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# Lipschitz stability in an inverse problem for the Kuramoto-Sivashinsky equation

Lucie Baudouin\*, Eduardo Cerpa†, Emmanuelle Crépeau‡ and Alberto Mercado†

## Abstract

This paper presents an inverse problem for the Kuramoto-Sivashinsky (K-S) equation. The problem of retrieving the anti-diffusion coefficient from a measurement of the solution is discussed. This measurement consists of the solution at some positive time and partial boundary data. Uniqueness and Lipschitz stability for this inverse problem are proven with the Bukhgeim-Klibanov method. The proof is based on a global Carleman inequality for the linearized K-S equation.

**Keywords:** Inverse problem, Kuramoto-Sivashinsky equation, Carleman estimate

**AMS subject classifications:** 35R30, 35K55.

## 1 Introduction

This paper considers an inverse problem that consists of the determination of a coefficient of a partial differential equation from the partial knowledge of a given single solution of the equation. For the solution of this class of problems (single-measurement inverse problems), the Bukhgeim-Klibanov method was introduced in [6] (see also [11, 12]). This method, which is based on Carleman estimates, allows uniqueness to be obtained, i.e., to prove that each measurement corresponds to only one coefficient. Regarding the continuity of this inverse problem, the first Lipschitz stability result for a multidimensional wave equation was obtained by Puel and Yamamoto [13] by using a method based on [6]. Since then, this method has been applied to other inverse problems including Lipschitz stability for the Schrödinger equation [1] and Logarithmic stability for the wave equation [2, 3].

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This approach was extended to a parabolic equation in [10]. Since then, this type of inverse problems for parabolic equations has received a large amount of attention. The primary difference with respect to hyperbolic inverse problems is that parabolic problems are not time-reversible: therefore, an additional measurement must be added if that method is applied. Knowledge of the full-state of the solution for some positive time is required (see the discussion in the introduction of [10]). To prove the Lipschitz stability without this assumption, which is usually needed when global Carleman inequalities are used, is still an open problem.

Recent results regarding linear parabolic problems can be found in [4] (discontinuous coefficient), [8] (systems), and the references therein. In [5, 9, 14], nonlinear parabolic equations were considered.

In this paper, the Kuramoto-Sivashinsky (K-S) equation is considered, which is a nonlinear fourth-order parabolic equation. This equation is used to model the physical phenomena of plane flame propagation: it describes the combined influence of diffusion and thermal conduction of gas on the stability of a plane flame front. In this nonlinear partial differential equation, the fourth-order term models the diffusion, and the second-order term models the incipient instabilities. To the knowledge of the authors there are no results in the literature concerning the determination of coefficients for this nonlinear equation. However, a Carleman estimate has been used to obtain the null-controllability of the K-S equation in reference [7] for the constant coefficient case.

In this paper, the inverse problem of retrieving the anti-diffusion coefficient  $\gamma$  from boundary measurements of the solution is considered. Since the linearized equation is parabolic, boundary measurements are not sufficient and we must consider an additional measurement of the full solution for a given time  $T_0$  (as in [4, 10] among others).

The K-S equation with non-constant coefficients describing the diffusion  $\sigma = \sigma(x)$ , and the anti-diffusion  $\gamma = \gamma(x)$ , is given as

$$\begin{cases} y_t + (\sigma(x)y_{xx})_{xx} + \gamma(x)y_{xx} + yy_x = g, & \forall (t, x) \in Q, \\ y(t, 0) = 0, \quad y(t, 1) = 0, & \forall t \in (0, T), \\ y_x(t, 0) = 0, \quad y_x(t, 1) = 0, & \forall t \in (0, T), \\ y(0, x) = y_0(x), & \forall x \in (0, 1), \end{cases} \quad (1)$$

where  $Q := (0, T) \times (0, 1)$ ,  $\sigma : [0, 1] \rightarrow \mathbb{R}_+^*$ , and the functions  $y_0, g$  are the initial data and the source term respectively. All of these terms are assumed to be known.

The first result involves the local well-posedness of the nonlinear equation (1). A less regular framework can be used for this equation. However, the method applied in this paper requires the solution and its time-derivative to be at least in  $L^2(0, T; H^4(0, 1))$ . Let us

introduce the following notations for the spaces appearing in this paper:

$$\begin{aligned}\mathcal{Y}_k &:= C([0, T]; H^k(0, 1)) \cap L^2(0, T; H^{k+2}), \quad \text{for } k = 0, 1, 2, \dots \\ \mathcal{F} &:= \{f \in L^2(0, T; H^4(0, 1)) / f_t \in L^2(0, T; L^2(0, 1))\} \\ \mathcal{Z} &:= \{z \in \mathcal{Y}_6 / z_t \in \mathcal{Y}_2\}\end{aligned}\tag{2}$$

**Theorem 1.1** *Let  $\gamma \in H^4(0, 1)$  and  $\sigma \in H^4([0, 1])$  be such that*

$$\forall x \in (0, 1), \sigma(x) \geq \sigma_0 > 0.\tag{3}$$

*Let  $y_0 \in H^6(0, 1)$  and  $g \in \mathcal{F}$ . There exists  $\varepsilon > 0$  such that if*

$$\|y_0\|_{H^6(0,1)} \leq \varepsilon, \quad \|g\|_{\mathcal{F}} \leq \varepsilon,\tag{4}$$

*then the K-S equation (1) has a unique solution  $y \in \mathcal{Z}$ .*

Once the existence of solutions to the K-S equation has been established, the following inverse problem is addressed:

Is it possible to retrieve the anti-diffusion coefficient  $\gamma = \gamma(x)$  from the measurement of  $y_{xx}(t, 0)$  and  $y_{xxx}(t, 0)$  on  $(0, T)$  and from the measurement of  $y(T_0, x)$  on  $(0, 1)$ , where  $y$  is the solution to Equation (1) and  $T_0 \in (0, T)$ ?

A local answer for this nonlinear inverse problem is given. To be more specific, let  $\tilde{\gamma}$  be fixed. We denote by  $\tilde{y}$  as the solution to Equation (1) with  $\gamma$  replaced by  $\tilde{\gamma}$ . This paper focuses on the following two questions.

**Uniqueness:** Do the equalities of the measurements  $\tilde{y}_{xx}(t, 0) = y_{xx}(t, 0)$  and  $\tilde{y}_{xxx}(t, 0) = y_{xxx}(t, 0)$  for  $t \in (0, T)$  and  $\tilde{y}(T_0, x) = y(T_0, x)$  for  $x \in (0, 1)$  imply  $\tilde{\gamma} = \gamma$  on  $(0, 1)$ ?

**Stability:** Is it possible to estimate  $\|\tilde{\gamma} - \gamma\|_{L^2(0,1)}$  by suitable norms  $\|\tilde{y}(T_0, x) - y(T_0, x)\|$  in space and  $\|\tilde{y}_{xx}(t, 0) - y_{xx}(t, 0)\|, \|\tilde{y}_{xxx}(t, 0) - y_{xxx}(t, 0)\|$  in time?

To answer these questions, the Bukhgeim-Klibanov method is used. First, a global Carleman estimate for the linearized K-S equation with non-constant coefficients is obtained. It is then used to prove the primary result which can be stated as follows.

**Theorem 1.2** *Let  $\mathcal{U}_1$  and  $\mathcal{U}_2$  be two bounded subsets of  $L^\infty(0, 1)$  and  $L^\infty(0, T; W^{1,\infty}(0, 1))$  respectively. Let  $\sigma \in H^4(0, 1)$  and  $\gamma, \tilde{\gamma} \in H^4(0, 1)$ . We consider data  $y_0$  and  $g$  regular enough such that the solutions of (1) belong to  $H^1(0, T; H^4(0, 1))$ .*

*Let us denote  $y$  the solution of equation (1) associated to  $\gamma$  and  $\tilde{y}$  the solution associated to  $\tilde{\gamma}$ . We assume that there exists  $T_0 \in (0, T)$  such that*

$$[\tilde{y}_{xx}(T_0, \cdot)] \geq r_0 > 0, \text{ almost everywhere in } (0, 1).$$

There exist a positive constant  $C$ , depending on  $T, \|\tilde{\gamma}\|_{L^\infty(0,1)}, r_0, g, \mathcal{U}_1$  and  $\mathcal{U}_2$ , such that for every  $\gamma \in \mathcal{U}_1$ , for every  $y \in \mathcal{U}_2$ ,

$$\begin{aligned} \frac{1}{C} \|\gamma - \tilde{\gamma}\|_{L^2(0,1)}^2 &\leq \|y_{xx}(\cdot, 0) - \tilde{y}_{xx}(\cdot, 0)\|_{H^1(0,T)}^2 + \|y_{xxx}(\cdot, 0) - \tilde{y}_{xxx}(\cdot, 0)\|_{H^1(0,T)}^2 \\ &\quad + \|y(T_0, \cdot) - \tilde{y}(T_0, \cdot)\|_{H^4(0,1)}^2 + \|y(T_0, \cdot) - \tilde{y}(T_0, \cdot)\|_{H^1(0,1)}^4 \\ &\leq C \left( \|y - \tilde{y}\|_{H^1(0,T;H^4(0,1))}^2 + \|y - \tilde{y}\|_{L^\infty(0,T;H^1(0,1))}^4 \right). \end{aligned} \quad (5)$$

This two-sided inequality gathers two complementary informations, namely, the stability of the inverse problem (with the first estimate) and the regularity of the measurements. Indeed, the second estimate in (5) indicates that the required measurements are finite if  $y$  and  $\tilde{y}$  belong to the space  $H^1(0, T; H^4(0, 1))$  and this is true if  $y$  and  $\tilde{y}$  are solutions provided by Theorem 1.1 under the hypothesis (3) and (4).

**Remark 1.3** *As stated above, an internal measurement at  $t = T_0$  is required if this method is used to solve this type of problem for parabolic equations. Nevertheless, this is probably a technical point and there is no counter-example that demonstrates whether this assumption is required. In [14], uniqueness (but not stability) is proven using a very different technique in an inverse problem for a parabolic equation and without any internal measurements in the entire space domain.*

**Remark 1.4** *In this paper, the boundary measurements are located at  $x = 0$ , but the result would be the same if we measure at  $x = 1$  instead. Indeed, the choice of a suitable weight function in the proof of the Carleman estimate in Section 3 is critical to impose the side of measurement.*

This paper is organized as follows. Section 2 proves the well-posedness result stated in Theorem 1.1. A global Carleman estimate for a general K-S equation is proved in Section 3. Finally, in Section 4 the Bukhgeim-Klibanov method is used to prove the Lipschitz stability of the inverse problem stated in Theorem 1.2.

## 2 On the Cauchy problem for KS equation

This section presents a proof of Theorem 1.1 in a more general case including time dependent low-order coefficients. The following system

$$\begin{cases} y_t + (\sigma(x)y_{xx})_{xx} + \gamma(x)y_{xx} + G_1y_x + G_2y + yy_x = g, & \forall (t, x) \in Q, \\ y(t, 0) = 0, \quad y(t, 1) = 0, & \forall t \in (0, T), \\ y_x(t, 0) = 0, \quad y_x(t, 1) = 0, & \forall t \in (0, T), \\ y(0, x) = y_0(x), & \forall x \in (0, 1), \end{cases} \quad (6)$$

can be considered, where  $G_1, G_2 \in H^1(0, T; H^4(0, 1))$ . Recall that the coefficients satisfy  $\gamma \in H^4(0, 1)$ ,  $\sigma \in H^4(0, 1)$  and hypothesis (3).

First, the main part of the linear differential operator is utilized in the next proposition.

**Proposition 2.1** *Let  $z_0 \in H^6(0, 1)$  and  $f \in \mathcal{F}$ . Then, the following equation*

$$\begin{cases} z_t + (\sigma(x)z_{xx})_{xx} = f, & \forall (t, x) \in Q, \\ z(t, 0) = 0, \quad z(t, 1) = 0, & \forall t \in (0, T), \\ z_x(t, 0) = 0, \quad z_x(t, 1) = 0, & \forall t \in (0, T), \\ z(0, x) = z_0(x), & \forall x \in (0, 1), \end{cases} \quad (7)$$

has a unique solution  $z \in \mathcal{Z}$ .

**Proof.** The operator

$$w \in H^4 \cap H_0^2(0, 1) \subset L^2(0, 1) \mapsto (\sigma(x)w''(x))'' \in L^2(0, 1)$$

is simultaneously positive, coercive and self-adjoint: thus, it generates a strongly continuous semigroup in  $L^2(0, 1)$ . Therefore, for each  $z_0 \in H^4 \cap H_0^2(0, 1)$  and  $f \in C^1([0, T]; L^2(0, 1))$ , Equation (7) has a unique solution  $z \in C([0, T]; H^4 \cap H_0^2(0, 1)) \cap C^1([0, T]; L^2(0, 1))$ .

We will demonstrate that the solutions  $z \in \mathcal{Z}$  (refer to the notation introduced in (2)), can be obtained by taking  $z_0$  and  $f$  sufficiently regular.

We now search for some energy estimates that indicate the space that the solutions lie on depending on the regularity of the data. Suppose that there are solutions sufficiently regular to perform the following computations. Equation (7) is multiplied by  $z$  and integrated over  $(0, 1)$ . After integrating by parts, we obtain

$$\frac{d}{dt} \left( \int_0^1 |z|^2 \right) + \int_0^1 |z_{xx}|^2 \leq C \left( \int_0^1 |f|^2 + \int_0^1 |z|^2 \right), \quad (8)$$

for some constant  $C > 0$ . Throughout this paper,  $C$  denotes a positive constant that varies from line to line. Gronwall's lemma is used to obtain

$$\int_0^1 |z|^2 \leq C \left( \iint_Q |f|^2 + \int_0^1 |z_0|^2 \right). \quad (9)$$

Then, (8) is integrated over  $[0, T]$  and (9) is used to obtain

$$\iint_Q |z_{xx}|^2 \leq C \left( \iint_Q |f|^2 + \int_0^1 |z_0|^2 \right). \quad (10)$$

Inequalities (9) and (10) finally imply that

$$\|z\|_{\mathfrak{Y}_0}^2 \leq C \left( \iint_Q |f|^2 + \int_0^1 |z_0|^2 \right). \quad (11)$$

Equation (7) is multiplied by  $(\sigma z_{xx})_{xx}$  and integrated over  $(0, 1)$ . After integrating by parts, we obtain

$$\frac{1}{2} \frac{d}{dt} \left( \int_0^1 \sigma |z_{xx}|^2 \right) + \int_0^1 |(\sigma z_{xx})_{xx}|^2 = \int_0^1 f (\sigma z_{xx})_{xx}. \quad (12)$$

We can then write

$$\frac{d}{dt} \left( \int_0^1 \sigma |z_{xx}|^2 \right) + \int_0^1 \sigma |(z_{xx})_{xx}|^2 \leq \int_0^1 |f|^2. \quad (13)$$

Thus, we obtain

$$\|z\|_{\mathfrak{Y}_2}^2 \leq C \left( \iint_Q |f|^2 + \int_0^1 |z_0''|^2 \right). \quad (14)$$

On the other hand, Equation (7) is derived with respect to time. Thus  $q := z_t$  satisfies

$$\begin{cases} q_t + (\sigma(x)q_{xx})_{xx} = f_t, & \forall (t, x) \in Q, \\ q(t, 0) = 0, \quad q(t, 1) = 0, & \forall t \in (0, T), \\ q_x(t, 0) = 0, \quad q_x(t, 1) = 0, & \forall t \in (0, T), \\ q(0, x) = f(0, x) - (\sigma z_0''(x))'', & \forall x \in (0, 1). \end{cases} \quad (15)$$

Using estimate (14), we obtain  $q \in \mathcal{Y}_2$  if  $(f(0, x) - (\sigma z_0''(x))'') \in H^2(0, 1)$  and  $f_t \in L^2(0, T; L^2(0, 1))$ . These hypotheses are fulfilled if  $z_0 \in H^6(0, 1)$  and  $f \in \mathcal{F}$ . Note that  $\mathcal{F} \subset C([0, T]; H^2(0, 1))$ . From the equation satisfied by  $z$  and the fact that  $f \in \mathcal{F}$  and  $z_t \in \mathcal{Y}_2$ , we determine that  $z \in \mathcal{Y}_6$ , which concludes the proof of Proposition 2.1. ■

Then, we focus on the linear problem with low-order coefficients that depend on time.

**Proposition 2.2** *Let  $z_0 \in H^6(0, 1)$ ,  $\hat{f} \in \mathcal{F}$  and  $G_1, G_2 \in H^1(0, T; H^4(0, 1))$ . Then, the equation*

$$\begin{cases} z_t + (\sigma(x)z_{xx})_{xx} + \gamma(x)z_{xx} + G_1 z_x + G_2 z = \hat{f}, & \forall (t, x) \in Q, \\ z(t, 0) = 0, \quad z(t, 1) = 0, & \forall t \in (0, T), \\ z_x(t, 0) = 0, \quad z_x(t, 1) = 0, & \forall t \in (0, T), \\ z(0, x) = z_0(x), & \forall x \in (0, 1), \end{cases} \quad (16)$$

has a unique solution  $z \in \mathcal{Z}$ .

**Proof.** For any  $\hat{z} \in \mathcal{Z}$ ,  $\Pi(\hat{z})$  is defined as the solution of (7) with  $f = (\hat{f} - \gamma(x)\hat{z}_{xx} - G_1\hat{z}_x - G_2\hat{z})$ . Note that  $f \in \mathcal{F}$  and therefore  $\Pi(\hat{z}) \in \mathcal{Z}$  is well defined.

If  $T$  is small enough, then  $\Pi$  is a contraction. Indeed, for any  $z, \hat{z} \in \mathcal{Z}$ , we have (the space  $L^m(0, T; H^n(0, 1))$  is denoted as  $L^m(H^n)$ )

$$\begin{aligned} \|\Pi(\hat{z}) - \Pi(z)\|_{\mathcal{Z}} &\leq C\|\gamma(x)(z_{xx} - \hat{z}_{xx}) + G_1(z_x - \hat{z}_x) + G_2(z - \hat{z})\|_{\mathcal{F}} \\ &\leq C\|z - \hat{z}\|_{L^2(H^6)} + C\|z_t - \hat{z}_t\|_{L^2(H^2)} \\ &\leq CT^{\frac{1}{4}}\|z - \hat{z}\|_{L^4(H^6)} + CT^{\frac{1}{4}}\|z_t - \hat{z}_t\|_{L^4(H^2)} \\ &\leq CT^{\frac{1}{4}}\|z - \hat{z}\|_{\mathcal{Y}_6} + CT^{\frac{1}{4}}\|z_t - \hat{z}_t\|_{\mathcal{Y}_2} \\ &\leq CT^{\frac{1}{4}}\|z - \hat{z}\|_{\mathcal{Z}}. \end{aligned} \quad (17)$$

Hence,  $\Pi$  has a unique fixed point in  $\mathcal{Z}$ , which is the solution of (16). Using standard arguments and the linearity of this equation, the solution can be extended to a larger interval. ■

**Remark 2.3** The third-order term  $z_{xxx}$  can be added to Equation (16). Indeed, in that case (17) becomes  $C\|z - \hat{z}\|_{L^2(H^7)} + C\|z_t - \hat{z}_t\|_{L^2(H^3)}$ , which is bounded by

$$CT^{\frac{1}{4}}\|z - \hat{z}\|_{L^\infty(H^6)}^{1/2}\|z - \hat{z}\|_{L^2(H^8)}^{1/2} + CT^{\frac{1}{4}}\|z_t - \hat{z}_t\|_{L^\infty(H^2)}^{1/2}\|z_t - \hat{z}_t\|_{L^2(H^4)}^{1/2}.$$

This last expression is bounded by (18). The remainder of the proof is the same.

Again, by using a fixed point theorem, we can prove Theorem 1.1 for equation (6). Let  $y_0 \in H^6(0, 1)$  and  $g \in \mathcal{F}$ . For any  $v \in \mathcal{Z}$ , we define  $\Lambda(v)$  as the solution of (16) with  $\hat{f} = (g - vv_x)$  and  $z_0 = y_0$ . Note that  $\hat{f} \in \mathcal{F}$  and therefore  $\Lambda(v) \in \mathcal{Z}$  is well defined. Indeed, if  $v \in \mathcal{Y}_3$  and  $v_t \in \mathcal{Y}_0$ , then we have

$$(vv_x)_{xxxx} = (10v_{xx}v_{xxx} + 5v_xv_{xxxx} + vv_{xxxx}) \in L^2(0, T; L^2(0, 1))$$

and

$$(vv_x)_t = v_tv_x + vv_{xt} \in L^2(0, T; L^2(0, 1)).$$

Furthermore, we can prove

$$\|\Lambda(v)\|_{\mathcal{Z}} \leq C\{\|g\|_{\mathcal{F}} + \|vv_x\|_{\mathcal{F}} + \|y_0\|_{H^6}\} \leq C\{\|g\|_{\mathcal{F}} + \|v\|_{\mathcal{Z}}^2 + \|y_0\|_{H^6}\}. \quad (19)$$

Let  $\varepsilon > 0$  to be chosen later and suppose that  $y_0$  and  $g$  satisfy (4). Consider  $v$  such that  $\|v\|_{\mathcal{Z}} \leq r$  with  $r > 0$  satisfying  $C(2\varepsilon + r^2) < r$ . From (19), we obtain  $\|\Lambda(v)\|_{\mathcal{Z}} < r$ . Thus, the application  $\Lambda$  maps the ball  $B_r := \{v \in \mathcal{Z} / \|v\|_{\mathcal{Z}} \leq r\}$  into itself.

We will now prove that  $\Lambda : B_r \rightarrow B_r$  is a contraction. For any  $z, v \in B_r$ ,  $\Lambda(z) - \Lambda(v)$  is the solution of (16) with  $\hat{f} = 0$  and  $z_0 = 0$ . We obtain the estimate

$$\|\Lambda(z) - \Lambda(v)\|_{\mathcal{Z}} \leq C\|vv_x - zz_x\|_{\mathcal{F}} \leq C\|(v - z)v_x\|_{\mathcal{F}} + \|z(v_x - z_x)\|_{\mathcal{F}}.$$

Using the definition of the space  $\mathcal{F}$ ,  $v, z \in L^\infty(0, T; W^{5, \infty}(0, 1)) \hookrightarrow C([0, 1]; H^6(0, 1))$  and  $v_t, z_t \in L^\infty(0, T; W^{1, \infty}(0, 1)) \hookrightarrow C([0, 1]; H^6(0, 1))$ , we obtain

$$\|\Lambda(z) - \Lambda(v)\|_{\mathcal{Z}} \leq C(\|v\|_{\mathcal{Z}} + \|z\|_{\mathcal{Z}})\|v - z\|_{\mathcal{Z}} \leq 2Cr\|v - z\|_{\mathcal{Z}},$$

which implies that  $\Lambda$  is a contraction if  $r$  is chosen small enough. More precisely, we can choose  $r, \varepsilon$  such that  $2Cr < 1$  and  $C(2\varepsilon + r^2) < r$ . Hence, the map  $\Lambda$  has a unique fixed point  $y \in \mathcal{Z}$ , which is the unique solution of (1).

Thus, we have proven Theorem 1.1.

### 3 Global Carleman inequality

In this section, a global Carleman inequality will be proven for the linearized K-S equation. We define the space

$$\mathcal{V} = \{v \in L^2(0, T; H^4(0, 1)) / Lv \in L^2((0, T) \times (0, 1))\} \quad (20)$$



where

$$Lv = v_t + (\sigma v_{xx})_{xx} + q_2 v_{xx} + q_1 v_x + q_0 v$$

with  $q_j \in L^\infty(\Omega)$  for  $j = 0, 1, 2$ .

Consider  $\beta \in C^4([0, 1])$  such that for some  $r > 0$  we have, for all  $x \in (0, 1)$ :

$$0 < r \leq \frac{d^k \beta}{dx^k}(x), \quad \text{for } k = 0, 1, \quad (21)$$

$$\frac{d^2 \beta}{dx^2}(x) \leq -r < 0, \quad (22)$$

$$|\sigma_x(x)\beta_x(x)| \leq \frac{r}{4} \min_{z \in [0, 1]} \{\sigma(z)\}. \quad (23)$$

For instance, if  $\sigma$  is constant, we can consider  $\beta(x) = \sqrt{1+x}$ .

On the other hand, given  $T_0 \in (0, T)$  we can choose  $\phi_0 \in C^1([0, T])$  such that

$$\phi_0(0) = \phi_0(T) = 0, \quad \text{and} \quad (24)$$

$$0 < \phi_0(t) \leq \phi_0(T_0) \quad \text{for each } t \in (0, T). \quad (25)$$

For example, if  $T_0 = T/2$ , we can use  $\phi_0(t) = t(T-t)$ .

We finally define the function

$$\phi(t, x) = \frac{\beta(x)}{\phi_0(t)}, \quad (26)$$

for  $(t, x) \in (0, T) \times [0, 1]$ , which is the weight function of the Carleman estimate and has a crucial role in the following result.

**Theorem 3.1** *Let  $\phi$  be a function as (26) and  $m > 0$ . Then there exists  $\lambda_0 > 0$  and a constant  $C = C(T, \lambda_0, r, m) > 0$  such that if  $\|q_i\|_{L^\infty((0, T) \times (0, 1))} \leq m$  for  $i = 0, 1, 2$  then we have*

$$\begin{aligned} & \iint_Q e^{-2\lambda\phi} \left( \frac{|v_t|^2 + |(\sigma v_{xx})_{xx}|^2}{\lambda\phi} + \lambda^7 \phi^7 |v|^2 + \lambda^5 \phi^5 |v_x|^2 + \lambda^3 \phi^3 |v_{xx}|^2 + \lambda\phi |v_{xxx}|^2 \right) dxdt \\ & \leq C \iint_Q e^{-2\lambda\phi} |Lv|^2 dxdt \\ & + C \int_0^T e^{-2\lambda\phi(t, 0)} \left( \lambda^3 \phi_x^3(t, 0) \sigma(0)^2 |v_{xx}(t, 0)|^2 + \lambda\phi_x(t, 0) \sigma^2(0) |v_{xxx}(t, 0)|^2 \right) dt \quad (27) \end{aligned}$$

for all  $v \in \mathcal{V}$ , for all  $\lambda \geq \lambda_0$ .

**Proof.** Consider the following operator  $P$  defined in  $\mathcal{W}_\lambda := \{e^{-\lambda\phi} v : v \in \mathcal{V}\}$  by

$$Pw = e^{-\lambda\phi} L(e^{\lambda\phi} w).$$

We then obtain the decomposition  $Pw = P_1w + P_2w + Rw$ , where

$$P_1w = 6\lambda^2\phi_x^2\sigma w_{xx} + \lambda^4\phi_x^4\sigma w + (\sigma w_{xx})_{xx} + 6\lambda^2(\phi_x^2\sigma)_x w_x \quad (28)$$

$$P_2w = w_t + 4\lambda^3\phi_x^3\sigma w_x + 4\lambda\phi_x\sigma w_{xxx} + 4\lambda^3\phi_x(\phi_x^2\sigma)_x w \quad (29)$$

$$\begin{aligned} Rw = & \lambda\phi_t w + 2\lambda\phi_x\sigma_{xx}w_x + \lambda^2\phi_x^2\sigma_{xx}w + \lambda\phi_{xx}\sigma_{xx}w \\ & + 6\lambda\phi_x\sigma_x w_{xx} + 6\lambda^2\phi_x\phi_{xx}\sigma_x w + 2\lambda\phi_{xx}\sigma_x w + 2\lambda\phi_{xxx}\sigma_x w \\ & + 4\lambda^2\phi_x\phi_{xxx}\sigma w + 6\lambda\phi_{xx}\sigma w_{xx} + 3\lambda^2\phi_{xx}^2\sigma w + 4\lambda\phi_{xxx}\sigma w_x \\ & + \lambda\phi_{xxxx}\sigma w + \lambda\phi_{xxxx}\sigma w + q_0 w + q_1 w_x + q_1\lambda\phi_x w \\ & + q_2 w_x + 2\lambda q_2\phi_x w_x + \lambda^2 q_2\phi_x^2 w + \lambda\phi_{xx}q_2 w \\ & + 2\lambda^3\phi_x^2\phi_{xx}\sigma w - 2\lambda^3\phi_x^3\sigma_x w. \end{aligned} \quad (30)$$

Then

$$\|Pw - Rw\|_{L^2(Q)}^2 = \|P_1w\|_{L^2(Q)}^2 + 2\langle P_1w, P_2w \rangle + \|P_2w\|_{L^2(Q)}^2$$

where  $\langle \cdot, \cdot \rangle$  is the  $L^2(Q)$  scalar product.

For any  $v \in \mathcal{V}$  we obtain  $v_t \in L^2(0, T; L^2(0, 1))$  and then  $v \in C([0, T]; L^2(0, 1))$ . From the construction of  $\phi$  (see (24)), we obtain  $w \in C([0, T]; L^2(0, 1))$  and  $w(x, 0) = w(x, T) = 0$  for any  $w \in \mathcal{W}_\lambda$ .

Let us define the notations

$$I(w) = -6\lambda^7 \iint_Q \phi_x^6 \phi_{xx} \sigma^2 |w|^2,$$

$$I(w_x) = -\lambda^5 \iint_Q \phi_x^4 \sigma (30\phi_{xx}\sigma + 12\phi_x\sigma_x) |w_x|^2,$$

$$I(w_{2x}) = -\lambda^3 \iint_Q \phi_x^2 \sigma (58\phi_{xx}\sigma + 40\phi_x\sigma_x) |w_{xx}|^2,$$

$$I(w_{3x}) = -\lambda \iint_Q \sigma (2\phi_{xx}\sigma - 4\phi_x\sigma_x) |w_{xxx}|^2,$$

and

$$I_x = \int_0^T (10\lambda^3\phi_x^3\sigma^2 |w_{xx}|^2 + 2\lambda\phi_x\sigma\sigma_{xx} |w_{xx}|^2 + 2\lambda\phi_x\sigma^2 |w_{xxx}|^2) \Big|_{x=0}^1.$$

The following weighted norm is defined, for any  $w \in \mathcal{W}_\lambda$ , as

$$\|w\|_{\lambda, \phi}^2 = \iint_Q (\lambda^7\phi^7 |w|^2 + \lambda^5\phi^5 |w_x|^2 + \lambda^3\phi^3 |w_{xx}|^2 + \lambda\phi |w_{xxx}|^2) dxdt.$$

We first require the following

**Lemma 3.2** *Under the same hypothesis of the Theorem 3.1, there exists  $\delta > 0$  such that*

$$\langle P_1w, P_2w \rangle_{L^2(Q)} \geq \delta \|w\|_{\lambda, \phi}^2 + I_x \quad (31)$$

for  $\lambda$  large enough, for all  $w \in \mathcal{W}_\lambda$ .

**Proof.** It is sufficient to prove that

$$\langle P_1 w, P_2 w \rangle_{L^2} = \sum_{k=0}^3 I(w_{kx}) + R_0(w) + I_x \quad (32)$$

for a large enough  $\lambda$ , for all  $w \in \mathcal{W}_\lambda$ , where  $|R_0(w)| \leq \lambda^{-1} \|w\|_{\lambda, \phi}^2$ .

Let us assume (32). From hypotheses (21) to (23) we know that there exists  $\varepsilon > 0$  such that  $\phi$  satisfies

$$\begin{aligned} \phi_{xx}(x) &\leq -\varepsilon\phi < 0, \\ 30\phi_{xx}(x)\sigma(x) + 12\phi_x(x)\sigma_x(x) &\leq -\varepsilon\phi < 0, \\ 58\phi_{xx}(x)\sigma(x) + 40\phi_x(x)\sigma_x(x) &\leq -\varepsilon\phi < 0, \text{ and} \\ 2\phi_{xx}(x)\sigma(x) - 4\phi_x(x)\sigma_x(x) &\leq -\varepsilon\phi < 0 \end{aligned} \quad (33)$$

for all  $x \in (0, 1)$ . Furthermore, from (21) we can prove that  $\phi \leq C\phi_x$ . Then from (32) we obtain, for a large enough  $\lambda$ ,

$$\begin{aligned} \langle P_1 w, P_2 w \rangle_{L^2} &= \sum_{k=0}^3 I(w_{kx}) + R_0(w) + I_x \\ &\geq 2\delta \|w\|_{\lambda, \phi}^2 - |R_0(w)| + I_x \\ &\geq \delta \|w\|_{\lambda, \phi}^2 + I_x. \end{aligned} \quad (34)$$

To prove (32), we write  $\langle P_1 w, P_2 w \rangle_{L^2(Q)} = \sum_{i,j=1}^4 I_{i,j}$  where  $I_{i,j}$  denotes the  $L^2$ -product between the  $i$ -th term of  $P_1 w$  in (28) and the  $j$ -th term of  $P_2 w$  in (29).

Integration by parts in time or space is performed on each expression  $I_{i,j}$ . Each resulting expression will be included in one of the terms on the right-hand side of (32). Each of the results is listed below, which indicates the term that it will be included in.

- $I_{1,1} = -I_{4,1} + \underbrace{3\lambda^2 \iint_Q (\phi_x^2 \sigma)_t |w_x|^2}_{R_0(w)}$
- $I_{1,2} = -12\lambda^5 \underbrace{\iint_Q (\phi_x^5 \sigma^2)_x |w_x|^2}_{I(w_x)}$
- $I_{1,3} = -12\lambda^3 \underbrace{\iint_Q (\phi_x^3 \sigma^2)_x |w_{xx}|^2}_{I(w_{2x})} + 12\lambda^3 \underbrace{\int_0^T \left[ \phi_x^3 \sigma^2 |w_{xx}|^2 \right]_0^1}_{I_x}$
- $I_{1,4} = 12\lambda^5 \underbrace{\iint_Q [\phi_x^3 \sigma (\phi_x^2 \sigma)_x]_{xx} |w|^2}_{R_0(w)} - 24\lambda^5 \underbrace{\iint_Q \phi_x^3 \sigma (\phi_x^2 \sigma)_x |w_x|^2}_{I(w_x)}$
- $I_{2,1} = -\frac{\lambda^4}{2} \underbrace{\iint_Q (\phi_x^4 \sigma)_t |w|^2}_{R_0(w)}$
- $I_{2,2} = -2\lambda^7 \underbrace{\iint_Q (\phi_x^7 \sigma^2)_x |w|^2}_{I(w)}$

- $I_{2,3} = \underbrace{-2\lambda^5 \iint_Q (\phi_x^5 \sigma^2)_{xxx} |w|^2}_{R_0(w)} + \underbrace{6\lambda^5 \iint_Q (\phi_x^5 \sigma^2)_x |w_x|^2}_{I(w_x)}.$
- $I_{2,4} = \underbrace{4\lambda^7 \iint_Q \phi_x^5 \sigma (\phi_x^2 \sigma)_x |w|^2}_{I(w)}.$
- $I_{3,1} = \frac{1}{2} \int_0^1 \left[ \sigma |w_{xx}|^2 \right]_0^T = 0.$
- $I_{3,2} = \underbrace{-2\lambda^3 \iint_Q [(\phi_x^3 \sigma)_{xx} \sigma]_x |w_x|^2}_{R_0(w)} + \underbrace{4\lambda^3 \iint_Q (\phi_x^3 \sigma)_x \sigma |w_{xx}|^2}_{I(w_{2x})}$   
 $+ \underbrace{2\lambda^3 \iint_Q (\phi_x^3)_x \sigma^2 |w_{xx}|^2}_{I(w_{2x})} - \underbrace{2\lambda^3 \int_0^T \left[ \phi_x^3 \sigma^2 |w_{xx}|^2 \right]_0^1}_{I_x}.$
- $I_{3,3} = \underbrace{2\lambda \int_0^T \left[ \phi_x \sigma \sigma_{xx} |w_{xx}|^2 \right]_0^1}_{I_x} - \underbrace{2\lambda \iint_Q (\phi_x \sigma \sigma_{xx})_x |w_{xx}|^2}_{R_0(w)} + \underbrace{8\lambda \iint_Q \phi_x \sigma \sigma_x |w_{3x}|^2}_{I(w_{xxx})}$   
 $+ \underbrace{2\lambda \int_0^T \left[ \phi_x \sigma^2 |w_{xxx}|^2 \right]_0^1}_{I_x} - \underbrace{2\lambda \iint_Q (\phi_x \sigma^2)_x |w_{xxx}|^2}_{I(w_{xxx})}.$
- $I_{3,4} = \underbrace{4\lambda^3 \iint_Q (\phi_x (\phi_x^2 \sigma)_x)_{xx} \sigma w w_{xx}}_{R_0(w)} - \underbrace{4\lambda^3 \iint_Q (\phi_x (\phi_x^2 \sigma)_x)_x \sigma |w_x|^2}_{R_0(w)}$   
 $+ \underbrace{4\lambda^3 \iint_Q \phi_x (\phi_x^2 \sigma)_x \sigma |w_{2x}|^2}_{I(w_{2x})}.$
- $I_{4,1} = 6\lambda^2 \iint_Q (\phi_x^2 \sigma)_x w_x w_t$ , which is canceled when adding with  $I_{1,1}$ .
- $I_{4,2} = \underbrace{24\lambda^5 \iint_Q (\phi_x^2 \sigma)_x \phi_x^3 \sigma |w_x|^2}_{I(w_x)}.$
- $I_{4,3} = \underbrace{12\lambda^3 \iint_Q [(\phi_x^2 \sigma)_x \phi_x \sigma]_{xx} |w_x|^2}_{R_0(w)} - \underbrace{24\lambda^3 \iint_Q (\phi_x^2 \sigma)_x \phi_x \sigma |w_{xx}|^2}_{I(w_{2x})}.$
- $I_{4,4} = \underbrace{-12\lambda^5 \iint_Q (\phi_x^2 \sigma)_x (\phi_x^3 \sigma)_x |w|^2}_{R_0(w)}.$

By summing all of the terms, we obtain (32). ■

Then, we will prove a Carleman inequality for the conjugated operator  $P$ .

**Lemma 3.3** *There exists  $\lambda_0 > 0$  such that for all  $\lambda \geq \lambda_0$  we have*

$$\iint_Q (\lambda^7 \phi^7 |w|^2 + \lambda^5 \phi^5 |w_x|^2 + \lambda^3 \phi^3 |w_{xx}|^2 + \lambda \phi |w_{xxx}|^2) dxdt$$

$$+ \|P_1 w\|_{L^2(Q)}^2 + \|P_2 w\|_{L^2(Q)}^2 \leq C \|Pw\|_{L^2(Q)}^2 - I_x$$

for all  $w \in \mathcal{W}_\lambda$ .

**Proof.**

From hypothesis (21) and inequalities (33), we know that there exists  $\delta > 0$  such that

$$\sum_{k=0}^3 I(w_{kx}) \geq \delta \|w\|_{\lambda, \phi}^2 \quad (35)$$

for a large enough  $\lambda$ .

On the other hand, due to the definition of  $Rw$  in (30) and  $\|q_i\|_{L^\infty((0,T) \times (0,1))} \leq m$  for  $i = 0, 1, 2$  it is trivial to check that

$$\begin{aligned} \|Rw\|_{L^2((0,T) \times (0,1))}^2 &\leq C \left( \lambda^6 \iint \phi^6 |w|^2 + \lambda^2 \iint \phi^2 |w_x|^2 + \lambda^2 \iint \phi^2 |w_{xx}|^2 \right) \\ &\leq C \lambda^{-1} \|w\|_{\lambda, \phi}^2. \end{aligned} \quad (36)$$

Thus, for a large enough  $\lambda$ , we have

$$\begin{aligned} \|P_1 w\|_{L^2}^2 + 2 \langle P_1 w, P_2 w \rangle + \|P_2 w\|_{L^2}^2 &= \|Pw - Rw\|_{L^2}^2 \\ &\leq 2 \|Pw\|_{L^2}^2 + 2 \|Rw\|_{L^2}^2 \\ &\leq 2 \|Pw\|_{L^2}^2 + C \lambda^{-1} \|w\|_{\lambda, \phi}^2. \end{aligned} \quad (37)$$

From estimates (37) and (35) and Lemma 3.2, we conclude the proof of Lemma 3.3.  $\blacksquare$

To complete the proof of Theorem 3.1, we have to deal with the norms for  $P_1 w$  and  $P_2 w$  appearing in Lemma 3.3. From the definition of  $P_2 w$ , and because (21) holds, we have

$$\frac{1}{\lambda \phi} |w_t|^2 \leq \frac{2}{\lambda \phi} |P_2 w|^2 + C (\lambda^5 \phi^5 |w|^2 + \lambda^5 \phi^5 |w_x|^2 + \lambda \phi |w_{xxx}|^2)$$

and

$$\iint \frac{1}{\lambda \phi} |w_t|^2 \leq C \iint |P_2 w|^2 + C \|w\|_{\lambda, \phi}^2$$

for a large enough  $\lambda$ . A similar result is proven for  $(\sigma w_{xx})_{xx}$  and  $P_1 w$ , and we then have

$$\iint \frac{1}{\lambda \phi} (|w_t|^2 + |(\sigma w_{xx})_{xx}|^2) \leq C \iint (|P_1 w|^2 + |P_2 w|^2) + C \|w\|_{\lambda, \phi}^2. \quad (38)$$

From (38) and Lemma 3.3 we obtain

$$\begin{aligned} \iint_Q \frac{1}{\lambda \phi} (|w_t|^2 + |(\sigma w_{xx})_{xx}|^2) + \lambda^7 \phi^7 |w|^2 + \lambda^5 \phi^5 |w_x|^2 + \lambda^3 \phi^3 |w_{xx}|^2 + \lambda \phi |w_{xxx}|^2 dx dt \\ \leq C \iint_Q |Pw|^2 dx dt - CI_x. \end{aligned} \quad (39)$$

To handle the terms in  $I_x$ , we note that

$$-C \lambda \int_0^T \phi_x(x, t) \sigma(x) \sigma_{xx}(x) |w_{xx}(x, t)|^2 dt \leq C \lambda^3 \int_0^T \phi_x(x, t)^3 \sigma(x)^2 |w_{xx}(x, t)|^2 dt$$

for any  $x \in (0, 1)$  and a large enough  $\lambda$ . Then

$$-CI_x \leq C \lambda^3 \int_0^T \phi_x(0, t)^3 \sigma(0)^2 |w_{xx}(0, t)|^2 dt + C \lambda \int_0^T \phi_x(0, t) \sigma(0)^2 |w_{xxx}(0, t)|^2 dt \quad (40)$$

and from (39) and (40) we obtain

$$\begin{aligned} \iint_Q \frac{1}{\lambda\phi} (|w_t|^2 + |(\sigma w_{xx})_{xx}|^2) + \|w\|_{\lambda,\varphi}^2 &\leq C \iint_Q |Pw|^2 dxdt \\ &+ C\lambda^3 \int_0^T \phi_x(0,t)^3 \sigma(0)^2 |w_{xx}(0,t)|^2 dt + C\lambda \int_0^T \phi_x(0,t) \sigma(0)^2 |w_{xxx}(0,t)|^2 dt. \end{aligned} \quad (41)$$

By calculating the derivatives of  $e^{\lambda\phi}w$  it is trivial to prove that

$$\left| \partial_x^k v \right|^2 = \left| \partial_x^k (e^{\lambda\phi}w) \right|^2 \leq C \sum_{j=0}^k \left| \lambda^{k-j} \phi^{k-j} \partial_x^j w \right|^2$$

for each  $k = 0, \dots, 3$ . Therefore

$$\begin{aligned} \iint_Q e^{-2\lambda\phi} \left( \lambda^7 \phi^7 |e^{\lambda\phi}w|^2 + \lambda^5 \phi^5 |(e^{\lambda\phi}w)_x|^2 + \lambda^3 \phi^3 |(e^{\lambda\phi}w)_{xx}|^2 + \lambda \phi |(e^{\lambda\phi}w)_{xxx}|^2 \right) dxdt \\ \leq C \|w\|_{\lambda,\phi}. \end{aligned}$$

Considering finally that  $Pw = e^{-\lambda\varphi}Lv$ , we obtain Carleman estimate (27). ■

**Remark 3.4** *We considered the function  $\beta$  to be increasing. This allows the Carleman inequality to be obtained with boundary terms at  $x = 0$ . If a decreasing function  $\beta$  was used instead, then an inequality with boundary terms at  $x = 1$  would have been obtained. As discussed in the following section, the boundary terms in the Carleman inequality are related to the location of the observations in the inverse problem.*

## 4 Inverse Problem

In this section, the local stability of the nonlinear inverse problem stated in Theorem 1.2 will be proved following the ideas of [6] and [12].

Let  $\gamma$ ,  $\tilde{\gamma}$ ,  $y$  and  $\tilde{y}$  be defined as in Theorem 1.2. If we set  $u = y - \tilde{y}$ ,  $f = \tilde{\gamma} - \gamma$  and  $R = \tilde{y}_{xx}$ , then  $u$  solves the following K-S equation:

$$\begin{cases} u_t + (\sigma(x)u_{xx})_{xx} + \gamma u_{xx} + \tilde{y}u_x + \tilde{y}_x u + uu_x = f(x)R(x,t), & \forall (t,x) \in Q, \\ u(t,0) = u(t,1) = 0, & \forall t \in (0,T), \\ u_x(t,0) = u_x(t,1) = 0, & \forall t \in (0,T), \\ u(0,x) = 0, & \forall x \in (0,1). \end{cases} \quad (42)$$

Then, to prove the stability of the inverse problem mentioned in the introduction, it is sufficient to prove an estimate of  $f$  in terms of  $u_{xx}(\cdot,0)$ ,  $u_{xxx}(\cdot,0)$  and  $u(T_0,\cdot)$ , where  $R$  and  $\tilde{\gamma}$  are given,  $\gamma \in \mathcal{U}_1$  and  $u$  is the solution of Equation (42).

We begin by deriving Equation (42) with respect to time. Thus,  $v = u_t$  satisfies the following equation:

$$\begin{cases} v_t + (\sigma v_{xx})_{xx} + \gamma v_{xx} + yv_x + y_x v = fR_t - g, & \forall (t,x) \in Q, \\ v(t,0) = v(t,1) = 0, & \forall t \in (0,T), \\ v_x(t,0) = v_x(t,1) = 0, & \forall t \in (0,T), \\ v(0,x) = fR(x,0), & \forall x \in (0,1), \end{cases} \quad (43)$$

where  $g(x, t) = u(x, t)\tilde{y}_{xt}(x, t) + u_x(x, t)\tilde{y}_t(x, t)$ .

The proof relies on the use of the Carleman estimate given in Theorem 3.1. This result will be used twice: first, Equation (43) will be used to estimate  $v$  in terms of  $f$ ,  $R$  and  $g$ ; then, Equation (42) will be used to handle the terms  $u$  and  $u_x$ , which appear in the expression for  $g$ .

Similar to the proof of the Carleman estimate, we set  $w = e^{-\lambda\phi}v$ . Then, we work on the term

$$I = 2 \int_0^1 \int_0^{T_0} w(t, x)w_t(t, x) dt dx.$$

On one hand, we can calculate  $I$  and bound it from below. Using  $w(0, x) = e^{-\lambda\phi(0, x)}v(0, x) = 0$  for all  $x \in (0, 1)$  and Equation (42), we can easily obtain

$$\begin{aligned} I &= \int_0^1 |w(T_0, x)|^2 dx \\ &= \int_0^1 e^{-2\lambda\phi(T_0, x)} |(fR - (\sigma u_{xx})_{xx} - \gamma u_{xx} - \tilde{y}u_x - \tilde{y}_x u - uu_x)(T_0, x)|^2 dx \\ &\geq \int_0^1 e^{-2\lambda\phi(T_0, x)} |f(x)|^2 |R(T_0, x)|^2 dx - C \|u(T_0)\|_{H^4(0,1)}^2 - C \|u(T_0)\|_{H^1(0,1)}^4 \end{aligned}$$

where  $C$  depends on  $\|\gamma\|_{L^\infty(0,1)}$ ,  $\|\tilde{y}(T_0)\|_{W^{1,\infty}(0,1)}$  and  $\|\sigma\|_{W^{2,\infty}(0,1)}$ .

On the other hand, to estimate  $I$  from above we apply the Carleman estimate (41) to Equation (43) using  $q_0 = y_x$  and  $q_1 = y$ , which are uniformly bounded in  $L^\infty((0, T) \times (0, 1))$  by the hypothesis in Theorem 1.2 (refer to the set  $\mathcal{U}_2$ ). We obtain that

$$\begin{aligned} I &= 2 \int_0^1 \int_0^{T_0} w(t, x)w_t(t, x) dt dx \\ &\leq \left( \int_0^1 \int_0^{T_0} \lambda\phi(t, x)|w(t, x)|^2 dt dx \right)^{\frac{1}{2}} \left( \int_0^1 \int_0^{T_0} \frac{|w_t(t, x)|^2}{\lambda\phi(t, x)} dt dx \right)^{\frac{1}{2}} \\ &\leq C\lambda^{-3} \int_0^1 \int_0^T e^{-2\lambda\phi} |f(x)R_t(x, t)|^2 dx dt + C\lambda^{-3} \int_0^1 \int_0^T e^{-2\lambda\phi} |g(x, t)|^2 dx dt \\ &\quad + C\lambda^{-3} \int_0^T e^{-2\lambda\phi(0, t)} (\lambda^3 \phi_x^3(0, t) \sigma^2(0) |v_{xx}(0, t)|^2 + \lambda \phi_x(0, t) \sigma^2(0) |v_{xxx}(0, t)|^2) dt. \end{aligned}$$

Considering that  $g = u\tilde{y}_{xt} + u_x\tilde{y}_t$ , we will now apply the Carleman estimate for Equation (42) to manage the term in  $g$ . The fixed trajectory  $\tilde{y}$  is such that  $\tilde{y}_{xt}$  and  $\tilde{y}_t$  belong to  $L^\infty(0, T; L^\infty(0, 1))$ . Thus, we have

$$\begin{aligned} \iint_Q e^{-2\lambda\phi} |g(x, t)|^2 dx dt &\leq 2 \iint_Q e^{-2\lambda\phi} |u|^2 |\tilde{y}_{xt}|^2 dx dt + 2 \iint_Q e^{-2\lambda\phi} |u_x|^2 |\tilde{y}_t|^2 dx dt \\ &\leq C \iint_Q e^{-2\lambda\phi} (|u|^2 + |u_x|^2) dx dt. \end{aligned}$$

Carleman estimate (41) is applied to equation (42), using the identity  $\tilde{y}_x u + uu_x = u y_x$ , and taking  $q_0 = y_x$  and  $q_1 = \tilde{y}$ , which are uniformly bounded in  $L^\infty((0, T) \times (0, 1))$ .

Therefore, we can choose  $\lambda_0$  as large as possible in Theorem 3.1: we then obtain

$$\begin{aligned}
& \iint_Q e^{-2\lambda\phi} |g(x,t)|^2 dxdt \\
& \leq C\lambda^{-5} \iint_Q e^{-2\lambda\phi} (\lambda^7 |u|^2 + \lambda^5 |u_x|^2) dxdt \\
& \leq C\lambda^{-5} \iint_Q e^{-2\lambda\phi} |f(x)R(x,t)|^2 dxdt \\
& \quad + C\lambda^{-5} \int_0^T e^{-2\lambda\phi(0,t)} (\lambda^3 \phi_x^3(0,t) \sigma^2(0) |u_{xx}(0,t)|^2 + \lambda \phi_x(0,t) \sigma^2(0) |u_{xxx}(0,t)|^2) dt.
\end{aligned}$$

By gathering all of the estimates of  $I$  and  $g$  that were obtained above, we have

$$\begin{aligned}
& \int_0^1 e^{-2\lambda\phi(T_0,x)} |f(x)|^2 |R(T_0,x)|^2 dx - C \|u(T_0)\|_{H^4(0,1)}^2 - C \|u(T_0)\|_{H^1(0,1)}^4 \\
& \leq C\lambda^{-3} \iint_Q e^{-2\lambda\phi} |f(x)R_t(x,t)|^2 dxdt + C\lambda^{-8} \iint_Q e^{-2\lambda\phi} |f(x)R(x,t)|^2 dxdt \\
& \quad + C\lambda^{-8} \int_0^T e^{-2\lambda\phi(0,t)} (\lambda^3 \phi_x^3(0,t) \sigma^2(0) |u_{xx}(0,t)|^2 + \lambda \phi_x(0,t) \sigma^2(0) |u_{xxx}(0,t)|^2) dt \\
& \quad + C\lambda^{-3} \int_0^T e^{-2\lambda\phi(0,t)} (\lambda^3 \phi_x^3(0,t) \sigma^2(0) |v_{xx}(0,t)|^2 + \lambda \phi_x(0,t) \sigma^2(0) |v_{xxx}(0,t)|^2) dt.
\end{aligned}$$

By using the fact that  $R \in H^1(0,T; L^\infty(0,1))$ ,  $0 < r < |R(T_0,x)|$  a.e. in  $(0,1)$  and  $e^{-2\lambda\phi(t,x)} \leq e^{-2\lambda\phi(T_0,x)}$  in  $(0,T) \times (0,1)$ , we obtain

$$\begin{aligned}
& \int_0^1 e^{-2\lambda\phi(T_0,x)} |f(x)|^2 dx \\
& \leq C \left( \lambda^{-3} \int_0^1 e^{-2\lambda\phi(T_0,x)} |f(x)|^2 dx + \|u(T_0)\|_{H^4(0,1)}^2 + \|u(T_0)\|_{H^1(0,1)}^4 \right. \\
& \quad + \lambda^{-8} \int_0^T e^{-2\lambda\phi(0,t)} (\lambda^3 \phi_x^3(0,t) \sigma^2(0) |u_{xx}(0,t)|^2 + \lambda \phi_x(0,t) \sigma^2(0) |u_{xxx}(0,t)|^2) dt \\
& \quad \left. + \lambda^{-3} \int_0^T e^{-2\lambda\phi(0,t)} (\lambda^3 \phi_x^3(0,t) \sigma^2(0) |v_{xx}(0,t)|^2 + \lambda \phi_x(0,t) \sigma^2(0) |v_{xxx}(0,t)|^2) dt \right).
\end{aligned}$$

Therefore, by choosing a sufficiently large enough  $\lambda$ , we prove that there exists a constant  $C$  that depends on  $r_0, \tilde{y}, T, \lambda_0, \mathcal{U}_1$  and  $\mathcal{U}_2$  such that

$$\begin{aligned}
\|f(x)\|_{L^2(0,1)}^2 & \leq C \left( \|u(T_0,x)\|_{H^4(0,1)}^2 + \|u(T_0,x)\|_{H^1(0,1)}^4 \right. \\
& \quad \left. + \|u_{xx}(t,0)\|_{H^1(0,T)}^2 + \|u_{xxx}(t,0)\|_{H^1(0,T)}^2 \right).
\end{aligned}$$

This estimate leads to the stability of the initial inverse problem and we have obtained the first estimate in Theorem 1.2. We will now show that the second estimate also holds. Indeed, using  $T_0 \in (0,T)$  and using that  $H^1(0,T; H^4(0,1)) \hookrightarrow C([0,T]; H^4(0,1))$ , we have  $\|y(T_0, \cdot) - \tilde{y}(T_0, \cdot)\|_{H^4(0,1)} \leq C \|y - \tilde{y}\|_{H^1(0,T; H^4(0,1))}$  for some constant  $C$ . In a similar manner, we can prove that  $\|y(T_0, \cdot) - \tilde{y}(T_0, \cdot)\|_{H^1(0,1)} \leq C \|y - \tilde{y}\|_{H^1(0,T; H^1(0,1))}$ .

However, using  $(y - \tilde{y}) \in L^2(0,T; H^4 \cap H_0^2(0,1))$ , we can integrate the following expressions by parts with respect to  $x$

$$\int_0^T \int_0^1 (y - \tilde{y})_{xxxx} (y - \tilde{y})_x (1-x) dxdt, \quad \text{and} \quad \int_0^T \int_0^1 (y - \tilde{y})_{xxxx} (y - \tilde{y})_{xxx} (1-x) dxdt$$

to obtain the desired traces. We obtain the estimates

$$\|y_{xx}(\cdot, 0) - \tilde{y}_{xx}(\cdot, 0)\|_{L^2(0,T)} \leq C \|y - \tilde{y}\|_{L^2(0,T; H^4(0,1))}$$



and

$$\|y_{xxxx}(\cdot, 0) - \tilde{y}_{xxxx}(\cdot, 0)\|_{L^2(0,T)} \leq C \|y - \tilde{y}\|_{L^2(0,T;H^4(0,1))}$$

for some positive constant  $C$ . Using the same approach, but starting with  $(y_t - \tilde{y}_t) \in L^2(0, T; H^4 \cap H_0^2(0, 1))$ , we obtain the estimates

$$\|y_{txx}(\cdot, 0) - \tilde{y}_{txx}(\cdot, 0)\|_{L^2(0,T)} \leq C \|y_t - \tilde{y}_t\|_{L^2(0,T;H^4(0,1))}$$

and

$$\|y_{txxx}(\cdot, 0) - \tilde{y}_{txxx}(\cdot, 0)\|_{L^2(0,T)} \leq C \|y_t - \tilde{y}_t\|_{L^2(0,T;H^4(0,1))}$$

These inequalities provide the second estimate in (5) and conclude the proof of Theorem (1.2).

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