_0 = 0.0163$. So, when we have observed two units up to time t and no failure has occurred, then we stop at time t iff $t \geq \gamma_0/2 = 0.00815$ and we take $\ln[1 + 1/(\gamma_0 + 1)]$ as the estimator of the parameter ϑ . If only one failure has occurred up to time t and we have not stopped yet, then we stop at time t iff $X_{(1)} + t \geq \gamma_1 = 0.0163$ and we take $2 \ln[1 + 1/(\gamma_1 + 1)]$ as the estimator of the parameter ϑ . If the second failure has occurred at time t and we have not stopped yet, then we stop immediately and we take $3 \ln[1 + 1/(x_1 + t + 1)]$ as the estimator of the parameter ϑ .

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