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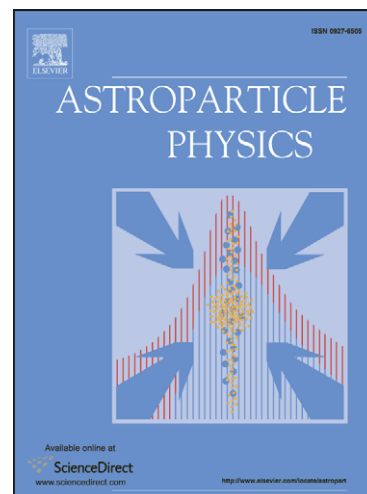
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# Linearized Treatment of Scalar perturbations in the Asymptotic Cosmological Model

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In this paper the implications of a recently proposed phenomenological model of cosmology, the Asymptotic Cosmological Model (ACM), on the behavior of scalar perturbations are studied. Firstly we discuss new fits of the ACM at the homogeneous level, including fits to the Type Ia Supernovae UNION dataset, first CMB peak of WMAP5 and BAOs. The linearized equations of scalar perturbations in the FRW metric are derived. A simple model is used to compute the CMB temperature perturbation spectrum. The results are compared with the treatment of perturbations in other approaches to the problem of the accelerated expansion of the universe.

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## INTRODUCTION

According to General Relativity (GR), if the universe is filled with the particles of the Standard Model of particle physics, gravity should lead to a decelerated expansion of the universe. However, in 1998 two independent evidences of present accelerated expansion were presented [1, 2] and later confirmed by different observations [3–5].

There is no compelling explanation for this cosmic acceleration, but many intriguing ideas are being explored. These ideas can be classified into three main groups: new exotic sources of the gravitational field with large negative pressure [6] (Dark Energy), modifications of gravity at large scales [7] and rejection of the spatial homogeneity as a good approximation in the description of the present universe [8].

Different models (none of them compelling) for the source responsible of acceleration have been considered. Einstein equations admit a cosmological constant  $\Lambda$ , which can be realized as the stress-energy tensor of empty space. This  $\Lambda$  together with Cold Dark Matter, Standard Model particles and General Relativity form the current cosmological model,  $\Lambda$ CDM. However, quantum field theory predicts a value of  $\Lambda$  which is 120 orders of magnitude higher than observed. Supersymmetry can lower this value 60 orders of magnitude, which is still ridiculous [9]. In order to solve this paradox, dynamical Dark Energy models have been proposed.

This has also lead to explore the possibility that cosmic acceleration arises from new gravitational physics. Again here several alternatives for a modification of the Einstein-Hilbert action at large and small curvatures [10], or even higher dimensional models [11, 12], producing an accelerated expansion have been identified. All these analyses include an *ad hoc* restriction to actions involving simple functions of the scalar curvature and or the Gauss-Bonnet tensor. This discussion is sufficient to establish the point that cosmic acceleration can be made compatible with a standard source for the gravitational field but it is convenient to consider a more general framework in order to make a systematic analysis of the cosmological

effects of a modification of general relativity.

The Asymptotic Cosmological Model (ACM) was presented [13] as a strictly phenomenological generalization of the Standard Cosmological Model including a past and a future epoch of accelerated expansion. It follows from the assumptions that GR is not a fundamental theory, but only a good approximation when the Hubble rate  $H$  is between but far away from two fundamental scales  $H_-$  and  $H_+$ , which act as bounds on  $H$ . A general covariant metric theory of gravity without spatial curvature is assumed. The model is well defined in the homogeneous approximation and includes  $\Lambda$ CDM as a particular case.

In next section we review the ACM and we provide new fits to the Type Ia Supernovae UNION dataset, first acoustic peak of CMB of WMAP5 and BAOs. In the third section we derive the linearized equations of the scalar perturbations of the metric, following from general covariance and a single new assumption on the perturbations. In the fourth section we will consider how to solve the system of equations for adiabatic perturbations in a given fluid. In the fifth section we derive the CMB spectrum in the ACM. In the sixth section we will compare the treatment of the scalar perturbations in the ACM with other models which are equivalent in the homogeneous approximation. The last section is devoted to the summary and conclusions.

## THE ASYMPTOTIC COSMOLOGICAL MODEL: HOMOGENEOUS BACKGROUND

The Asymptotic Cosmological Model (ACM) was introduced in Ref. [13]. In this model the universe is filled with photons and neutrinos (massless particles), baryons (electrically charged massive particles) and Dark Matter (electrically neutral massive particles), but General Relativity (GR) is only a good approximation to the gravitational interaction in a certain range of the Hubble rate  $H$ , between but far from its two bounds,  $H_-$  and  $H_+$ . General Covariance and the absence of spatial curvature are assumed.

The gravitational part of the action might include

derivatives of the metric of arbitrarily high order, and therefore arbitrarily high derivatives of the scale factor should appear in the Friedman Equations. We should start by considering a generalized first Friedman equation

$$8\pi G\rho(t) = 3f(H(t), \dot{H}(t), \dots, H^{(n)}(t), \dots), \quad (1)$$

and the corresponding equation for the pressure will be derived using the continuity equation

$$\dot{\rho} = -3H(\rho + p). \quad (2)$$

However, as the resulting differential equations should be solved and only one of its solutions deserves interest (the one describing the evolution of the universe), we can use the one to one correspondence between time  $t$  and Hubble parameter  $H$  (assuming  $\dot{H} < 0$ ) to write the modified First Friedman Equation as a bijective map linking the total energy density  $\rho$  with  $H$  in our universe

$$8\pi G\rho = 3g(H). \quad (3)$$

The use of the continuity equation (2) enables us to write the modified Friedman Equation for the pressure evaluated at the solution corresponding to the cosmic evolution

$$-8\pi Gp = 3g(H) + g'(H)\dot{H}/H. \quad (4)$$

The evolution of the universe will be determined by the concrete form of the function  $g(H)$ , which we assume to be smooth. However, the most significant features of the evolution at a given period can be described by some simple approximation to  $g(H)$  and the matter content. Those are a pole at  $H = H_+$  of order  $\alpha_+$  and a zero of order  $\alpha_-$  at  $H = H_-$ . The energy density of the universe has a contribution from both massless particles (radiation) and massive ones (matter).

For the sake of simplicity, the history of the universe can be then divided into three periods. In the first period,  $H \gg H_-$  so the effect of the lower bound can be neglected and the universe is radiation dominated. The universe undergoes (and exits from) an accelerated expansion which we call inflation. The simplest parameterization is

$$g(H) \approx H^2 \left(1 - \frac{H^2}{H_+^2}\right)^{-\alpha_+}. \quad (5)$$

In the second period we cannot neglect neither the effect of radiation nor the effect of nonrelativistic matter, but  $H_+ \gg H \gg H_-$ . Then GR offers a good description of the gravitational interaction,  $g(H) \approx H^2$  in this region, and the universe performs a decelerated expansion.

In the third period,  $H_+ \gg H$  so the effect of the upper bound can be neglected, and the universe is matter

dominated. This period corresponds to the present time in which the universe also undergoes an accelerated expansion. The simplest choice is

$$g(H) \approx H^2 \left(1 - \frac{H^2}{H_-^2}\right)^{\alpha_-}. \quad (6)$$

We will assume that the transitions between these three periods are smooth and that the details about these transitions are unimportant.

This model preserves the successes of the Cosmological Standard Model, while giving a description of the early accelerated expansion (inflation) and of the present one, including  $\Lambda$ CDM as a particular case ( $\alpha_- = 1$ ).

Without any knowledge of the evolution of perturbations in this model, the background evolution at late times can be used to fit the Type Ia Supernovae UNION dataset [14], the first acoustic peak in the Cosmic Microwave (CMB) Background [15], and the Baryon Acoustic Oscillations (BAOs) [16], via the parameters  $H_0$  (the present Hubble parameter),  $H_-$  and  $\alpha_-$ .

We use Monte Carlo Markov Chains to explore the likelihood of the fit of the supernovae UNION dataset [14] to the ACM. The dataset provides the luminosity distance

$$d_l(z) = c(1+z) \int_0^z dz'/H(z') \quad (7)$$

and redshift of 307 supernovae. A  $\chi^2$  analysis have been performed, where  $\chi^2$  has been marginalized over the nuisance parameter  $H_0$  using the method described in [17]. The resulting parameter space is spanned by the values of  $\alpha_-$  and  $H_-/H_0$  (FIG. 1).

The constraints from the CMB data follow from the reduced distance to the surface of last scattering at  $z = 1089$ . The reduced distance  $R$  is often written as

$$R = \Omega_m^{1/2} H_0 \int_0^{1089} dz/H(z). \quad (8)$$

The WMAP-5 year CMB data alone yield  $R_0 = 1.715 \pm 0.021$  for a fit assuming a constant equation of state  $\omega$  for the dark energy [21]. We will take this value as a first approximation to the fit assuming ACM. We can define the corresponding  $\chi^2$  as  $\chi^2 = [(R - R_0)/\sigma_{R_0}]^2$ , and find the confidence regions of the joint constraints (FIG. 2).

BAO measurements from the SDSS data provide a constraint on the distance parameter  $A(z)$  at redshift  $z = 0.35$ ,

$$A(z) = \Omega_m^{1/2} H_0 H(z)^{-1/3} z^{-2/3} \left[ \int_0^z dz'/H(z') \right]^{2/3}. \quad (9)$$

Ref. [16] gives  $A_0 = 0.469 \pm 0.17$ . We can define the corresponding  $\chi^2$  as  $\chi^2 = [(A(z = 0.35) - A_0)/\sigma_{A_0}]^2$ . The confidence regions resulting from adding this constraint are shown in FIG. 3.

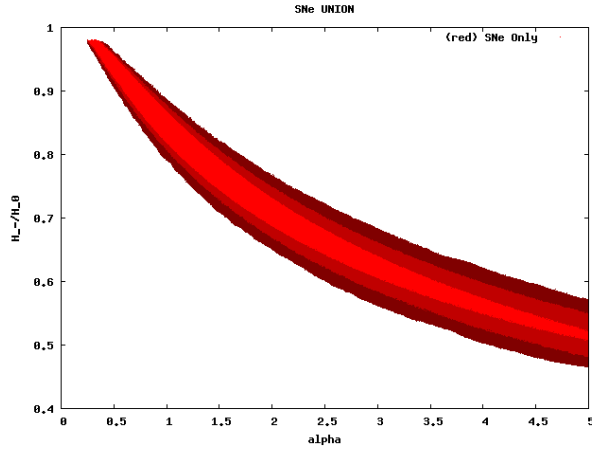


FIG. 1: Confidence regions in parameter space of the Asymptotic Cosmological Model (ACM) from the fit of the supernovae UNION dataset without priors at  $1\sigma$ ,  $2\sigma$  and  $3\sigma$ . The  $\Lambda$ CDM is inside the  $1\sigma$  region. The best fit to ACM lies in  $\alpha_- = 0.35$ ,  $H_-/H_0 = 0.97$  ( $\chi^2 = 310.5$ ) in contrast to the best fit to  $\Lambda$ CDM, which lies in  $\alpha_- \equiv 1$ ,  $H_-/H_0 = 0.84$  ( $\chi^2 = 311.9$ ).

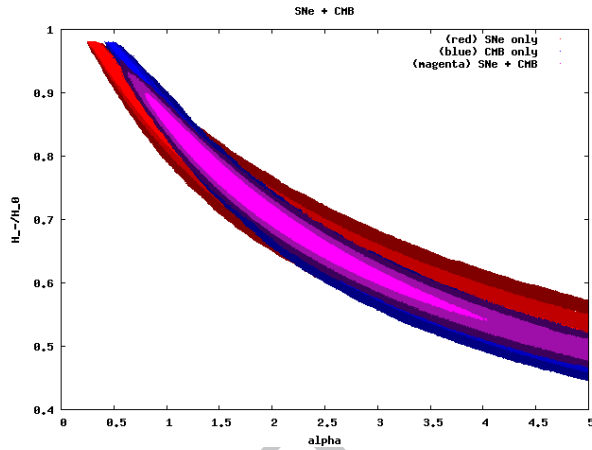


FIG. 2: Confidence regions in parameter space of the Asymptotic Cosmological Model (ACM) from the fit of the supernovae UNION dataset (red), of the distance to the surface of last scattering from WMAP-5 (blue) and from the joint fit (magenta) at  $1\sigma$ ,  $2\sigma$  and  $3\sigma$ . Significantly, the  $\Lambda$ CDM is still inside the  $1\sigma$  region, unlike in our previous study. This is due to the change in the Supernovae dataset. The best fit to ACM lies in  $\alpha_- = 1.50$ ,  $H_-/H_0 = 0.77$  ( $\chi^2 = 312.5$ ) in contrast to the best fit to  $\Lambda$ CDM, which lies in  $\alpha_- \equiv 1$ ,  $H_-/H_0 = 0.86$  ( $\chi^2 = 313.2$ ).

In our previous work the use of the supernovae Gold dataset [18] and of the WMAP-3 data [19] led us to the conclusion that the  $\Lambda$ CDM was at  $3\sigma$  level in the parameter space of the ACM. The position of the confidence regions seems to depend very tightly on the dataset that is being used. However, the value of the combination of

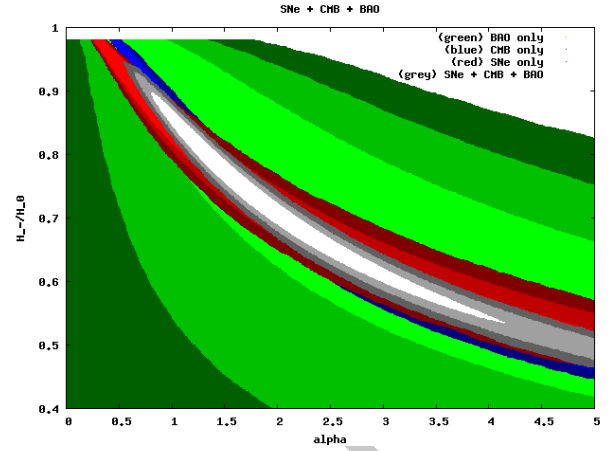


FIG. 3: Confidence regions in parameter space of the Asymptotic Cosmological Model (ACM) from the fit of the supernovae UNION dataset (red), of the distance to the surface of last scattering from WMAP-5 (blue), of the Baryon Acoustic Oscillations peak (green) and from the joint fit (grey) at  $1\sigma$ ,  $2\sigma$  and  $3\sigma$ . Measurements of the BAOs peak do not add significant information with their present precision. The best fit to ACM lies in  $\alpha_- = 1.50$ ,  $H_-/H_0 = 0.77$  ( $\chi^2 = 312.7$ ) in contrast to the best fit to  $\Lambda$ CDM, which lies in  $\alpha_- \equiv 1$ ,  $H_-/H_0 = 0.86$  ( $\chi^2 = 313.7$ ).

parameters

$$\Omega_m \equiv \left(1 - \frac{H_-^2}{H_0^2}\right)^{\alpha_-} \quad (10)$$

does not depend much neither on the value of  $\alpha_-$  or the dataset used (FIG. 4).

Moreover, we can conclude from the figures that BAO's do not provide much information in order to constrain the confidence regions of the ACM, unlike in other models such as  $\Lambda$ CDM with nonzero spatial curvature.

## SCALAR PERTURBATIONS

The lack of an action defining the ACM is a serious obstacle in the derivation of the equations governing the behavior of the perturbations. Given a background behavior described by the ACM, what can be said about the evolution of perturbations on top of this background? We will find that general covariance together with an additional assumption fixes completely the set of equations for the scalar perturbations in the linearized approximation and in the region close to  $H_-$ .

The key point will be the following. Knowing the exact Friedman equations in the homogeneous approximation gives us a clue on the form of the equations for the scalar perturbations. In particular, we can formally describe perturbations over the FRW metric which do not depend on the spatial coordinates,  $\{\phi(\mathbf{x}, t) = \phi(t), \psi(\mathbf{x}, t) = \psi(t)\}$ . Thus the perturbed metric becomes the FRW

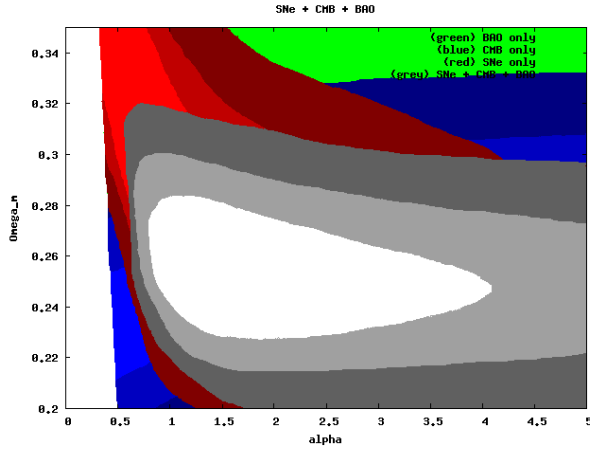


FIG. 4: Confidence regions in parameter space of the Asymptotic Cosmological Model (ACM) from the fit of the supernovae UNION dataset (red), of the distance to the surface of last scattering from WMAP-5 (blue), of the Baryon Acoustic Oscillations peak (green) and from the joint fit (grey) at  $1\sigma$ ,  $2\sigma$  and  $3\sigma$ . The plot shows  $\alpha_+$  in the horizontal axis and  $\Omega_m$  in the vertical axis. Noticeably,  $\Omega_m = 0.26 \pm 0.04$  almost independently of the value of  $\alpha_+$ .

metric written in a new coordinate frame. With a change of coordinates one can derive, starting from the Friedman equations, the terms containing only time derivatives of the scalar perturbations. Next an assumption on the validity of the GR description of scalar perturbations when

$H_- \ll H \ll H_+$ , together with the relations, valid for any general covariant theory, between terms with time derivatives and those involving spatial derivatives, allow to derive the evolution of scalar perturbations.

After this short sketch, we will perform the derivation of the equations for the scalar perturbations in detail. We can write the metric of spacetime with scalar perturbations in the Newtonian gauge,

$$ds^2 = (1 + 2\phi(\mathbf{x}, t))dt^2 - a^2(t)(1 - 2\psi(\mathbf{x}, t))d\mathbf{x}^2, \quad (11)$$

and the stress-energy tensor of the source fields will be

$$\begin{aligned} T_0^0 &= \rho_{(0)}(t) + \delta\rho(\mathbf{x}, t), \\ T_i^0 &= (\rho_{(0)}(t) + p_{(0)}(t))\partial_i\theta(\mathbf{x}, t), \\ T_j^i &= -(p_{(0)}(t) + \delta p(\mathbf{x}, t))\delta_j^i + \partial_i\partial_j\Pi(\mathbf{x}, t), \end{aligned} \quad (12)$$

where  $\theta$  is the velocity potential of the fluid,  $\Pi$  is the anisotropic stress tensor (shear) potential,  $\delta\rho$  and  $\delta p$  are small perturbations on top of the background homogeneous density  $\rho_{(0)}$  and pressure  $p_{(0)}$ , respectively, and  $\phi \sim \psi \ll 1$ . We have used the Newtonian gauge because the gauge invariant scalar perturbations of the metric ( $\Phi$  and  $\Psi$ ) and gauge invariant perturbations in the stress energy tensor coincide with the perturbations explicitly written in this gauge.

The general form of the equations for the perturbations in a metric theory with arbitrarily high derivatives is

$$8\pi G\delta\rho = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[ \frac{a_{nm}}{H^{2n+m-2}} \left(\frac{\Delta}{a^2}\right)^n \partial_t^m \phi + \frac{b_{nm}}{H^{2n+m-2}} \left(\frac{\Delta}{a^2}\right)^n \partial_t^m \psi \right], \quad (13)$$

$$8\pi G(\rho_{(0)} + p_{(0)})\partial_i\theta = \partial_i \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[ \frac{c_{nm}}{H^{2n+m-1}} \left(\frac{\Delta}{a^2}\right)^n \partial_t^m \phi + \frac{d_{nm}}{H^{2n+m-1}} \left(\frac{\Delta}{a^2}\right)^n \partial_t^m \psi \right], \quad (14)$$

$$\begin{aligned} 8\pi G(-\delta p \delta_j^i + \partial_i\partial_j\Pi) &= \delta_j^i \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[ \frac{e_{nm}}{H^{2n+m-2}} \left(\frac{\Delta}{a^2}\right)^n \partial_t^m \phi + \frac{f_{nm}}{H^{2n+m-2}} \left(\frac{\Delta}{a^2}\right)^n \partial_t^m \psi \right] \\ &+ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{a^2} (\delta_j^i \Delta - \partial_i\partial_j) \left[ \frac{g_{nm}}{H^{2n+m}} \left(\frac{\Delta}{a^2}\right)^n \partial_t^m \phi + \frac{h_{nm}}{H^{2n+m}} \left(\frac{\Delta}{a^2}\right)^n \partial_t^m \psi \right]. \end{aligned} \quad (15)$$

The coefficients  $a_{nm}, \dots, h_{nm}$  are adimensional functions of the Hubble parameter and its time derivatives, and can be turned into functions of just the Hubble parameter using the bijection explained in the beginning of the previous section. We have mentioned we are going to be able to determine exactly the terms with only time derivatives, that is, the precise form of the coefficients  $a_{0m}, b_{0m}, e_{0m}$  and  $f_{0m}$ . This shows that there is much freedom of choosing a covariant linearized theory of cosmological perturbations, even for a given solution of the

homogeneous equations.

However, under a single assumption, it is possible to greatly reduce this freedom. The assumption is that the standard linearized equations of the perturbations of General Relativity are effectively recovered in the limit  $H_+ \gg H \gg H_-$  for all the Fourier modes with  $H_+ \gg k > H_-$  (subhorizon modes in the present time).

Let us work out the implications of this assumption. In the equation for the perturbation of the energy density



in General relativity,

$$8\pi G\delta\rho = -6H^2\phi - 6H\dot{\psi} + \frac{2}{a^2}\Delta\psi, \quad (16)$$

there is only a term proportional to  $\Delta\psi$ . However, for subhorizon modes ( $k \gg H$ ) the dominant terms are those with the highest number of spatial derivatives. If we demand the terms with spatial derivatives of order greater than 2 not to spoil the behavior of these modes in the period in which GR is a good approximation, they must be negligible at least for the observable modes.

For instance, they should be negligible for the modes responsible of the acoustic peaks of the CMB spectrum, and therefore for the modes which have entered the horizon after recombination (which have even lower  $k$ ). It is possible that these terms are suppressed by inverse powers of the UV scale  $H_+$ , becoming irrelevant for current tests of gravity, although they may be relevant for the physics of quantum fluctuations in the very early universe.

Therefore, at times when  $H \ll H_+$ , the equations can be approximated by

$$8\pi G\delta\rho = \sum_{m=0}^{\infty} \left[ \frac{a_{0m}}{H^{m-2}} \partial_t^m \phi + \frac{b_{0m}}{H^{m-2}} \partial_t^m \psi + \frac{a_{1m}}{H^m} \frac{\Delta}{a^2} \partial_t^m \phi + \frac{b_{1m}}{H^m} \frac{\Delta}{a^2} \partial_t^m \psi \right], \quad (17)$$

$$8\pi G(\rho_{(0)} + p_{(0)})\partial_i\theta = \partial_i \sum_{m=0}^{\infty} \left[ \frac{c_{0m}}{H^{m-1}} \partial_t^m \phi + \frac{d_{0m}}{H^{m-1}} \partial_t^m \psi \right], \quad (18)$$

$$\begin{aligned} 8\pi G(-\delta p \delta_j^i + \partial_i \partial_j \Pi) &= \delta_j^i \sum_{m=0}^{\infty} \left[ \frac{e_{0m}}{H^{m-2}} \partial_t^m \phi + \frac{f_{0m}}{H^{m-2}} \partial_t^m \psi \right] \\ &+ \sum_{m=0}^{\infty} \frac{1}{a^2} (\delta_j^i \Delta - \partial_i \partial_j) \left[ \frac{g_{0m}}{H^m} \partial_t^m \phi + \frac{h_{0m}}{H^m} \partial_t^m \psi \right], \end{aligned} \quad (19)$$

where general covariance has been used to exclude the terms with coefficients  $e_{1m}, f_{1m}$  which are not compatible with (17), (18).

We must also take into account that the modes which are responsible of the acoustic peaks of the CMB undergo a phase in which they oscillate as sound waves, i.e.:  $\partial_t \phi \sim k\phi$ . In order to explain the acoustic peaks of the spectrum of the CMB it is required that the acoustic oscillations of the modes of the gravitational potentials,  $\phi_{\mathbf{k}}$  and  $\psi_{\mathbf{k}}$ , which lead to the acoustic peaks of the CMB spectrum, have a frequency  $\sim k$ . When radiation domi-

nates and we consider modes well inside the horizon, two of the solutions of the system of differential equations of arbitrary order have this property <sup>1</sup>.

However, in order for nondecaying superhorizon modes to evolve into oscillating subhorizon modes with frequency  $\sim k$ , a fine-tuning of coefficients is required unless the equations are of order two in time derivatives. Then the acoustic peaks in the CMB spectrum are reproduced when terms with more than two derivatives in the equations for the perturbations can be neglected. Then

$$8\pi G\delta\rho = \sum_{m=0}^2 [a_{0m}H^{2-m}\partial_t^m\phi + b_{0m}H^{2-m}\partial_t^m\psi] + a_{10}\frac{\Delta}{a^2}\phi + b_{10}\frac{\Delta}{a^2}\psi, \quad (20)$$

$$8\pi G(\rho_{(0)} + p_{(0)})\partial_i\theta = \partial_i \sum_{m=0}^1 [c_{0m}H^{1-m}\partial_t^m\phi + d_{0m}H^{1-m}\partial_t^m\psi], \quad (21)$$

$$\begin{aligned} 8\pi G(-\delta p \delta_j^i + \partial_i \partial_j \Pi) &= \delta_j^i \sum_{m=0}^2 [e_{0m}H^{2-m}\partial_t^m\phi + f_{0m}H^{2-m}\partial_t^m\psi] \\ &+ \frac{1}{a^2} (\delta_j^i \Delta - \partial_i \partial_j) [g_{00}\phi + h_{00}\psi], \end{aligned} \quad (22)$$

<sup>1</sup> As we will see below, if the term with the highest time derivative

is of order  $D$ ,  $a_{1,D-2} = -e_{0D}$  as a consequence of general co-

and the arbitrariness in the evolution equations for the scalar perturbations has been reduced to twenty undetermined dimensionless coefficients at this level.

The equations (20),(21),(22) are in principle valid for any perturbation mode  $\{\phi_{\mathbf{k}}, \psi_{\mathbf{k}}\}$  as long as  $k \ll H_+$ . In particular, it must be valid for the mode  $\mathbf{k} = \mathbf{0}$ , which corresponds formally to a perturbation with no spatial dependence. In practice, we will be able to neglect the spatial dependence of perturbation whose spatial dependence is sufficiently smooth, i.e.: its wavenumber  $k$  is sufficiently low. In GR it suffices for a mode to be super-horizon  $k \ll H$  in order to neglect its spatial dependence in a first approximation.

If we consider the FRW metric perturbed by one of these modes we have,

$$ds^2 = (1 + 2\phi(t))dt^2 - a^2(t)(1 - 2\psi(t))d\mathbf{x}^2. \quad (23)$$

From now on the dot will represent derivative with respect to the cosmic time. By means of the invariance under time reparameterizations we can introduce a new time variable  $dt' = (1 + \phi(t))dt$ , a new scale factor  $a'(t') = a(t)(1 - \psi(t))$  and a new energy density  $\rho'(t') = \rho_0(t) + \delta\rho(t)$  leading us back to the ACM in a homogeneous background. The Hubble rate  $H = \frac{\dot{a}}{a}$  and its time derivatives  $H^{(n)}$  will change as

$$\begin{aligned} H^{(n)'} &= \frac{d^n}{dt'^n} \left[ \frac{1}{a'(t')} \frac{da'}{dt'} \right] = H^{(n)} + \delta H^{(n)}, \\ \delta H^{(n)} &= -\psi^{(n+1)} - \sum_{m=0}^n \binom{n+1}{m+1} H^{(n-m)} \phi^{(m)}. \end{aligned} \quad (24)$$

Noticeably, this procedure must be applied to the generalized Friedman equation before one of its homogeneous solutions is used to write the time variable  $t$  as a function of  $H$ . Therefore we should start by considering a generalized first Friedman equation in primed coordinates

$$8\pi G\rho'(t') = 3f(H'(t'), \dot{H}'(t'), \dots, H^{(n)'}(t'), \dots), \quad (25)$$

and the corresponding equation for the pressure will be derived using the continuity equation. The linearized equations for the sufficiently smooth scalar perturbations in the Newtonian gauge (23) can then be derived with the help of (24). The result is

$$8\pi G\delta\rho(t) = \sum_{n=0}^{\infty} g_{|n}(H(t))\delta H^{(n)}(t), \quad (26)$$

$$\begin{aligned} -8\pi G\delta p &= \frac{1}{H} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} g_{|nm}(H)H^{n+1}\delta H^{(m)} \\ &+ \sum_{n=0}^{\infty} g_{|n}(H) \left( \frac{\delta H^{(n+1)}}{H} - \frac{H^{(n+1)}}{H^2}\delta H \right) \\ &+ 3 \sum_{n=0}^{\infty} g_{|n}(H)\delta H^{(n)}, \end{aligned} \quad (27)$$

where

$$g_{|n}(H) = \left[ \frac{\partial f}{\partial H^{(n)}} \right]_{H^{(n)}=H^{(n)}(t(H))}, \quad (28)$$

$$g_{|nm}(H) = \left[ \frac{\partial^2 f}{\partial H^{(n)}\partial H^{(m)}} \right]_{H^{(n)}=H^{(n)}(t(H))}, \quad (29)$$

and the time dependence of  $H$  is derived from the homogeneous evolution (3). This procedure fixes exactly the coefficients  $a_{0m}, b_{0m}, e_{0m}$  and  $f_{0m}$  in (17),(18),(19), i.e. all the terms with no spatial derivatives in the equations for the scalar perturbations, as functions of  $f(H, \dot{H}, \dots)$  and its partial derivatives.

We can now count the number of functional degrees of freedom in the superhorizon modes of the linearized theory. Let us assume that there is a maximum number  $D$  of time derivatives in the equations for the perturbations<sup>2</sup>. The number of derivatives, with the use of (24),(26),(27), fixes  $f$  to be a function of at most  $H^{(D-2)}$ . When particularized to a solution of the homogeneous Friedman equations,  $f$  and its first and second partial derivatives become functions of the Hubble rate  $H$ :  $g(H), g_{|n}(H)$  and  $g_{|nm}(H)$  respectively. That makes  $1 + (D-1) + D(D-1)/2 = D(D+1)/2$  functional degrees of freedom. The first derivative of  $g(H)$  can be written in terms of the  $g_{|n}(H)$ , and the first derivative of the later can be also written in terms of the  $g_{|nm}(H)$ . That makes  $1 + (D-1)$  conditions, so the result depends on  $D(D-1)/2$  independent functions of the Hubble parameter  $H$  in the terms without any spatial derivatives.

Notice however that if we restrict the number of derivatives appearing in the equations for the perturbations to two as in Eqs. (20),(21),(22), then Eqs. (24),(26),(27) tell us that we must consider just functions  $f(H, \dot{H}, \ddot{H}, \dots) = f(H) = g(H)$ , at least as an approximation at times  $H \ll H_+$  and modes  $k \ll H_+$ . Therefore the number of functional degrees of freedom in the terms with no spatial derivatives is just one: the homogeneous evolution  $f(H) = g(H)$ .

variance. Well inside the horizon these are the dominant terms appearing in the wave equation resulting when calculating the adiabatic perturbations in the radiation domination epoch.

<sup>2</sup> We are forgetting here the restriction on the number of derivatives required in order to reproduce the acoustic peaks of CMB.



Our assumption might be relaxed and we could impose that general relativity should be valid for all the modes  $H_- < k < k_{obs}$  that have been observed in the spectrum of CMB and matter perturbations. This could lead to new terms in the equations for the perturbations (suppressed not necessarily by the UV scale  $H_+$ ) which have been negligible for the observed modes but that could lead to ultraviolet deviations from the spectrum derived in the general relativistic cosmology which have not yet been observed. However, terms with more than two spatial derivatives will be very tightly constrained by solar system experiments.

In this article we will restrict ourselves to the system of second order differential equations (20), (21), (22). Let

us now derive the linearized equations for the rest of the modes under this assumption. In the  $(t', \mathbf{x})$  coordinate system, the metric is Friedman Robertson Walker, and we know that for this metric, we can use equation (3). Thus,

$$8\pi G\rho' = 3g(H'), \quad (30)$$

with

$$H' = \frac{1}{a'} \frac{da'}{dt'} = H(1 - \phi) - \dot{\psi} \quad (31)$$

at linear order in perturbations. Therefore, we can deduce from (3) that in the  $(t, \mathbf{x})$  coordinate system

$$8\pi G\delta\rho = -3g'(H) (H\phi + \dot{\psi}) + a_{10} \frac{\Delta}{a^2} \phi + b_{10} \frac{\Delta}{a^2} \psi, \quad (32)$$

$$8\pi G(\rho_{(0)} + p_{(0)})\partial_i\theta = \partial_i \sum_{m=0}^1 \left[ \frac{c_{0m}}{H^{m-1}} \partial_t^m \phi + \frac{d_{0m}}{H^{m-1}} \partial_t^m \psi \right], \quad (33)$$

$$\begin{aligned} 8\pi G(-\delta p \delta_j^i + \partial_i \partial_j \Pi) &= \delta_j^i \left[ g'(H) \left( 3H\phi + 3\dot{\psi} - \frac{\dot{H}}{H^2} \dot{\psi} + \frac{\dot{H}}{H} \phi + \dot{\phi} + \frac{1}{H} \ddot{\psi} \right) \right. \\ &\quad \left. + \frac{\dot{H}}{H} g''(H) (H\phi + \dot{\psi}) \right] \\ &\quad + \frac{1}{a^2} (\delta_j^i \Delta - \partial_i \partial_j) [g_{00}\phi + h_{00}\psi], \end{aligned} \quad (34)$$

and the twelve coefficients of terms with no spatial derivatives are fixed by the function  $g(H)$  which defines the homogeneous cosmological model.

The last requirement comes again from general covariance. If the tensor  $T_{\mu\nu}$  comes from the variation of a certain matter action  $S_m$  with respect to the inverse of the metric  $g^{\mu\nu}$ , and  $S_m$  is a scalar under general coordinate transformations, then  $T_\nu^\mu$  must be a 1-covariant, 1-contravariant divergenceless tensor, i.e.  $\nabla_\mu T_\nu^\mu = 0$ . As the stress-energy tensor is proportional to  $G_\nu^\mu$ , the later must also be divergenceless. In the linearized approximation  $G_\nu^\mu = G_{(0)\nu}^\mu + \delta G_\nu^\mu$  and the Christoffel symbols  $\Gamma_{\mu\nu}^\lambda = \Gamma_{(0)\mu\nu}^\lambda + \delta\Gamma_{\mu\nu}^\lambda$ , and we can write the linearized version of this requirement as

$$\begin{aligned} \delta G_{\mu|0}^0 + \delta G_{\mu|i}^i + \delta G_{\mu}^\lambda \Gamma_{(0)\lambda\nu}^\nu - \delta G_{\nu}^\lambda \Gamma_{(0)\lambda\mu}^\nu \\ + G_{(0)\mu}^\lambda \delta\Gamma_{\lambda\nu}^\nu - G_{(0)\nu}^\lambda \delta\Gamma_{\lambda\mu}^\nu = 0. \end{aligned} \quad (35)$$

Let us study this condition in order to see if we can further limit the number of independent coefficients of the equations for scalar perturbations. For  $\mu = i$  the term  $\delta G_{i|0}^0$  will give at most a term proportional to  $d_{01}\dot{\psi}_i$ , which will not be present in other terms except  $\delta G_{i|j}^j$ .

This term will give at most a term  $g'\dot{\psi}_i/H$  which fixes the exact value of  $d_{01}$  for all  $H \ll H_+$ . For the same reason, the term  $c_{01}\dot{\phi}_i$  in  $\delta G_{i|0}^0$  can not be canceled and the term  $c_{00}H\dot{\phi}_i$  can only be canceled by the term  $g'\dot{\phi}_i$  in  $\delta G_{i|j}^j$ , which means that  $c_{01} = 0$  and fixes the exact value of  $c_{00}$  for all  $H \ll H_+$ . The value of  $d_{00}$  can be fixed in terms of  $d_{01}$  and the terms proportional to  $\dot{\psi}$  in (34). This fixes (33) completely in terms of the function describing the homogeneous evolution,  $g(H)$ :

$$8\pi G(\rho_0 + p_0)\partial_i\bar{\theta} = \frac{g'(H)}{H} \partial_i (H\Phi + \dot{\Psi}). \quad (36)$$

In the previous equation and from now on we will refer directly to the gauge invariant counterparts of the variables in the Newtonian gauge, which we will denote by  $\Phi, \Psi, \bar{\theta}, \bar{\delta}\rho$  and  $\bar{\delta}p$  ( $\Pi$  is already gauge invariant). Now we can use the divergenceless condition for  $\mu = 0$ . The term  $\delta G_{0|0}^0$  will give at most a term proportional to  $a_{10}\Delta\dot{\phi}$  and a term proportional to  $b_{10}\Delta\dot{\psi}$ , which will not be present in other terms except  $\delta G_{0|j}^j$ . This term will give at most a term proportional to  $\Delta\phi$  and a term proportional to  $\Delta\dot{\psi}$ .

Therefore  $a_{10} = 0$  and the value of  $b_{10}$  is set completely in terms of  $g(H)$ :

$$8\pi G\bar{\delta\rho} = -3g'(H)(H\Phi + \dot{\Psi}) + \frac{g'(H)}{a^2 H} \Delta\Psi. \quad (37)$$

$$8\pi G(\bar{\delta\rho} \delta_{ij} + \partial_i \partial_j \bar{\Pi}) = \delta_{ij} \left\{ g'(H) \left[ 3H\Phi + 3\dot{\Psi} - \frac{\dot{H}}{H^2} \dot{\Psi} + \frac{\dot{H}}{H} \Phi + \dot{\Phi} + \frac{1}{H} \ddot{\Psi} \right] + \frac{\dot{H}}{H} g''(H) [H\Phi + \dot{\Psi}] \right\} + \frac{1}{a^2} (\delta_{ij} \Delta - \partial_i \partial_j) \left( \frac{g'(H)}{2H} (\Phi - \Psi) + \frac{\dot{H}}{2H^3} (g'(H) - Hg''(H)) \Psi \right), \quad (38)$$

and we have finally a set of equations for the scalar perturbations (37), (36), (38) which are determined by the function  $g(H)$  which defined the ACM in the homogeneous approximation.

Had we relaxed our assumption, we could work with the system of equations (13),(14),(15) restricted to a maximum number of derivatives  $D$  as an approximation. Together with the condition  $G_{\nu;\mu}^\mu = 0$ , the system of equations for the perturbations define a set of coupled differential equations for the coefficients of the terms with at least one spatial derivative. The set of equations coming from  $G_{i;\mu}^\mu = 0$  define the terms proportional to  $e_{mn}$  and  $f_{mn}$  in (15) as a function of the terms in (14). The set of equations coming from  $G_{0;\mu}^\mu$  define the rest of the terms in (15) as a function of the terms in (14) and (13). Therefore, the only freedom, for what concerns linearized perturbations, is that of choosing the set of functions  $\{a_{nm}, b_{nm}, c_{nm}, d_{nm}\}$ , with some of them fixed by the homogeneous dynamics (1). An interesting case is found if  $D = 4$  is imposed. This includes a description of  $f(R)$ -theories [10], bigravity theories [20], and other of the most studied modified gravity theories. We will study further this case in the subsection devoted to the comparison of the model with  $f(R)$  theories.

The rhs of the equations derived are the equivalent to the components of the linearized Einstein tensor derived in a general covariant theory whose Friedman equation is exactly (3). Aside from General Relativity with or without a cosmological constant, i.e. for  $\alpha_- \neq 1, 0$ , it will be necessary to build an action depending on arbitrarily high derivatives of the metric in order to derive a theory such that both the equations in the homogeneous approximation and the linearized equations for the scalar perturbations are of finite order.

Noticeably, the nonlinear equations for the perturbations will include derivatives of arbitrary order of the metric perturbations, with the exceptional case of General Relativity with a cosmological constant. This makes problematic the consistency of the ACM beyond the linear approximation. This issue will be further studied in a future work.

The complete knowledge of (37) and (36) when  $H \ll H_+$  gives a complete knowledge of (34) in terms of  $g(H)$  when  $H \ll H_+$ :

To summarize, what we have shown in this section is that the behavior of scalar perturbations in a general covariant theory is intimately connected to the background evolution. Except in the very early universe, the linearized equations for the scalar perturbations are determined by the equations in the homogeneous approximation if one assumes that there are no terms with more than two spatial derivatives. This assumption could be relaxed in order to include more general theories.

There will be a subset of general covariant theories with a background evolution given by (3) that verify these equations for the perturbations. We may be able to distinguish among these at the linearized level by means of the vector and tensor perturbations.

## HYDRODYNAMICAL PERTURBATIONS

Vector perturbations represent rotational flows which decay very quickly in the General Relativistic theory. As we expect a small modification of the behavior of perturbations just in the vicinity of the lower bound  $H_-$ , we will assume that the vector perturbations have decayed to negligible values when the scale  $H_-$  begins to play a role and therefore they will be ignored.

Tensor perturbations correspond to gravitational waves, which at the present have been not observed, their effect being far beyond the resolution of current observations.

Present measurements restrict their attention to perturbations in the photon sector (CMB) and the matter sector (matter power spectrum, both dark and baryonic). These observations can be computed taking into account only the effect of scalar perturbations. Therefore, we can use the result of the previous section to study the deviations from  $\Lambda$ CDM in the spectrum of CMB. A comprehensive study of the matter power spectrum predicted by the ACM would require a detailed knowledge of the nonlinear regime, which we lack at present.

Let us start by deriving the behavior of scalar perturbations in a universe whose gravitation is described by

the ACM and filled with a perfect fluid ( $\Pi = 0$ ) in (12). Therefore, (38) with  $i \neq j$  fixes  $\Phi$  as a function of  $\Psi$ ,

$$\Phi = \left\{ 1 + \frac{\dot{H}}{2H^2} \left( 1 - \frac{Hg''(H)}{g'(H)} \right) \right\} \Psi. \quad (39)$$

Then, this expression can be used to substitute  $\Phi$  in the remaining three differential equations for  $\delta\rho$ ,  $\delta p$  and  $\theta$ . Given the equation of state of the fluid  $p = p(\rho, S)$ , with  $S$  the entropy density, the perturbation of the pressure can be written as

$$\delta p = c_s^2 \delta\rho + \tau \delta S, \quad (40)$$

where  $c_s^2 \equiv (\partial p / \partial \rho)_S$  is the square of the speed of sound in the fluid and  $\tau \equiv (\partial p / \partial S)_\rho$ . Substituting the pressure and the energy density perturbations for the corresponding functions of  $\Psi$  and its derivatives into (40), we arrive at the equation for the entropy density perturbation  $\delta S$ . The perturbations we are interested in are adiabatic, i.e.:  $\delta S = 0$  and therefore, the equation for the entropy perturbations turns into an equation of motion of the adiabatic perturbations of the metric. If we want to consider the effect of the lower bound of the Hubble rate,  $H_-$ , on the evolution of perturbations, we must consider a matter dominated universe ( $c_s^2 = 0$ ),

$$g'(H) \left[ 3H\Phi + 3\dot{\Psi} - \frac{\dot{H}}{H^2} \dot{\Psi} + \frac{\dot{H}}{H} \Phi + \dot{\Phi} + \frac{1}{H} \ddot{\Psi} \right] + \frac{\dot{H}}{H} g''(H) \left[ H\Phi + \dot{\Psi} \right] = 0. \quad (41)$$

This equation is a second order linear differential equation with non-constant coefficients. Thus it is useful to work with the Fourier components of the metric perturbation  $\Psi_k$ . The equation of motion for the Fourier components is just a second order linear ODE with non-constant coefficients, which will be analytically solvable just for some simple choices of  $g(H)$ . However, in general, it will be mandatory to perform a numerical analysis.

## THE CMB SPECTRUM IN ACM VERSUS $\Lambda$ CDM

The main observation that can be confronted with the predictions of the theory of cosmological perturbations at the linearized level is the spectrum of temperature fluctuations of the Cosmic Microwave Background (CMB), from which WMAP has recorded accurate measurements for five years [21].

The temperature fluctuations  $\frac{\delta T}{T}$  are connected to the metric perturbations via the Sachs-Wolfe effect [22], which states that, along the geodesic of a light ray  $\frac{dx^i}{dt} = l^i(1 + \Phi + \Psi)$ , characterized by the unit three-vector  $l^i$ , the temperature fluctuations evolve according to

$$\left( \frac{\partial}{\partial t} + \frac{l^i}{a} \frac{\partial}{\partial x^i} \right) \left( \frac{\delta T}{T} + \Phi \right) = \frac{\partial}{\partial t} (\Psi + \Phi). \quad (42)$$

Neglecting a local monopole and dipole contribution, taking recombination to be instantaneous at a certain time and assuming that the reionization optical depth is negligible, the present temperature fluctuation of a distribution of photons coming from a given direction of the sky  $l^i$  can be related to the temperature perturbation and metric perturbation  $\Psi$  in the last scattering surface plus a line integral along the geodesic of the photons of the derivatives of  $\Phi$  and  $\Psi$  (Integrated Sachs Wolfe effect, ISW).

These relations are purely kinematical and remain unchanged in the ACM. In order to take into account all the effects involved in the calculation of the CMB spectrum the code of CAMB [23] has been adapted to the ACM. (see the appendix for details).

We have compared the late time evolution of the perturbations described by the ACM (6) for several values of  $\alpha_-$  (taking into account that  $\alpha_- = 1$  corresponds to  $\Lambda$ CDM) for constant  $H_0$  and  $\Omega_m$  defined in Eq. (10). We have assumed a Harrison-Zel'dovich scale invariant spectrum of scalar perturbations as initial condition ( $n_s = 0$ ) in the region in which the universe is radiation dominated but  $H \ll H_+$ . Thus the study of the effect of the scale  $H_+$  is postponed. We have taken  $H_0 = 72 \pm 8 \text{ km/sMpc}$  from the results of the Hubble Key Project [24] and  $\Omega_m = 0.26 \pm 4$  from our previous analysis of the evolution at the homogeneous level. The effect of ACM as compared to  $\Lambda$ CDM is twofold (FIG. 5).

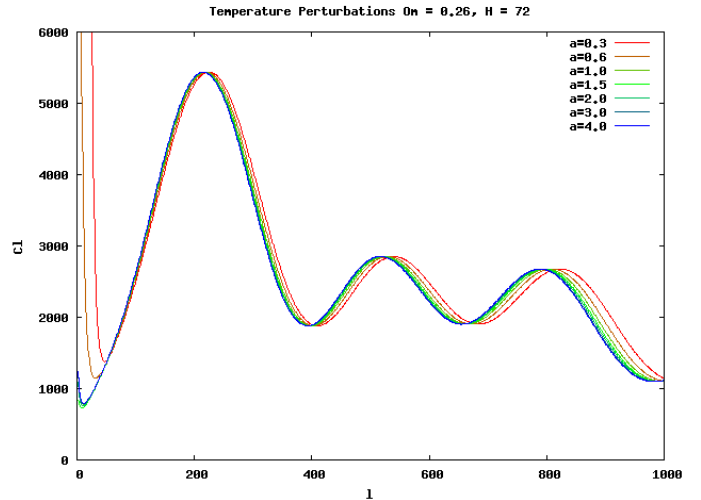


FIG. 5: Spectrum of temperature fluctuations of the Cosmic Microwave Background for  $\Omega_m = 0.26$ ,  $H_0 = 72 \text{ km/sMpc}$  and varying  $\alpha_- = 0.3$  (red), 0.6, 1.0, 1.5, 2.0, 3.0, 4.0 (blue). The lowest values of  $\alpha_-$  show extreme ISW effects and therefore will be ruled out also by the CMB spectrum. Unfortunately, the cosmic variance masks the effect of  $1.0 < \alpha_- < 4.0$  at large angular scales and therefore we can obtain little information about the ACM from the late ISW effect.

There is a shift in the peak positions due to the difference in the distance to the last scattering surface for

different values of  $\alpha_-$ . This shift increases with the peak number. In particular, more precise measurements of the position of the third peak could be used to estimate the value of  $\alpha_-$ .

There is also an increase of the lower multipoles due to the late Integrated Sachs-Wolfe effect, which is particularly extreme for  $\alpha_- \lesssim 1.0$  (which corresponds to  $H_- \simeq H_0$  for constant  $\Omega_m$ ). This large deviation is clearly not present in the experimental data. The much smaller deviation for  $\alpha_- \gtrsim 1.0$  is within the error bands due to the cosmic variance, and therefore the value of the lower multipoles cannot be used to exclude values of  $\alpha_-$  greater than but of the order of one.

## COMPARISON WITH THE TREATMENT OF PERTURBATIONS IN OTHER MODELS

### Perturbations in $f(R)$ theories

In our previous work [13] we found that given an expansion history parameterized by a modified Friedman equation in a universe filled by a given component, it

was always possible to find a biparametric family of  $f(R)$  theories (see Ref. [27] for a review) which had the same homogeneous evolution as a solution of their equations of motion. We wonder now if these theories have also the same linearized equations for the scalar perturbations of the metric.

An  $f(R)$  theory is defined by its action and therefore the equations for the perturbations are uniquely determined. The action is given by

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} [f(R) + 16\pi G L_m], \quad (43)$$

where  $R$  is the curvature scalar and  $f(R)$  is an arbitrary function of this scalar. The equations governing the evolution of the perturbations will be derived from the Einstein equations of the  $f(R)$  theory,

$$f'(R)R_\nu^\mu - \frac{1}{2}f(R)\delta_\nu^\mu + (\delta_\nu^\mu \square - \nabla^\mu \nabla_\nu) f'(R) = 8\pi G T_\nu^\mu. \quad (44)$$

At the homogeneous level, it is possible to write the Friedman equations of the  $f(R)$  theory,

$$8\pi G \rho_{(0)} = -3(\dot{H} + H^2)f'(R_{(0)}) - \frac{1}{2}f(R_{(0)}) - 3Hf''(R_{(0)})\dot{R}_{(0)}, \quad (45)$$

$$-8\pi G p_{(0)} = -(\dot{H} + 3H^2)f'(R_{(0)}) - \frac{1}{2}f(R_{(0)}) + f^{(3)}(R_{(0)})\dot{R}_{(0)}^2 + f''(R_{(0)})\ddot{R}_{(0)}, \quad (46)$$

where

$$R_{(0)}(t) = -6(\dot{H} + 2H^2) \quad (47)$$

is the curvature scalar of the Friedman-Robertson-Walker (FRW) metric (zeroth order in perturbations). The solution of the system of equations (45),(46) together with the equation of state of the dominant component of the stress-energy tensor, defines  $R_{(0)}$  and its derivatives as functions of the Hubble parameter  $H$  [13].

If we expand (44) in powers of the scalar perturbations of the FRW metric (11), the first order term gives the linearized equations for the perturbations. This will be a system of differential equations of fourth order, but it will be possible to turn it into a system of differential equations of second order, as we will see below.

If the stress-energy tensor has no shear, the Einstein equation for  $\mu = i \neq \nu = j$  reduces to

$$f''(R_{(0)})\delta R + f'(R_{(0)})(\Phi - \Psi) = 0, \quad (48)$$

where  $\delta R$  is the perturbation of the curvature scalar

$$\delta R = 12(\dot{H} + 2H^2)\Phi + 6H\dot{\Phi} + 24H\dot{\Psi} + 6\ddot{\Psi} + \frac{2}{a^2}\Delta(\Phi - 2\Psi). \quad (49)$$

Eq. (48) is a second order differential equation for the scalar perturbations except in the case of General Relativity ( $f'' = 0$ ) where it becomes an algebraic equation ( $\Phi = \Psi$ ). Eq. (48) can be used to turn the remaining components of the equations of  $f(R)$  theories (44) into second order equations.

In the case of adiabatic perturbations in a matter dominated epoch one has

$$\begin{aligned}
-8\pi G\bar{\delta p} &= f'(R_{(0)})(a^{-2}\Delta(\Phi - 2\Psi) + \ddot{\Psi} + 6H\dot{\Psi} + H\dot{\Phi} + 2(\dot{H} + 3H^2)\Phi - \frac{1}{2}\delta R) \\
&+ f''(R_{(0)})(-\dot{H} + 3H^2)\delta R - 2\dot{R}_{(0)}\dot{\Psi} - 8H\dot{R}_{(0)}\Psi - 4H\dot{R}_{(0)}\Phi - 2\dot{R}_{(0)}\Phi - 2f^{(3)}(R_{(0)})\dot{R}_{(0)}^2\Phi \\
&- \partial_t^2 [f'(R_{(0)})(\Phi - \Psi)] - 2H\partial_t [f'(R_{(0)})(\Phi - \Psi)] = 0.
\end{aligned} \tag{50}$$

If we compare the equations governing the behavior of adiabatic perturbations in a  $f(R)$  theory (48), (50) with those coming from the ACM, (39),(41), we find that they describe very different behaviors. In one case we have a system of two coupled second order differential equations for  $\Phi$  and  $\Psi$ . In the ACM case we got a single second order differential equation and an algebraic equation between the two gravitational potentials. For a given  $f(R)$  theory with an associated  $g(H)$  homogeneous behavior, we find that the behavior of linearized perturbations differs from the one defined by the linearized perturbations in the ACM. This means that  $f(R)$  theories break in general our assumption in the third section (there are terms with more than two derivatives in the equations for the perturbations, and therefore the behavior of perturbations in General Relativity is not recovered when  $H_+ \gg H \gg H_-$  for all modes).

Let us see this in a simple example. We will choose the easiest biparametric family of  $f(R)$  theories: the one that, under matter domination, gives the same background evolution as General Relativity without a cosmological constant [13],

$$f(R) = R + c_1|R|^{\frac{1}{12}(7-\sqrt{73})} + c_2|R|^{\frac{1}{12}(7+\sqrt{73})}. \tag{51}$$

These kind of models were introduced in Ref. [28].

Let us first check if the solutions of the equations for the perturbations in General Relativity in the presence of pressureless matter,

$$\begin{aligned}
\Psi = \Phi &= const, \\
\Psi = \Phi &\propto t^{-5/3},
\end{aligned} \tag{52}$$

are solutions of the equations of motion in the case of the biparametric family of  $f(R)$  theories (51). The easiest is to verify if Eq. (44) with  $\mu = i \neq \nu = j$ ,

$$\begin{aligned}
2f''(R_{(0)})(3H^2\Phi + 3H\dot{\Phi} + 12H\dot{\Psi} + 3\ddot{\Psi} + \frac{1}{a^2}\Delta(\Phi - 2\Psi)) \\
+ f'(R_{(0)})(\Phi - \Psi) = 0,
\end{aligned} \tag{53}$$

where  $R_{(0)}(H) = -3H^2$  is the value of the homogeneous curvature scalar as a function of  $H$  in this family of theories, is also fulfilled by the solutions (52). The result is obviously not.

The second order differential equations for the perturbations in the  $f(R)$  theories (51) in the presence of matter can be found by substituting (53) and its derivatives into

the other equations of the system (44). The equation for the pressure perturbations then gives

$$\begin{aligned}
-f'(R_{(0)})[\ddot{\Psi} + \ddot{\Phi} + 4H\dot{\Psi} + 4H\dot{\Phi} + 3H^2\Phi] \\
-27H^3f''(R_{(0)})\dot{\Phi} + \frac{f(R_{(0)})}{2}(\Psi - 3\Phi) = 8\pi G\bar{\delta p} = 0.
\end{aligned} \tag{54}$$

It is obvious that the  $i \neq j$  equation of the cosmologic perturbations in General Relativity,

$$\Phi - \Psi = 0 \tag{55}$$

is not recovered from (48) in the  $H \gg H_-$  limit for all  $k \ll H_+$ . In fact the deviation would be significant for modes with  $k \gtrsim Ha \left(\frac{H}{H_-}\right)^{\frac{1}{12}(7-\sqrt{73})} \sim Ha$ . Therefore, the assumption we have made in order to derive the equations for the perturbations in the ACM is broken.

It is a known problem that  $f(R)$  theories are unable to pass cosmological and astrophysical tests involving perturbations [29], unless the function involved is properly fine-tuned. In particular, if the conformal equivalence between  $f(R)$  theories and scalar-tensor theories is used, it is necessary that the effective mass acquired by the new scalar degree of freedom is unnaturally large [30]. Some  $f(R)$  theories that pass cosmological and solar system tests have been proposed [31–33]. However, it is also subject of debate if the conformal equivalence can be used in order to extract physical predictions from this models, especially predictions which involve perturbations [34]. A mathematically rigorous treatment of perturbations in  $f(R)$ -gravity can be found in Ref. [35].

## Quintessence Models

Another class of models used to describe the acceleration of the universe are those in which a so called quintessence field, typically of scalar type  $\varphi$ , is added as a component of the universe [36]. The only functional degree of freedom in most models is just the scalar potential  $V(\varphi)$ , which can be tuned to fit the homogeneous expansion of the universe. In our previous work [13] we found the correspondence between a given homogeneous evolution parameterized by  $g(H)$  and the potential for the quintessence which drives under General Relativity this evolution. We now wonder if the behavior of perturbations in quintessence models also resembles the behavior in ACM.

The field action is given by



$$S_\varphi = \int d^4x \sqrt{-g} [g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi)]. \quad (56)$$

Let us assume that the field can be split in two components: one which is only time dependent and which drives the homogeneous evolution of the universe,  $\varphi_{(0)}(t)$ , and a small perturbation which is inhomogeneous,  $\delta\varphi(\mathbf{x}, t)$ . We will consider linearized perturbations of the metric, the field, and the other components of the universe. The resulting linearized stress-energy tensor of the field is

$$\delta T_{00}^\varphi = \dot{\varphi}_{(0)} \delta \dot{\varphi} - \Phi V(\varphi_{(0)}) + \frac{1}{2} V'(\varphi_{(0)}) \delta \varphi, \quad (57)$$

$$\delta T_{0i}^\varphi = \dot{\varphi}_{(0)} \partial_i \delta \varphi, \quad (58)$$

$$\delta T_{ij}^\varphi = a^{-2} \delta_{ij} \left[ \dot{\varphi}_{(0)}^2 (\Phi - \Psi) + \dot{\varphi}_{(0)} \delta \dot{\varphi} + \Psi V(\varphi_{(0)}) - \frac{1}{2} V'(\varphi_{(0)}) \delta \varphi \right]. \quad (59)$$

By virtue of the Einstein's equations, it is always possible to turn a modification in the Einstein tensor into a new component of the stress-energy momentum tensor of the sources of the gravitational field. We wonder if it is possible to account for the modification of the effective Einstein tensor described in the third section with a new component described by this quintessence field. However, it is not hard to see that the effective Einstein tensor that we are proposing has a modified  $i \neq j$  component, while the  $i \neq j$  component of the stress-energy tensor of the quintessence field  $\varphi$  is zero. Therefore, it is not possible to describe the evolution of perturbations in the ACM as driven by an effective scalar field component (56).

### Dark Fluid Models

A deformation of the gravitational physics can be also made equivalent to the addition of a non-standard fluid component to the cosmic pie at the homogeneous level. The fluid component used to explain the present accelerated expansion of the universe is typically taken to be a perfect fluid with large negative pressure.

One of the most popular parameterizations of this fluid is the so called equation of state  $\omega = p/\rho$  [37]. The simplest models are  $\omega = -1$ , which is equivalent to a cosmological constant, or a constant  $\omega$ , but recently a possible time dependence of  $\omega$  has been considered [38]. Needless to say, these *ad hoc* parameterizations are subsets of the more general equation of state of a perfect fluid,  $p = p(\rho, s)$ , where  $s$  is the entropy density. It is also questionable why must we restrict ourselves to perfect fluids and not include a possible anisotropic stress tensor, possibly depending on the energy and entropy densities [39].

As in the previous subsection, it is always possible to use the Einstein's equations to turn a modification in the

gravitational physics into a new fluid component of the universe. In the case of the equations derived in the third section, the equivalent Dark Energy Fluid would have the following properties:

$$8\pi G \bar{\delta} \rho_X = (2H - g') \left[ \frac{1}{a^2 H} \Delta \Psi - 3(H\Phi + \dot{\Psi}) \right], \quad (60)$$

$$8\pi G \bar{\theta}_X = \frac{(2H - g')}{H} (H\Phi + \dot{\Psi}), \quad (61)$$

$$8\pi G \Pi_X = \frac{1}{a^2} \left\{ \frac{2H - g'}{2H} (\Phi - \Psi) - \frac{\dot{H}}{2H^3} (g' - Hg'') \Psi \right\}, \quad (62)$$

$$8\pi G \bar{\delta} p_X = (2H - g') (3H\Phi + 3\dot{\Psi} - \frac{\dot{H}}{H^2} \dot{\Psi} + \frac{\dot{H}}{H} \Phi + \dot{\Phi} + \frac{1}{H} \ddot{\Psi}) + \frac{\dot{H}}{H} (2 - g'') (H\Phi + \dot{\Psi}) - 8\pi G \Delta \Pi_X. \quad (63)$$

The resulting fluid is not an ideal fluid ( $\Pi_X = 0$ ) or even a Newtonian fluid ( $\Pi_X \propto \bar{\theta}_X$ ). With the use of (3), (39) and (41), it will be possible to write  $\bar{\delta} \rho_X$  and  $\bar{\delta} p_X$  as a combination of  $\bar{\theta}_X$ ,  $\Pi_X$  and their derivatives,

$$\bar{\delta} \rho_X = f_1(\rho_{(0)X}) \Delta \Pi_X + f_2(\rho_{(0)X}) \bar{\theta}_X, \quad (64)$$

$$\bar{\delta} p_X = f_3(\rho_{(0)X}) \Pi_X - \Delta \Pi_X + f_4(\rho_{(0)X}) \bar{\theta}_X. \quad (65)$$

These very non-standard properties show that, although it is formally possible to find a fluid whose consequences mimic the ones of such a modification of the gravitational physics, this fluid is very exotic.

### SUMMARY AND CONCLUSIONS

In view of recent data, an updated comparison of cosmological observations with a phenomenological model proposed in a recent work has been presented.

An extension of this phenomenological model (ACM) beyond the homogeneous approximation has been introduced allowing us to describe the evolution of scalar perturbations at the linear level.

A comparison with the spectrum of thermal fluctuations in CMB has been used to explore the possibility to determine the parameters of the ACM through its role in the evolution of scalar perturbations. The results of this comparison does not further restrict the parameters of the model, due to the masking of the associated late ISW effect by the cosmic variance. However, better measurements of the position of the third acoustic peak should improve the constraints significantly.

It has been shown that the equivalence of different formulations of the accelerated expansion of the universe in the homogeneous approximation is lost when one considers inhomogeneities. In particular we have shown that the general structure of the evolution equations for scalar perturbations in the ACM differs from the structure of



the equations corresponding to modified  $f(R)$  theories of gravity, to quintessence models or to a dark fluid with standard properties.

The possibility of going beyond the linearized approximation for the scalar perturbations and to consider vector and tensor perturbations will be the subject of a future work.

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### Appendix: Covariant Perturbation Equations for the ACM

The code of CAMB [23] makes use of the equations for the perturbations of the metric in the covariant approach. The quantities can be computed in a given “frame”, labeled by a 4-velocity  $u^\mu$ . In particular, CAMB uses the dark matter frame, in which the velocity of the dark matter component is zero (the dark matter frame). Furthermore, it parameterizes the time evolution with the conformal time  $a(\tau)d\tau = dt$ .

In order to apply CAMB to the ACM model it is nec-

essary to identify frame invariant quantities, and then relate them to gauge invariant quantities [25]. The following comoving frame quantities are used in the CAMB code:  $\eta$  (the curvature perturbation),  $\sigma$  (the shear scalar),  $z$  (the expansion rate perturbation),  $A$  (the acceleration,  $A = 0$  in the dark matter frame),  $\phi$  (the Weyl tensor perturbation),  $\chi^{(i)}$  (the energy density perturbation of the species  $i$ ),  $q^{(i)}$  (the heat flux of the species  $i$ ), and  $\Pi^{(i)}$  (the anisotropic stress of the species  $i$ ). All of these quantities are defined in [26]. These variables are related to the gauge invariant variables via the following dictionary,

$$-\frac{\eta}{2} - \frac{\mathcal{H}\sigma}{k} \equiv \Psi, \quad (66)$$

$$-A + \frac{\sigma' + \mathcal{H}\sigma}{k} \equiv \Phi, \quad (67)$$

$$\chi + \frac{\rho'\sigma}{k} \equiv \bar{\delta}\rho, \quad (68)$$

$$\rho q + (\rho + p)\sigma \equiv \frac{k}{a}(\rho + p)\bar{\theta}, \quad (69)$$

where prime denotes derivatives with respect to conformal time, except when acting on  $g(H)$  where it denotes a derivative with respect to the Hubble rate  $H$ , and  $\mathcal{H} = a'/a = aH$ . On the other hand  $z = \sigma + \frac{3}{2k}(\eta' + 2\mathcal{H}A)$  and  $\phi = (\Phi + \Psi)/2$ . The anisotropic stress  $\Pi$  is already frame invariant.

Written in the dark matter frame, the equations for the scalar perturbations read

$$\begin{aligned} g' \left( \frac{k^2 \eta}{2\mathcal{H}} + kz \right) &= 8\pi G a \Sigma_i \chi^{(i)}, \\ \frac{g' k^2}{3\mathcal{H}} (\sigma - z) - \frac{g' k}{2\mathcal{H}} \eta' &= 8\pi G a \Sigma_i \rho_i q^{(i)}, \\ -\mathcal{H} k g' (z' + \mathcal{H}z) &= 8\pi G a \Sigma_i \left[ \mathcal{H}^2 (1 + 3c_s^{(i)2}) - (\mathcal{H}' - \mathcal{H})^2 (1 - \frac{\mathcal{H}g''}{ag'}) \right] \chi^{(i)}, \\ \frac{g'}{\mathcal{H}} \left( \frac{\sigma' + \mathcal{H}\sigma}{k} - \phi \right) - \frac{\mathcal{H}' - \mathcal{H}^2}{2\mathcal{H}^3} (g' - \mathcal{H}g''/a) \left( \frac{\eta}{2} + \frac{\mathcal{H}\sigma}{k} \right) &= -8\pi G a \Sigma_i \Pi^{(i)} / k^2. \end{aligned} \quad (70)$$

The following combination of the constraint equations is also useful:

$$k^2 \phi = -\frac{8\pi G a \mathcal{H}}{g'} \Sigma_i \left[ \Pi^{(i)} + \left( 1 - \frac{\mathcal{H}' - \mathcal{H}^2}{2\mathcal{H}^2} \left( 1 - \frac{\mathcal{H}g''}{ag'} \right) \right) (\chi^{(i)} + 3\mathcal{H}\rho_i q^{(i)} / k) \right]. \quad (71)$$

These equations are plugged into the Maple files provided with the CAMB code and run to get the ISW effect.

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