Dynamics of solids with microperiodic nonconnected fluid inclusions

Francesco Dell'Isola, Luigi Rosa, Cz. Wozniak

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Summary A new model for the description of micro- and macro-dynamics of linear elastic solid body with micro-periodic nonconnected fluid filled inclusions is proposed. Using the model we are able to account for the microstructural length-scale effect on the global dynamic behavior of the body. The effect falls out of the scope of the known homogenized models obtained by asymptotic approaches. The general equations obtained in the model proposed here are applied in order to investigate dispersion phenomena in wave propagation problems. The obtained results are compared with those for the asymptotic model when the length-scale effect vanishes. It is shown that this effect plays an important role in the analysis of dynamic problems.

Key words elastodynamics, microperiodic media, two-phase media

1
Introduction

The main aim of this paper is to propose and apply a new model describing the dynamical behavior of a linear elastic solid matrix with a microperiodical system of cells containing non connected fluid inclusions. The characteristic feature of the proposed model is that it is able to describe the microstructural length-scale effect on the global dynamic behavior of the body. Effects of this kind cannot be taken into account when the well-known homogenized models are used. Indeed, they are based either on the asymptotic procedure in which all quantities of an order higher or equal to the cell length dimensions are neglected, [1, 2, 3], or on the ideal mixture assumptions introduced in the known theories of saturated porous media [4, 5, 6].

The procedure used here takes into account the approach proposed in [7], and recently developed in a series of papers [8, 9, 10]. The general idea relies on the introduction of so called "microshape functions" (*msf*) and "macro functions" (*mf*). They allow to represent the displacement field in terms of some macrodescriptors, one of which being the average (on a cell) displacement, and the others describing, among the microdisturbances in displacements caused by inhomogeneity of the medium, those which are relevant in the considered phenomena. We assume that displacements of the fluid inclusions are described by the same *msf* and macrodescriptors as those of the solid matrix. Because of this hypothesis, our analysis is limited to the case of a porous matrix with fully saturated nonconnected fluid inclusions. For the sake of simplicity the investigations are carried out in the framework of the linear elasticity theory and for a perfect fluid; more general cases will be treated separately.

The general evolution equations determining the model which we propose are applied to the analysis of some dynamical problems. The main results are:

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i) by comparing the proposed model with the asymptotic one, it is shown that in investigating high-frequency vibrations of the fluid-saturated solid matrix the length scale effect cannot be neglected;
ii) the incompressibility of the fluid filling the inclusions implies that the highest frequency of free vibration becomes infinity.

The proposed approach allows to obtain the aforementioned results using analytical methods only, without any numerical simulation.

2
Fundamental notations and concepts

We start describing the geometry of the typical cell V, which periodically reproduced will allow for the construction of the considered body. Let

$$V: = \left(-\frac{l_1}{2}, \frac{l_1}{2}\right) \times \left(-\frac{l_2}{2}, \frac{l_2}{2}\right) \times \left(-\frac{l_3}{2}, \frac{l_3}{2}\right) ,$$

and define

$$l := \sqrt{\Sigma_{i=1}^{3} l_i^2}$$

as the microstructure length parameter (cf. [7] and references quoted there). It is assumed that the smallest characteristic length dimension L_Ω of Ω satisfies condition:

$$\frac{l}{L_\Omega} << 1.$$

Let V_F be a subset of V whose boundary does not intersect with the boundary of V; we will assume that V_F is a regular region. We remark that V_F can be eventually partitioned in different simply-connected subsets. We define $V_S: = V \backslash \overline{V}_F$. In what follows, we will assume that V_S is occupied by a solid medium while V_F by a fluid one. For an arbitrary point $\mathbf{x}$ in the physical 3-space E, we will use the following notation:

$$V(\mathbf{x}): = \mathbf{x} + V, \quad V_S(\mathbf{x}): = \mathbf{x} + V_S, \quad V_F(\mathbf{x}): = \mathbf{x} + V_F.$$

We now define the geometry of the region Ω occupied by the body B in the reference configuration. We assume that there exists a lattice Λ of places in E such that

$$\left[\forall(\mathbf{x}_1, \mathbf{x}_2) \in \Lambda^2\right]\left[\mathbf{x}_1 \neq \mathbf{x}_2 \rightarrow V(\mathbf{x}_1) \cap V(\mathbf{x}_2) = \emptyset\right] ,$$

$$\Omega = int \left[\bigcup_{\mathbf{x} \in \Lambda} \bar{V}(\mathbf{x})\right] ,$$

is a regular region, where *int* (A) means the interior topological part of A.

In the following we will also set

$$\Omega_S: = int \left[\bigcup_{\mathbf{x} \in \Lambda} \bar{V}_S(\mathbf{x})\right], \quad \Omega_F: = int \left[\bigcup_{\mathbf{x} \in \Lambda} \bar{V}_F(\mathbf{x})\right], \quad \Omega^o: = \{\mathbf{x} \in \Omega: V(\mathbf{x}) \subset \Omega\} ,$$

which represent respectively the placement of solid matrix, of fluid inclusions and of what is called the macrointerior of Ω. For a generic function f defined respectively in $V(\mathbf{x})$, $V_S(\mathbf{x})$ and $V_F(\mathbf{x})$, we denote the mean values as follows:

$$\langle f \rangle_{\mathbf{x}} = \frac{1}{|V|} \int_{V_{\mathbf{x}}} f \mathrm{d}V, \langle f \rangle_{\mathbf{x}}^{S} = \frac{1}{|V|} \int_{V_S(\mathbf{x})} f \mathrm{d}V, \langle f \rangle_{\mathbf{x}}^{F} = \frac{1}{|V|} \int_{V_F(\mathbf{x})} f \mathrm{d}V ,$$

If f is a periodic function then for every $\mathbf{x} \in E$ we obtain

$$< f >_{\mathbf{x}} = < f >,\ < f >^{S}_{\mathbf{x}} = < f >^{S}, < f >^{F}_{\mathbf{x}} = < f >^{F}$$

where $< f >, < f >^{S}, < f >^{F}$ are constants.

An exact dynamical description of the body B at a microlevel is obtained by introducing displacement fields whose variations in every cell $V(\mathbf{x})$ can be relevant. Different coarse descriptions, i.e. descriptions at a macrolevel (in which only some overall averaged properties of the body are to be considered) can be obtained. The "homogenized" description accounts for the average displacement in V only. More refined descriptions are obtained by introducing "macrodescriptors" of displacement which are nearly constant within every $V(\mathbf{x})$.

Following [7] we introduce two concepts which we will need in the subsequent sections. We call microshape functions (*msf*) a system of sufficiently regular V-periodic linearly independent functions $h^a_l(\cdot),\ a = 1, \ldots, n$ such that:

$$h^a_l(\mathbf{x}) \in O(l),\ \nabla h^a_l(\mathbf{x}) \in O(1),\ < h^a_l >^{S} = < h^a_l >^{F} = 0,\ \mathbf{x} \in E\ , \tag{1}$$

where the index l displays the dependence of the *msf* on the microstructure length parameter; $O(l)$ represents a set of functions of l which are infinitesimal of equal or higher order than l, and $O(1)$ is a set of functions of l whose maximum value remains finite when $l \to 0$.
For some $\varepsilon > 0$, we call a real-valued function F defined on Ω an ε-macrofunction if

$$(\forall x \in \Omega^o)[\forall z \in V(\mathbf{x})]\ F\ (\mathbf{z}) \simeq F(\mathbf{x})\ , \tag{2}$$

where
$a \simeq b$ means $\|a - b\| \leq \varepsilon$.

The parameter ε is assigned to $F(\mathbf{x}), \varepsilon = \varepsilon_F$, and will be interpreted as an error implied by the choice of giving up the microscopically accurate description of the system. In the sequel, it will be assumed that for every ε-macrofunction $F(\mathbf{x})$ the accuracy parameter $\varepsilon = \varepsilon_F$ is known, and $F(\mathbf{x})$ will be referred to as a macrofunction.

3
Kinematical macrodescriptors and their evolution equations

In this paper, we will assume that the solid matrix is linear elastic and that the inclusions are constituted by a perfect fluid. Therefore, on the macrolevel, we assume that, with

$$\mathbf{e} := sym(\nabla \mathbf{u})\ ,$$

the following constitutive equations hold:

$$\mathbf{s} = \mathbf{C} : \mathbf{e};\quad \mathbf{t} = p\mathbf{I}\ . \tag{3}$$

Here and in the sequel, we use the conventions

$$\mathbf{C}.\mathbf{v} := \sum_{l=1}^{3} C_{ijkl} v_1,\ \mathbf{C} : \mathbf{w} : = \sum_{k,l=1}^{3} C_{ijkl} w_{kl}$$

where $\mathbf{u}$ is the microdisplacement vector field defined in Ω_S for every substantial point belonging to the matrix, $\mathbf{C}$ is the microlinear elasticity fourth-order tensor, p is the pressure field, defining in the fluid inclusions the Cauchy tensor $\mathbf{t}$, and $\mathbf{I}$ is the identity tensor.

Moreover, let ρ_S, ρ_F represent mass density fields of the solid and fluid components, respectively, and let $\mathbf{b}$ stand for an external body force. We will assume that the following weak form of balance equations holds, see for instance [11]:

$$\int_{\Omega_S} \mathbf{s} : \boldsymbol{\delta}\mathbf{e} \mathrm{d}V + \int_{\Omega_F} p(\mathrm{div}(\boldsymbol{\delta}\mathbf{u}))\mathrm{d}V + \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega_S} \rho_S \dot{\mathbf{u}} \cdot \boldsymbol{\delta}\mathbf{u} \mathrm{d}V + \int_{\Omega_F} \rho_F \dot{\mathbf{u}} \cdot \boldsymbol{\delta}\mathbf{u} \mathrm{d}V\right)$$
$$= \int_{\Omega_S} \rho_S \mathbf{b} \cdot \boldsymbol{\delta}\mathbf{u} \mathrm{d}V + \int_{\Omega_F} \rho_F \mathbf{b} \cdot \boldsymbol{\delta}\mathbf{u} \mathrm{d}V \tag{4}$$

for every $\delta\mathbf{u}$ admissible by possible constraints, continuous in Ω and of class C^1 in $\Omega_S^o \cup \Omega_F^o$, and such that $\delta\mathbf{u}|_{\partial\Omega} = 0$, where $\delta\mathbf{e} = sym\,(\nabla\delta\mathbf{u})$. The dynamical problem for the body B is given by Eqs. (3) and (4), together with a constitutive equation for the fluid. It describes completely the microbehavior of considered continuum. Its solution – because of the micro periodical shape of Ω_S and Ω_F – is not feasible, even using complicated numerical methods. For these reasons we resign from describing the details of the considered system, and introduce the two following hypotheses:

Macro Constraint Hypothesis (MCH)
The displacement field $\mathbf{u}$ is represented as

$$\mathbf{u}(\mathbf{x},t) = \mathbf{U}(\mathbf{x},t) + h_l^a(\mathbf{x})\mathbf{Q}^a(\mathbf{x},t), \quad \mathbf{x} \in \Omega, \quad t \in (t_0,t_f) \ , \tag{5}$$

where summation over a holds and $h_l^a(\cdot), a = 1,\dots n$, are *msf* describing the kinematics of the system at the microscopic level. These *msf* have to be postulated *a priori* in order to account for those features of the solutions at the microlevel which are more relevant at the macrolevel in the considered phenomena. Moreover, functions $\mathbf{U}$ and $\mathbf{Q}^a$ are sufficiently regular macrofunctions which will be treated as new kinematical independent variables. They will be called macrodisplacements and macrointernal variables respectively.

Macro-Modelling Approximation (MMA)
In the integrals appearing in Eq. (4) we will neglect all addends which are of the same or higher order as the accuracy parameter ε related to the macrofunctions $\mathbf{U}(\cdot,t), \mathbf{Q}^a(\cdot,t)$ and all their derivatives. In the sequel, this kind of approximation will be denoted by $\simeq$, cf. Eq. (2). We remark explicitly that in the proposed approach the microstructure length parameter l is assumed to be a known physical constant independent of ε and will be not neglected.

Because of the definition of *msf* given in Sec. 2, for every $\mathbf{x} \in \Omega^o$, we obtain

$$< \mathbf{u} >_{\mathbf{x}}^S = < \mathbf{u} >_{\mathbf{x}}^F = \mathbf{U}(\mathbf{x},t), t \in (t_0,t_f) \ .$$

Therefore, the macrodisplacements of the matrix are described in the same way as the macrodisplacements of the fluid inclusions. The hypotheses formulated for *msf* are therefore suitable for describing the behavior of a solid matrix with nonconnected fluid inclusions. Because of *MCH* we see that on a microlevel the motion, described by $\mathbf{u}$, is a superposition of microdisturbances $h_l^a\ \mathbf{Q}^a$ over the macromotion $\mathbf{U}$. Hence $\mathbf{Q}^a(\cdot,t)$ are macroquantities describing those disturbances on the microlevel which are relevant at the macrolevel. Choosing different *msf* allows for the study of different aspects of the considered process, whose more detailed description can be obtained increasing the number n of *msf*. In what follows, we will assume, coherently with *MCM* and *MCH*, that

$$\mathbf{e} = \mathbf{E} + sym\big[\nabla(h_l^a\mathbf{Q}^a)\big] = \mathbf{E} + sym(\nabla h_l^a \otimes \mathbf{Q}^a + h_l^a\nabla\mathbf{Q}^a) \simeq \mathbf{E} + sym(\nabla h_l^A \otimes \mathbf{Q}^a) \ ,$$

where we have denoted

$$\mathbf{E} := sym(\nabla\mathbf{U}) \ ,$$

and

$$\delta\mathbf{u} = \delta\mathbf{U} + h_l^a\delta\mathbf{Q}^a \ ,$$

$$\delta\mathbf{e} = \delta\mathbf{E} + sym\big(\nabla(h_l^a \otimes \mathbf{Q}^a\big) + sym(h_l^a\nabla\delta\mathbf{Q}^a) \simeq \delta\mathbf{E} + sym(\nabla h_l^a \otimes \mathbf{Q}^a) \ ,$$

with $\delta\mathbf{U}, \delta\mathbf{Q}^a$ as ε-macrofunctions together with all derivatives.
Using the above relations we obtain, manipulating each term in Eq. (4), the weak form of balance equations involving $\mathbf{U}$ and $\mathbf{Q}^a$. To this aim observe that after denotations

$$\mathbf{S}(\mathbf{x},t) := < \mathbf{s} >_{\mathbf{x}}^S, \quad \mathbf{H}^a(\mathbf{x},t) := < \mathbf{s}\cdot\nabla h^a >_{\mathbf{x}}^S \tag{6}$$

for every $\mathbf{x} \in \Omega^o$ we obtain from (4)

$$\int_{\Omega_S} \mathbf{s} : \boldsymbol{\delta}\mathbf{e} \mathrm{d}V \simeq \sum_{\mathbf{x}\in\Lambda} \int_{V_S(\mathbf{x})} \mathbf{s} : \boldsymbol{\delta}\mathbf{e} \mathrm{d}V \simeq \sum_{\mathbf{x}\in\Lambda} \left\{ \int_{V_S(\mathbf{x})} [\mathbf{s} : \boldsymbol{\delta}\mathbf{E} + \mathbf{s} : (\nabla h^a \otimes \boldsymbol{\delta}\mathbf{Q}^a)] \mathrm{d}V \right\}$$

$$\sum_{\mathbf{x}\in\Lambda} [< \mathbf{s} >_{\mathbf{x}} : \boldsymbol{\delta}\mathbf{E} + < \mathbf{s} \cdot \nabla h^a >_{\mathbf{x}} \cdot \boldsymbol{\delta}\mathbf{Q}^a] |V| \simeq \int_{\Omega} (\mathbf{S} : \boldsymbol{\delta}\mathbf{E} + \mathbf{H}^a \cdot \boldsymbol{\delta}\mathbf{Q}^a) \mathrm{d}V \ . \tag{7}$$

Because of Eq. (3) we also have

$$\mathbf{S} = < \mathbf{C} >^S : \mathbf{E} + < \mathbf{C} \cdot \nabla h^a >^S \cdot \mathbf{Q}^a \ , \tag{8}$$

$$\mathbf{H}^a = < \mathbf{C} \cdot \nabla h^a >^S : \mathbf{E} + < \mathbf{C} : (\nabla h^a \otimes \nabla h_l^B) >^S \cdot \mathbf{Q}^B \ . \tag{9}$$

The above equations are to be referred to as macroconstitutive equations.
We remark that the last two equations imply that both $\mathbf{S}$ and H^a are ε-macrofunctions.
In the same way we find:

$$\int_{\Omega_F} \rho \dot{\mathbf{u}} \cdot \boldsymbol{\delta}\mathbf{u} \mathrm{d}V + \int_{\Omega_S} \rho \dot{\mathbf{u}} \cdot \boldsymbol{\delta}\mathbf{u} \mathrm{d}V \simeq$$
$$\sum_{\mathbf{x}\in\Lambda} \int_{V_F(\mathbf{x})} \rho \left(\dot{\mathbf{U}} + h^a \dot{\mathbf{Q}}^a\right) \cdot \left(\boldsymbol{\delta}\mathbf{U} + h_l^b \boldsymbol{\delta}\mathbf{Q}^b\right) \mathrm{d}V + \sum_{\mathbf{x}\in\Lambda} \int_{V_S(\mathbf{x})} \rho \left(\dot{\mathbf{U}} + h^a \dot{\mathbf{Q}}^a\right) \cdot \left(\boldsymbol{\delta}\mathbf{U} + h_l^b \boldsymbol{\delta}\mathbf{Q}^b\right) \mathrm{d}V$$
$$\simeq \int_{\Omega} \left\{ \left(< \rho > \dot{\mathbf{U}} + < \rho h^a > \dot{\mathbf{Q}}^a \right) \cdot \boldsymbol{\delta}\mathbf{U} \right.$$
$$\left. + \left(< \rho h^a > \dot{\mathbf{U}} + < \rho h^a h_l^b > \dot{\mathbf{Q}}^b \right) \cdot \boldsymbol{\delta}\mathbf{Q}^a \right\} \mathrm{d}V \ , \tag{10}$$

$$\int_{\Omega_F} \rho \mathbf{b} \cdot \boldsymbol{\delta}\mathbf{u} \mathrm{d}V + \int_{\Omega_S} \rho \mathbf{b} \cdot \boldsymbol{\delta}\mathbf{u} \mathrm{d}V \simeq$$
$$\sum_{\mathbf{x}\in\Lambda} \int_{V_F(\mathbf{x})} [\rho \mathbf{b} \cdot \boldsymbol{\delta}\mathbf{U} + \rho \mathbf{b} \cdot h^a \boldsymbol{\delta}\mathbf{Q}^a] \mathrm{d}V + \sum_{\mathbf{x}\in\Lambda} \int_{V_S(\mathbf{x})} [\rho \mathbf{b} \cdot \boldsymbol{\delta}\mathbf{U} + \rho \mathbf{b} \cdot h^a \boldsymbol{\delta}\mathbf{Q}^a] \mathrm{d}V$$
$$\simeq \int_{\Omega} [< \rho \mathbf{b} > \cdot \boldsymbol{\delta}\mathbf{U} + < \rho \mathbf{b} h^a > \cdot \boldsymbol{\delta}\mathbf{Q}^a] \mathrm{d}V \ , \tag{11}$$

$$\int_{\Omega_F} p \mathrm{div}(\boldsymbol{\delta}\mathbf{u}) \mathrm{d}V \simeq \int_{\Omega} (-\nabla P \cdot \boldsymbol{\delta}\mathbf{U} + \mathbf{P}^a \cdot \boldsymbol{\delta}\mathbf{Q}^a) \ \mathrm{d}V \ , \tag{12}$$

where we have defined

$$P := < p >_{\mathbf{x}}^F, \ \mathbf{P}^a := < p \nabla h^a >_{\mathbf{x}}^F \ , \tag{13}$$

for every $\mathbf{x} \in \Omega^o$. Finally

$$\int_{\Omega} (\mathbf{S} : \boldsymbol{\delta}\mathbf{E} + \mathbf{H}^a \cdot \boldsymbol{\delta}\mathbf{Q}^a) \ \mathrm{d}V + \int_{\Omega} \left\{ \left(< \rho > \ddot{\mathbf{U}} + < \rho h^a > \ddot{\mathbf{Q}}^a \right) \cdot \boldsymbol{\delta}\mathbf{U} \right.$$
$$\left. + \left(< \rho h^a > \ddot{\mathbf{U}} + < \rho h^a h_l^b > \ddot{\mathbf{Q}}^b \right) \boldsymbol{\delta}\mathbf{Q}^a \right\} \mathrm{d}V$$
$$= \int_{\Omega} [(< \rho \mathbf{b} > \cdot \boldsymbol{\delta}\mathbf{U} + < \rho \mathbf{b} h^a > \cdot \boldsymbol{\delta}\mathbf{Q}^a) + (\nabla P \cdot \boldsymbol{\delta}\mathbf{U} + \mathbf{P}^a \cdot \boldsymbol{\delta}\mathbf{Q}^a)] \mathrm{d}V \tag{14}$$

for every $\boldsymbol{\delta}\mathbf{U}$, $\boldsymbol{\delta}\mathbf{Q}^a$, with

$$\boldsymbol{\delta}\mathbf{U}|_{\partial\Omega} = \mathbf{0}, \ \boldsymbol{\delta}\mathbf{Q}^a|_{\partial\Omega} = \mathbf{0} \ .$$

In this way we obtain the following system of equations of motion for macrodisplacements

$$\mathrm{div}\mathbf{S} - < \rho > \ddot{\mathbf{U}} - < \rho h^a > \ddot{\mathbf{Q}}^a + \nabla P + < \rho \mathbf{b} > = \mathbf{0} \ , \tag{15}$$

and evolution equations for macrointernal variables

$$< \rho h^a h_l^b > \ddot{\mathbf{Q}}^b + < \rho h^a > \ddot{\mathbf{U}} + \mathbf{H}^a + \mathbf{P}^a - < \rho \mathbf{b} h^a > = \mathbf{0} \ . \tag{16}$$

Eqs. (15), (16) together with (8), (9) represent the system of equations for macrodisplacements $\mathbf{U}$ and macrointernal variables $\mathbf{Q}^a$, provided that the fluid pressure in the inclusions is known. We remark that the coefficients of the system are constant. This fact is a consequence of the periodicity of the medium. Note that some of the above coefficients depend on the size of the microstructure since

$$< \rho h_l^a h_l^B > \in O(l^2) \text{ and } < \rho h_l^a > \in O(l) \ .$$

In order to obtain a full description of the medium, we have to introduce some extra constitutive specification relative to the fluid filling the cavities. This will be done in the subsequent section.

4
Evolution equations for a micro-structured model

In the sequel, we neglect the body forces and assume ρ to be piecewise constant so that

$$< \rho h_l^a > = 0 \ .$$

Under these assumptions equations (14) and (15) become:

$$\mathrm{div}\mathbf{S} - < \rho > \ddot{\mathbf{U}} + \nabla P = \mathbf{0} \ , \tag{17}$$

$$< \rho h^a h_l^B > \ddot{\mathbf{Q}}^b + \mathbf{H}^a + \mathbf{P}^a = \mathbf{0} \ , \tag{18}$$

with $\mathbf{S}$ and $\mathbf{H}^a$ given by (8) and (9).

We will distinguish two cases:

Compressible fluid

Denoting by p^0 and α the initial pressure and the compressibility parameter, respectively, we have

$$p = p^0 + \alpha \mathrm{div}\mathbf{u} \simeq p^0 + \alpha(\mathrm{div}\mathbf{U} + \nabla h_l^a \cdot \mathbf{Q}^a) \ .$$

Incompressible fluid

In this case the macroincompressibility condition will be derived by

$$< \mathrm{div}\mathbf{u} > = 0 \Rightarrow < 1 >^F \mathrm{div}\mathbf{U} + < \nabla h_l^a >^F \cdot \mathbf{Q}^a = 0 \ ,$$

and considerations will be restricted to the case in which p can be treated as an unknown macrofunction; hence

$$< p >^F = < 1 >^F p, \ < p \nabla h_l^a >^F = < \nabla h_l^a >^F p \ .$$

We remark that (see also [7]):

1. Equations (8) and (9) are regarded as macroconstitutive equations. It means that once the *msf* are chosen, all coefficients in (8) and (9) can be analytically determined by performing integration on the typical cell V. Therefore Eqs. (17) and (18) combined with (8) and (9) lead to a system of differential equations whose coefficients can be explicitly calculated. This is not the case when homogenized models [1, 2, 3] are considered: indeed the formulation of those models requires the solution of a special boundary-value problem at the microlevel, i.e. inside a cell V which is called the unit cell.

2. Since $< \rho h_l^a h_l^\beta > \in O(l^2)$ then Eqs. (18) take into account the effect of the microstructure size on a global behavior.
3. Equations (18) are ordinary differential equations in $\mathbf{Q}^a$. Hence, for the determination of the evolution of the macrointernal variable $\mathbf{Q}^a$ no boundary conditions are necessary.

5
The asymptotic model equations

In this section, we assume that the microstructure length parameter l tends to zero and hence we pass to what can be called an asymptotic approximation of the model proposed. Then Eqs. (14) and (15) combined with (7) and (8) become:

$$div(< \mathbf{C} >: \mathbf{E} + < \mathbf{C} \cdot \nabla h^a > \cdot \mathbf{Q}^a) - < \rho > \ddot{\mathbf{U}} + \nabla P = \mathbf{0} \ , \tag{19}$$

$$< \mathbf{C} \cdot \nabla h^a >: \mathbf{E} + < \mathbf{C} : (\nabla h^a \otimes \nabla h_l^b) > \cdot \mathbf{Q}^b + \mathbf{P}^a = \mathbf{0} \ . \tag{20}$$

In Eq. (20) the time derivatives have disappeared and therefore it allows for the determination of macrointernal variables $\mathbf{Q}^a$ in terms of macrodeformation $\mathbf{E}$. Using the positive-definiteness condition of strain-energy function it can be proved that $< \mathbf{C} : (\nabla h_l^a \otimes \nabla h_l^b) >$ represents an invertible linear transformation. Hence, from Eq. (20) we obtain

$$\mathbf{Q}^a = - < \mathbf{C} : (\nabla h_l^b \otimes \nabla h^a) >^{-1} \cdot (< \mathbf{C}.\nabla h_l^b >: \mathbf{E}) - < \mathbf{C} : (\nabla h_l^b \otimes \nabla h^a) >^{-1} \cdot \mathbf{P}^b \ . \tag{21}$$

After defining

$$\mathbf{C}^* = < \mathbf{C} > - < \mathbf{C}.\nabla h^a > \cdot < \mathbf{C} : (\nabla h_l^b \otimes \nabla h^a) >^{-1} \cdot < \mathbf{C}.\nabla h_l^b > \ , \tag{22}$$

Eq. (19) becomes

$$div(\mathbf{C}^* : \mathbf{E}) - div\left[< \mathbf{C} : (\nabla h_l^b \otimes \nabla h^a) >^{-1} \cdot \mathbf{P}^b\right] - < \rho > \mathbf{U} + \nabla P = 0 \ . \tag{23}$$

Hence, the asymptotic approximation of the model proposed leads to a system of partial differential equations for macrodisplacement $\mathbf{U}$. It can be also proved that the quadratic form $\mathbf{E} : (\mathbf{C}^* : \mathbf{E})$ with $\mathbf{C}^*$ defined by (22) is positive-definite.

6
Applications

We consider a two-dimensional problem, see Fig. 1, assuming that all fields are independent of the x_3 coordinate. Also assume that the solid phase is homogeneous and isotropic, hence

$$\mathbf{C} : \mathbf{E} = 2\mu\mathbf{E} + \lambda tr(\mathbf{E})\mathbf{I}$$

with constants μ and λ. To study the dispersion relation analyze a free vibration problem, setting:

$$\begin{aligned} \mathbf{U} &= a \sin(kx) \cos(\omega t)\mathbf{e}_1 \ , \\ \mathbf{Q}^1 &= a_1 \cos(kx) \cos(\omega t)\mathbf{e}_1 \ , \\ \mathbf{Q}^2 &= a_2 \cos(kx) \cos(\omega t)\mathbf{e}_2 \ , \end{aligned} \tag{24}$$

where $k = 2\pi/L$ is the wave number, ω is the angular frequency, and $\mathbf{e}_1$ and $\mathbf{e}_2$ are versors related to the x and x axes, respectively. Fields $\mathbf{U}, \mathbf{Q}^1$ and $\mathbf{Q}^2$ as functions of x are ε-macro-functions only if $l/L < \varepsilon$ which is the applicability condition of the model proposed. Using the denotation shown in Fig. 1 we choose as *msf* the following ones:

$$h^1(x,y) \ := h^1(x) = \begin{cases} h_1 - 2\frac{h_1}{g_1}x & \text{if } 0 \le x \le g_1 \ , \\ -h_1 + 2h_1\frac{x-g_1}{l_1-g_1} & \text{if } g_1 \le x \le l_1 \ , \end{cases}$$

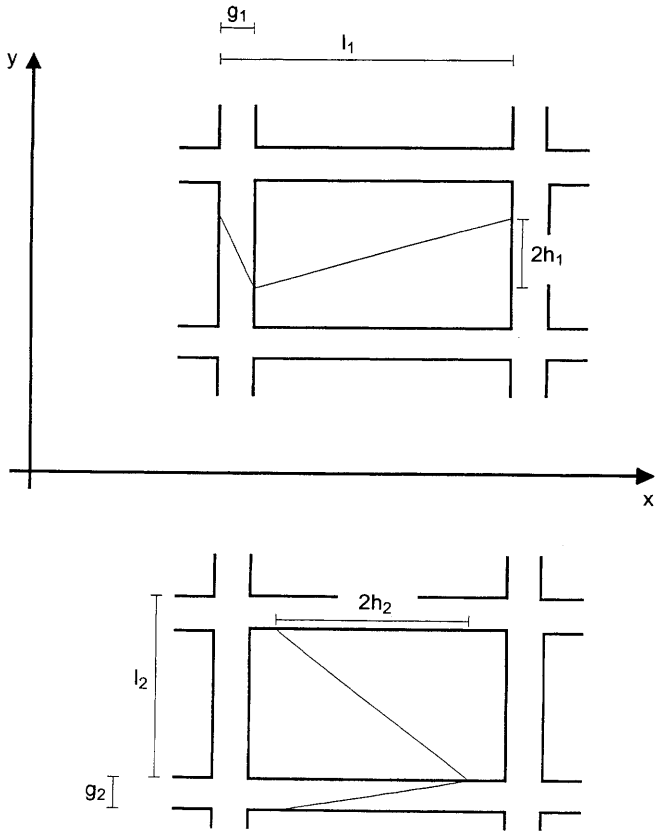


Fig. 1. Shape and size of a typical cell, plots of used microshape functions

$$h^2(x,y) \; := h^2(y) = \begin{cases} h_2 - 2\frac{h_2}{g_2}y & \text{if } 0 \le y \le g_2 \;, \\ -h_2 + 2h_2\frac{y-g_2}{l_2-g_2} & \text{if } g_2 \le y \le l_2 \;. \end{cases} \tag{25}$$

Substituting the right-hand side of (24) into Eqs. (17) and (18) combined with Eq. (8) and (9), the following system of equations for amplitudes a, a_1, a_2 can be derived for the compressible fluid:

$$\begin{aligned} (< c_1 > k^2 - < \rho > \omega^2)a + c_2 k a_1 + c_3 k a_2 = 0 \;, \\ c_2 k a + (c_5 - l^2 < \rho_1 > \omega^2)a_1 + c_6 k a_2 = 0 \;, \\ c_3 k a + c_6 a_1 + (c_9 - l^2 < \rho_2 > \omega^2)a_2 = 0 \;, \end{aligned} \tag{26}$$

where

$$< \rho > = \frac{1}{l_1 l_2}(\rho_s g_1 g_2 + \rho_F(l_1 - g_1)(l_2 - g_2));$$

$$\rho_1 = \frac{h_1^2}{3l_1}(\rho_S g_2 + \rho_F(l_1 - g_1))l^{-2} \;;$$

$$\rho_2 = \frac{h_2^2}{3l_2}(\rho_S g_2 + \rho_F(l_2 - g_2))l^{-2} \;;$$

$$c_1 = < 1 >^S (2\mu + \lambda) + < 1 >^F \alpha \;;$$

$$c_2 = 2 < \frac{h_1}{l_1 - g_1} >^F \alpha - 2 < \frac{h_1}{g_1} >^S (2\mu + \lambda) \;;$$

$$c_3 = 2 < \frac{h_2}{l_2 - g_2} >^F \alpha - 2 < \frac{h_2}{g_2} >^S \lambda \;;$$

$$c_5 = 4(< \frac{h_1}{l_1 - g_1} >^F)^2 \alpha + 4(< \frac{h_1}{g_1} >^S)^2 (2\mu + \lambda) \;;$$

$$c_6 = 2 < \frac{h_1}{l_1 - g_1} >^F \alpha - 2 < \frac{h_1}{g_1} >^S \lambda \ ;$$

$$c_9 = 4(< \frac{h_2}{l_2 - g_2} >^F)^2 \alpha + 4(< \frac{h_2}{g_2} >^S (2\mu + \lambda) \ .$$

Note that the values of $< \rho >, \rho_1, \rho_2, c_1 \ldots c_9$ are independent of l. The characteristic equation for the system (26) leads to the following polynomial equation in ω^2:

$$-l^4 d_0 (\omega^2)^3 + (d_1 l^2 + k^2 l^4 d_2)(\omega^2)^2 + (d_3 + k^2 l^2 d_4)\omega^2 + k^2 d_5 = 0 \ , \tag{27}$$

with the extra moduli

$$d_0 = < \rho > \rho_1 \rho_2 \ ;$$

$$d_1 = < \rho > (\rho_1 c_9 + \rho_2 c_5) \ ;$$

$$d_2 = c_1 \rho_1 \rho_2 \ ;$$

$$d_3 = < \rho > (c_6^2 - c_5 c_9) \ ;$$

$$d_4 = \rho_1 (c_3^2 - c_1 c_9) + \rho_2 (c_2^2 - c_5 c_1) \ ;$$

$$d_5 = -c_3^2 c_5 + 2 c_2 c_3 c_6 - c_1 c_6^2 - c_2^2 c_9 + c_1 c_5 c_9 \ ,$$

which are independent of the microstructure length parameter l.

Defining $q = kl := 2\pi \frac{l}{L}$, as a consequence of (25) we have $q \ll 1$. Expanding solutions of Eq. (27) in terms of q we can represent them in the following asymptotic form

$$\omega_1^2 = b_1 k^2 (1 + b_2 q^2) + O(q^4) \ ,$$
$$\omega_2^2 = l^{-2} b_3 + b_4 k^2 + O(q^4) \ , \tag{28}$$
$$\omega_3^2 = l^{-2} b_5 + b_6 k^2 + O(q^4) \ ,$$

with

$$b_1 = -\frac{d_5}{d_3}, \ b_2 = \frac{d_1 d_5 - d_3 d_4}{d_3^2} \ ,$$

$$b_3 = \frac{d_1 - \sqrt{d_1^2 + 4 d_0 d_3}}{2 d_0} \ ,$$

$$b_4 = \frac{d_1^2 d_2 + 2 d_0 d_2 d_3 + d_0 d_1 d_4 + 2 d_0^2 d_5 - (d_1 d_2 + d_0 d_4)\sqrt{d_1^2 + 4 d_0 d_3}}{d_0 (d_1^2 + 4 d_0 d_3 - d_1 \sqrt{d_1^2 + 4 d_0 d_3})} \ ,$$

$$b_5 = \frac{d_1 + \sqrt{d_1^2 + 4 d_0 d_3}}{2 d_0} \ ,$$

$$b_6 = \frac{d_1^2 d_2 + 2 d_0 d_2 d_3 + d_0 d_1 d_4 + 2 d_0^2 d_5 + (d_1 d_2 + d_0 d_4)\sqrt{d_1^2 + 4 d_0 d_3}}{d_0 (d_1^2 + 4 d_0 d_3 + d_1 \sqrt{d_1^2 + 4 d_0 d_3})} \ ,$$

where always $d_1^2 + 4 d_0 d_3 > 0$.

we note that solutions ω_2 and ω_3 depend on l^{-2} and represent higher vibration frequencies of the solid-fluid system under consideration.

This kind of behavior cannot be revealed in the framework of the asymptotic model described in Sec. 5.

In the case of inclusions filled by an incompressible fluid we must regard the pressure as an extra unknown. We have stated before that in this case the pressure can be treated as a macrofunction. Assuming $p = a_p \cos(kx_1) \sin(\omega t)$ after simple manipulations we obtain homogeneous system of equations:

$$(< c_1' > k^2 - < \rho > \omega^2) a + c_2' k a_1 + c_3' k a_2 + < 1 >^F k a_p = 0 \ ,$$

$$c'_2 ka + (c'_5 - l^2 < \rho_1 > \omega^2) a_1 + c'_6 k a_2 + < 2 \frac{h_1}{l_1 - g_1} >^F a_p = 0 \ ,$$

$$c'_3 ka + c'_6 a_1 + (c'_9 - l^2 < \rho_2 > \omega^2) a_2 + < 2 \frac{h_2}{l_2 - g_2} >^F a_p = 0 \ ,$$

$$ka + < 2 \frac{h_1}{l_1 - g_1} >^F a_1 + < 2 \frac{h_2}{l_2 - g_2} >^F a_2 = 0 \ , \tag{29}$$

where

$c'_i := c_i|_{\alpha=0}$, $i = 1, \ldots 6$, and 9 .

The non trivial solutions to (29) exist only if

$$(t_1 l^2 + k^2 l^4 t_2)(\omega^2)^2 + (t_3 + k^2 l^2 t_4)\omega^2 + k^2 t_5 = 0 \tag{30}$$

where

$$t_1 = < \rho > \left[< \rho_1 > 4 \left(\frac{h_2}{l_2 - g_2} \right)^2 + < \rho_2 > 4 \left(\frac{h_1}{l_1 - g_1} \right)^2 \right] \ ;$$

$$t_2 = < 1 >^F < \rho_1 >< \rho_2 > \ ;$$

$$t_3 = < \rho > \left[4 \left(\frac{h_2}{l_2 - g_2} \right)^2 c'_5 - 8 \frac{h_1 h_2}{(l_1 - g_1)(l_2 - g_2)} c'_6 + 4 \left(\frac{h_1}{l_1 - g_1} \right)^2 c'_9 \right] \ ;$$

$$t_4 = < \rho_1 > \left[c'_9 - 2 \left(\frac{h_2}{l_2 - g_2} \right) c'_3 + 4 \left(\frac{h_1}{l_1 - g_1} \right)^2 c'_1 \right]$$

$$+ < \rho_2 > \left[c'_5 - 2 \left(\frac{h_1}{l_1 - g_1} \right) c'_2 + 4 \left(\frac{h_2}{l_2 - g_2} \right)^2 c'_1 \right] \ ;$$

$$t_5 = (c'_6)^2 - c'_5 c'_9 + 2 \left(\frac{h_2}{l_2 - g_2} \right)(c'_3 c'_5 - c'_2 c'_6) + 2 \left(\frac{h_1}{l_1 - g_1} \right)(c'_2 c'_9 - c'_3 c'_6)$$

$$+ 4 \left(\frac{h_2}{l_2 - g_2} \right)^2 ((c'_2)^2 - c'_1 c'_5) + 4 \left(\frac{h_1}{l_1 - g_1} \right)^2 ((c'_3)^2 - c'_1 c'_9)$$

$$+ 8 \frac{h_1 h_2}{(l_1 - g_1)(l_2 - g_2)} (c'_1 c'_6 - c'_2 c'_3) \ .$$

Using the same asymptotic expansion as before, solutions to (30) can be given in the explicit form:

$$\omega_4^2 = -k^2 \frac{t_5}{t_3} \left[1 + q^2 \frac{t_1 t_5 - t_3 t_4}{t_3^2} \right] + O(q^4) \ , \tag{31}$$

$$\omega_5^2 = \frac{1}{l^2} \frac{t_3}{t_1} + k^2 \left[\frac{t_1^2 t^5 + t_1 t_3 t_4 - t_2 t_3^2}{t_1^2 t_3} \right] + O(q^4) \ . \tag{32}$$

We see that ω_5 represents the higher free vibration frequency.

7
Conclusions and numerical example

We start comparing the results obtained for the compressible case with the incompressible one. To this aim we have to calculate the values of ω_1^2, ω_2^2 and ω_3^2 when α tends to infinity. Indeed we observe that

$$t_i = \lim_{\alpha \to \infty} \frac{\mathrm{d_i}}{< 1 >^F \alpha}, i = 1, \ldots 5 \ .$$

This can be easily seen after dividing both sides of Eq. (27) by $< 1 >^F \alpha$ and then letting α go to infinity. Therefore we prove that the coefficient d_0 in Eq. (27) vanishes, and Eq. (27) turns into Eq. (30).

Consequently, if $\alpha \to \infty$ then $\omega_1^2 \to \omega_4^2$ and $\omega_2^2 \to \omega_5^2$. Moreover, we remark that the condition $\omega_3^2 \to \infty$ is consistent with the lowering of the order of the characteristic equations in the incompressible case.

We underline that:

i) the limit case $\alpha \to 0$ is physically meaningless with the present choice of *msf*.

ii) In the case of the asymptotic model the characteristic equation for ω^2 is of the first order. It can be obtained easily by setting $h_i = g_i = 0$, $i = 1, 2$ in Eqs. (27) and (30). In this case, the free vibration frequency for the compressible fluid is:

$$\omega_6^2 = -k^2 \frac{d_5}{d_3} \quad , \tag{33}$$

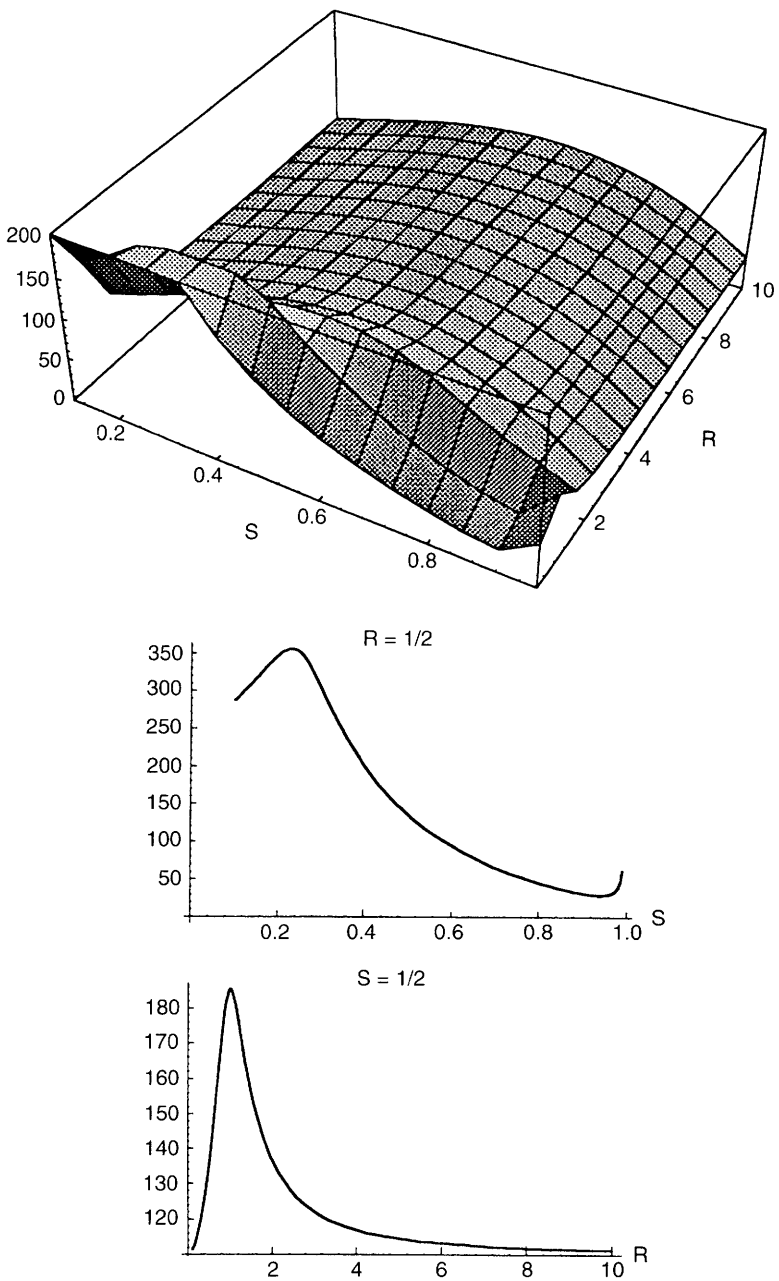


Fig. 2. Plots of b_3/b_1 as a function of the ratios $S: = g_1/l_1$ and $R: = l_2/l_1$ with $g_2/l_2 = 1/2$

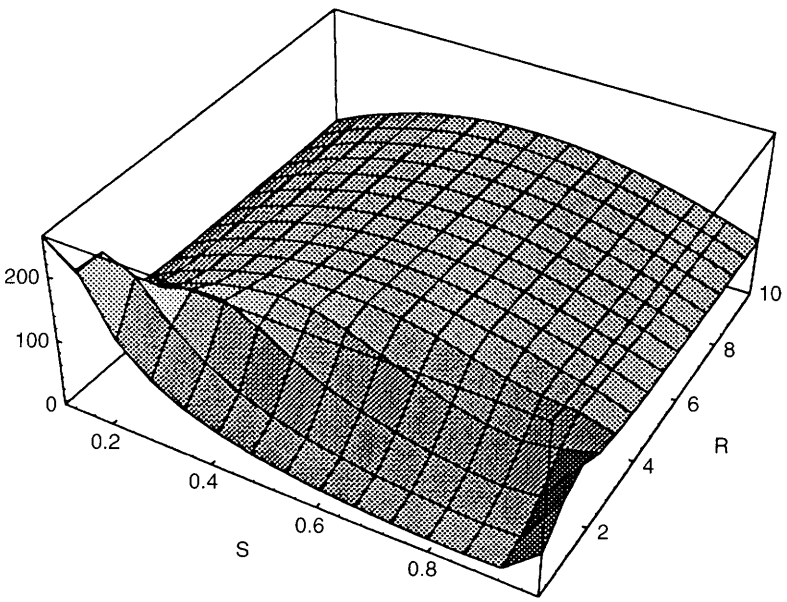
and for the incompressible one we obtain

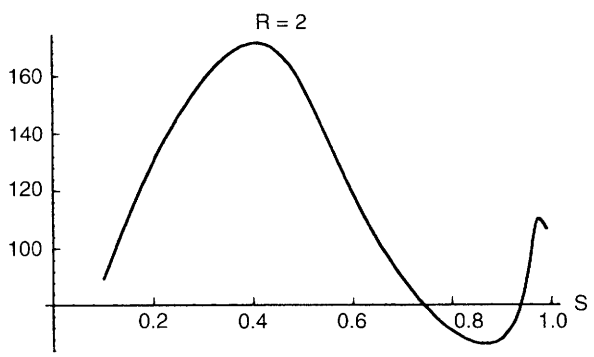
$$\omega_7^2 = -k^2 \frac{t_5}{t_3} \quad . \tag{34}$$

Let us compare the asymptotic solutions with the complete ones. To this extent we study the ratios:

$$\left(\frac{\omega_1}{\omega_6}\right)^2 = 1 + b_2 q^2 + \frac{1}{k^2} O(q^4) \ ,$$

$$\left(\frac{\omega_2}{\omega_6}\right)^2 = \frac{b_4}{b_1} + \frac{b_3}{b_1}\frac{1}{q^2} + \frac{1}{k^2} O(q^4) \ ,$$





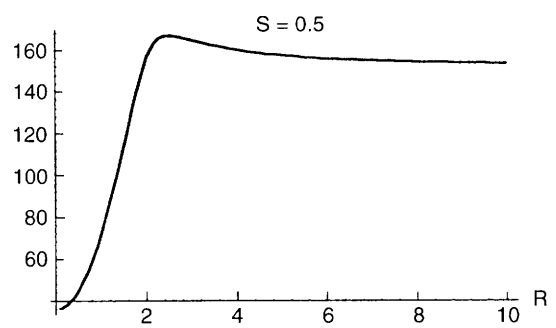


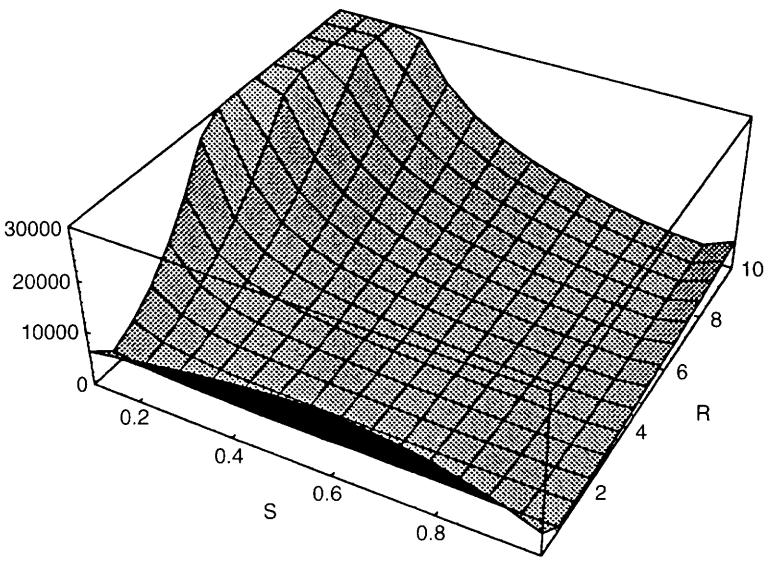
Fig. 3. The same as Fig. 2, with $g_2/l_2 = 1/4$

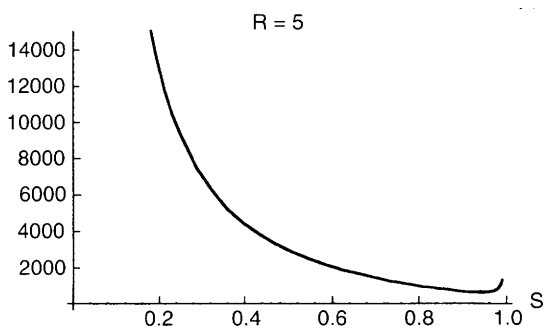
$$\left(\frac{\omega_3}{\omega_6}\right)^2 = \frac{b_6}{b_1} + \frac{b_5}{b_1}\frac{1}{q^2} + \frac{1}{k^2}O(q^4). \tag{35}$$

In this paper, we confine ourselves to waves verifying the so called 'macro-wave' approximation [12], i.e. we deal with waves for which not only $q \ll 1$, but also terms $\frac{1}{k^2}O(q^4)$ in Eqs. (35) are neglected.

To supply a numerical application we consider compressed concrete with water inclusions, setting: $\lambda = 4.91 \cdot 10^5/cm^2$, $\mu = 1.52 \cdot 10^6 N/cm^2$, $\alpha = 0.22 \cdot 10^6 N/cm^2$, $\rho_6/\rho_F = 2.2$. It is easily seen that the term b_2q^2 is negligible with respect to 1, and that b_4/b_1 (resp. b_6/b_1) are negligible with respect to $\frac{b_3}{b_1}q^{-2}$ (resp. $\frac{b_5}{b_1}q^{-2}$).

In Fig. 2–5 we show how the coefficients b_3/b_1 and b_5/b_1 vary in terms of the geometrical shape of the typical cell. Finally, we note that the used macromodelling procedure allows to obtain the aforementioned results using analytical methods only, without any numerical simulation. It is worth to emphasize the fact how relatively easy it is to obtain meaningful applications of the proposed model in many technologically interesting problem which cannot be investigated within the framework of asymptotic models.





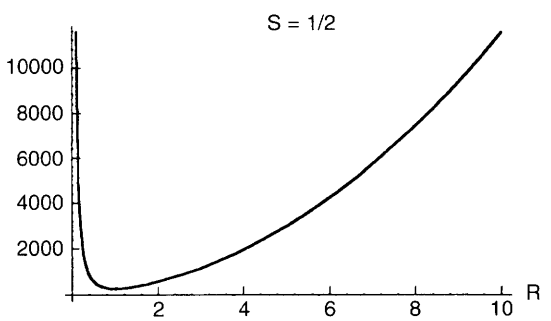


Fig. 4. Plots of b_5/b_1 as a function of the ratios $S: = g_1/l_1$ and $R: = l_2/l_1$ with $g_2/l_2 = 1/2$

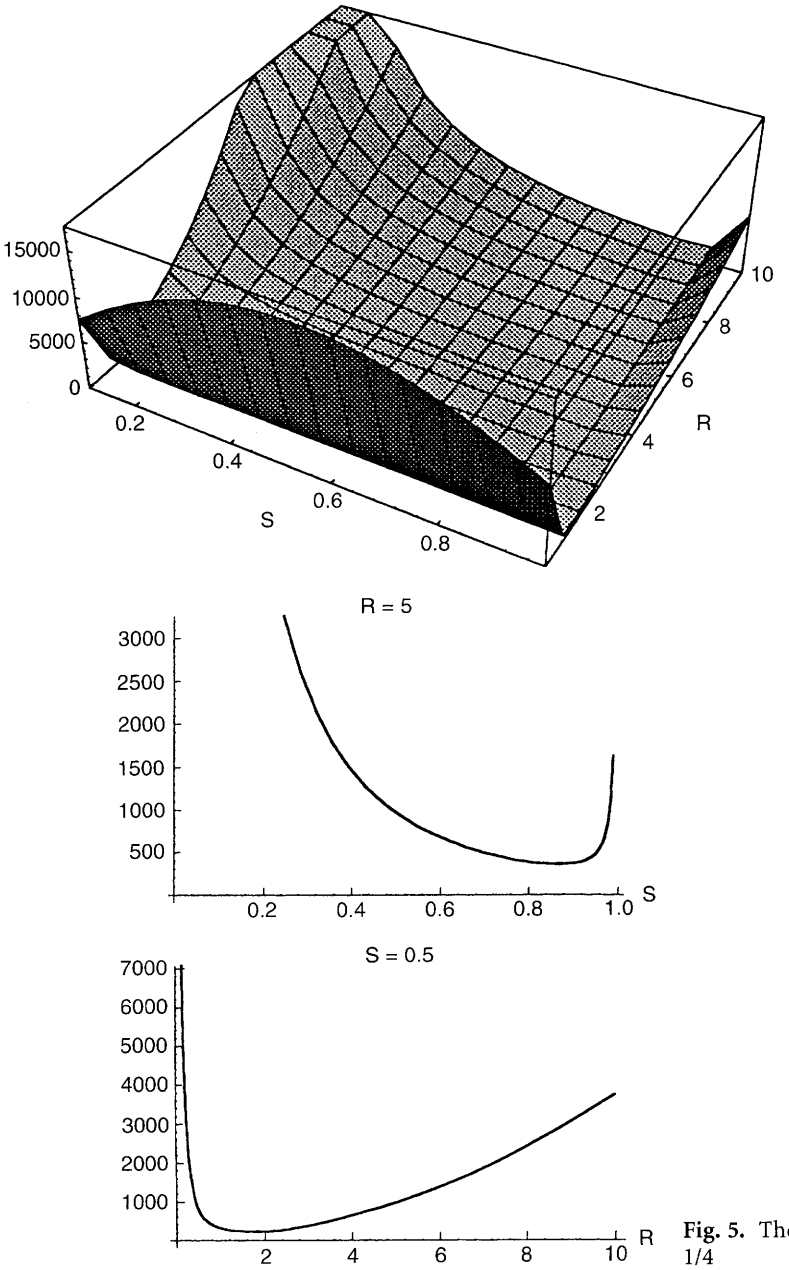


Fig. 5. The same as Fig. 4, with $g_2/l_2 = 1/4$

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