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A GENERALIZATION OF A RESULT CONCERNING THE ASYMPTOTIC BEHAVIOR OF FINITE MARKOV CHAINS

ALINA NICOLAIE

ABSTRACT. In this paper we give, in a more general context than previous studies, sufficient conditions for weak, strong and C-strong ergodicity of a finite state nonhomogeneous Markov chain in terms of similar properties of a certain Markov chain of smaller size.

AMS Subject Classification: 60J10.

Key words: weak ergodicity; strong ergodicity; C-strong ergodicity; finite Markov chain.

1. PRELIMINARIES

Consider a finite homogeneous Markov chain with state space $S = \{1, \dots, r\}$ and transition matrix P . We shall refer to it as the finite Markov chain P . Assume that the stochastic matrix P has $p \geq 1$ irreducible aperiodic closed classes or equivalently, ergodic classes, and, perhaps transient states, so that it has the form

$$(1.1) \quad P = \begin{pmatrix} S_{(1)} & 0 & \dots & 0 & 0 \\ 0 & S_{(2)} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & S_{(p)} & 0 \\ L_{(1)} & L_{(2)} & \dots & L_{(p)} & T \end{pmatrix},$$

where $S_{(i)}$ are $r_i \times r_i$ transition matrices, $1 \leq i \leq p$, associated with the p irreducible aperiodic closed classes, T concerns the transitions of the chain as long as it stays in the $r - \sum_{t=1}^p r_t$ transient states and the $L_{(i)}$ concern transitions from the transient states into the ergodic sets S_i , $1 \leq i \leq p$.

Definition 1.1. A probability distribution $\mu = (\mu_1, \dots, \mu_r)$ is said to be invariant with respect to an $r \times r$ stochastic matrix P if $\mu P = \mu$.

We shall need the following result

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THEOREM 1.2. Consider a finite homogeneous Markov chain with state space S and transition matrix P of the form (1.1). Then

$$(1.2) \quad \lim_{n \rightarrow \infty} P^n = \begin{pmatrix} \Gamma_1 & 0 & \dots & 0 & 0 \\ 0 & \Gamma_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \Gamma_p & 0 \\ \Omega_1 & \Omega_2 & \dots & \Omega_p & 0 \end{pmatrix},$$

where Γ_i is a strictly positive $r_i \times r_i$ matrix, $1 \leq i \leq p$; each row of the matrix Γ_i is the invariant probability vector $\mu^{(i)} = (\mu_1^{(i)}, \dots, \mu_{r_i}^{(i)})$ with respect to $S_{(i)}$, $1 \leq i \leq p$, and

$$\Omega_i = \begin{pmatrix} \mu_1^{(i)} a_{r_1+r_2+\dots+r_p+1,i} & \dots & \mu_{r_i}^{(i)} a_{r_1+r_2+\dots+r_p+1,i} \\ \dots & \dots & \dots \\ \mu_1^{(i)} a_{r,i} & \dots & \mu_{r_i}^{(i)} a_{r,i} \end{pmatrix}$$

is an $(r - \sum_{t=1}^p r_t) \times r_i$ matrix, where a_{ji} = probability that the chain will enter and thus, will be absorbed in S_i given that the initial state is j , $\sum_{t=0}^p r_t \leq j \leq r$, $1 \leq i \leq p$ [with convention $r_0 = 1$].

Proof. For the form of Γ_i , $1 \leq i \leq p$, see, e.g., Iosifescu (1980, p. 123) and for Ω_i , $1 \leq i \leq p$, see, e.g., Karlin et Taylor (1975, p. 91). \square

Remark 1.3. Clearly, $a_{ji} \geq 0$, $\sum_{t=0}^p r_t \leq j \leq r$, $1 \leq i \leq p$ and

$$(1.3) \quad \sum_{i=1}^p a_{ji} = 1, \sum_{t=0}^p r_t \leq j \leq r.$$

A vector $x \in \mathbf{C}^n$ will be understood as a row vector and x' denotes the transpose of x . Set $\mathbf{e} = (1, 1, \dots, 1)$ and let $(e_i)_{1 \leq i \leq r}$ be the canonical basis of the linear space \mathbf{R}^r .

THEOREM 1.4 (Nicolaie (2008a)). Let $A = -I_r + P$ with P of the form (1.1). Then there exists a nonsingular complex $r \times r$ matrix Q such that

$$(1.4) \quad A = QJQ^{-1},$$

where J is a Jordan $r \times r$ matrix, and Q reads as

$$Q = \begin{matrix} r_1 \\ r_2 \\ \vdots \\ r_p \\ r - \sum_{t=0}^p r_t \end{matrix} \left\{ \begin{array}{cccccc} 1 & 0 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & 0 & \dots \\ 0 & 1 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 & \dots \\ 0 & 0 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \dots & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & \dots \\ a_{r_1+r_2+\dots+r_p+1,1} & a_{r_1+r_2+\dots+r_p+1,2} & \dots & a_{r_1+r_2+\dots+r_p+1,p} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{r,1} & a_{r,2} & \dots & a_{r,p} & \dots \end{array} \right\},$$

where the first column contains 1 in the first r_1 rows, the next $p-1$ columns contain 1 in the $r_{i-1}+1, \dots, r_i$ rows, $2 \leq i \leq p$, and the last $r-p$ columns comprise complex numbers. For a_{ji} , $\sum_{t=0}^p r_t \leq j \leq r$, $1 \leq i \leq p$, we have the meaning given in Theorem 1.2. The inverse Q^{-1} has the form

$$Q^{-1} = \begin{pmatrix} \mu_1^{(1)} & \dots & \mu_{r_1}^{(1)} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & \dots & 0 & \mu_1^{(p)} & \dots & \mu_{r_p}^{(p)} & 0 & \dots & 0 \\ q_{p+1,1} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & q_{p+1,r} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ q_{r,1} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & q_{rr} \end{pmatrix},$$

where $\mu^{(i)} = (\mu_1^{(i)}, \dots, \mu_{r_i}^{(i)})$ is the invariant probability vector with respect to $S_{(i)}$, $1 \leq i \leq p$, and the last $r-p$ rows comprise complex numbers.

Proof. See Nicolaie (2008a) and also Gidas (1985). \square

Remark 1.5 (Nicolaie (2008a)). (i) We shall need some spectral properties of A , where $A = -I_p + P$ with P of the form (1.1). We have

$$(1.5) \quad \lambda_1 = 0$$

is an eigenvalue of A whose algebraic multiplicity is equal to its geometric multiplicity and equal to p . All other distinct eigenvalues $\lambda_2, \dots, \lambda_{l+s}$ of A satisfy

$$(1.6) \quad |\lambda_i + 1| < 1, \quad 2 \leq i \leq l + s.$$

(ii) From (1.5) (see Horn et Johnson (1985, pp. 129-131)) we have

$$J = \begin{pmatrix} J_1 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & J_2 & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & J_l & 0 & \dots & \dots \\ 0 & 0 & \dots & 0 & J_{l+1} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & J_{l+s} \end{pmatrix},$$

where $J_1 = \mathbf{0}_{p \times p}$, J_k is a diagonal $m_k \times m_k$ matrix with entries the eigenvalues λ_k whose algebraic and geometric multiplicities are identical, $2 \leq k \leq l$, and

$$J_{l+i} = \begin{pmatrix} \lambda_{l+i} & \varepsilon_1^{(i)} & 0 & \dots & \dots & 0 \\ 0 & \lambda_{l+i} & \varepsilon_2^{(i)} & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \lambda_{l+i} & \varepsilon_{m_{l+i}-1}^{(i)} \\ 0 & 0 & \dots & \dots & 0 & \lambda_{l+i} \end{pmatrix}$$

are $m_{l+i} \times m_{l+i}$ matrices corresponding to eigenvalues whose geometric multiplicities are smaller than their algebraic multiplicities and $\varepsilon_t^{(i)} \in \{0, 1\}$, $1 \leq t \leq m_{l+i} - 1$, $1 \leq i \leq s$. Clearly, $p + m_2 + \dots + m_{l+s} = r$.

If $A = (A_{ij})$ is an $m \times n$ matrix, then for $M \subseteq \{1, \dots, m\}$, $N \subseteq \{1, \dots, n\}$, $M, N \neq \emptyset$, define $A_{M \times N} = (A_{ij})_{i \in M, j \in N}$. Define, also, the matrix norm (see, e.g., Horn et Johnson (1985, p. 295)) $\|A\|_\infty = \max_{1 \leq i \leq m} \sum_{j=1}^n |A_{ij}|$. Note that such a norm tends to zero if and only if all elements of A tend to zero.

Let $(P_n)_{n \geq 1}$ be a sequence of stochastic $r \times r$ matrices. Set $S = \{1, 2, \dots, r\}$. For all integers $m \geq 0$, $n > m$, define $P_{m,n} = P_{m+1}P_{m+2}\dots P_n = ((P_{m,n})_{ij})_{i,j \in S}$. In the context of finite Markov chains, the matrix $P_{m,n}$ has a very simple probabilistic interpretation, namely being the $n - m$ step transition matrix starting at m .

Definition 1.6 (see, e.g., Isaacson et Madsen (1976, p. 144)). Let P be a stochastic $r \times r$ matrix. The ergodic coefficient of P , denoted by $\delta(P)$, is defined by $\delta(P) = 1 - \min_{1 \leq i, k \leq r} \sum_{j=1}^r \min(P_{ij}, P_{kj})$.

THEOREM 1.7 (see, e.g., Isaacson et Madsen (1976, p. 147)). *Let R be a real $m \times n$ matrix with $Re' = \mathbf{0}$ and P be a stochastic $n \times p$ matrix. Then $\|RP\|_\infty \leq \|R\|_\infty \delta(P)$.*

Proof. See, e.g., Isaacson et Madsen (1976, p. 147). \square

Definition 1.8 (see, e.g., Iosifescu (1980, p. 217)). A sequence of stochastic $r \times r$ matrices $(P_n)_{n \geq 1}$ is said to be weakly ergodic if $\forall m \geq 0, \forall i, j, k \in S, \lim_{n \rightarrow \infty} [(P_{m,n})_{ik} - (P_{m,n})_{jk}] = 0$.

A stochastic matrix whose rows are identical is said to be stable.

THEOREM 1.9. *Let $(P_n)_{n \geq 1}$ be a sequence of stochastic $r \times r$ matrices. Then the following statements are equivalent:*

- (i) *the sequence $(P_n)_{n \geq 1}$ is weakly ergodic;*
- (ii) *there exist stable stochastic $r \times r$ matrices $\Pi_{m,n}$, $m \geq 0$, $n \geq 1$ such that $\lim_{n \rightarrow \infty} (P_{m,n} - \Pi_{m,n}) = \mathbf{0}$, $\forall m \geq 0$;*
- (iii) *$\lim_{n \rightarrow \infty} \delta(P_{m,n}) = 0$, $\forall m \geq 0$.*

Proof. For (i) \Leftrightarrow (ii) see, e.g., Iosifescu (1980, p. 218) and for (i) \Leftrightarrow (iii) see, e.g., Isaacson et Madsen (1976, p. 149). \square

Definition 1.10 (see, e.g., Iosifescu (1980, p. 223)). A sequence of stochastic $r \times r$ matrices $(P_n)_{n \geq 1}$ is said to be strongly ergodic if $\forall m \geq 0$, $\forall i, j \in S$, the limit $\lim_{n \rightarrow \infty} (P_{m,n})_{ij} =: (\pi_m)_j$ exists and does not depend on i .

Remark 1.11 (see, e.g., Iosifescu (1980, p. 223)). It is easy to prove that if $\lim_{n \rightarrow \infty} (P_{m,n})_{ij}$ exists as stated in Definition 1.10, then the $(\pi_m)_j$ are also independent of $m \geq 0$. Therefore, a sequence of stochastic matrices $(P_n)_{n \geq 1}$ is strongly ergodic if and only if there exists a stable stochastic matrix Π such that $\lim_{n \rightarrow \infty} (P_{m,n} - \Pi) = \mathbf{0}$, $\forall m \geq 0$.

Definition 1.12 (Isaacson et Madsen (1976)). A sequence of stochastic matrices $(P_n)_{n \geq 1}$ is said to be C-strongly ergodic if there exists a stable stochastic matrix Π such that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n P_{0,k} = \Pi$.

THEOREM 1.13 (Isaacson et Madsen (1976)). *Let $(P_n)_{n \geq 1}$ be a sequence of stochastic matrices such that, for some fixed d , $\lim_{n \rightarrow \infty} P_{0,nd+t} = W_t$, $1 \leq t \leq d$. Then the limit $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n P_{0,k}$ exists and equals $W = \frac{1}{d}[W_1 + \dots + W_d]$.*

Proof. See Isaacson et Madsen (1976, p. 184). \square

Remark 1.14. Note that in Theorem 1.13 the limiting matrices W_t of the Cesaro averages are not necessarily stable matrices. But if they are stable, this theorem provides sufficient conditions for the C-strong ergodicity of the sequence $(P_n)_{n \geq 1}$.

The next two results will only be used in Section 2 of this paper.

PROPOSITION 1.15 (see, e.g., Isaacson et Madsen (1976, p. 29)). *Let $(a_{nk})_{n,k \geq 1}$ be a doubly indexed sequence of real numbers such that $\lim_{n \rightarrow \infty} a_{nk} = a_k$ exists, for all $k \geq 1$. If there exists a sequence of nonnegative numbers $(b_k)_{k \geq 1}$ such that $|a_{nk}| \leq b_k$, for all $n \geq 1$, $k \geq 1$ and $\sum_{k=1}^{\infty} b_k < \infty$, then $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} a_{nk} = \sum_{k=1}^{\infty} a_k$.*

Proof. See, e.g., Isaacson et Madsen (1976, p. 29). \square

PROPOSITION 1.16 (see, e.g., Isaacson et Madsen (1976, pp. 34-36)). *Let $(a_n)_{n \geq 0}$ be a sequence of real numbers convergent to 0 and $\sum_{n=0}^{\infty} b_n$ an absolute convergent series. Then $\lim_{n \rightarrow \infty} \sum_{i=0}^n a_i b_{n-i} = 0$.*

Proof. See, e.g., Isaacson et Madsen (1976, pp. 34-36). \square

2. SOME RESULTS ON RECURRENCE RELATIONS

In this section we give some results related to sequences defined by recurrence relations. First, we shall need the following

PROPOSITION 2.1. *Let $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ be two sequences of nonnegative real numbers with $\lim_{n \rightarrow \infty} b_n = 0$. If*

$$(2.1) \quad a_{s+d} \leq \alpha a_s + b_s, \quad s \geq 0,$$

where d stands for a natural number, $d \geq 1$ and $\alpha \in [0, 1)$, then

$$(2.2) \quad \lim_{n \rightarrow \infty} a_n = 0.$$

Proof. Applying the recurrence inequality (2.1) successively we obtain

$$(2.3) \quad a_{td+l} \leq \alpha^t a_l + \sum_{k=0}^{t-1} \alpha^{t-k-1} b_{kd+l}, \quad t \geq 0, \quad 0 \leq l \leq d-1.$$

Since $\alpha \in [0, 1)$, we have $\lim_{t \rightarrow \infty} \alpha^t = 0$. Moreover, the series $\sum_{t=0}^{\infty} \alpha^t < \infty$ (the geometrical series). Also, since $\lim_{n \rightarrow \infty} b_n = 0$, all the subsequences of $(b_n)_{n \geq 0}$ are convergent with limit zero; in particular, $\lim_{t \rightarrow \infty} b_{td+l} = 0$, $0 \leq l \leq d-1$. Applying Proposition 1.16 it follows that $\lim_{t \rightarrow \infty} \sum_{k=0}^{t-1} \alpha^{t-k-1} b_{kd+l} = 0$. Next, letting $t \rightarrow \infty$ in (2.3), it follows $a_{td+l} \rightarrow 0$, $0 \leq l \leq d-1$. The conclusion follows. \square

The next result is a generalization of Proposition 2.2 from Nicolaie (2008b).

PROPOSITION 2.2. *Let $(X_n)_{n \geq 1}$ and $(R_n)_{n \geq 1}$ be two sequences of real vectors, each vector X_n and R_n having p components, $R_n \mathbf{e}' = \mathbf{0}$, $n \geq 1$ and $\sum_{n=1}^{\infty} \|R_n\|_{\infty} < \infty$. Let $(C_n)_{n \geq 1}$ be a sequence of stochastic $p \times p$ matrices and set $\tilde{C}_n^{(l)} = C_{(n-1)d+l+1}$, for some fixed $d, n \geq 1, 0 \leq l \leq d-1$. Suppose that*

$$(2.4) \quad X_{s+d} = X_s C_s + R_s, \quad s \geq 1,$$

Then the following statements hold:

(i) *If $X_n \mathbf{e}' = \mathbf{0}$, $n \geq 1$, and $(\tilde{C}_n^{(l)})_{n \geq 1}$ is weakly ergodic, $0 \leq l \leq d-1$, then $\lim_{n \rightarrow \infty} X_n =$*

$\mathbf{0}$.

(ii) If $X_n \mathbf{e}' = 1$, $n \geq 1$ and $(\tilde{C}_n^{(l)})_{n \geq 1}$ is strongly ergodic with $\lim_{n \rightarrow \infty} \tilde{C}_{m,n}^{(l)} = \Pi = \mathbf{e}' \cdot \pi$, $m \geq 0$, $0 \leq l \leq d-1$, then $\lim_{n \rightarrow \infty} X_n = \pi$.

(iii) If $X_n \mathbf{e}' = 1$, $n \geq 1$ and $(\tilde{C}_n^{(l)})_{n \geq 1}$ is strongly ergodic with $\lim_{n \rightarrow \infty} \tilde{C}_{m,n}^{(l)} = \Pi_{(l+1)} = \mathbf{e}' \cdot \pi^{(l+1)}$, $m \geq 0$, $0 \leq l \leq d-1$, then $\lim_{n \rightarrow \infty} X_{nd+l+1} = \pi^{(l+1)}$, $0 \leq l \leq d-1$.

Proof. Applying the recurrence relation (2.4) successively we obtain

$$(2.5) \quad X_{(t+1)d+l+1} = X_{l+1} \tilde{C}_{0,t+1}^{(l)} + \left[R_{td+l+1} + \sum_{k=0}^{t-1} R_{kd+l+1} \tilde{C}_{k+1,t+1}^{(l)} \right], t \geq 1, 0 \leq l \leq d-1.$$

First, assuming that $(\tilde{C}_n^{(l)})_{n \geq 1}$ is weakly ergodic, $0 \leq l \leq d-1$, we shall prove that

$$(2.6) \quad \lim_{t \rightarrow \infty} \sum_{k=0}^{t-1} R_{kd+l+1} \tilde{C}_{k+1,t+1}^{(l)} = \mathbf{0}, 0 \leq l \leq d-1.$$

We have

$$\left\| \sum_{k=0}^{t-1} R_{kd+l+1} \tilde{C}_{k+1,t+1}^{(l)} \right\|_{\infty} \leq \sum_{k=0}^{t-1} \left\| R_{kd+l+1} \tilde{C}_{k+1,t+1}^{(l)} \right\|_{\infty} \leq$$

(using Theorem 1.7)

$$\leq \sum_{k=0}^{t-1} \left\| R_{kd+l+1} \right\|_{\infty} \delta(\tilde{C}_{k+1,t+1}^{(l)}).$$

Next, choose $a_{tk}^{(l)} = \left\| R_{kd+l+1} \right\|_{\infty} \delta(\tilde{C}_{k+1,t+1}^{(l)})$, $t, k \geq 1$ (take $a_{tk}^{(l)} = 0$ if $k > t$), $0 \leq l \leq d-1$. Then, by Theorem 1.9 (iii), it follows that $\lim_{t \rightarrow \infty} a_{tk}^{(l)} = 0$, $k \geq 1$, $0 \leq l \leq d-1$. Moreover, $|a_{tk}^{(l)}| \leq b_k^{(l)} := \left\| R_{kd+l+1} \right\|_{\infty}$, $t \geq 1$, since $\delta(\tilde{C}_{k,t}^{(l)}) \leq 1$, $0 \leq k < t$, $0 \leq l \leq d-1$. The conditions of Proposition 1.15 are fulfilled, so $\lim_{t \rightarrow \infty} \sum_{k=1}^t a_{tk}^{(l)} = \lim_{t \rightarrow \infty} \sum_{k=1}^{\infty} a_{tk}^{(l)} = 0$, $0 \leq l \leq d-1$, which means (2.6).

(i) Let $l \in \{0, 1, \dots, d-1\}$. By Theorem 1.9 (ii), it follows that there exists a sequence of stable stochastic matrices $\Pi_{m,t}^{(l)}$ such that $\lim_{t \rightarrow \infty} (\tilde{C}_{m,t}^{(l)} - \Pi_{m,t}^{(l)}) = \mathbf{0}$.

Letting $t \rightarrow \infty$ in (2.5), using $X_t \mathbf{e}' = 0$, $t \geq 1$, $\sum_{n=1}^{\infty} \left\| R_n \right\|_{\infty} < \infty$ and (2.6), we get $\lim_{t \rightarrow \infty} X_{(t+1)d+l+1} = X_{l+1} \lim_{t \rightarrow \infty} (\tilde{C}_{0,t+1}^{(l)} - \Pi_{0,t+1}^{(l)}) = \mathbf{0}$. (because $X_{l+1} \Pi_{m,t}^{(l)} = \mathbf{0}$, $0 \leq m < t$). Since l was chosen arbitrarily, it follows $\lim_{t \rightarrow \infty} X_{(t+1)d+l+1} = \mathbf{0}$, $0 \leq l \leq d-1$. Therefore, the sequence $(X_n)_{n \geq 1}$ is convergent and $\lim_{n \rightarrow \infty} X_n = \mathbf{0}$.

(ii) Letting $t \rightarrow \infty$ in (2.5), using $X_t \mathbf{e}' = 1$, $t \geq 1$, $\sum_{n=1}^{\infty} \left\| R_n \right\|_{\infty} < \infty$ and (2.6), we get $\lim_{t \rightarrow \infty} X_{(t+1)d+l+1} = X_{l+1} \lim_{t \rightarrow \infty} \tilde{C}_{0,t+1}^{(l)} = X_{l+1} \Pi = \pi$, $0 \leq l \leq d-1$. Therefore, the sequence $(X_n)_{n \geq 1}$ is convergent and $\lim_{n \rightarrow \infty} X_n = \pi$.

(iii) Letting $t \rightarrow \infty$ in (2.5), using $X_t \mathbf{e}' = 1$, $t \geq 1$, $\sum_{n=1}^{\infty} \left\| R_n \right\|_{\infty} < \infty$ and (2.6), we get $\lim_{t \rightarrow \infty} X_{(t+1)d+l+1} = X_{l+1} \lim_{t \rightarrow \infty} \tilde{C}_{0,t+1}^{(l)} = X_{l+1} \Pi_{(l+1)} = \pi^{(l+1)}$, $0 \leq l \leq d-1$. \square

3. WEAK, STRONG AND C-STRONG ERGODICITY RESULTS

In this section a previous study of the author from Nicolaie (2008a) and Nicolaie (2008b) is continued. We give sufficient conditions for weak, strong and C-strong ergodicity of a finite state nonhomogeneous Markov chain in terms of similar behavior of a certain nonhomogeneous Markov chain of smaller size. Our main result is given in Theorem 3.4.

In the sequel, we shall consider a nonhomogeneous Markov chain with state space $S = \{1, 2, \dots, r\}$ and transition matrices $(P_n)_{n \geq 1}$. We shall refer to it as the finite Markov chain $(P_n)_{n \geq 1}$. Assume that $\exists d \geq 1$ such that $\lim_{n \rightarrow \infty} P_{n, n+d} = P$.

Suppose that P has $p \geq 1$ ergodic classes S_i , $1 \leq i \leq p$, and, perhaps, transient states, i.e., P is of the form (1.1). Let $\mu^{(i)}$ be the invariant probability vector with respect to S_i , $1 \leq i \leq p$, and a_{ji} , $\sum_{t=0}^p r_t \leq j \leq r$, $1 \leq i \leq p$ as in Theorem 1.2.

The following Remark will illustrate several aspects that characterize this type of nonhomogeneous Markov chains.

Remark 3.1. (i) Obviously, the condition $\lim_{n \rightarrow \infty} P_{n, n+d} = P$, for some $d > 1$, where P is of the form (1.1), does not imply the existence of $\lim_{n \rightarrow \infty} P_n$. An example is the chain $(P_n)_{n \geq 1}$, where

$$P_{2n-1} = \begin{pmatrix} \frac{6}{8} - \frac{1}{8n} & \frac{2}{8} + \frac{1}{8n} & 0 \\ \frac{1}{8} + \frac{2}{8n} & \frac{7}{8} - \frac{2}{8n} & 0 \\ 0 & \frac{1}{n} & 1 - \frac{1}{n} \end{pmatrix}, \quad P_{2n} = \begin{pmatrix} \frac{4}{10} - \frac{1}{10n} & \frac{6}{10} + \frac{1}{10n} & 0 \\ \frac{3}{10} + \frac{3}{10n^2} & \frac{7}{10} - \frac{3}{10n^2} & 0 \\ 0 & \frac{1}{n} & 1 - \frac{1}{n} \end{pmatrix},$$

$\forall n \geq 1$. We see that $\lim_{n \rightarrow \infty} P_n$ does not exist, but there exists $\lim_{n \rightarrow \infty} P_{n, n+2} = P$, where

$$P = \begin{pmatrix} \frac{3}{8} & \frac{5}{8} & 0 \\ \frac{25}{80} & \frac{55}{80} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(ii) A remarkable example for the case when $\lim_{n \rightarrow \infty} P_n$ exists is $\lim_{n \rightarrow \infty} P_n =: R$, where R has $p \geq 1$ irreducible closed periodic classes S_i , $1 \leq i \leq p$, and, possibly, transient states, S_i having period $d_i \geq 1$, $1 \leq i \leq p$. There exists d (e.g., $d = \text{lcm}\{d_i | 1 \leq i \leq p\} \geq 1$) such that $\lim_{n \rightarrow \infty} P_{n, n+d} = R^d$, where R^d has only ergodic classes and, perhaps, transient states, as the next example shows

Let $(P_n)_{n \geq 1}$ given by

$$P_n = \begin{pmatrix} 0 & 1 - \frac{1}{n} & 0 & \frac{1}{n} \\ 1 - \frac{1}{n} & 0 & 0 & \frac{1}{n} \\ 0 & 0 & 1 - \frac{1}{n} & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}, n \geq 1.$$

In this section we prove an ergodic theorem for nonhomogeneous Markov chains of this types.

Let $m \geq 0$. By the Chapman-Kolmogorov equation we have $P_{m,n+d} = P_{m,n}P_{n,n+d}$, $n > m$. Subtracting $P_{m,n}$ from both sides, we obtain

$$(3.1) \quad P_{m,n+d} - P_{m,n} = P_{m,n}[-I_r + P_{n,n+d}], n > m.$$

Setting

$$(3.2) \quad z_{m,n}^{(i)} = ((P_{m,n})_{i,1}, \dots, (P_{m,n})_{i,r}), n > m, i \in S,$$

then equations (3.1) read as $z_{m,n+d}^{(i)} - z_{m,n}^{(i)} = z_{m,n}^{(i)}[-I_r + P_{n,n+d}]$, $n > m, i \in S$. We remark that $z_{m,n}^{(i)}$ defined in (3.2) are solutions of equations of the type

$$(3.3) \quad x_{m,n+d} - x_{m,n} = x_{m,n}[-I_r + P_{n,n+d}], n > m,$$

under the conditions

$$(3.4) \quad (x_{m,n})_i \in [0, 1], i \in S, \sum_{i=1}^r (x_{m,n})_i = 1, n > m,$$

or

$$(3.5) \quad (x_{m,n})_i \in [-1, 1], i \in S, \sum_{i=1}^r (x_{m,n})_i = 0, n > m,$$

where $x_{m,n} = [(x_{m,n})_1, \dots, (x_{m,n})_r]$, $n > m$.

We are interested in the asymptotic behavior of the proposed solutions of (3.3), under conditions (3.4) or (3.5). A first result is given in the next

LEMMA 3.2.

$$(3.6) \quad \lim_{n \rightarrow \infty} (x_{m,n})_i = 0, p + 1 \leq i \leq r,$$

uniformly with respect to $m \leq 0$.

Proof. A key step is the choice of

$$(3.7) \quad A = -I_r + P.$$

We can benefit of the result given in Theorem 1.4. Let Q and Q^{-1} as in Theorem 1.4. A second key step is the choice of $V_n = P_{n,n+d} - P$, $n \geq 1$, and

$$(3.8) \quad \tilde{V}_n = Q^{-1}V_nQ, \quad n \geq 1.$$

Also, setting

$$(3.9) \quad y_{m,n} = x_{m,n}Q, \quad n > m,$$

equations (3.3) amount to

$$(3.10) \quad y_{m,n+d} - y_{m,n} = y_{m,n}J + y_{m,n}\tilde{V}_n, \quad n > m.$$

We shall study the asymptotic behavior of $(y_{m,n})_i$, $p+1 \leq i \leq r$. First, note that

$$\| \|y_{m,n}\| \|_\infty \leq \| \|x_{m,n}\| \|_\infty \| \|Q\| \|_\infty \leq 2 \| \|Q\| \|_\infty =: \beta, \quad 0 \leq m < n.$$

Using the boundedness of $y_{m,n}$, $0 \leq m < n$, and $\lim_{n \rightarrow \infty} V_n = \mathbf{0}$, it follows that

$$(W_{m,n})_i := \sum_{j=1}^r (y_{m,n})_j (\tilde{V}_n)_{ji} \rightarrow 0 \text{ as } n \rightarrow \infty, \quad m \geq 0, \quad p+1 \leq i \leq r.$$

Moreover, the convergence is uniform with respect to $m \geq 0$, for $p+1 \leq i \leq r$.

In the system (3.10) we shall be concerned with the equations corresponding to $i \in \{p+1, \dots, p+\sum_{t=2}^l m_t+1\}$. Equivalently, we can write $(y_{m,n+d})_i = (\lambda_2+1)(y_{m,n})_i + (W_{m,n})_i$, $n > m$, $p+1 \leq i \leq p+m_2$, and

$$(y_{m,n+d})_i = (\lambda_t+1)(y_{m,n})_i + (W_{m,n})_i, \quad n > m, \quad p+\sum_{s=2}^{t-1} m_s+1 \leq i \leq p+\sum_{s=2}^t m_s, \quad 3 \leq t \leq l,$$

respectively. The conditions of Proposition 2.1 are verified, so

$$\lim_{n \rightarrow \infty} (y_{m,n})_i = 0, \quad p+1 \leq i \leq p+\sum_{t=2}^l m_t+1.$$

Moreover, this convergence is uniform with respect to $m \geq 0$, for $p+1 \leq i \leq p+\sum_{t=2}^l m_t+1$ since $\exists \beta \geq 0$ such that $\| \|y_{m,n}\| \|_\infty \leq \beta$, $0 \leq m < n$, and the convergence to zero of $(W_{m,n})_i$ is uniform with respect to $m \geq 0$, for $p+1 \leq i \leq r$.

Next, we pay attention to the equations corresponding to i from $p+\sum_{t=2}^l m_t+2$ to $p+\sum_{t=2}^{l+1} m_t$. The case $\varepsilon_k^{(i)} = 0$ for some $1 \leq k \leq m_{l+1}-1$ is similar to the preceding one. We are interested now in cases for which $\varepsilon_k^{(i)} = 1$. Set $k_* = \min\{k \in \{1, \dots, m_{l+1}-1\} | \varepsilon_k^{(i)} = 1\}$ (k_* is well defined since the set $\{k \in \{1, \dots, m_{l+1}-1\} | \varepsilon_k^{(i)} = 1\}$ is nonempty, see Remark 1.5 (ii)). If $k_* = 1$, then $(y_{m,n+d})_{p+\sum_{t=2}^l m_t+2} = (\lambda_1+1)(y_{m,n})_{p+\sum_{t=2}^l m_t+2} + (y_{m,n})_{p+\sum_{t=2}^l m_t+1} + (W_{m,n})_{p+\sum_{t=2}^l m_t+2}$, $n > m$. Using $\lim_{n \rightarrow \infty} (y_{m,n})_{p+\sum_{t=2}^l m_t+1} = 0$ uniformly with respect to $m \geq 0$, the conditions of Proposition 2.1 are verified, so we get $\lim_{n \rightarrow \infty} (y_{m,n})_{p+\sum_{t=2}^l m_t+2} =$

0 uniformly with respect to $m \geq 0$. If $k_* > 1$, then $\lim_{n \rightarrow \infty} (y_{m,n})_{p+\sum_{t=2}^l m_t+k_*-1} = 0$ uniformly with respect to $m \geq 0$, since $\varepsilon_{k_*-1}^{(i)} = 0$. Also,

$$(y_{m,n+d})_{p+\sum_{t=2}^l m_t+k_*} = (\lambda_l + 1)(y_{m,n})_{p+\sum_{t=2}^l m_t+k_*} + (y_{m,n})_{p+\sum_{t=2}^l m_t+k_*-1} + (W_{m,n})_{p+\sum_{t=2}^l m_t+k_*},$$

$n > m$. The conditions of Proposition 2.1 are fulfilled, so $\lim_{n \rightarrow \infty} (y_{m,n})_{p+m_2+\dots+m_l+k_*} = 0$, uniformly with respect to $m \geq 0$. Similarly, $\lim_{n \rightarrow \infty} (y_{m,n})_i = 0$ uniformly with respect to $m \geq 0$, for $p + \sum_{t=2}^l m_t + 3 \leq i \leq p + \sum_{t=2}^l m_t$.

Moreover, we can conclude that

$$(3.11) \quad \lim_{n \rightarrow \infty} (y_{m,n})_i = 0,$$

regardless of the initial data y_0 and uniformly with respect to $m \geq 0$, for $p+1 \leq i \leq r$. The conclusion follows. \square

In the following, we shall prepare the context for the examination of the remaining components $(x_{m,n})_i$, $1 \leq i \leq p$, of the vector $x_{m,n}$. Set

$$(R_{m,n})_i = \sum_{j=p+1}^r (y_{m,n})_j (\tilde{V}_n)_{ji}, \quad n > m, \quad 1 \leq i \leq p.$$

If the equality (3.9) is multiplied on the right side by \tilde{V}_n , we get $y_{m,n} \tilde{V}_n = x_{m,n} V_n Q$. Set \tilde{q}_i for the i th column of the matrix Q . From

$$\begin{aligned} \sum_{i=1}^p (y_{m,n} \tilde{V}_n)_i &= \sum_{j=1}^r \left(\sum_{i=1}^p (y_{m,n})_j (\tilde{V}_n)_{ji} \right) = \\ &= \sum_{j=1}^p (y_{m,n})_j \left(\sum_{i=1}^p (\tilde{V}_n)_{ji} \right) + \sum_{j=p+1}^r \left(\sum_{i=1}^p (y_{m,n})_j (\tilde{V}_n)_{ji} \right) = \\ &\text{(using } \sum_{i=1}^p (\tilde{V}_n)_{ji} = 0, \quad 1 \leq j \leq p \text{ (see Nicolaie (2008b), Proposition 3.1))} \\ &= \sum_{i=1}^p (R_{m,n})_i, \quad n > m, \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^p (x_{m,n} V_n Q)_i &= \sum_{i=1}^p x_{m,n} V_n \tilde{q}_i = x_{m,n} V_n \sum_{i=1}^p \tilde{q}_i = \\ &= x_{m,n} V_n \mathbf{e}' = x_{m,n} \mathbf{0}_{p \times 1} = 0, \quad n > m, \end{aligned}$$

it follows

$$(3.12) \quad R_{m,n} \mathbf{e}' = 0, \quad n > m.$$

Setting $Y_{m,n} = ((y_{m,n})_1, \dots, (y_{m,n})_p)$, $R_{m,n} = ((R_{m,n})_1, \dots, (R_{m,n})_p)$, $n > m$, and

$$(3.13) \quad C_n = I_p + (\tilde{V}_n)_{M \times M}, \quad n \geq 1, \quad \text{where } M = \{1, \dots, p\},$$

we have

$$(3.14) \quad Y_{m,n+d} = Y_{m,n}C_n + R_{m,n}, n > m.$$

We need a preliminary result on C_n .

PROPOSITION 3.3. $(C_n)_{n \geq 1}$ is a sequence of stochastic matrices.

Proof. Similar to the proof of Proposition 3.1 from Nicolaie (2008b). \square

We are now ready to give the main result of this paper. The next theorem is a generalization of Theorem 3.2 from Nicolaie (2008b).

THEOREM 3.4. Let $(P_n)_{n \geq 1}$ be a Markov chain with state space S for which $\exists d \geq 1$ such that $\lim_{n \rightarrow \infty} P_{n,n+d} = P$, where P is of the form (1.1) with $p \geq 1$. Let $V_n = P_{n,n+d} - P$, $n \geq 1$, and $\tilde{V}_n = Q^{-1}V_nQ$, $n \geq 1$, with Q and Q^{-1} as in Theorem 1.4. Then

$$(3.15) \quad \lim_{n \rightarrow \infty} (P_{m,n})_{ij} = 0, \quad i \in S, \quad \sum_{t=0}^p r_t \leq j \leq r,$$

uniformly with respect to $m \geq 0$.

Moreover, let C_n , $n \geq 1$ as in (3.13) and set $\tilde{C}_n^{(l)} = C_{(n-1)d+l+1}$, $n \geq 1$, $0 \leq l \leq d-1$. Suppose that

$$(3.16) \quad \sum_{n=1}^{\infty} \|\tilde{V}_n\|_{(S \setminus M) \times M} < \infty.$$

Then the following statements hold:

- (i) If $(\tilde{C}_n^{(l)})_{n \geq 1}$ is weakly ergodic, $0 \leq l \leq d-1$, then $(P_n)_{n \geq 1}$ is weakly ergodic, i.e., the chain $(P_n)_{n \geq 1}$ is weakly ergodic ;
- (ii) If $(\tilde{C}_n^{(l)})_{n \geq 1}$ is strongly ergodic, $0 \leq l \leq d-1$, such that $\lim_{n \rightarrow \infty} \tilde{C}_{m,n}^{(l)} = \Pi$, $m \geq 0$, $0 \leq l \leq d-1$, then $(P_n)_{n \geq 1}$ is strongly ergodic, i.e., the chain $(P_n)_{n \geq 1}$ is strongly ergodic .
- (iii) If $(\tilde{C}_n^{(l)})_{n \geq 1}$ is strongly ergodic, $0 \leq l \leq d-1$, such that $\lim_{n \rightarrow \infty} \tilde{C}_{m,n}^{(l)} = \Pi_{(l+1)} = \mathbf{e}'\pi^{(l+1)}$, $m \geq 0$, $0 \leq l \leq d-1$, then $(P_n)_{n \geq 1}$ is C -strongly ergodic, i.e., the chain $(P_n)_{n \geq 1}$ is C -strongly ergodic.

Proof. For the first conclusion of the theorem, we will use Lemma 3.1 taking $x_{m,n} = z_{m,n}^{(i)}$ in equation (3.3), where $z_{m,n}^{(i)}$ is defined in (3.2), $n > m$, $i \in S$. From (3.9) we get $x_{m,n} = y_{m,n}Q^{-1}$, $n > m$. Next, letting $n \rightarrow \infty$ in the last equality, on account of (3.11), we get (3.15).

(i) We shall take $x_{m,n} = z_{m,n}^{(i)} - z_{m,n}^{(j)}$ in equation (3.3), $n > m$, $i, j \in S$, $i \neq j$, where $z_{m,n}^{(i)}$ is defined in (3.2). Starting from (3.9), using (1.3), we have (for $y_{m,n} := x_{m,n}Q$)

$$\sum_{k=1}^p (y_{m,n})_k = \sum_{k=1}^p x_{m,n} \tilde{q}_k = x_{m,n} \sum_{k=1}^p \tilde{q}_k = x_{m,n} \mathbf{e}' = 0, \quad n > m.$$

Let $i, j \in S$. By (3.16) and the fact that $\exists \beta \geq 0$ such that $\|y_{m,n}\|_\infty \leq \beta$, $0 \leq m < n$, it follows that $\sum_{n=m}^\infty \|R_{m,n}\|_\infty < \infty$, $m \geq 0$; as concerns equation (3.14), by Proposition 2.2 (i), we get $\lim_{n \rightarrow \infty} Y_{m,n} = \mathbf{0}$, $m \geq 0$.

This, (3.11) and $x_{m,n} = y_{m,n}Q^{-1}$ give us

$$\lim_{n \rightarrow \infty} [(P_{m,n})_{ik} - (P_{m,n})_{jk}] = 0, \quad m \geq 0, \quad k \in S.$$

Therefore, $(P_n)_{n \geq 1}$ is weakly ergodic.

(ii) We shall take $x_{m,n} = z_{m,n}^{(i)}$ in equation (3.3), where $z_{m,n}^{(i)}$ is defined in (3.2), $n > m$, $i \in S$. Starting from (3.9), using (1.3), we have (for $y_{m,n} := x_{m,n}Q$)

$$\sum_{k=1}^p (y_{m,n})_k = \sum_{k=1}^p x_{m,n} \tilde{q}_k = x_{m,n} \sum_{k=1}^p \tilde{q}_k = x_{m,n} \mathbf{e}' = 1, \quad n > m.$$

Let $i \in S$. By (3.16) and the fact that $\exists \beta \geq 0$ such that $\|y_{m,n}\|_\infty \leq \beta$, $0 \leq m < n$, it follows that $\sum_{n=m}^\infty \|R_{m,n}\|_\infty < \infty$, $m \geq 0$; as concerns the equation (3.14), by Proposition 2.2 (ii), it follows that $\lim_{n \rightarrow \infty} Y_{m,n} = \pi$, $m \geq 0$.

This and $x_{m,n} = y_{m,n}Q^{-1}$ give us

$$\lim_{n \rightarrow \infty} ((P_{m,n})_{i,r_0+\dots+r_{t-1}}, \dots, (P_{m,n})_{i,r_1+\dots+r_t}) = (\mu_1^{(t)} \pi_t, \dots, \mu_{r_t}^{(t)} \pi_t).$$

$m \geq 0$, $1 \leq t \leq p$. Therefore, using (3.15), $(P_n)_{n \geq 1}$ is strongly ergodic.

(iii) Making the same choice in equation (3.3) as in the proof of (ii) above and applying Proposition 2.3 (iii), it follows $\lim_{n \rightarrow \infty} Y_{m,nd+l+1} = \pi^{(l+1)}$, $m \geq 0$, $0 \leq l \leq d-1$. This and $x_{m,n} = y_{m,n}Q^{-1}$ give us

$$\lim_{n \rightarrow \infty} ((P_{m,nd+l+1})_{i,r_0+\dots+r_{t-1}}, \dots, (P_{m,nd+l+1})_{i,r_1+\dots+r_t}) = (\mu_1^{(t)} \pi_t^{(l+1)}, \dots, \mu_{r_t}^{(t)} \pi_t^{(l+1)}).$$

$m \geq 0$, $1 \leq t \leq p$ and $0 \leq l \leq d-1$. Therefore, using (3.15), $\lim_{n \rightarrow \infty} P_{m,nd+l+1}$ exists and is a stable stochastic matrix for $m \geq 0$, $0 \leq l \leq d-1$. Set $W_{l+1} = \lim_{n \rightarrow \infty} P_{0,nd+l+1}$, $0 \leq l \leq d-1$. The conditions of Theorem 1.13 are fulfilled; it follows that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n P_{0,k} = \frac{1}{d} [W_1 + \dots + W_d]$, i.e., the chain $(P_n)_{n \geq 1}$ is C-strongly ergodic. \square

Remark 3.5. (i) We can prove that the chains $(P_n)_{n \geq 1}$ from Remark 3.1 (i) and (ii), respectively, are strongly ergodic using Theorem 3.4 (ii) (for the second chain take $d = 2$).

(ii) Obviously, in the hypothesis (3.16), the condition that any subsequence $(\tilde{C}_n^{(l)})_{n \geq 1}$ is strongly ergodic, $0 \leq l \leq d-1$, does not imply that $(P_n)_{n \geq 1}$ is strongly ergodic. An example is the chain $(P_n)_{n \geq 1}$, where

$$P_{2n-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P_{2n} = \begin{pmatrix} 0 & 1 \\ 1 - \frac{1}{2n} & \frac{1}{2n} \end{pmatrix}, \quad n \geq 1.$$

The chain $(\tilde{C}_n^{(0)})_{n \geq 1}$ is strongly ergodic and

$$\lim_{n \rightarrow \infty} \tilde{C}_{m,n}^{(0)} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} := \Pi_1, \quad m \geq 0.$$

Also, the chain $(\tilde{C}_n^{(1)})_{n \geq 1}$ is strongly ergodic and

$$\lim_{n \rightarrow \infty} \tilde{C}_{m,n}^{(1)} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} := \Pi_2, \quad m \geq 0.$$

The chain $(P_n)_{n \geq 1}$ is not strongly ergodic, but using Theorem 3.4 (iii) we can prove that the chain $(P_n)_{n \geq 1}$ is C-strongly ergodic. Moreover, $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n P_{0,k} = \frac{1}{2} [\Pi_1 + \Pi_2]$.

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