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# Generalized stochastic target problems for pricing and partial hedging under loss constraints - Application in optimal book liquidation

Bruno Bouchard

CEREMADE, Université Paris Dauphine  
and CREST-ENSAE

bouchard@ceremade.dauphine.fr

Ngoc Minh Dang

CEREMADE, Université Paris Dauphine  
and CA Cheuvreux

dang@ceremade.dauphine.fr

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## Abstract

We consider a singular with state constraints version of the stochastic target problems studied in [14], [16] and more recently [1], among others. This provides a general framework for the pricing of contingent claims under risk constraints. Our extended version perfectly suits to market models with proportional transaction costs and to order book liquidation issues. Our main result is a PDE characterization of the associated pricing function. As an example of application, we discuss the evaluation of VWAP-guaranteed type book liquidation contracts, for a general class of risk functions.

**Key words:** Stochastic target problems, state constraints, pricing under risk constraint, book liquidation.

**Mathematical subject classifications:** Primary 49L25, 60J60; secondary 49L20, 35K55.

## 1 Introduction

Stochastic target technics have been originally introduced in mathematical finance by Soner and Touzi [13] in order to provide a PDE characterization of the super-hedging price of an European claim under gamma constraints.

The classical super-hedging problem takes the general form: find the minimal  $Y^\phi(0)$  such that there exists a control  $\phi$ , in a suitable admissibility set  $\mathcal{A}$ , satisfying  $Y^\phi(T) \geq g(X^\phi(T))$   $\mathbb{P}$  – a.s., where  $g$  is the payoff function of an European claim,  $\phi$  stands for the financial strategy,  $Y^\phi$  for the wealth process and  $X^\phi$  for the stock price process, which may be influenced by the financial strategy, as in large investor models for instance. In general, such a problem is treated in mathematical finance via the dual formulation approach which allows one to relate the minimal  $Y^\phi(0)$  to a stochastic control problem in standard form. However, this approach heavily relies on the fact that the wealth dynamics is linear in the control and that the stocks prices are not influenced by the trading strategy. In particular, it does not apply to large investor models or to more general dynamics or constraints, such as gamma constraints. This was the motivation of Soner and Touzi for introducing the so-called stochastic target approach.

Their main discovery is a dynamic programming principle which is directly written on the associated stochastic target problem, and therefore does not appeal to any form of dual formulation, see Theorem 2.3 below. It turns out to be sufficient to provide a PDE characterization for the associated value function. This approach led to a series of papers providing a direct way to characterize super-hedging prices, see e.g. [2], [4], [15] and [17].

Up to the recent work of Bouchard, Elie and Touzi [1], this approach was however limited to super-hedging problems which in turn typically lead to high prices which are not reasonable in practice, see e.g. [6] and [7]. Apart from technical improvements, the main result of Bouchard, Elie and Touzi [1] is that pricing problems under risk constraints of the form: find the minimal  $Y^\phi(0)$  such that there exists a control  $\phi \in \mathcal{A}$  satisfying  $\mathbb{E}[\ell(Y^\phi(T) - g(X^\phi(T)))] \geq p$ , for some “loss function”  $\ell$  and a threshold  $p$ , can actually be treated via the stochastic target approach of Soner and Touzi [13] and [14]. For  $\ell$  of the form  $\ell(r) = \mathbf{1}_{r \geq 0}$  and  $p \in (0, 1)$ , one retrieves the quantile hedging problem of Follmer and Leukert [9]. When  $\ell$  stands for a utility function and  $p := \sup\{\mathbb{E}[\ell(Y^\phi(T))] : Y^\phi(0) = y_0, \phi \in \mathcal{A}\}$ , this corresponds to a utility indifference pricing problem. More generally, one can treat risk constraints of the form  $\mathbb{E}[\Psi(X^\phi(T), Y^\phi(T))] \geq p$ , for a general class of “risk functions”  $-\Psi$ . The success ratio hedging problem of Follmer and Leukert [9] enters into this framework. Finally, American type constraints can be introduced, see Bouchard and Vu [3]. This provides a general framework for a direct characterization of risk based prices of contingent contracts.

In Bouchard, Elie and Touzi [1], the authors restrict to dynamics given by Brownian SDEs in which only the drift and the volatility coefficients are controlled. In this paper, we show how their results can be extended to the case where the dynamics are controlled by processes with bounded variations and state constraints have to be satisfied. This extension is mainly motivated by the pricing of a VWAP-type<sup>1</sup> book liquidation contract, however the domain of application is vast, in particular it perfectly suits to partial hedging problems under proportional transaction costs, see Example 2.3 below.

We therefore first consider a general abstract formulation that could be used in many different practical situations/models. It is presented in Section 2 together with examples of application. The associated general PDE characterization is provided in Section 3. The pricing problem of a VWAP-type book liquidation contract is fully discussed in Section 4. The proofs of our abstract results are collected in Section 5.

Notations: We denote by  $x^i$  the  $i$ -th component of a vector  $x \in \mathbb{R}^d$ , which will always be viewed as a column vector, with transposed vector  $x^\top$ , and Euclidean norm  $|x|$ . The element  $e_i \in \mathbb{R}^d$  is the  $i$ -th unit vector:  $e_i^j = \mathbf{1}_{i=j}$ ,  $i, j \leq d$ . The set  $\mathbb{M}^d$  is the collection of  $d$ -dimensional square matrices  $M$  with coordinates  $M^{ij}$ , and norm  $|M|$  defined by viewing  $M$  as an element of  $\mathbb{R}^{d \times d}$ . We denote by  $\mathbb{S}^d$  the subset of elements of  $\mathbb{M}^d$  that are symmetric. For a subset  $\mathcal{O}$  of  $\mathbb{R}^d$ , we denote by  $\bar{\mathcal{O}}$  its closure, by  $\text{int}(\mathcal{O})$  its interior, by  $\partial\mathcal{O}$  its boundary, and by  $\mathbf{dist}(x, \mathcal{O})$  the Euclidean distance from  $x$  to  $\mathcal{O}$  with the convention  $\mathbf{dist}(x, \emptyset) = \infty$ . We denote by  $B_r(x)$  the open ball of radius  $r > 0$  centered at  $x \in \mathbb{R}^d$ . If  $B = [s, t] \times \mathcal{O}$  for  $s \leq t$  and  $\mathcal{O} \subset \mathbb{R}^d$ , we write  $\partial_p B := ([s, t] \times \partial\mathcal{O}) \cup (\{t\} \times \bar{\mathcal{O}})$  for its parabolic boundary. Given a smooth function  $\varphi : (t, x_1, \dots, x_k) \in \mathbb{R}_+ \times \mathbb{R}^{kd} \rightarrow \mathbb{R}$ , we denote by  $\partial_t \varphi$  its derivative with respect to its first variable, we write  $D\varphi$  and  $D^2\varphi$  for the Jacobian and Hessian matrix with respect to  $(x_1, \dots, x_k)$ , and  $D_{x_i}\varphi$  and  $D_{x_i}^2\varphi$  the Jacobian and Hessian matrix with respect to  $x_i$ ,  $i \geq 1$ . Any inequality or inclusion involving random variables has to be taken in the a.s. sense. For a process  $L$  with bounded variations, we write  $|L|$  to denote its total variation.

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<sup>1</sup>VWAP means Volume Weighted Average Price, see Section 4 for a detailed presentation.

## 2 Abstract formulation and dynamic programming

### 2.1 The general singular stochastic target problem with state constraints

We first describe the abstract model. We refer to Section 2.2 for examples of typical dynamics in finance, and to Section 4 for a full discussion of its application to the pricing of VWAP-type book liquidation contracts.

From now on, we let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space supporting a  $d$ -dimensional Brownian motion  $W$ ,  $d \geq 1$ ,  $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$  denote the right-continuous completed filtration generated by  $W$ , and  $T > 0$  be a finite time horizon.

The abstract stochastic target problem is defined as follows.

Our set of controls is  $\mathcal{U} \times \mathcal{L}$ , where  $\mathcal{U}$  stands for the set of all progressively measurable process  $\nu$  in  $L^2([0, T] \times \Omega)$  taking values in a given closed subset  $U$  of  $\mathbb{R}^d$ , and  $\mathcal{L}$  denotes the set of continuous  $\mathbb{R}^d$ -valued adapted processes  $L$  which are non-decreasing (component by component) and such that  $\mathbb{E}[|L|_T^2] < \infty$ .

For  $t \in [0, T]$ ,  $z := (x, y) \in \mathbb{R}^d \times \mathbb{R}$  and  $\phi := (\nu, L) \in \mathcal{A} := \mathcal{U} \times \mathcal{L}$ , the controlled process  $Z_{t,z}^\phi := (X_{t,x}^\phi, Y_{t,x,y}^\phi)$  is defined as the  $\mathbb{R}^d \times \mathbb{R}$ -valued unique strong solution of the stochastic differential equation

$$\begin{aligned} X_{t,x}^\phi(s) &= x + \int_t^s \mu_X(X_{t,x}^\phi(r), \nu_r) dr + \beta_X(X_{t,x}^\phi(r)) dL_r + \int_t^s \sigma_X(X_{t,x}^\phi(r), \nu_r) dW_r \\ Y_{t,x,y}^\phi(s) &= y + \int_t^s \mu_Y(Z_{t,x,y}^\phi(r), \nu_r) dr + \beta_Y(Z_{t,x,y}^\phi(r))^\top dL_r + \int_t^s \sigma_Y(Z_{t,x,y}^\phi(r), \nu_r)^\top dW_r, \end{aligned} \quad (2.1)$$

where  $(\mu_X, \sigma_X) : (x, u) \in \mathbb{R}^d \times U \mapsto \mathbb{R}^d \times \mathbb{M}^d$ ,  $(\mu_Y, \sigma_Y) : (z, u) \in \mathbb{R}^{d+1} \times U \mapsto \mathbb{R} \times \mathbb{R}^d$ ,  $\beta_X \in C^2(\mathbb{R}^d, \mathbb{M}^d)$  and  $\beta_Y \in C^2(\mathbb{R}^{d+1}, \mathbb{R}^d)$  are assumed to be Lipschitz continuous.

Given a family of non-empty Borel subsets  $(O(t))_{t \leq T}$  of  $\mathbb{R}^{d+1}$ , the stochastic target problem consists in characterizing the value function

$$(t, x) \in [0, T] \times \mathbb{R}^d \mapsto v(t, x) := \inf \{y \in \mathbb{R} : (x, y) \in V(t)\}, \quad (2.2)$$

where the set valued map  $V$  is defined as

$$t \in [0, T] \mapsto V(t) := \{z \in \mathbb{R}^{d+1} : \mathcal{A}_{t,z} \neq \emptyset\}, \quad (2.3)$$

and

$$\mathcal{A}_{t,z} := \{\phi \in \mathcal{A} : Z_{t,z}^\phi(s) \in O(s) \text{ for all } s \in [t, T] \text{ } \mathbb{P} - \text{a.s.}\}. \quad (2.4)$$

In order to fully characterize the set valued map  $V$  in terms of the value function  $v$ , we shall assume all over this paper the following:

**Standing Assumption 1:** For all  $(t, x) \in [0, T] \times \mathbb{R}^d$ :  $(x, y) \in O(t)$  and  $y' \geq y \Rightarrow (x, y') \in O(t)$ .

**Remark 2.1.** It follows from Standing Assumption 1 and standard comparison arguments for stochastic differential equations that  $\mathcal{A}_{t,x,y'} \subset \mathcal{A}_{t,x,y}$  for  $y' \geq y$ . In particular,  $(x, y') \in V(t)$  whenever  $(x, y) \in V(t)$  and  $y' \geq y$ , so that  $V(t)$  can be (at least when the infimum in the definition of  $v$  is achieved) identified to  $\{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y \geq v(t, x)\}$ .

In order to give a sense to the following discussions, we also assume that:

**Standing Assumption 2:**  $v$  is locally bounded on  $\bar{D}_Y$  where

$$D_Y := \{(t, x) \in [0, T] \times \mathbb{R}^d : \exists y \in \mathbb{R} \text{ s.t. } (x, y) \in O(t)\}.$$

**Remark 2.2.** Obviously, the fact that all the relevant quantities take values in  $\mathbb{R}^d$  is used to save notations. One could without difficulty restrict to the case where some components of  $X$  only take positive values, which is typically the case for prices of stocks or bonds. By putting to 0 part of the coefficients, one can also retrieve situations where  $W$ ,  $L$  and  $X$  do not have the same effective dimension. One could similarly add a time dependence in the coefficients, e.g. by considering the first component of  $X$  as a time parameter.

## 2.2 Examples of application

Before to go further in the general treatment, let us immediately discuss some typical examples of application that motivate this work (see also Section 4 for an application to optimal book liquidation that will be studied in details).

**Example 2.1.** Let us first consider the case where  $(\beta_X, \beta_Y) = 0$ ,

$$\mu_X(x, u) = \text{diag}[x]\mu, \quad \sigma_X(x, u) = \text{diag}[x]\sigma$$

and

$$\mu_Y(x, y, u) = u^\top \text{diag}[x]\mu, \quad \sigma_Y(x, y, u) = u^\top \text{diag}[x]\sigma$$

where  $\text{diag}[x]$  stands for the diagonal matrix with  $x^i$  as the  $i$ -th diagonal element,  $\mu \in \mathbb{R}^d$  and  $\sigma \in \mathbb{M}^d$ . The dynamics (2.1) then read, for  $s \in [t, T]$ :

$$\begin{aligned} X_{t,x}(s) &= x + \int_t^s \text{diag}[X_{t,x}(r)] \mu dr + \int_t^s \text{diag}[X_{t,x}(r)] \sigma dW_r \\ Y_{t,x,y}^\nu(s) &= y + \int_t^s \nu_r^\top dX_{t,x}(r), \end{aligned}$$

where we only write  $X$  for  $X^{\nu,L}$  and  $Y^\nu$  for  $Y^{\nu,L}$  because  $X$  is not affected by the control and  $Y$  depends on  $(\nu, L)$  only through  $\nu$ .

Restricting to initial condition  $x \in (0, \infty)^d$ , this corresponds to the  $d$ -dimensional Black and Scholes model:  $X^i$  models the dynamics of a financial asset, the risk free interest rate is 0,  $\nu_t^i$  stands for the number of units of  $X^i$  held in a financial portfolio at time  $t$ , and  $Y$  is the associated wealth process starting from the initial endowment  $y$ .

If we now take  $O$  of the form:

$$O(t) = (0, \infty)^d \times \mathbb{R} \mathbf{1}_{t < T} + \mathbf{1}_{t=T} \left\{ (x, y) \in (0, \infty)^d \times \mathbb{R} : y \geq g(x) \right\}, \quad t \leq T,$$

for some measurable map  $g : \mathbb{R}^d \mapsto \mathbb{R}$ , the value function can be written as

$$v(t, x) := \inf \left\{ y \in \mathbb{R} : Y_{t,x,y}^\nu(T) \geq g(X_{t,x}(T)) \text{ for some } \nu \in \mathcal{U} \right\}.$$

This corresponds to the usual definition of the super-hedging price of an European option of payoff function  $g$ .

For  $O$  of the form

$$O(t) = \left\{ (x, y) \in (0, \infty)^d \times \mathbb{R} : y \geq g(x) \right\}, \quad t \leq T,$$

this corresponds to the super-hedging price of an American option.

**Example 2.2.** Let us now consider a two-dimensional model  $d = 2$  with the following parameters

$$\mu_X^1(x, u) = x^1 \mu, \quad \mu_X^2(t, u) = x^2 \mu, \quad \sigma_X^{11}(x, u) = x^1 \sigma, \quad \sigma_X^{21}(x, u) = x^2 \sigma$$

and

$$\beta_X^{21}(x) = -1, \quad \beta_X^{22}(x) = 1, \quad \beta_Y(x, y) = (1 - \lambda, -1 - \lambda),$$

where  $\mu \in \mathbb{R}$ ,  $\sigma > 0$  and  $\lambda \in (0, 1)$ , and the other parameters are equal to 0. The dynamics (2.1) then read, for  $s \in [t, T]$ :

$$\begin{aligned} X_{t,x}^1(s) &= x^1 + \int_t^s X_{t,x}^1(r) \mu dr + \int_t^s X_{t,x}^1(r) \sigma dW_r^1 \\ X_{t,x}^{2,L}(s) &= x^2 + \int_t^s \frac{X_{t,x}^{2,L}(r)}{X_{t,x}^1(r)} dX_{t,x}^1(r) - \int_t^s dL_r^1 + \int_t^s dL_r^2 \\ Y_{t,y}^L(s) &= y + \int_t^s (1 - \lambda) dL_r^1 - \int_t^s (1 + \lambda) dL_r^2. \end{aligned}$$

This corresponds to the one-dimensional model with proportional transaction costs studied in [6]. More precisely, there is only one risky asset,  $X^1$ , with a Black and Scholes type dynamics. When buying or selling this risky asset, the investor pays a proportional transaction costs  $\lambda \in (0, 1)$ . The process  $\bar{L} := L^2 - L^1$  stands for the cumulative net amount of money invested in the risky asset from time 0, i.e.  $L_r^2$  (resp.  $L_r^1$ ) is the cumulated value of bought (resp. sold) shares of  $X^1$ . Each time a buying or selling operation  $d\bar{L}$  is done, the investor pays, in money, a proportional transaction cost  $\lambda d|\bar{L}|$ . The wealth process is described by the two dimensional process  $(Y, X^2)$  where  $Y$  models the evolution of the cash account, and  $X^2$  corresponds to the value of the part of the portfolio invested in  $X^1$ , when taking into account the transaction costs.

For  $O$  defined as

$$O(t) = (0, \infty) \times \mathbb{R}^2 \mathbf{1}_{t < T} + \mathbf{1}_{t=T} \{ (x, y) \in (0, \infty) \times \mathbb{R}^2 : \Lambda(y, x) \geq g(x) \}, \quad t \leq T,$$

where

$$\Lambda(y, x) := y + x^1 x^2 - \lambda |x^2| x^1$$

provides the value in cash of a terminal position  $(y, x^2)$  if the value of the stock is  $x^1$ , one retrieves the notion of super-hedging price of an European option with cash delivery, in the financial market model with proportional transaction costs. Obviously, one can consider similarly markets with more than one risky asset, see e.g. [2].

**Example 2.3.** As shown in [6] and [2], the super-hedging criteria is much too strict in markets with proportional transaction, as it leads to degenerate strategies of buy-and-hold type which do not reflect the market behavior. It follows that it should be relaxed by using, for instance, quantile or expected loss approaches as studied in [9], [10], for frictionless markets.

The loss function pricing approach consists in choosing a non-decreasing (typically concave) function  $\ell : \mathbb{R} \mapsto \mathbb{R}$  and defining the price at time  $t$  of an European option of payoff (say with cash delivery)  $g(X_{t,x}^1(T))$  as  $\hat{v}(t, x^1, 0; p)$  with

$$\hat{v}(t, x; p) := \inf \{ y \in \mathbb{R} : \mathbb{E} [\ell (\Lambda(Y_{t,y}^L(T), X_{t,x}^L(T))) - g(X_{t,x}^1(T))] \geq p \text{ for some } L \in \mathcal{L} \},$$

where  $p$  is a given threshold in  $\mathbb{R}$ , and  $\Lambda$  is defined as in the previous example.

Assuming that  $\Xi_{t,x,y}^L(T) := \ell(\Lambda(Y_{t,y}^L(T), X_{t,x}^L(T)) - g(X_{t,x}^1(T))) \in L^2$  for all initial conditions and control  $L$ , the arguments of Proposition 3.1 in [1] then show that

$$\hat{v}(t, x; p) = \inf \left\{ y \in \mathbb{R} : \Xi_{t,x,y}^L(T) \geq X_{t,p}^{3,\nu}(T) \text{ for some } (L, \nu) \in \mathcal{L} \times \mathcal{U} \right\},$$

with  $U = \mathbb{R}^2$  and

$$X_{t,p}^{3,\nu} := p + \int_t^\cdot \nu_s^1 dW_s^1.$$

Indeed, if  $\Xi_{t,x,y}^L(T) \geq X_{t,p}^{3,\nu}(T)$  then taking expectation leads to  $\mathbb{E}[\Xi_{t,x,y}^L(T)] \geq p$ , while, if  $p_0 := \mathbb{E}[\Xi_{t,x,y}^L(T)] \geq p$ , then the martingale representation theorem implies that we can find  $\nu \in \mathcal{U}$  such that  $\Xi_{t,x,y}^L(T) = X_{t,p_0}^{3,\nu}(T) \geq X_{t,p}^{3,\nu}(T)$ .

Hence, this last example enters into our general framework with the dynamics given in Example 2.2 and an additional controlled process  $X^{3,\nu}$  defined as above.

**Example 2.4.** Influence of the trading strategies can be incorporated in the previous example without much difficulties. It suffices to consider more general models in which the dynamics of  $X^1$  depends on  $L$ . It can for instance take the form

$$X_{t,x}^{1,L}(s) = x^1 + \int_t^s X_{t,x}^{1,L}(r) \mu dr + \int_t^s X_{t,x}^{1,L}(r) \sigma dW_r^1 - \int_t^s X_{t,x}^{1,L}(r) \beta_- dL_r^1 + \int_t^s X_{t,x}^{1,L}(r) \beta_+ dL_r^2$$

with  $\beta_-, \beta_+ \geq 0$ . In this case, a buying order drives the price up, while a selling order pushes the price down. Note that constraints on the liquidation value of the portfolio  $(X^{2,L}, Y^L)$  could also be incorporated by playing with the definition of  $O$ . For instance,

$$\begin{aligned} O(t) = & \left\{ (x, y) \in (0, \infty) \times \mathbb{R}^2 : \Lambda(y, x) \geq -c \right\} \mathbf{1}_{t < T} \\ & + \left\{ (x, y) \in (0, \infty) \times \mathbb{R}^2 : \Lambda(y, x) \geq g(x) \right\} \mathbf{1}_{t=T}, t \leq T, \end{aligned}$$

means that the liquidation value of the portfolio should never be less than  $-c$ .

## 2.3 Dynamic programming

We now come back to the abstract problem (2.2).

In order to provide a PDE characterization of the value function  $v$ , we shall appeal to the *geometric dynamic programming principle* introduced in [14] and [16] in the case  $O(t) = \mathbb{R}^{d+1}$  for  $t < T$ , and extended in [3] in the general case.

It expresses the fact that  $z \in V(t)$  if and only if one is able to find a control  $\phi$  such that  $Z_{t,z}^\phi \in O(\cdot) \cap V(\cdot)$  on  $[t, T]$ , i.e.  $Z_{t,z}^\phi(s)$  lies in the domain  $O(s)$ , which is our constraint, and  $Z_{t,z}^\phi(s)$  is such that, starting from this point at time  $s$ , one can find a control on  $[s, T]$  such that the state process remains in the domains  $O(\cdot)$  on  $[s, T]$ , i.e.  $Z_{t,z}^\phi(s) \in V(s)$  by definition of  $V$ .

This heuristical reasoning can be made rigorous under the following right-continuity assumption:

**Standing Assumption 3.** [Right-continuity of the target] For all sequence  $(t_n, z_n)_n$  of  $[0, T] \times \mathbb{R}^{d+1}$  such that  $(t_n, z_n) \rightarrow (t, z)$ , we have

$$t_n \geq t_{n+1} \text{ and } z_n \in O(t_n) \forall n \geq 1 \implies z \in O(t).$$

In the statement below, we denote by  $\mathcal{T}_{[t,T]}$  the set of stopping times with values in  $[t, T]$ , for  $t \leq T$ , and use the notation

$$O \bigoplus^{\tau, \theta} V := O(\tau) 1_{\tau \leq \theta} + V(\theta) 1_{\tau > \theta} \text{ for } \theta, \tau \in \mathcal{T}_{[0,T]} .$$

**Theorem 2.3** (Geometric Dynamic Programming Principle). *For all  $t \leq T$ ,*

$$V(t) = \left\{ z \in \mathbb{R}^{d+1} : \exists \phi \in \mathcal{A} \text{ s.t. } Z_{t,z}^\phi(\theta \wedge \tau) \in O \bigoplus^{\tau, \theta} V \text{ for all } \theta, \tau \in \mathcal{T}_{[t,T]} \right\} .$$

**Proof.** Note that the formulation is slightly different from Theorem 2.1 in [3], however it should be clear that their result can be stated in the above form, see the proofs of Lemma 2.1 and 2.2 in [3]. It thus suffices to check that the conditions **A1-A2** and **Z1-Z5** of Section 2.1 in [3] hold. Clearly, **A1** holds. Also note that  $\mathcal{A}$  is a separable metric space so that **A2** holds by Lemma 2.1 and the discussion in Section 2.5 in [16]. The condition **Z1** is satisfied with the additional convention  $Z_{t,z}^\phi(s) = 0$  for  $s < t$ . The verification of **Z2** is standard in our Brownian diffusion framework, up to passing to the canonical space, see e.g. Section 3.2 of [3]. The flow and causality property **Z3** and **Z4** follow from the uniqueness of the solution to (2.1) for any  $\phi \in \mathcal{A}$  and any initial condition. As for the last condition **Z5**, one easily deduces from the Lipschitz continuity of the coefficients in (2.1) that, for all  $s \geq t$ ,  $|Z_{t,z}^\phi(s) - Z_{t',z'}^{\phi'}(s)|$  converges to 0 in  $L^2(\Omega)$  when  $(t', z') \rightarrow (t, z)$ ,  $\nu' \rightarrow \nu$  in  $L^2(\Omega \times [0, T])$ , and  $|L' - L|_T \rightarrow 0$  in  $L^2(\Omega)$ , where we used the identification  $\phi = (\nu, L)$  and  $\phi' = (\nu', L')$ .  $\square$

Under our Standing Assumption 1, recall Remark 2.1, Theorem 2.3 translates in terms of the value function  $v$  as follows:

**Corollary 2.4.** *Fix  $(t, x, y) \in [0, T] \times \mathbb{R}^{d+1}$ .*

**(GDP1):** *If  $y > v(t, x)$ , then there exists  $\phi \in \mathcal{A}$  such that, for all  $\theta \in \mathcal{T}_{[t,T]}$ ,*

$$Y_{t,x,y}^\phi(\theta) \geq v(\theta, X_{t,x}^\phi(\theta)) \text{ and } Z_{t,x,y}^\phi(s \wedge \theta) \in O(s \wedge \theta) \text{ for all } s \in [t, T] .$$

**(GDP2):** *If  $y < v(t, x)$ , then*

$$\mathbb{P} \left[ Y_{t,x,y}^\phi(\theta) > v(\theta, X_{t,x}^\phi(\theta)) \text{ and } Z_{t,x,y}^\phi(s \wedge \theta) \in O(s \wedge \theta) \forall s \in [t, T] \right] < 1 \quad \forall (\phi, \theta) \in \mathcal{A} \times \mathcal{T}_{[t,T]} .$$

### 3 PDE characterization in the abstract model

Our main result is a direct PDE characterization of the risk constraint based pricing function  $v$ .

#### 3.1 Formal derivation

Before to state our main result rigorously, let us first explain formally how it can be deduced from Corollary 2.4.



### 3.1.1 Interior of the domain

In the case where  $O(t) = \mathbb{R}^{d+1}$  for all  $t < T$  and  $\beta_X = \beta_Y = 0$ , it is shown in [1] and [16] that  $v$  should (in a suitable sense) solve on  $[0, T) \times \mathbb{R}^d$

$$\sup \{ F^u(x, v(t, x), \partial_t v(t, x), Dv(t, x), D^2v(t, x)), u \in \mathcal{N}_0(x, v(t, x), Dv(t, x)) \} = 0 \quad (3.1)$$

where, for  $\Theta = (x, y, r, p, Q) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ ,  $u \in U$ , and  $\varepsilon \geq 0$ ,

$$\begin{aligned} F^u(\Theta) &:= \mu_Y(x, y, u) - r - p^\top \mu_X(x, u) - \frac{1}{2} \text{Tr} \left[ \sigma_X \sigma_X^\top(x, u) Q \right], \\ \mathcal{N}_\varepsilon(\Theta) &:= \{ u \in U : |N^u(x, y, p)| \leq \varepsilon \} \\ &\quad \text{with } N^u(x, y, p) := \sigma_Y(x, y, u) - \sigma_X(x, u)^\top p. \end{aligned} \quad (3.2)$$

The reasoning behind the above result is the following. If  $y = v(t, x)$ , if the infimum in the definition of  $v$  is achieved, and if  $v$  is smooth, then Theorem 2.3 implies that there exists  $\nu \in \mathcal{U}$  such that, with  $\phi = (\nu, 0)$ ,  $dY_{t,x,y}^\phi(t) \geq dv(t, X_{t,x}^\phi(t))$ . Formally, this implies that  $\nu_t$  should thus be such that  $\sigma_Y(X_{t,x}^\phi(t), Y_{t,x,y}^\phi(t), \nu_t) = \sigma_X(X_{t,x}^\phi(t), \nu_t)^\top Dv(t, X_{t,x}^\phi(t))$  and  $\mu_Y(X_{t,x}^\phi(t), Y_{t,x,y}^\phi(t), \nu_t) \geq \mathcal{L}_X^{\nu_t} v(t, X_{t,x}^\phi(t))$  where, for a smooth function  $\varphi$  and  $u \in U$ ,

$$\mathcal{L}_X^u \varphi := \partial_t \varphi + D\varphi^\top \mu_X(\cdot, u) + \frac{1}{2} \text{Tr} \left[ \sigma_X \sigma_X^\top(\cdot, u) D^2 \varphi \right].$$

Since  $(X_{t,x}^\phi(t), Y_{t,x,y}^\phi(t)) = (x, y) = (x, v(t, x))$ , this imposes that the left-hand side of (3.1) is non-negative. On the other hand, the ‘‘optimality’’ of  $v$  should lead to equality in (3.1).

In our situation where  $\beta_X, \beta_Y \neq 0$ , one can also use the bounded variation process  $L$  in the dynamics (2.1) to insure that  $dY_{t,x,y}^\phi(t) \geq dv(t, X_{t,x}^\phi(t))$ . It suffices to find a direction  $\ell \in \Delta_+ := [0, \infty)^d \cap B_1(0)$  such that  $G^\ell(x, y, Dv(t, x)) > 0$ , where

$$G^\ell(x, y, p) := \left( \beta_Y(x, y)^\top - p^\top \beta_X(x) \right) \ell,$$

and to ‘‘push in this direction’’. This corresponds to reflecting the process  $(s, X(s), Y(s))_s$  on the boundary of the set  $\{(t', x', y') : y' \geq v(t', x')\}$ . Assuming  $v$  smooth enough, it is possible only if such a  $\ell$  exists.

It thus follows that  $v$  should satisfy either (3.1) or  $G^\ell(x, v(t, x), Dv(t, x)) > 0$  for some  $\ell \in \Delta_+$ , i.e., at least,

$$H_0(x, v(t, x), \partial_t v(t, x), Dv(t, x), D^2v(t, x)) \geq 0 \quad (3.3)$$

where, for  $\Theta = (x, y, r, p, Q) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ , and  $\varepsilon \geq 0$ ,

$$H_\varepsilon(\Theta) := \max \{ F_\varepsilon(\Theta), G(\Theta) \}$$

with

$$F_\varepsilon(\Theta) := \sup \{ F^u(\Theta), u \in \mathcal{N}_\varepsilon(\Theta) \}, \quad G(\Theta) := \max \left\{ \left( \beta_Y(x, y)^\top - p^\top \beta_X(x) \right) \ell, \ell \in \Delta_+ \right\}.$$

### 3.1.2 Space boundary

We also have to take care of the state constraint  $(X, Y) \in O$ . To this purpose, we shall assume that the set

$$D := \{(t, x, y) \in [0, T] \times \mathbb{R}^{d+1} : (x, y) \in O(t)\} . \quad (3.4)$$

is smooth enough:

**Standing Assumption 4:** There exists a locally  $C^{1,2}$  function  $\delta$  on  $[0, T] \times \mathbb{R}^{d+1}$  such that  $\delta > 0$  in  $\text{int}D$ ,  $\delta = 0$  on  $\partial_0 D := \partial D \cap ([0, T] \times \mathbb{R}^{d+1})$ , and  $\delta < 0$  on  $([0, T] \times \mathbb{R}^{d+1}) \setminus D$ .

For  $(t, x)$  such that  $(t, x, y) = (t, x, v(t, x)) \in \partial_0 D$ , we can then follow the same reasoning as above, taking into account the fact that now, the control  $\phi = (\nu, L)$  should be such that, at the same time,  $d\delta(t, X_{t,x}^\phi(t), Y_{t,x,y}^\phi(t)) \geq 0$  and  $dY_{t,x,y}^\phi(t) \geq dv(t, X_{t,x}^\phi(t))$ . As above, this can be achieved either through the drift parts, once the Brownian parts are cancelled, or through the bounded variation process, in the case where a suitable inward direction is available. This leads to

$$H_0^{\text{in}}(x, v(t, x), \partial_t v(t, x), Dv(t, x), D^2 v(t, x)) \geq 0 \quad (3.5)$$

where, for  $\Theta = (t, x, y, r, p, Q) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ ,  $u \in U$ , and  $\varepsilon \geq 0$ ,

$$H_\varepsilon^{\text{in}}(\Theta) := \max \{ F_\varepsilon^{\text{in}}(\Theta), G^{\text{in}}(\Theta) \}$$

with

$$\begin{aligned} F_\varepsilon^{\text{in}}(\Theta) &:= \sup_{u \in \mathcal{N}_\varepsilon^{\text{in}}(\Theta)} \min \{ F^u(\Theta), \mathcal{L}_Z^u \delta(t, x, y) \} \\ G^{\text{in}}(\Theta) &:= \max_{\ell \in \Delta_+} \min \left\{ \left( \beta_Y(x, y)^\top - p^\top \beta_X(t, x) \right) \ell, D\delta(t, x, y)^\top \beta_Z(x, y) \ell \right\} . \end{aligned}$$

and

$$\begin{aligned} \mathcal{N}_\varepsilon^{\text{in}}(\Theta) &:= \left\{ u \in \mathcal{N}_\varepsilon(\Theta) : |D\delta^\top \sigma_Z(x, y, u)| \leq \varepsilon \right\} , \\ \mathcal{L}_Z^u \varphi &:= \partial_t \varphi + D\varphi^\top \mu_Z(\cdot, u) + \frac{1}{2} \text{Tr} \left[ \sigma_Z \sigma_Z^\top(\cdot, u) D^2 \varphi \right] \\ \mu_Z &:= (\mu_X^\top, \mu_Y^\top)^\top, \sigma_Z := [\sigma_X^\top \ \sigma_Y^\top]^\top, \beta_Z := [\beta_X^\top \ \beta_Y^\top]^\top \end{aligned}$$

for  $\varphi$  smooth.

### 3.1.3 Terminal condition

In order to fully characterize the value function  $v$ , it remains to define appropriate boundary conditions.

We first note that  $(x, v(T-, x)) \in O(T)$  can be expressed as

$$v(T-, x) \geq w(x) := \inf \{ y \in \mathbb{R} : (x, y) \in O(T) \} .$$

It follows that  $v(T-, \cdot)$  should formally satisfy  $v(T-, \cdot) \geq w$ .

On the other hand, the fact that  $v$  satisfies (3.3) imposes a constraint on  $v$  and its gradient through  $\mathcal{N}$  and  $G$ :  $\mathcal{N}_0(x, v(t, x), Dv(t, x)) = 0$  or  $G(t, x, v(t, x), Dv(t, x)) \geq 0$ . As usual it

should propagate up to the boundary. In order to take care of this constraint, we follow [1] and introduce the set valued map

$$\mathbf{N}(x, y, p) := \left\{ r \in \mathbb{R}^d : r = N^u(x, y, p) \text{ for some } u \in U \right\}, \quad (3.6)$$

together with the signed distance function from its complement set  $\mathbf{N}^c$  to the origin:

$$R := \mathbf{dist}(0, \mathbf{N}^c) - \mathbf{dist}(0, \mathbf{N}). \quad (3.7)$$

Then,

$$0 \in \text{int}(\mathbf{N}(x, y, p)) \quad \text{iff} \quad R(x, y, p) > 0. \quad (3.8)$$

With these notations, the terminal condition formally reads:

$$\min \{v(T-, x) - w(x), M(x, v(T, x), Dv(T, x))\} = 0, \quad (3.9)$$

where

$$M(x, v(T, x), Dv(T, x)) := \max\{R(x, v(T-, x), Dv(T-, x)), G(x, v(T-, x), Dv(T-, x))\}.$$

However, the above expression does not incorporate the part of the state constraint that may be imposed on  $(t, x)$ . In order to take care of this, we shall make the following assumption.

**Standing Assumption 5:** The function  $\delta$  admits a locally  $C^{1,2}$  extension on  $[0, T] \times \mathbb{R}^{d+1}$ .

Under the above additional condition, we shall show that  $v(T-, \cdot)$  indeed satisfies the constrained boundary condition

$$\min \{v(T-, x) - w(x), M^{\text{in}}(T, x, v(T-, x), Dv(T-, x))\} = 0, \quad (3.10)$$

when

$$(T, x, v(T-, x)) \in \partial D_T := \{T\} \times \left( \bigcap_{0 < r < T} \bigcup_{0 < \varepsilon \leq r} \partial O(T - \varepsilon) \right), \quad (3.11)$$

where

$$M^{\text{in}} := \max\{R^{\text{in}}, G^{\text{in}}\}$$

with  $R^{\text{in}}$  defined as  $R$  with  $\mathbf{N}^{\text{in}}$  in place of  $\mathbf{N}$  and

$$\mathbf{N}^{\text{in}}(t, x, y, p) := \{r \in \mathbb{R}^d : r = N^u(x, y, p) \text{ and } D\delta(t, x, y)^\top \sigma_Z(x, y, u) = 0 \text{ for some } u \in U\}.$$

For later use, we set

$$\text{int}D_T := (\{T\} \times O(T)) \setminus \partial D_T. \quad (3.12)$$

### 3.2 Main results

As in [1], the operators  $F$  and  $F^{\text{in}}$  are in general neither upper-semicontinuous nor lower-semicontinuous and need to be relaxed, i.e. we have to consider their semi-relaxed upper- and lower-semicontinuous envelopes.

$$F^*(\Theta) := \limsup_{\varepsilon \searrow 0, \Theta' \rightarrow \Theta} F_\varepsilon(\Theta'), F_*(\Theta) := \liminf_{\varepsilon \searrow 0, \Theta' \rightarrow \Theta} F_\varepsilon(\Theta'), H^* := \max\{F^*, G\}, H_* := \max\{F_*, G\}$$

and we define similarly  $F^{\text{in}*}, F_*^{\text{in}}, H^{\text{in}*}, H_*^{\text{in}}$  from  $F^{\text{in}}, H^{\text{in}}$  as well as  $R^*, R_*, R^{\text{in}*}, R_*^{\text{in}}, M^{\text{in}*}, M_*^{\text{in}}, M^*, M_*$ . For ease of notations, we shall simply write  $H_*\varphi$  for  $H_*(\cdot, \varphi, \partial_t \varphi, D\varphi, D^2\varphi)$ , and use similar notations for all the above defined operators. We shall also write  $w_*$  and  $w^*$  for the lower- and upper-semicontinuous envelopes of  $w$ .

**Remark 3.1.** (i) It follows from the convention  $\sup \emptyset = -\infty$  that  $F_\varepsilon(\Theta) = -\infty$  whenever  $\mathcal{N}_\varepsilon(x, y, q) = \emptyset$ .

(ii) Since  $F_\varepsilon$  is non-decreasing in  $\varepsilon \geq 0$ , we have  $H_*(\Theta) = \liminf_{\Theta' \rightarrow \Theta} H_0(\Theta')$ . In particular,  $F_*(\Theta) > -\infty$  implies that there exists a neighborhood of  $\Theta$  on which  $\mathcal{N}_0 \neq \emptyset$ .

(iii) The same reasoning holds for  $F^{\text{in}}$ .

Since the value function  $v$  may not be continuous, we also introduce the corresponding semi-continuous envelopes:

$$v_*(t, x) := \liminf_{\substack{(t', x') \rightarrow (t, x) \\ (t', x') \in D_Y}} v(t', x'), v^*(t, x) := \limsup_{\substack{(t', x') \rightarrow (t, x) \\ (t', x') \in D_Y}} v(t', x'), (t, x) \in \bar{D}_Y, \quad (3.13)$$

where  $D_Y$  is defined as in Standing Assumption 2 and  $\bar{D}_Y$  denotes its closure.

Before to state our main results, we need to introduce the following continuity assumption, compare with Assumption 2.1 in [1], which will be used to prove the subsolution property.

**Assumption 3.2.** *Let  $(t_0, z_0, q_0)$  be an element of  $D \times \mathbb{R}^d$ .*

(i) *If  $\mathcal{N}_0 \neq \emptyset$  on a neighborhood  $B$  of  $(z_0, q_0)$ , then for every  $\varepsilon > 0$  and  $u_0 \in \mathcal{N}_0(z_0, q_0)$  there exists a locally Lipschitz map  $\hat{v}$  defined on a neighborhood of  $B'$  of  $(z_0, q_0)$  such that  $|\hat{v}(z_0, q_0) - u_0| \leq \varepsilon$  and  $\hat{v} \in \mathcal{N}_0$  on  $B'$ .*

(ii) *If  $\mathcal{N}_0^{\text{in}} \neq \emptyset$  on a neighborhood  $B$  of  $(t_0, z_0, q_0)$ , then for every  $\varepsilon > 0$  and  $u_0 \in \mathcal{N}_0^{\text{in}}(t_0, z_0, q_0)$  there exists a locally Lipschitz map  $\hat{v}$  defined on a neighborhood of  $B'$  of  $(t_0, z_0, q_0)$  such that  $|\hat{v}(t_0, z_0, q_0) - u_0| \leq \varepsilon$  and  $\hat{v} \in \mathcal{N}_0^{\text{in}}$  on  $B'$ .*

Under the above assumption, we shall show that  $v$  is a discontinuous viscosity solution of (3.3)-(3.5)-(3.9)-(3.10) in the following sense.

**Theorem 3.3.**  $v_*$  is a viscosity super-solution on  $\bar{D}_Y$  of

$$\begin{cases} H^*\varphi \geq 0 & \text{on } \bar{D}_Y \cap ([0, T] \times \mathbb{R}^d) \\ \min\{(\varphi - w_*)\mathbf{1}_{\{F^*\varphi < \infty, G\varphi < 0\}}, M^*\varphi\} \geq 0 & \text{on } \bar{D}_Y \cap (\{T\} \times \mathbb{R}^d) \end{cases}. \quad (3.14)$$

If Assumption 3.2 holds, then  $v^*$  is a viscosity sub-solution on  $\bar{D}_Y \cap ([0, T] \times \mathbb{R}^d)$  of

$$\begin{cases} H_*\varphi \leq 0 & \text{if } (\cdot, \varphi) \in \text{int}D \\ H_*^{\text{in}}\varphi \leq 0 & \text{if } (\cdot, \varphi) \in \partial D \\ \min\{\varphi - w^*, M_*\varphi\} \leq 0 & \text{if } (\cdot, \varphi) \in \text{int}D_T \\ \min\{\varphi - w^*, M_*^{\text{in}}\varphi\} \leq 0 & \text{if } (\cdot, \varphi) \in \partial D_T \end{cases} \quad \text{on } \bar{D}_Y \cap ([0, T] \times \mathbb{R}^d) \quad (3.15)$$

Note that, as usual, the state constraints appear only on the subsolution property, see e.g. [11] and [12]. The proof of this result is reported in Section 5.

## 4 Application in optimal book liquidation

In this section, we study an application of our general model to the pricing of a book liquidation contract under a VWAP (volume weighted average price) constraint.

For sake of simplicity, we shall restrict to the case where  $W$  is a one-dimensional Brownian motion although  $d \neq 2$ , which amounts to set part of the coefficients equal to 0. We shall also consider time-dependent coefficients, which corresponds to adding a component interpreted as time in the process  $X$  and can always be done by suitably choosing the drift parameter.

Moreover, the dynamics will be only controlled by a real valued non-decreasing process  $L$ . We shall therefore only write  $X^L$  and  $Y^L$ , and now consider  $\mathcal{L}$  as the set of continuous real-valued non-decreasing adapted processes  $L$  satisfying  $\mathbb{E}[L_T^2] < \infty$ . Still, similar arguments as those used in Example 2.3 will lead to the introduction of an additional control in  $\mathcal{U}$ , see Proposition 4.2 below.

### 4.1 Description of the model

The optimal book liquidation problem is the following. A financial agent asks a broker to sell on the market a total of  $K > 0$  stocks on a time interval  $[0, T]$ . The broker takes the engagement that he will obtain a mean selling price which corresponds to (at least)  $\gamma \in (0, 1)$  times the mean price of the market, i.e. the observed selling prices weighted by the volume of the corresponding transactions initiated by all the traders that are acting on the market on  $[0, T]$ . Such contracts are referred to as *VWAP guaranteed*. The financial agent pays to the broker a premium  $y$  at time 0.

The cumulated number of stocks sold by the broker on the market since time 0 is described by a continuous real-valued non-decreasing adapted process  $L$ . Given  $L \in \mathcal{L}$ , the dynamic of the broker's portfolio  $Y^L$  is given by

$$dY^L(t) = X^{L,1}(t)dL_t, \quad Y^L(0) = 0$$

where  $X^{L,1}$  represents the stock's selling price dynamics and is assumed to solve

$$dX^{L,1}(t) = X^{L,1}(t)\mu(t, X^{L,1}(t))dt + X^{L,1}(t)\sigma(t, X^{L,1}(t))dW_t - X^{L,1}(t)\beta(t, X^{L,1}(t))dL_t$$

where  $\mu, \sigma, \beta : [0, T] \times \mathbb{R} \mapsto \mathbb{R}$  are continuous functions satisfying

$$\begin{aligned} x \in [0, \infty) \mapsto x(\mu(t, x), \sigma(t, x), \beta(t, x)) & \text{ is uniformly Lipschitz, uniformly in } t \in [0, T] \quad (4.1) \\ \text{and } \beta : [0, T] \times \mathbb{R} \mapsto \mathbb{R}_+ & \text{ is } C^2 \text{ in space, uniformly in time.} \end{aligned}$$

Note that we allow the trading strategy of the broker to have an impact on the price dynamics if  $\beta \neq 0$ .

For sake of simplicity, we model the intensity of all the transactions on the market by a deterministic non-negative integrable process  $\vartheta$ , so that

$$\Theta(t) := \int_0^t \vartheta(s)ds$$

denotes the cumulated number of stocks sold on the market since time 0. Then, the mean price of selling orders in the market, denoted by  $X^{L,2}$ , has the dynamics

$$dX^{L,2}(t) = X^{L,1}(t)\vartheta(t)dt, \quad X^{L,2}(0) = 0.$$

In order to accept the contract, a highly risk adverse broker should ask for an initial premium  $y$  such that

$$y + (Y^L(T)/K - \gamma X^{L,2}(T)/\Theta(T)) K \geq 0 \text{ for some } L \in \mathcal{L} \text{ s.t. } L_T - L_0 = K,$$

i.e. which a.s. compensates the loss made if the mean selling price of the market is not matched.

In practice, it is clear that the above problem does not make sense and needs to be relaxed. We shall therefore consider problems of the form

$$\begin{aligned} & \text{Find the minimal } y \text{ s.t., for some } L \in \mathcal{L} \text{ with } L_0 = 0, \\ & L_T = K \text{ and } \mathbb{E} [\ell (y + [Y^L(T)/K - \gamma X^{L,2}(T)/\Theta(T)] K)] \geq p, \end{aligned}$$

for  $p \in \mathbb{R}$  and  $\ell : \mathbb{R} \mapsto \mathbb{R}$  non-decreasing.

Moreover, practitioners typically impose bounds on the cumulated number of sold stocks  $X^{L,3} := L - L_0$ . We shall therefore restrict to strategies  $L \in \mathcal{L}$  such that

$$X^{L,3}(s) \in [\underline{\Lambda}(s), \bar{\Lambda}(s)] \text{ for all } s \leq T,$$

where  $\underline{\Lambda}$  and  $\bar{\Lambda}$  are assumed here to be  $C^1$  deterministic functions such that

$$\underline{\Lambda} < \bar{\Lambda} \text{ on } [0, T].$$

**Remark 4.1.** Up to an obvious change of variables, the initial premium  $y$  can be incorporated in the initial condition  $Y(0)$  of  $Y$ . Similarly, the constant  $\gamma K/\Theta(T)$  can be simply written  $\gamma > 0$  up to a change of variable. It follows that the above problem could be alternatively written as

$$\begin{aligned} & \text{Find the minimal } Y^L(0) \text{ s.t., for some } L \in \mathcal{L} \text{ with } L_0 = 0, \\ & X_T^{L,3} = K, X^{L,3}(s) \in [\underline{\Lambda}(s), \bar{\Lambda}(s)] \text{ for all } s \leq T, \text{ and } \mathbb{E} [\ell (Y^L(T) - \gamma X^{L,2}(T))] \geq p, \end{aligned}$$

with  $\gamma > 0$ .

## 4.2 Value function and problem reduction

In order to define the associated value function, we now extend the above dynamics to arbitrary initial conditions. Given  $L \in \mathcal{L}$ , we set  $X^L := (X^{L,1}, X^{L,2}, X^{L,3})$ . We write  $Z_{t,x,y}^L = (X_{t,x}^L, Y_{t,x,y}^L)$  the corresponding processes satisfying the initial condition  $Z_{t,x,y}^L(t) = (x, y)$ . In the following, we restrict to initial conditions  $y \geq 0$  and  $x = (x^1, x^2, x^3) \in (0, \infty) \times [0, \infty)^2$  to be consistent with the fact that the above quantities should be non-negative and that the process  $X^{L,1}$  takes positive values if  $X^{L,1}(0) > 0$ . In order to simplify our analysis, we make the following assumption:

$$\underline{\Lambda}(T) = \bar{\Lambda}(T) = K \text{ and } D\underline{\Lambda}, D\bar{\Lambda} \in (0, M] \text{ on } [0, T] \text{ for some } M > 0. \quad (4.2)$$

The first condition allows us to impose the constraint  $X^{3,L}(T) = K$  via the simpler one  $X^{3,L} \in [\underline{\Lambda}, \bar{\Lambda}]$ , while the assumption on the right-hand side will be used in the proof of Proposition 4.7 below in order to provide boundary conditions which will turn easier to handle.

In view of Remark 4.1 and the left-hand side of (4.2), the value function associated to the above stochastic target problem can then be written as

$$v(t, x, p) := \inf\{y \geq 0 : \exists L \in \mathcal{L} \text{ s.t. } X_{t,x}^{3,L} \in [\underline{\Lambda}, \bar{\Lambda}] \text{ and } \mathbb{E} [\Psi(Z_{t,x,y}^L(T))] \geq p\} ,$$

with  $\Psi(x, y) = \ell(y - \gamma x^2)$  and  $\gamma > 0$ .

In order to convert the above problem into a stochastic target problem in the form of the one studied in the previous sections, we use the key argument of [1] as explained in Example 2.3. In the following we set  $\mathcal{A} := \mathcal{U} \times \mathcal{L}$ , where  $\mathcal{U}$  denotes the set of all progressively measurable process  $\nu$  in  $L^2([0, T] \times \Omega)$  taking values in  $\mathbb{R}$ .

**Proposition 4.2.** *Assume that  $\ell$  has polynomial growth. Then, for all  $(t, x, p) \in [0, T] \times (0, \infty) \times [0, \infty)^2 \times \mathbb{R}$ ,*

$$v(t, x, p) := \inf\{y \geq 0 : \mathcal{A}_{t,x,y,p} \neq \emptyset\} ,$$

where  $\mathcal{A}_{t,x,y,p}$  denotes the set of processes  $(\nu, L) \in \mathcal{A}$  such that  $(Z_{t,x,y}^L, P_{t,p}^\nu) \in V$  on  $[t, T]$  with

$$\begin{aligned} V := & \{(x, y, p) \in (0, \infty) \times [0, \infty)^2 \times \mathbb{R}^2 : x^3 \in [\underline{\Lambda}, \bar{\Lambda}]\} \mathbf{1}_{[0,T]} \\ & + \{(x, y, p) \in (0, \infty) \times [0, \infty)^2 \times \mathbb{R}^2 : x^3 = K \text{ and } \ell(y - \gamma x^2) \geq p\} \mathbf{1}_{\{T\}} , \end{aligned}$$

and

$$P_{t,p}^\nu := p + \int_t^\cdot \nu_s dW_s .$$

**Proof.** If  $\ell$  has polynomial growth, then it is clear that  $\ell(Y_{t,x,y}^L(T) - \gamma X_{t,x}^{2,L}(T)) \in L^2$  for all  $L \in \mathcal{L}$  such that  $X_{t,x}^{3,L}(T) \leq K$ . It then suffices to reproduce the arguments used in Example 2.3 or in the proof of Proposition 3.1 in [1].  $\square$

In the following, we set

$$D_Y := \{(t, x, p) \in [0, T] \times (0, \infty) \times [0, \infty)^2 \times \mathbb{R} : x^3 \in [\underline{\Lambda}(t), \bar{\Lambda}(t)]\} ,$$

which is the natural domain on which our problem is stated. It satisfies the Standing Assumption 2 under the additional condition (4.4) below, see Proposition 4.5 below. In the following,  $v_*$  and  $v^*$  are defined as in (3.13) for  $D_Y$  as above.

### 4.3 Additional assumptions and a-priori estimates

In the context of the above problem, the sets  $(\mathcal{N}_\varepsilon)_\varepsilon$  reads

$$\mathcal{N}_\varepsilon \varphi = \{u \in \mathbb{R} : |uD_p \varphi + x^1 \sigma D_{x^1} \varphi| \leq \varepsilon\}$$

and

$$F_* \varphi = F^* \varphi = F_0 \varphi \text{ if } D_p \varphi \neq 0 , \tag{4.3}$$

where

$$F_0 \varphi := -\mathcal{L}_X \varphi - \frac{(x^1 \sigma)^2}{2} \left( |D_{x^1} \varphi / D_p \varphi|^2 D_p^2 \varphi - 2(D_{x^1} \varphi / D_p \varphi) D_{(x^1, p)}^2 \varphi \right)$$

with

$$\mathcal{L}_X \varphi := \partial_t \varphi + x^1 \mu D_{x^1} \varphi + x^1 \vartheta D_{x^2} \varphi + \frac{1}{2} (x^1 \sigma)^2 D_{x^1}^2 \varphi .$$

In order to provide a PDE characterization in terms of the continuous operator  $F_0$  rather than in terms of  $F^*$  and  $F_*$ , we need to ensure that  $D_p\varphi \neq 0$  for any test function for  $v_*$  or  $v^*$ . Moreover, the proof of a comparison principle will require a control of the ratio  $D_{x^1}\varphi/D_p\varphi$  which appears in  $F_0\varphi$ . In order to control the last term, we shall assume from now on that:

$$\ell \text{ admits right- and left-derivatives, there exists } \epsilon > 0 \text{ s.t. } \epsilon \leq D^-\ell, D^+\ell \leq \epsilon^{-1}, \quad (4.4)$$

$$\text{and } \lim_{r \rightarrow \infty} D^+\ell(r) = \lim_{r \rightarrow \infty} D^-\ell(r) =: D\ell(\infty), \quad (4.5)$$

where  $D^+$  and  $D^-$  denote the right- and left-derivatives respectively.

The above conditions indeed induce the following controls on  $v$ , in which we use the notation  $e_1 := (1, 0, 0)$ .

**Proposition 4.3.** *For all  $(t, x, p) \in \bar{D}_Y$  and  $h \in (-(x^1 \wedge 1), 1)$*

$$v(t, x, p) \geq \max\{v(t, x, p - \epsilon^{-1}|h|) + |h|, v(t, x + he_1, p - C(x)|h|)\},$$

where

$$C : [0, \infty)^3 \rightarrow \mathbb{R}_+ \text{ is a continuous map.} \quad (4.6)$$

**Proof. a.** We start with the first inequality  $v(t, x, p) \geq v(t, x, p - \epsilon^{-1}|h|) + |h|$ . Fix  $y > v(t, x, p)$ . Then, there exists  $\phi = (\nu, L) \in \mathcal{A}$  such that  $(Z_{t,x,y}^L, P_{t,p}^\nu) \in V$  on  $[t, T]$ . Since, by (4.4),  $\ell(r - |h|) \geq \ell(r) - \epsilon^{-1}|h|$ , we have

$$\mathbb{E} \left[ \ell \left( Y_{t,x,y}^L(T) - |h| - \gamma X_{t,x}^{L,2}(T) \right) \right] \geq p - \epsilon^{-1}|h|.$$

Since  $Y_{t,x,y}^L - |h| = Y_{t,x,y-|h|}^L$  and  $X^L$  does not depend on the initial value of  $Y^L$ , this implies the required result by arbitrariness of  $y > v(t, x, p)$  and the definition of the value function  $v$ .

**b.** Before to prove the second inequality, let us observe that standard computations based on Burkholder-Davis-Gundy's inequality, Gronwall's Lemma and the Lipschitz continuity assumption on our coefficients imply that there exists a continuous map  $C : [0, \infty)^3 \rightarrow \mathbb{R}_+$  such that, for all  $h \in [-1, 1]$  and  $L \in \mathcal{L}$  such that  $|L|_T \leq K$ ,

$$\mathbb{E} \left[ \left| \Psi \left( Z_{t,x+he_1,y}^L(T) \right) - \Psi \left( Z_{t,x,y}^L(T) \right) \right| \right] \leq C(x)|h|.$$

**c.** We now turn to the second inequality  $v(t, x, p) \geq v(t, x + he_1, p - C(x)|h|)$ . Fix  $y > v(t, x, p)$  and consider  $\phi = (\nu, L) \in \mathcal{A}$  such that  $(Z_{t,x,y}^L, P_{t,p}^\nu) \in V$  on  $[t, T]$ . It follows from b. above that

$$\mathbb{E} \left[ \Psi \left( Z_{t,x+he_1,y}^L(T) \right) \right] \geq \mathbb{E} \left[ \Psi \left( Z_{t,x,y}^L(T) \right) \right] - C(x)|h| \geq p - C(x)|h|.$$

As above, the required result then follows from the arbitrariness  $y > v(t, x, p)$ .  $\square$

The immediate consequence of the above estimates is a control on  $D_{x^1}\varphi/D_p\varphi$  for test functions of  $v^*$  or  $v_*$ .

**Corollary 4.4.** *The function  $v_*$  is a viscosity supersolution of*

$$\min \{ D_p\varphi - \epsilon, (D_{x^1}\varphi - C(x)D_p\varphi)\mathbf{1}_{x^1 > 0}, -D_{x^1}\varphi + C(x)D_p\varphi \} = 0 \text{ on } \bar{D}_Y \quad (4.7)$$

and  $v^*$  is a viscosity subsolution of

$$\max \{ -D_p\varphi + \epsilon, (D_{x^1}\varphi - C(x)D_p\varphi)\mathbf{1}_{x^1 > 0}, -D_{x^1}\varphi + C(x)D_p\varphi \} = 0 \text{ on } \bar{D}_Y. \quad (4.8)$$



We now provide additional estimates that will be used later on to establish a comparison principle on the PDE associated to  $v$ . We first show that the conditions (4.2)-(4.4) allows us to deduce a classical growth condition on  $v$ .

**Proposition 4.5.** *There exists  $\eta > 0$  such that*

$$0 \leq v(t, x, p) \leq \epsilon^{-1}|p - \ell(0)| + \gamma\eta(1 + |x|) \text{ for all } (t, x, p) \in \bar{D}_Y. \quad (4.9)$$

**Proof.** Define  $L := \max\{x^3, \underline{\Lambda}\}$ , which belongs to  $\mathcal{L}$  by (4.2). Then, it follows from (4.2) again that  $X_{t,x}^{L,3}(T) = K$  and  $X_{t,x}^{L,3} \in [\underline{\Lambda}, \bar{\Lambda}]$  on  $[t, T]$ . Moreover, standard estimates imply

$$\mathbb{E} [ |X_{t,x}^L(T)|^2 ]^{\frac{1}{2}} \leq \eta(1 + |x|)$$

for some  $\eta > 0$  which does not depend on  $(t, x)$ . In particular, for  $y > 0$ , the above inequality combined with our assumption (4.4) leads to

$$\begin{aligned} \mathbb{E} \left[ \ell \left( Y_{t,x,y}^L(T) - \gamma X_{t,x}^{L,2}(T) \right) \right] &\geq \mathbb{E} \left[ \ell \left( y - \gamma X_{t,x}^{L,2}(T) \right) \right] \\ &\geq \epsilon(y - \gamma\eta(1 + |x|)) + \ell(0). \end{aligned}$$

By choosing  $y$  equal to the right-hand side of (4.9), we obtain  $\mathbb{E} \left[ \ell \left( Y_{t,x,y}^L(T) - \gamma X_{t,x}^{L,2}(T) \right) \right] \geq p$ . The required result follows from the definition of  $v$ .  $\square$

We finally provide suitable boundary conditions for  $v$ .

**Proposition 4.6.** *Fix  $(t, x, p) \in \bar{D}_Y$ . Then,*

$$v_*(t, 0, x^2, x^3, p) = v^*(t, 0, x^2, x^3, p) = \Psi^{-1}(0, x^2, x^3, p), \quad (4.10)$$

where

$$\Psi^{-1}(x, p) := \inf\{y \geq 0 : \Psi(x, y) \geq p\}.$$

Moreover, for all sequence  $(t_n, x_n, p_n)_n \subset D_Y$  such that  $(t_n, x_n) \rightarrow (t, x) \in [0, T] \times [0, \infty)^3$ ,

$$\lim_{n \rightarrow \infty} v_*(t_n, x_n, p_n) = \lim_{n \rightarrow \infty} v^*(t_n, x_n, p_n) = 0 \text{ if } p_n \rightarrow -\infty, \quad (4.11)$$

$$\lim_{n \rightarrow \infty} v_*(t_n, x_n, p_n)/p_n = \lim_{n \rightarrow \infty} v^*(t_n, x_n, p_n)/p_n = 1/D\ell(\infty) \text{ if } p_n \rightarrow \infty. \quad (4.12)$$

**Proof. a.** We start with the first assertion. Let  $(t_n, x_n, p_n)_n$  be a sequence in  $D_Y$  that converges to  $(t, 0, x^2, x^3, p)$  and fix  $y > \Psi^{-1}(x, p)$  with  $x = (0, x^2, x^3)$ . Then, the Lipschitz continuity of the coefficients implies that  $X_{t_n, x_n}^0(T) \rightarrow (0, x^2, x^3)$   $\mathbb{P}$ -a.s. and in  $L^q$  for any  $q \geq 2$ . Since  $\Psi$  is Lipschitz continuous, it follows that  $\lim_{n \rightarrow \infty} \mathbb{E} [\Psi(Z_{t_n, x_n, y}^0(T))] = \Psi(x, y) > p$ . Hence,  $\mathbb{E} [\Psi(Z_{t_n, x_n, y}^0)] \geq p_n$  for  $n$  large enough, and therefore  $v(t_n, x_n, p_n) \leq y$ . The arbitrariness of  $y$  thus implies that  $\limsup_{n \rightarrow \infty} v(t_n, x_n, p_n) \leq \Psi^{-1}(x, p)$ . We next deduce from the Lipschitz continuity of the coefficients again that, for any  $L^n \in \mathcal{L}$  such that  $L_T^n \leq K$  and  $(y_n)_{n \geq 1}$  such that  $\mathbb{E} [\Psi(Z_{t_n, x_n, y_n}^{L^n}(T))] \geq p_n$ , we have  $Z_{t_n, x_n, y_n}^{L^n}(T) \rightarrow (0, x^2, x^3, y)$   $\mathbb{P}$ -a.s. and in  $L^q$  for any  $q \geq 2$ , whenever  $y_n \rightarrow y$ . Hence,  $\lim_{n \rightarrow \infty} \mathbb{E} [\Psi(Z_{t_n, x_n, y}^{L^n}(T))] = \Psi(x, y) \geq p$  so that  $y \geq \Psi^{-1}(x, p)$ . Taking  $y_n := v(t_n, x_n, p_n) + 1/n$  with  $(t_n, x_n, p_n)_n$  such that  $v(t_n, x_n, p_n) \rightarrow v_*(t, 0, x^2, x^3, p)$  then shows that  $v_*(t, 0, x^2, x^3, p) \geq \Psi^{-1}(x, p)$ .

**b.** We now turn to the second assertion. It follows from the following easy observation. Fix  $(t, x) \in [0, T] \times (0, \infty) \times [0, \infty)^2$  such that  $x^3 \in [\underline{\Lambda}(t), \bar{\Lambda}(t)]$ . Then, for  $L$  defined by  $L = \underline{\Lambda} + x^3 -$

$\underline{\Lambda}(t)$ , one obtains  $X_{t,x}^{L,3} \in [\underline{\Lambda}, \bar{\Lambda}]$  on  $[t, T]$ , recall (4.2), and  $\underline{p}(t, x) := \mathbb{E} [\Psi(Z_{t,x,0}^L(T))] > -\infty$ . It follows that  $v(t, x, p) = 0$  for  $p \leq \underline{p}(t, x)$ , where the function  $\underline{p}$  is clearly locally bounded.

**c.** We finally prove the last assertion. Since  $p_n \rightarrow \infty$  and, for any strategy  $L^n \in \mathcal{L}$  such that  $L_T^n \leq K$ ,  $Z_{t_n, x_n, 0}^{L^n}(T)$  is uniformly bounded in any  $L^q$ ,  $q \geq 2$ , one has  $y_n \rightarrow \infty$  whenever  $\mathbb{E} [\Psi(Z_{t_n, x_n, y_n}^{L^n}(T))] \geq p_n$  for all  $n$ . Using (4.5), one deduces that, for all  $\varepsilon > 0$ ,  $\exists r_\varepsilon \in \mathbb{R}$  such that

$$\Psi(Z_{t_n, x_n, y_n}^{L^n}(T)) \leq \ell(r_\varepsilon) + (D\ell(\infty) + \varepsilon) \left( y_n + Y_{t_n, x_n, 0}^{L^n}(T) - \gamma X_{t_n, x_n}^{2, L^n}(T) - r_\varepsilon \right) \quad \text{for } n \text{ large } \mathbb{P}\text{-a.s.}$$

It follows that

$$1 \leq \limsup_{n \rightarrow \infty} \mathbb{E} [\Psi(Z_{t_n, x_n, y_n}^{L^n}(T))] / p_n \leq (D\ell(\infty) + \varepsilon) \limsup_{n \rightarrow \infty} y_n / p_n .$$

This implies that  $\liminf_{n \rightarrow \infty} v_*(t_n, x_n, p_n) / p_n \geq 1 / (D\ell(\infty) + \varepsilon)$ . Choosing  $\varepsilon$  arbitrarily small leads to the required result. On the other hand, for  $y_n := p_n (D\ell(\infty) - \varepsilon)^{-1}$  with  $\varepsilon \in (0, D\ell(\infty))$ , we have, by similar arguments,

$$\Psi(Z_{t_n, x_n, y_n}^0(T)) / p_n \rightarrow D\ell(\infty) (D\ell(\infty) - \varepsilon)^{-1} > 1$$

so that

$$\liminf_{n \rightarrow \infty} \mathbb{E} [\Psi(Z_{t_n, x_n, y_n}^0(T))] / p_n > 1$$

and therefore  $v(t_n, x_n, p_n) \leq p_n (D\ell(\infty) - \varepsilon)^{-1}$  for  $n$  large enough. This implies that  $\limsup_{n \rightarrow \infty} v^*(t_n, x_n, p_n) / p_n \leq (D\ell(\infty) - \varepsilon)^{-1}$ , which yields the required result by arbitrariness of  $\varepsilon > 0$ .  $\square$

#### 4.4 PDE characterization

We can now provide the main results of this section. We first report the PDE characterization of  $v$ .

**Proposition 4.7.** *The functions  $v_*$  is a viscosity supersolution on  $D_Y$  of*

$$\max \{ F_0 \varphi, x^1 + x^1 \beta D_{x^1} \varphi - D_{x^3} \varphi \} = 0 . \quad (4.13)$$

*The function  $v^*$  is a subsolution on  $D_Y$  of*

$$\begin{aligned} \min \{ \varphi, \max \{ F_0 \varphi, x^1 + x^1 \beta D_{x^1} \varphi - D_{x^3} \varphi \} \} = 0 & \quad \text{if } \underline{\Lambda} < x^3 < \bar{\Lambda} \\ \min \{ \varphi, x^1 + \beta D_{x^1} \varphi - D_{x^3} \varphi \} = 0 & \quad \text{if } \underline{\Lambda} = x^3 \\ \min \{ \varphi, F_0 \varphi \} = 0 & \quad \text{if } x^3 = \bar{\Lambda} . \end{aligned} \quad (4.14)$$

Moreover,

$$v_*(T, x, p) = v^*(T, x, p) = \Psi^{-1}(x, p) \text{ for all } (x, p) \in [0, \infty)^2 \times \{K\} \times \mathbb{R} . \quad (4.15)$$

**Proof. a.** We first discuss the PDE characterization. In view of Theorem 3.3, we already know that  $v_*$  is a supersolution on  $D_Y$  of

$$\max \{ F^* \varphi, x^1 + x^1 \beta D_{x^1} \varphi - D_{x^3} \varphi \} = 0$$

and that  $v^*$  is a subsolution on  $D_Y \cap \{v^* > 0\}$  of

$$\begin{aligned} \max \{F_*\varphi, x^1 + x^1\beta D_{x^1}\varphi - D_{x^3}\varphi\} &= 0 & \text{if } \underline{\Lambda} < x^3 < \bar{\Lambda} \\ \max \{\min\{F_*\varphi, -D\underline{\Lambda}\}, \min\{x^1 + x^1\beta D_{x^1}\varphi - D_{x^3}\varphi, 1\}\} &= 0 & \text{if } \underline{\Lambda} = x^3 \\ \max \{\min\{F_*\varphi, D\bar{\Lambda}\}, \min\{x^1 + x^1\beta D_{x^1}\varphi - D_{x^3}\varphi, -1\}\} &= 0 & \text{if } x^3 = \bar{\Lambda}. \end{aligned}$$

Note that the boundary  $x^2 = 0$  does not play any role here since the process  $X^{2,L}$  is non-decreasing.

Since, by Proposition 4.3,  $v$  is strictly increasing in its  $p$  variable, any test function  $\varphi$  such that  $(t_0, x_0, p_0)$  achieves a local maximum (resp. minimum) of  $v^* - \varphi$  (resp.  $v_* - \varphi$ ) must then satisfy  $D_p\varphi(t_0, x_0, p_0) > 0$ . It follows that  $F^*$  and  $F_*$  can be replaced by  $F_0$ , see (4.3).

In order to simplify the subsolution property for  $t < T$ , it then suffices to use the fact that  $D\underline{\Lambda} > 0$  and  $D\bar{\Lambda} > 0$  by assumption (4.2).

**b.** It remains to prove the boundary condition at  $T$ .

**b.1.** We first discuss the supersolution property at  $T$ . Let  $(t_n, x_n, p_n)_{n \geq 1}$  be a sequence in  $D_Y$ , with  $t_n < T$  for all  $n$ , such that  $(t_n, x_n, p_n) \rightarrow (T, x_0, p_0) \in \bar{D}_Y$  and  $v(t_n, x_n, p_n) \rightarrow v_*(T, x_0, p_0)$ . Set  $y_n := v(t_n, x_n, p_n) + 1/n$  and let  $L^n$  be such that  $\mathbb{E}[\Psi(Z^n(T))] \geq p_n$  where  $Z^n = (X^n, Y^n) := (X_{t_n, x_n}^{L^n}, Y_{t_n, x_n, y_n}^{L^n})$ . Since  $X^{n,2}(T) = K$ , we have  $L^n(T) - L^n(t_n) = K - x_n^3$ . Since  $L^n$  is non-decreasing, this shows that  $\sup_{t_n \leq t \leq T} L^n(t) - L^n(t_n) \rightarrow 0$  in  $L^\infty$ . This implies that  $Z^n(T) \rightarrow z_0 := (x_0, y_0)$  in any  $L^q$ ,  $q \geq 2$ . It then follows from the dominated convergence theorem that

$$\Psi(z_0) - p_0 = \lim_{n \rightarrow \infty} \mathbb{E}[\Psi(Z^n(T))] - p_n \geq 0.$$

This shows that  $v_*(T, x_0, p_0) \geq \Psi^{-1}(x_0, p_0)$ .

**b.2.** We finally prove the subsolution property. Let  $(T, x_0, p_0) \in \bar{D}_Y$  and  $\varphi$  be a smooth function such that

$$\begin{aligned} (T, x_0, p_0) \text{ achieves a strict local maximum of } v^* - \varphi \text{ such that} \\ (v^* - \varphi)(T, x_0, p_0) = 0 \text{ and } \varphi(T, x_0, p_0) > \Psi^{-1}(x_0, p_0). \end{aligned} \quad (4.16)$$

Let  $(t_n, x_n, p_n)_{n \geq 1}$  be a sequence in  $D_Y$ , with  $t_n < T$  for all  $n$ , which converges to  $(T, x_0, p_0)$  and such that  $v(t_n, x_n, p_n) \rightarrow v^*(T, x_0, p_0)$ . Set  $y_n := v(t_n, x_n, p_n) - 1/n$ . Since  $\underline{\Lambda}$  and  $\bar{\Lambda}$  are  $C^1$ , there exists  $L^n$  such that  $X_{t_n, x_n}^{L^n, 2}$  is reflected on the boundary of  $[\underline{\Lambda}, \bar{\Lambda}]$ . It takes the form

$$dL_t^n = \alpha_t^n dt \quad (4.17)$$

where  $\alpha^n$  is a predictable process satisfying, recall (4.2),

$$\sup_{t \in [t_n, T]} |\alpha_t^n| \leq \sup_{n \geq 1} \sup_{t \in [t_n, T]} (D\underline{\Lambda}(t) \vee D\bar{\Lambda}(t)) \leq M. \quad (4.18)$$

Since  $\varphi$  is smooth, we can also define the control  $\nu^n := -\sigma(\cdot, X_{t_n, x_n}^{L^n, 1})(D_{x^1}\varphi/D_p\varphi)(\cdot, X_{t_n, x_n}^{L^n, 1})$ , recall from the above discussion that we must have  $D_p\varphi > 0$  on a neighborhood of  $(T, x_0, p_0)$ . For ease of notations, we write  $Z^n = (X^n, Y^n) := (X_{t_n, x_n}^{L^n}, Y_{t_n, x_n, y_n}^{L^n})$  and  $P^n := P_{t_n, p_n}^{\nu^n}$ .

Let  $\tilde{\varphi}$  be defined by  $\tilde{\varphi}(t, x, p) := \varphi(t, x, p) + \sqrt{T - t} + \iota - \sqrt{\iota}$  for some  $\iota > 0$ . It follows from the identity  $v(T, \cdot) = \Psi^{-1}$  and (4.16) that

$$\max_{(\{T\} \times \bar{B}_\varepsilon(x_0, p_0)) \cup ([T - \varepsilon, T] \times \partial B_\varepsilon(x_0, p_0))} (v^* - \tilde{\varphi}) := -\zeta < 0,$$

for  $\varepsilon > 0$  small enough. Moreover, for  $\varepsilon, \iota > 0$  small enough, one has

$$\inf_{|\alpha| \leq M} (x^1 \alpha + F_0 \tilde{\varphi} + \beta(x^1) \alpha D_{x^1} \tilde{\varphi} - \alpha D_{x^3} \tilde{\varphi}) \geq 0 \text{ on } [T - \varepsilon, T] \times \partial B_\varepsilon(x_0, p_0), \quad (4.19)$$

since  $\partial_t \tilde{\varphi} \rightarrow -\infty$  as  $t \rightarrow T$  and  $\iota \rightarrow 0$ . We next define the stopping times

$$\begin{aligned} \theta_n^o &:= \inf \{s \geq t_n : (s, X^n(s), P^n(s)) \notin [T - \varepsilon, T] \times B_\varepsilon(x_0, p_0)\}, \\ \theta_n &:= \inf \{s \geq t_n : |Y^n(s) - \varphi(s, X^n(s), P^n(s))| \geq \varepsilon\} \wedge \theta_n^o. \end{aligned}$$

Using (4.17)-(4.18)-(4.19), the same arguments as in the proof of Proposition 5.2 below show that

$$Y^n(\theta_n) - v(\theta_n, X^n(\theta_n), P^n(\theta_n)) > 0$$

for  $n$  large enough. Recalling that  $y_n = v(t_n, x_n, p_n) - n^{-1} < v(t_n, x_n, p_n)$ , this is in contradiction with (GDP2) of Corollary 2.4.  $\square$

## 4.5 Comparison principle and uniqueness

In order to complete the characterization of Proposition 4.7, it remains to provide a comparison theorem for (4.13)-(4.14).

Note that the term  $D_{x^1} \varphi / D_p \varphi$  which appears in the definition of  $F_0$  can be shown to be bounded because viscosity super- and subsolution of (4.13)-(4.14) have to satisfy  $D_p \varphi \geq \varepsilon$  and  $|D_{x^1} \varphi / D_p \varphi| \leq C$  in the viscosity sense. Still obtaining a general comparison theorem for the above PDE in an unbounded domain remains an open question.

In what follows, we shall therefore reduce to a bounded domain by adding the following condition:

$$\exists \hat{x}^1 > 0 \text{ s.t. } \mu(\cdot, \hat{x}^1) = \sigma(\cdot, \hat{x}^1) = 0. \quad (4.20)$$

This implies that  $\hat{x}^1$  is an absorbing point for  $X^{L,1}$ . In particular, it remains bounded as well as  $X^{L,2}$  which is bounded by  $\Theta(T)\hat{x}^1$ . In particular, we can then restrict to the bounded domain

$$\hat{D}_Y := D_Y \cap ([0, T] \times (0, 2\hat{x}^1) \times [0, 2\Theta(T)\hat{x}^1] \times [0, K] \times \mathbb{R}).$$

For  $x^1 > \hat{x}^1$ , the value function  $v$  can be easily computed explicitly, since the problem becomes deterministic, and is continuous:

$$v(t, x, p) = \Psi^{-1} \left( x^1, \gamma \left( x^2 + \int_t^T x^1 \vartheta(s) ds \right) - x^1(K - x^3), x^3, p \right) \text{ for } x^1 > \hat{x}^1. \quad (4.21)$$

Note that it is not a real limitation for practical applications, since  $\hat{x}^1$  can be arbitrary large.

**Proposition 4.8.** *Assume that (4.20) holds. Let  $U$  (resp.  $V$ ) be a non-negative lower-semicontinuous supersolution of (4.13) (resp. upper-semicontinuous subsolution of (4.14)) on  $\hat{D}_Y$ , such that  $U$  and  $V$  are continuous in  $x^3$ . Assume that*

$$U(t, x, p) \geq V(t, x, p) \text{ if } t = T \text{ or } x^1 \in \{0, 2\hat{x}^1\}, \quad (4.22)$$

and that there exists  $c_+ > 0$  and  $c_- \in \mathbb{R}$  such that

$$\limsup_{(t',x',p') \rightarrow (t,x,\infty)} V(t',x',p')/p' \leq c_+ \leq \liminf_{(t',y',p') \rightarrow (t,y,\infty)} U(t',y',p')/p', \quad (4.23)$$

$$\limsup_{(t',x',p') \rightarrow (t,x,-\infty)} V(t',x',p') \leq c_- \leq \liminf_{(t',y',p') \rightarrow (t,y,-\infty)} U(t',y',p') \quad (4.24)$$

$\forall (t, x) \in [0, T] \times [0, \infty)^3$ . If either  $U$  is a viscosity supersolution of (4.7) on  $\hat{D}_Y$  which is continuous in  $p$ , or that  $V$  is a viscosity subsolution of (4.8) on  $\hat{D}_Y$  which is continuous in  $p$ , then

$$U \geq V \text{ on } \hat{D}_Y.$$

Before to provide the proof of the above result, we state the following immediate corollary which shows that the characterization of  $v$  in Proposition 4.7 is indeed sharp.

**Corollary 4.9.** *Assume that (4.20) holds. Then,  $v$  is continuous on the closure of  $\hat{D}_Y$ . Moreover, it is the unique non-negative viscosity solution of (4.13) in the class of continuous functions that are either supersolutions of (4.7) or subsolutions of (4.8), and satisfy the boundary conditions (4.10)-(4.11)-(4.12)-(4.15)-(4.21).*

This follows from Proposition 4.6, Corollary 4.4, Proposition 4.7, Proposition 4.8 and the following continuity result.

**Proposition 4.10.** *The function  $v$  is continuous in its  $p$  and  $x^3$  variables, and, therefore, so are  $v_*$  and  $v^*$ .*

**Proof.** Since  $\ell$  is non-decreasing, so is  $v$ , in the  $p$ -variable. It thus suffices to show that  $\limsup_{|h| \rightarrow 0} v(t, x, p + |h|) \leq v(t, x, p)$ . To see this fix  $y > v(t, x, p)$  and  $L \in \mathcal{L}$  such that  $Z_{t,x,y}^L \in V$  and  $\mathbb{E}[\Psi(Z_{t,x,y}^L(T))] > p$ , which is possible since  $\ell$  is strictly increasing and  $y > v(t, x, p)$ . It follows that  $y \geq v(t, x, p + |h|)$  for  $h$  small enough. Sending  $|h| \rightarrow 0$  and then  $y \rightarrow v(t, x, p)$  leads to the required result.

We now turn to the continuity with respect to  $x^3$ . Fix  $h \in \mathbb{R}$  such that  $|h| \leq \min\{x^3 - \underline{\Lambda}(t), \bar{\Lambda}(t) - x^3\}$ . Denote  $e_3 := (0, 0, 1)$ , and let  $L \in \mathcal{L}$  and  $y \geq 0$  be such that  $Z_{t,x,y}^L \in V$ . Then,  $L_T - L_t \leq K$  and  $\sup_{t \leq s \leq T} |Z_{t,x,0}^L| \leq c$  for some  $c > 0$ . Set  $L^h := \mathbf{1}_{[t,T]}((L - L_t + x^3 + h) \wedge \bar{\Lambda}) \vee \underline{\Lambda}$ . Then,  $Z_{t,x+he_3,y}^{L^h} \in V$  and  $\sup_{t \leq s \leq T} |Z_{t,x,0}^L - Z_{t,x+he_3,0}^{L^h}| \leq c|h|$  where  $c > 0$  does not depend on  $(t, x, p)$ . This implies that  $v(t, x + he_3, p) \leq v(t, x, p) + c|h|$ . Similarly,  $v(t, x, p) \leq v(t, x + he_3, p) + c|h|$ .  $\square$

We conclude with the proof of Proposition 4.8.

**Proof of Proposition 4.8.** We assume that  $U$  is a viscosity supersolution of (4.7) which is continuous in  $p$ . The case where  $V$  is a viscosity subsolution of (4.8) which is continuous in  $p$  is treated similarly. As usual, we argue by contradiction and assume that there exists  $(t_0, x_0, p_0) \in \hat{D}_Y$  such that

$$(V - U)(t_0, x_0, p_0) =: 4\eta_0 > 0.$$

Given  $\kappa > 0$  and  $\zeta > 1$ , we define  $\tilde{U}_\zeta$  and  $\tilde{V}$  by  $\tilde{U}_\zeta(t, x, p) := e^{\kappa(t+x^3)}U(t, x, p\zeta)$  and  $\tilde{V}(t, x, p) := e^{\kappa(t+x^3)}V(t, x, p)$ . Since  $U$  is continuous in its  $p$ -variable, one has

$$\sup_{\hat{D}_Y} (\tilde{V} - \tilde{U}_\zeta) =: 2\eta \geq 2\eta_0 > 0, \quad (4.25)$$

for  $\kappa > 0$  and  $\zeta > 1$  small enough.

1. Note that  $\eta < \infty$  and that

$$2\eta = \sup_{\hat{D}_Y} (\tilde{V} - \tilde{U}_\zeta) = (\tilde{V} - \tilde{U}_\zeta)(\hat{t}_\zeta, \hat{x}_\zeta, \hat{p}_\zeta) \quad (4.26)$$

for some  $\hat{z}_\zeta := (\hat{t}_\zeta, \hat{x}_\zeta, \hat{p}_\zeta)$  in the closure of  $\hat{D}_Y$ . Indeed, if  $(t_k, x_k, p_k)_{k \geq 1}$  is a maximizing sequence, then  $(t_k, x_k)_{k \geq 1}$  is bounded and therefore converges along a subsequence. If  $p_k \rightarrow -\infty$ , we obtain a contradiction by appealing to (4.24). If  $p_k \rightarrow \infty$ , then  $\limsup_{k \rightarrow \infty} V(t_k, x_k, p_k)/p_k \leq c_+ < c_+\zeta \leq \zeta \limsup_{k \rightarrow \infty} U(t_k, x_k, p_k\zeta)/(p_k\zeta)$ , which also leads to a contradiction. The fact that the supremum is achieved then follows from the upper-semicontinuity of  $\tilde{V} - \tilde{U}_\zeta$ . From now on, we assume that

$$\hat{x}_\zeta^3 = \underline{\Lambda}(\hat{t}_\zeta) \quad (4.27)$$

for all  $\zeta > 1$  small enough. The case where  $\hat{x}_\zeta^3 = \bar{\Lambda}(\hat{t}_\zeta)$  (resp.  $\hat{x}_\zeta^3 \in (\underline{\Lambda}(\hat{t}_\zeta), \bar{\Lambda}(\hat{t}_\zeta))$ ) can be treated similarly by replacing  $|x^3 - y^3 - \delta|^2$  in the definition of  $\Lambda_n$  below by  $|y^3 - x^3 - \delta|^2$  (resp. 0).

2. For  $n \geq 1$ , we now set

$$\Theta_n : (t, x, y, p, q) \mapsto \tilde{V}(t, x, p) - \tilde{U}_\zeta(t, y, q) - \Lambda_n(t, x, y, p, q)$$

where

$$\begin{aligned} \Lambda_n(t, x, y, p, q) &:= \frac{n}{2} (|p - q|^2 + |x^1 - y^1|^2 + |x^3 - y^3 - \delta|^2) \\ &\quad + \frac{1}{2} (|t - \hat{t}_\zeta|^2 + |x^3 - \hat{x}_\zeta^3|^4) \end{aligned}$$

for some  $\delta > 0$ . Note that, by continuity of  $V$  in  $x^3$  and (4.26),

$$\liminf_{\delta \rightarrow 0, n \geq 1} \Theta_n(\hat{t}_\zeta, \hat{x}_\zeta + \delta e_3, \hat{x}_\zeta, \hat{p}_\zeta, \hat{p}_\zeta) = 2\eta,$$

with  $e_3 := (0, 0, 1)$ . It follows that

$$\sup_{\hat{D}_Y^2} \Theta_n \geq \eta \text{ for all } n \geq 1, \quad (4.28)$$

for  $\delta > 0$  small enough, where

$$\hat{D}_Y^2 := \{(t, x, y, p, q) \in [0, T] \times [0, \infty)^6 \times \mathbb{R}^2 : ((t, x, p), (t, y, q)) \in \hat{D}_Y \times \hat{D}_Y, x^2 = y^2\}.$$

Moreover, the same arguments as in step 1. above show that there exists  $\bar{z}_n := (t_n, x_n, y_n, p_n, q_n)$  in the closure of  $\hat{D}_Y$  satisfying

$$\Theta_n(\bar{z}_n) = \sup_{\hat{D}_Y^2} \Theta_n \geq \eta \text{ for all } n \geq 1. \quad (4.29)$$

It then follows from standard arguments, combined with the ones used in step 1. above, see e.g. [5], that

$$\bar{z}_n \rightarrow \bar{z}_{\zeta, \delta} := (t_{\zeta, \delta}, x_{\zeta, \delta}, y_{\zeta, \delta}, p_{\zeta, \delta}, q_{\zeta, \delta}) \text{ in the closure of } \hat{D}_Y^2 \text{ as } n \rightarrow \infty \quad (4.30)$$

where

$$\begin{aligned} & \lim_{n \rightarrow \infty} n|x_n^1 - y_n^1|^2 + n|p_n - q_n|^2 + n|x_n^3 - y_n^3 - \delta|^2 = 0 \\ & \lim_{n \rightarrow \infty} (\tilde{V} - \tilde{U}_\zeta)(\bar{z}_n) = (\tilde{V} - \tilde{U}_\zeta)(\bar{z}_{\zeta, \delta}) \\ \text{and } & \lim_{\delta \rightarrow 0} (t_{\zeta, \delta}, x_{\zeta, \delta}^3) = (t_\zeta, \hat{x}_\zeta^3), \quad \lim_{\delta \rightarrow 0} (\tilde{V} - \tilde{U}_\zeta)(\bar{z}_{\zeta, \delta}) = (\tilde{V} - \tilde{U}_\zeta)(\hat{z}_\zeta) = 2\eta > \eta_0. \end{aligned} \quad (4.31)$$

Note that, combined with (4.27), this implies that

$$x_{\zeta, \delta}^1 = y_{\zeta, \delta}^1, \quad x_{\zeta, \delta}^2 = y_{\zeta, \delta}^2, \quad p_{\zeta, \delta} = q_{\zeta, \delta} \quad \text{and} \quad \underline{\Lambda}(t_{\zeta, \delta}) < y_{\zeta, \delta}^3 + \delta = x_{\zeta, \delta}^3 < \bar{\Lambda}(t_{\zeta, \delta}), \quad (4.32)$$

for  $\delta > 0$  small enough.

**3. a.** Clearly, we cannot have  $\tilde{V}(t_{\zeta, \delta}, x_{\zeta, \delta}, p_{\zeta, \delta}) = 0$  since  $\tilde{U}_\zeta \geq 0$  by assumption.

**b.** We can neither have  $t_{\zeta, \delta} = T$  since this would imply  $x_{\zeta, \delta}^3 = y_{\zeta, \delta}^3$  by definition of  $\hat{D}_Y$  and (4.2), a contradiction to (4.32).

**c.** We can also not have  $x_{\zeta, \delta}^1 \in \{0, 2\hat{x}^1\}$  for all  $\delta > 0$  small enough. To see this assume the contrary and note that the fact that  $U$  is a supersolution of (4.7) implies that it is non-decreasing in  $p$ . First assume that  $q_{\zeta, \delta} \geq 0$  for all  $\zeta > 1$  small enough. Since  $\zeta > 1$ , it then follows from the upper-semicontinuity of  $V, -U$  and from (4.32) that  $V(t_{\zeta, \delta}, x_{\zeta, \delta}, p_{\zeta, \delta}) - U(t_{\zeta, \delta}, y_{\zeta, \delta}, \zeta q_{\zeta, \delta}) \leq (V - U)(t_{\zeta, \delta}, x_{\zeta, \delta}, p_{\zeta, \delta}) + O(\delta)$ . Recalling (4.22) and the definition of  $(\tilde{V}, \tilde{U}_\zeta)$ , we obtain  $\tilde{V}(t_{\zeta, \delta}, x_{\zeta, \delta}, p_{\zeta, \delta}) - \tilde{U}_\zeta(t_{\zeta, \delta}, y_{\zeta, \delta}, q_{\zeta, \delta}) \leq O(\delta) < \eta_0$  for  $\delta$  small enough whenever  $x_{\zeta, \delta}^1 \in \{0, 2\hat{x}^1\}$ , a contradiction to (4.31).

Now assume that  $p_{\zeta, \delta} < 0$  for all  $\zeta > 1$  small enough. Then (4.24), the fact that  $(t_{\zeta, \delta}, x_{\zeta, \delta})$  takes values in a compact set and the definition of  $\eta \geq \eta_0 > 0$  imply that  $|p_{\zeta, \delta}| \leq \xi$  for some  $\xi > 0$  which does not depend on  $\zeta$  or  $\delta$ . In particular, as  $\zeta \rightarrow 1$ ,  $(t_{\zeta, \delta}, x_{\zeta, \delta}, y_{\zeta, \delta}, p_{\zeta, \delta}, q_{\zeta, \delta})_{\zeta > 1}$  converges to some  $(t_{1, \delta}, x_{1, \delta}, y_{1, \delta}, p_{1, \delta}, q_{1, \delta})$  such that  $x_{1, \delta}^1 \in \{0, 2\hat{x}^1\}$ , (4.32) holds at the limit  $\zeta = 1$ , and  $\limsup_{\zeta \rightarrow 1} (\tilde{V}(t_{\zeta, \delta}, x_{\zeta, \delta}, p_{\zeta, \delta}) - \tilde{U}_\zeta(t_{\zeta, \delta}, y_{\zeta, \delta}, q_{\zeta, \delta})) \leq \tilde{V}(t_{1, \delta}, x_{1, \delta}, p_{1, \delta}) - \tilde{U}_1(t_{1, \delta}, x_{1, \delta} - \delta e_3, p_{1, \delta}) \leq O(\delta)$  by upper-semicontinuity of  $V, -U$ , and (4.22). This shows that  $\tilde{V}(t_{\zeta, \delta}, x_{\zeta, \delta}, p_{\zeta, \delta}) - \tilde{U}_\zeta(t_{\zeta, \delta}, y_{\zeta, \delta}, q_{\zeta, \delta}) < \eta_0 < 2\eta$  for  $\zeta$  sufficiently close to 1 and  $\delta > 0$  small enough, a contradiction to (4.31).

**4.** Now observe that, by assumption,  $\tilde{U}_\zeta$  and  $\tilde{V}$  are super and subsolutions on  $\hat{D}_Y$  of

$$\max \left\{ \kappa\varphi + \tilde{F}_0\varphi, \kappa\varphi + x^1 + x^1\beta D_{x^1}\varphi - D_{x^3}\varphi \right\} \geq 0 \quad (4.33)$$

and

$$\max \left\{ \kappa\varphi + \tilde{F}_0\varphi, \kappa\varphi + x^1 + x^1\beta D_{x^1}\varphi - D_{x^3}\varphi \right\} \leq 0 \quad \text{if } \underline{\Lambda} < x^3 < \bar{\Lambda} \quad \text{and} \quad \tilde{V} > 0 \quad (4.34)$$

respectively, with

$$\begin{aligned} \tilde{F}_0\varphi & := -\partial_t\varphi - x^1\mu D_{x^1}\varphi - x^1\vartheta(t)D_{x^2}\varphi \\ & \quad - \frac{(x^1\sigma)^2}{2} \left( D_{x^1}^2\varphi + |D_{x^1}\varphi/D_p\varphi|^2 D_p^2\varphi - 2(D_{x^1}\varphi/D_p\varphi)D_{(x^1, p)}^2\varphi \right), \end{aligned}$$

and that  $\tilde{U}_\zeta$  is a viscosity supersolution of

$$\min \left\{ \zeta^{-1}D_p\varphi - \epsilon, D_{x^1}\varphi - C(x)\zeta^{-1}D_p\varphi, -D_{x^1}\varphi + C(x)\zeta^{-1}D_p\varphi \right\} = 0 \quad . \quad (4.35)$$

Let  $\bar{\mathcal{P}}^+\tilde{V}$  and  $\bar{\mathcal{P}}^-\tilde{U}_\zeta$  denote the super- and subjets of  $\tilde{V}$  and  $\tilde{U}_\zeta$ , with  $(t, x^2, x^3)$  taken as a first order term. It then follows from Ishii's Lemma, see e.g. [5], that we can find 2-dimensional symmetric matrices  $(X_n, Y_n) \in \mathbb{S}^2$  such that

$$\begin{aligned} (D_t\Lambda_n, (D_x\Lambda_n, D_p\Lambda_n), X_n)(t_n, x_n, y_n, p_n, q_n) &\in \bar{\mathcal{P}}^+\tilde{V}(t_n, x_n, p_n) \\ (-D_t\Lambda_n, -(D_y\Lambda_n, D_q\Lambda_n), Y_n)(t_n, x_n, y_n, p_n, q_n) &\in \bar{\mathcal{P}}^-\tilde{U}_\zeta(t_n, y_n, q_n), \end{aligned}$$

where the super- and subjets are defined with  $(t, x^2, x^3)$  viewed as a first order term, that satisfy

$$\begin{pmatrix} X_n & 0 \\ 0 & -Y_n \end{pmatrix} \leq 3n \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \quad (4.36)$$

where  $I$  denotes the 2-dimensional identity matrix.

We now study two cases:

**Case 1.** We first assume that

$$\kappa\tilde{U}_\zeta(t_n, y_n, q_n) + y_n^1 - y_n^1\beta(t_n, y_n^1)D_{y^1}\Lambda_n(z_n) + D_{y^3}\Lambda_n(z_n) \geq 0,$$

along a subsequence. Then, by (4.29), (4.30), (4.32), (4.34), step 3. and the Lipschitz of  $x \mapsto x\beta(t, x)$ ,

$$\begin{aligned} -\kappa\eta &\geq \kappa(\tilde{U}_\zeta(t_n, y_n, q_n) - \tilde{V}(t_n, x_n, p_n)) \\ &\geq -(y_n^1 - x_n^1) + y_n^1\beta(t_n, y_n^1)D_{y^1}\Lambda_n(z_n) + x_n^1\beta(t_n, x_n^1)D_{x^1}\Lambda_n(z_n) \\ &\quad - D_{y^3}\Lambda_n(z_n) - D_{x^3}\Lambda_n(z_n) \\ &\geq O(n|x_n^1 - y_n^1|^2 + |x_n^1 - y_n^1|) + O(|x_n^3 - \hat{x}_\zeta^3|). \end{aligned}$$

It then follows from (4.30)-(4.31) that

$$-\kappa\eta \geq 0$$

which leads to a contradiction since  $\eta > 0$ .

**Case 2.** We now assume that

$$\kappa\tilde{U}_\zeta(t_n, y_n, q_n) + y_n^1 - y_n^1\beta(t_n, y_n^1)D_{y^1}\Lambda_n(z_n) + D_{y^3}\Lambda_n(z_n) < 0,$$

along a subsequence. Then, using (4.30), (4.32), (4.33), (4.34), step 3. again, we deduce that

$$\begin{aligned} 0 &\leq \left( \kappa\tilde{U}_\zeta + D_t\Lambda_n + y_n^1\mu(t_n, y_n^1)D_{y^1}\Lambda_n + y_n^1\vartheta(t_n)D_{y^2}\Lambda_n \right) (\bar{z}_n) \\ &\quad - \frac{(y_n^1\sigma(t_n, y_n^1))^2}{2} \left( Y_n^{11} + |A_n|^2 Y_n^{22} - 2A_n Y_n^{12} \right) \end{aligned} \quad (4.37)$$

and

$$\begin{aligned} 0 &\geq \left( \kappa\tilde{V} - D_t\Lambda_n - x_n^1\mu(t_n, x_n^1)D_{x^1}\Lambda_n - x_n^1\vartheta(t_n)D_{x^2}\Lambda_n \right) (\bar{z}_n) \\ &\quad - \frac{(x_n^1\sigma(t_n, x_n^1))^2}{2} \left( X_n^{11} + |A_n|^2 X_n^{22} - 2A_n X_n^{12} \right), \end{aligned} \quad (4.38)$$

where

$$A_n := \frac{-D_{y^1}\Lambda_n(\bar{z}_n)}{-D_q\Lambda_n(\bar{z}_n)} = \frac{D_{x^1}\Lambda_n(\bar{z}_n)}{D_p\Lambda_n(\bar{z}_n)}.$$



Moreover, the continuity of  $C$ , recall (4.6), and the viscosity supersolution properties of  $\tilde{U}_\zeta$  in (4.35), together with (4.30), imply that

$$|A_n| \leq \zeta^{-1} C(y_{\zeta, \delta}) + 1 \quad (4.39)$$

for  $n$  large. We now use (4.36), the Lipschitz continuity of the coefficients and (4.39) to obtain

$$\begin{aligned} \kappa(\tilde{V} - \tilde{U}_\zeta)(\bar{z}_n) &\leq n|x_n^1 \mu(t_n, x_n^1) - y_n^1 \mu(t_n, y_n^1)| |x_n^1 - y_n^1| + O(|\hat{z}_\zeta - (t_n, x_n, p_n)|) \\ &\quad + O((1 + |A_n|^2) |x_n^1 \sigma(t_n, x_n^1) - y_n^1 \sigma(t_n, y_n^1)|^2) \\ &\leq O(n|x_n^1 - y_n^1|^2) + O(|\hat{z}_\zeta - (t_n, x_n, p_n)|). \end{aligned}$$

Recalling (4.30)-(4.31) and sending  $n \rightarrow \infty$  and then  $\delta \rightarrow 0$  then leads to a contradiction since  $\eta > 0$ .  $\square$

## 5 Proof of the viscosity property in the abstract model

We now provide the proof of Theorem 3.3. It is divided in several subsections.

### 5.1 Viscosity solution property on $[0, T)$

#### 5.1.1 Supersolution property on $[0, T)$

We first consider the case  $(t_0, x_0) \in \bar{D}_Y$  with  $t_0 < T$ . The proof follows from almost exactly the same arguments as in [1]. The only difference comes from the part of the control with bounded variations, however it is easily handled. We provide it for completeness.

**Proposition 5.1.** *Let  $(t_0, x_0) \in \bar{D}_Y$ , with  $t_0 < T$ , and let  $\varphi$  be a smooth function such that*

$$(\text{strict}) \min_{[0, T] \times \mathbb{R}^d} (v_* - \varphi) = (v_* - \varphi)(t_0, x_0) = 0. \quad (5.1)$$

Then,  $H^* \varphi(t_0, x_0) \geq 0$ .

**Proof.** We assume to the contrary that

$$H^* \varphi(t_0, x_0) \leq -2\eta \quad (5.2)$$

for some  $\eta > 0$ , and work towards a contradiction. It follows from (5.2) and the definition of  $H^*$  that we may find  $\varepsilon > 0$  such that

$$\begin{aligned} \max_{\xi \in \Delta_+} [\beta_Y(x, y)^\top - D\varphi(t, x)^\top \beta_X(x)] \xi &\leq -\eta \\ \mu_Y(x, y, u) - \mathcal{L}^u \varphi(t, x) &\leq -\eta \quad \forall u \in \mathcal{N}_\varepsilon(x, y, D\varphi(t, x)) \\ \forall (t, x, y) \in [0, T] \times \mathbb{R}^{d+1} \text{ s.t. } (t, x) \in B_\varepsilon(t_0, x_0) \cap \bar{D}_Y, |y - \varphi(t, x)| &\leq \varepsilon. \end{aligned} \quad (5.3)$$

For later use, observe that, by (5.1) and the definition of  $\varphi$ ,

$$\zeta := \min_{\partial_p B_\varepsilon(t_0, x_0)} (v_* - \varphi) > 0, \quad (5.4)$$

where  $\partial_p B_\varepsilon(t_0, x_0)$  denotes the parabolic boundary of  $B_\varepsilon(t_0, x_0)$ .

Let  $(t_n, x_n)_{n \geq 1}$  be a sequence in  $D_Y$  which converges to  $(t_0, x_0)$  and such that  $v(t_n, x_n) \rightarrow v_*(t_0, x_0)$ . Set  $y_n = v(t_n, x_n) + n^{-1}$  and observe that

$$\gamma_n := y_n - \varphi(t_n, x_n) \rightarrow 0. \quad (5.5)$$

For each  $n \geq 1$ , we have  $y_n > v(t_n, x_n)$ . It thus follows from (GDP1) of Corollary 2.4, that there exists some  $\phi^n = (\nu^n, L^n) \in \mathcal{A}$  such that

$$Y^n(t \wedge \theta_n) \geq v(t \wedge \theta_n, X^n(t \wedge \theta_n)) \quad \text{for } t \geq t_n, \quad (5.6)$$

where

$$Z^n := (X^n, Y^n) := (X_{t_n, x_n}^{\nu^n}, Y_{t_n, x_n, y_n}^{\nu^n}) \quad \text{and } \theta_n := \theta_n^o \wedge \theta_n^1,$$

with

$$\theta_n^o := \left\{ s \geq t_n : (s, X_{t_n, x_n}^{\phi^n}(s)) \notin B_\varepsilon(t_0, x_0) \right\}, \quad \theta_n^1 := \left\{ s \geq t_n : |Y_{t_n, x_n, y_n}^{\phi^n}(s) - \varphi(s, X_{t_n, x_n}^{\phi^n}(s))| \geq \varepsilon \right\}.$$

Let us define

$$A_n := \left\{ s \in [t_n, \theta_n] : \mu_Y(Z^n(s), \nu_s^n) - \mathcal{L}^{\nu_s^n} \varphi(s, X^n(s)) > -\eta \right\}, \quad (5.7)$$

and observe that (5.3) implies that the process

$$\psi_s^n := N^{\nu^n}(Z_s^n, D\varphi(s, X_s^n)) \quad \text{satisfies } |\psi_s^n| > \varepsilon \text{ for } s \in A_n, \quad (5.8)$$

recall (3.2). Since  $L^n \in \mathcal{L}$  is continuous, so is the path of  $Z^n$ . Using (5.6), the definition of  $\zeta$  in (5.4) and the definition of  $\theta_n$ , thus leads to

$$\begin{aligned} Y^n(t \wedge \theta_n) &\geq \varphi(t \wedge \theta_n, X^n(t \wedge \theta_n)) + (\zeta \mathbf{1}_{\{\theta_n^o = \theta_n\}} + \varepsilon \mathbf{1}_{\{\theta_n^o > \theta_n\}}) \mathbf{1}_{\{t = \theta_n\}} \\ &\geq \varphi(t \wedge \theta_n, X^n(t \wedge \theta_n)) + (\zeta \wedge \varepsilon) \mathbf{1}_{\{t = \theta_n\}}, \quad t \geq t_n. \end{aligned}$$

Since  $\varphi$  is smooth, it then follows from Itô's Lemma, (5.3), (5.5) and the definition of  $\psi^n$  that

$$\begin{aligned} -(\zeta \wedge \varepsilon) \mathbf{1}_{\{t < \theta_n\}} &\leq K_t^n + \int_{t_n}^{t \wedge \theta_n} b_s^n \mathbf{1}_{A_n^c}(s) ds \\ &\quad + \int_{t_n}^{t \wedge \theta_n} \left[ \beta_Y^\top(Z^n(s)) - D\varphi(s, X^n(s))^\top \beta_X(X^n(s)) \right] dL_s^n \\ &\leq K_t^n, \end{aligned} \quad (5.9)$$

where

$$K_t^n := \gamma_n - (\zeta \wedge \varepsilon) + \int_{t_n}^{t \wedge \theta_n} b_s^n \mathbf{1}_{A_n}(s) ds + \int_{t_n}^{t \wedge \theta_n} \psi_s^n dW_s,$$

and

$$b_s^n := \left[ \mu_Y(Z^n(s), \nu_s^n) - \mathcal{L}^{\nu_s^n} \varphi(s, X^n(s)) \right].$$

Let  $M^n$  be the exponential local martingale defined by  $M_{t_n}^n = 1$  and, for  $s \geq t_n$ ,

$$dM_s^n = -M_s^n \left( b_s^n |\psi_s^n|^{-2} (\psi_s^n)^\top \right) \mathbf{1}_{A_n}(s) dW_s,$$

which is well defined by (5.8), the Lipschitz continuity of the coefficients and our definition of the set of admissible controls  $\mathcal{U}$ . By Itô's formula and (5.9), we see that  $M^n K^n$  is a local martingale which is bounded from below by the submartingale  $-(\zeta \wedge \varepsilon) M^n$ . Then,  $M^n K^n$  is a supermartingale, and it follows from (5.9) that

$$0 = \mathbb{E} \left[ -(\zeta \wedge \varepsilon) \mathbf{1}_{\{\theta_n < \theta_n\}} \right] \leq \mathbb{E} \left[ M_{\theta_n}^n K_{\theta_n}^n \right] \leq \gamma_n - (\zeta \wedge \varepsilon) < 0,$$

for  $n$  large enough, recall (5.5), which leads to a contradiction.  $\square$

### 5.1.2 Subsolution property on $[0, T)$

We first consider the case where  $(t_0, x_0, v^*(t_0, x_0)) \in \text{int}D$ . The first part of the proof is similar to those provided in [1]. The novelty comes from the second part where we play with the part of the control with bounded variations to obtain a contradiction.

**Proposition 5.2.** *Let  $(t_0, x_0) \in \bar{D}_Y$ , with  $t_0 < T$ , and  $\varphi$  be a smooth function such that*

$$(\text{strict}) \max_{\bar{D}_Y} (v^* - \varphi) = (v^* - \varphi)(t_0, x_0) = 0 . \quad (5.10)$$

*Assume that  $(t_0, x_0, v^*(t_0, x_0)) \in \text{int}D$ . Then,  $H_*\varphi(t_0, x_0) \leq 0$ .*

**Proof.** We assume to the contrary that

$$(t_0, x_0, v^*(t_0, x_0)) \in \text{int}D \quad \text{and} \quad H_*\varphi(t_0, x_0) \geq 2\eta \quad (5.11)$$

for some  $\eta > 0$ , and work towards a contradiction. For later use note that (5.11) implies that, for  $\varepsilon > 0$  small enough,

$$\left\{ (t, x, y) \in [0, T] \times \mathbb{R}^{d+1} \text{ s.t. } (t, x) \in B_\varepsilon(t_0, x_0), |y - \varphi(t, x)| \leq \varepsilon \right\} \subset \text{int}D . \quad (5.12)$$

Also observe that, by (5.10) and the definition of  $\varphi$ ,

$$-\zeta := \max_{\partial_p B_\varepsilon(t_0, x_0)} (v^* - \varphi) < 0 . \quad (5.13)$$

Moreover, we can find a sequence  $(t_n, x_n)_{n \geq 1}$  in  $D_Y$  which converges to  $(t_0, x_0)$  and such that  $v(t_n, x_n) \rightarrow v^*(t_0, x_0)$ . Set  $y_n = v(t_n, x_n) - n^{-1}$  and observe that

$$\gamma_n := y_n - \varphi(t_n, x_n) \rightarrow 0 . \quad (5.14)$$

We now consider two cases.

First case. We first assume that

$$F_*\varphi(t_0, x_0) \geq 2\eta . \quad (5.15)$$

Then it follows from Assumption 3.2 and Remark 3.1 that may find  $\varepsilon > 0$  such that

$$\begin{aligned} \mu_Y(\cdot, \hat{v}) - \mathcal{L}^{\hat{v}}\varphi &> \eta \\ \forall (t, x, y) \in [0, T] \times \mathbb{R}^{d+1} \text{ s.t. } (t, x) \in B_\varepsilon(t_0, x_0), |y - \varphi(t, x)| &\leq \varepsilon , \end{aligned} \quad (5.16)$$

where  $\hat{v}$  is a locally Lipschitz map satisfying

$$\hat{v}(x, y, D\varphi(t, x)) \in \mathcal{N}_0(x, y, D\varphi(t, x)) \text{ if } (t, x) \in B_\varepsilon(t_0, x_0) \text{ and } |y - \varphi(t, x)| \leq \varepsilon . \quad (5.17)$$

We now fix  $n$  large enough so that  $(t_n, x_n) \in B_\varepsilon(t_0, x_0)$  and let  $Z^n := (X^n, Y^n)$  denote the solution of (2.1) associated to the Markovian control  $\hat{\phi}^n$  and the initial condition  $Z^n(t_n) = (x_n, y_n)$ , where

$$\hat{\phi}^n = (\hat{v}^n, \hat{L}^n) := (\hat{v}(\cdot, X^n, Y^n), 0) .$$

We next define the stopping times

$$\begin{aligned} \theta_n^o &:= \inf \{s \geq t_n : (s, X^n(s)) \notin B_\varepsilon(t_0, x_0)\} , \\ \theta_n &:= \inf \{s \geq t_n : |Y^n(s) - \varphi(s, X^n(s))| \geq \varepsilon\} \wedge \theta_n^o . \end{aligned}$$

Note that, by definition of  $\hat{\phi}^n$  and (5.16),  $Y^n - \varphi(\cdot, X^n)$  is non-decreasing on  $[t_n, \theta_n]$ , so that

$$Y^n(\theta_n) - \varphi(\theta_n, X^n(\theta_n)) \geq y_n - \varphi(t_n, x_n) = \gamma_n > -(\varepsilon \wedge \zeta)/2 \quad (5.18)$$

for  $n$  large enough, recall (5.14). Since  $\varphi \geq v^* \geq v$ , it follows that

$$\begin{aligned} Y^n(\theta_n) - v(\theta_n, X^n(\theta_n)) &\geq \mathbf{1}_{\{\theta_n < \theta_n^o\}} \{Y^n(\theta_n) - \varphi(\theta_n, X^n(\theta_n))\} \\ &\quad + \mathbf{1}_{\{\theta_n = \theta_n^o\}} \{Y^n(\theta_n^o) - v^*(\theta_n^o, X^n(\theta_n^o))\} \\ &= \varepsilon \mathbf{1}_{\{\theta_n < \theta_n^o\}} + \mathbf{1}_{\{\theta_n = \theta_n^o\}} \{Y^n(\theta_n^o) - v^*(\theta_n^o, X^n(\theta_n^o))\} \\ &\geq \varepsilon \mathbf{1}_{\{\theta_n < \theta_n^o\}} + \mathbf{1}_{\{\theta_n = \theta_n^o\}} \{Y^n(\theta_n^o) + \zeta - \varphi(\theta_n^o, X^n(\theta_n^o))\} \\ &\geq \varepsilon \wedge \zeta + \mathbf{1}_{\{\theta_n = \theta_n^o\}} \{Y^n(\theta_n^o) - \varphi(\theta_n^o, X^n(\theta_n^o))\} . \end{aligned}$$

In view of (5.18), this leads to

$$Y^n(\theta_n) - v(\theta_n, X^n(\theta_n)) \geq (\varepsilon \wedge \zeta)/2$$

for  $n$  large enough. Recalling (5.12) and the fact that  $y_n = v(t_n, x_n) - n^{-1} < v(t_n, x_n)$ , this is clearly in contradiction with (GDP2) of Corollary 2.4.

Second case. If (5.15) does not hold, then it follows from (5.11) that we can find  $\hat{\ell} \in \Delta_+$  such that, for  $\varepsilon > 0$  small enough,

$$[\beta_Y^\top - D\varphi^\top \beta_X] \hat{\ell} > \eta \quad (5.19)$$

$$\forall (t, x, y) \in [0, T] \times \mathbb{R}^{d+1} \text{ s.t. } (t, x) \in B_\varepsilon(t_0, x_0), |y - \varphi(t, x)| \leq \varepsilon . \quad (5.20)$$

Set  $\mathcal{O}_n := \{(t, x, y) : (t, x) \in B_{2\varepsilon}(t_0, x_0), |y - \varphi(t, x)| < 2\varepsilon, y - \varphi(t, x) > -|\gamma_n|\}$ . It follows from (5.19) that we can find  $r > 0$  such that

$$\bigcup_{0 \leq \lambda \leq r} B_{\lambda r}((t, x, y) - \lambda \hat{\gamma}(t, x, y)) \subset \mathcal{O}_n^c \text{ for all } (t, x, y) \in \partial \mathcal{O}_n \text{ satisfying (5.20) ,}$$

where  $\hat{\gamma}(t, x, y)^\top := (0, \beta_X(x), \beta_Y^\top(x, y)) \hat{\ell}$ . Given  $u \in U$ , we thus deduce from Theorem 4.8 of [8] and the assumption made on our coefficients, that there exists an adapted process  $Z^n = (X^n, Y^n)$  and a continuous real-valued adapted non-decreasing process  $M^n$  satisfying

$$\begin{aligned} X^n(s) &= x_n + \int_{t_n}^s \mu_X(X^n(r), u) dr + \int_{t_n}^s \beta_X(X^n(r)) \hat{\ell} dM_r^n + \int_{t_n}^s \sigma_X(X^n(r), u) dW_r \\ Y^n(s) &= y_n + \int_{t_n}^s \mu_Y(Z^n(r), u) dr + \int_{t_n}^s \beta_Y(Z^n(r))^\top \hat{\ell} dM_r^n + \int_{t_n}^s \sigma_Y(Z^n(r), u)^\top dW_r , \\ Y^n(s \wedge \theta_n) &\geq \varphi(s \wedge \theta_n, X^n(s \wedge \theta_n)) - 2|\gamma_n| \text{ for all } s \geq t_n , \end{aligned} \quad (5.21)$$

where

$$\begin{aligned} \theta_n^o &:= \inf \{s \geq t_n : (s, X^n(s)) \notin B_\varepsilon(t_0, x_0)\} , \\ \theta_n &:= \inf \{s \geq t_n : |Y^n(s) - \varphi(s, X^n(s))| \geq \varepsilon\} \wedge \theta_n^o . \end{aligned}$$

Observe that  $Z^n$  coincides with the solution of (2.1) for the control  $(u, \hat{\ell} M^n)$ . In view of (5.14) and (5.21), we have  $Y^n(\theta_n) - \varphi(\theta_n, X^n(\theta_n)) \geq -2|\gamma_n| > -\varepsilon$  for  $n$  large enough. Following the arguments after (5.18) above then leads to the required contradiction.  $\square$

We now turn to the case where  $(t_0, x_0, v^*(t_0, x_0)) \in \partial D$ .

**Proposition 5.3.** *Let  $(t_0, x_0) \in \bar{D}_Y$ , with  $t_0 < T$ , and  $\varphi$  be a smooth function such that*

$$(\text{strict}) \max_{\bar{D}_Y} (v^* - \varphi) = (v^* - \varphi)(t_0, x_0) = 0. \quad (5.22)$$

*Assume that  $(t_0, x_0, v^*(t_0, x_0)) \in \partial D$ . Then,  $H_*^{\text{in}}\varphi(t_0, x_0) \leq 0$ .*

**Proof.** The fact that

$$(t_0, x_0, v^*(t_0, x_0)) \in \partial D \quad \text{and} \quad H_*^{\text{in}}\varphi(t_0, x_0) \geq 2\eta \quad (5.23)$$

for some  $\eta > 0$ , leads to a contradiction to (GDP2) of Corollary 2.4 follows exactly from the same arguments as in the proof of Proposition 5.2. We therefore only sketch the case where  $F_*^{\text{in}}\varphi(t_0, x_0) \geq 2\eta$ . In this case, it follows from the definition of  $F_*^{\text{in}}$  and (ii) of Assumption 3.2, see also Remark 3.1, that we may find  $\varepsilon > 0$  such that

$$\begin{aligned} \min\{\mu_Y(\cdot, \hat{v}) - \mathcal{L}^{\hat{v}}\varphi, \mathcal{L}_Z^{\hat{v}}\delta\} &> \eta \\ \forall (t, x, y) \in [0, T] \times \mathbb{R}^{d+1} \text{ s.t. } (t, x) \in B_\varepsilon(t_0, x_0), |y - \varphi(t, x)| &\leq \varepsilon, \end{aligned} \quad (5.24)$$

where  $\hat{v}$  is a locally Lipschitz map satisfying

$$\hat{v}(t, x, y) \in \mathcal{N}_0^{\text{in}}(t, x, y, D\varphi(t, x)) \text{ for } (t, x) \in B_\varepsilon(t_0, x_0) \text{ and } |y - \varphi(t, x)| \leq \varepsilon. \quad (5.25)$$

Let  $(t_n, x_n)_{n \geq 1}$  be a sequence in  $D_Y$  which converges to  $(t_0, x_0)$  and such that  $v(t_n, x_n) \rightarrow v^*(t_0, x_0)$ . Set  $y_n = v(t_n, x_n) - n^{-1}$  and observe that

$$\gamma_n := y_n - \varphi(t_n, x_n) \rightarrow 0. \quad (5.26)$$

Let  $Z^n := (X^n, Y^n)$  denote the solution of (2.1) associated to the Markovian control  $(\hat{v}^n, 0)$  and the initial condition  $Z^n(t_n) = (x_n, y_n)$ , where

$$\hat{v}^n = \hat{v}(\cdot, X^n, Y^n).$$

We next define the stopping times

$$\begin{aligned} \theta_n^o &:= \inf \{s \geq t_n : (s, X^n(s)) \notin B_\varepsilon(t_0, x_0)\}, \\ \theta_n &:= \inf \{s \geq t_n : |Y^n(s) - \varphi(s, X^n(s))| \geq \varepsilon\} \wedge \theta_n^o. \end{aligned}$$

The same arguments as in the proof of Proposition 5.2 show that

$$Y^n(\theta_n) - v(\theta_n, X^n(\theta_n)) > 0$$

for  $n$  large enough. Moreover, it follows from (5.24), (5.25) and Itô's Lemma that

$$\delta(\cdot, X^n, Y^n) \geq 0 \quad \text{on } [t_n, \theta_n].$$

Recalling that  $y_n = v(t_n, x_n) - n^{-1} < v(t_n, x_n)$ , this is in contradiction with (GDP2) of Corollary 2.4.  $\square$

## 5.2 Viscosity solution property at $T$

In this part we follow standard arguments which consist in propagating the boundary condition backward on some small time  $[T - \varepsilon, T]$  so as to be in position to repeat the arguments used to derive the viscosity solution property on  $[0, T]$ . The arguments being standard, see e.g. [1] or [16], we only sketch them.

We begin with the supersolution property.

**Proposition 5.4.** *Fix  $(T, x_0) \in \bar{D}_Y$ , and let  $\varphi$  be a smooth function such that*

$$(\text{strict}) \min_{[0, T] \times \mathbb{R}^d} (v_* - \varphi) = (v_* - \varphi)(T, x_0) = 0. \quad (5.27)$$

*Then,  $M^*\varphi(T, x_0) \geq 0$ . If moreover  $F^*\varphi(T, x_0) < \infty$  and  $G\varphi(T, x_0) < 0$ , then  $\varphi(T, x_0) - w_*(x_0) \geq 0$ .*

**Proof.** The fact that  $M^*\varphi(T, x_0) \geq 0$  is deduced from Proposition 5.1 and the upper-semicontinuity of  $M^*$  by standard arguments, see e.g. the proof of Lemma 5.2 in [14]. We now prove the second assertion. Assume that

$$F^*\varphi(T, x_0) < \infty, \quad G\varphi(T, x_0) < 0 \quad \text{and} \quad \varphi(T, x_0) = v_*(T, x_0) < w_*(x_0),$$

and let us work towards a contradiction. Since  $v(T, \cdot) = w$  by the definition of the problem, there is a constant  $\eta > 0$  such that  $\varphi - v(T, \cdot) \leq \varphi - w_* \leq -\eta$  on  $B_\varepsilon(x_0)$  for some  $\varepsilon > 0$ . Since  $x_0$  is a strict minimizer,  $2\zeta := \min_{x \in \partial B_\varepsilon(x_0)} (v_*(T, x) - \varphi(T, x)) > 0$  and it follows that there exists  $r > 0$  such that  $v(t, x) - \varphi(t, x) \geq \zeta > 0$  for all  $(t, x) \in [T - r, T] \times \partial B_\varepsilon(x_0)$ . Hence,

$$v(t, x) - \varphi(t, x) \geq \zeta \wedge \eta > 0 \quad \text{for } (t, x) \in ([T - r, T] \times \partial B_\varepsilon(x_0)) \cup (\{T\} \times B_\varepsilon(x_0)). \quad (5.28)$$

Since  $F^*\varphi(T, x_0) < \infty$  and  $G\varphi(T, x_0) < 0$ , we can assume, after possibly changing  $\varepsilon > 0$ , that

$$\begin{aligned} G\varphi(t, x) \leq 0 \quad \text{and} \quad \mu_Y(x, y, u) - \mathcal{L}_X^u \varphi(t, x) \leq C \quad \text{for all } u \in \mathcal{N}_\varepsilon(x, y, D\varphi(t, x)) \\ \text{and } (t, x, y) \in [0, T] \times \mathbb{R}^{d+1} \quad \text{s.t. } (t, x) \in B_\varepsilon(T, x_0) \quad \text{and } |y - \varphi(x)| \leq \varepsilon \end{aligned}$$

for some constant  $C > 0$ . Let  $\tilde{\varphi}(t, x) := \varphi(x) - \sqrt{T - t + \varepsilon} + \sqrt{\varepsilon}$ . Then, for sufficiently small  $\varepsilon > 0$ , we have

$$\begin{aligned} v(t, x) - \tilde{\varphi}(t, x) \geq \frac{1}{2}(\zeta \wedge \eta) > 0 \quad \text{for } (t, x) \in ([T - \varepsilon, T] \times \partial B_\varepsilon(x_0)) \cup (\{T\} \times B_\varepsilon(x_0)) \\ G\tilde{\varphi}(t, x) \leq 0 \quad \text{and} \quad \mu_Y(x, y, u) - \mathcal{L}_X^u \tilde{\varphi}(t, x) \leq 0 \quad \text{for all } u \in \mathcal{N}_\varepsilon(x, y, D\tilde{\varphi}(t, x)) \\ (t, x, y) \in [T - \varepsilon, T] \times \mathbb{R}^{d+1} \quad \text{s.t. } x \in B_\varepsilon(x_0) \quad \text{and } |y - \tilde{\varphi}(t, x)| \leq \varepsilon. \end{aligned}$$

By following the arguments in the proof of Proposition 5.1, the latter inequalities lead to a contradiction of (GDP1) of Corollary 2.4.  $\square$

We now turn to the subsolution property. As in the previous section, we first consider the case where  $(t_0, x_0, v^*(t_0, x_0)) \in \text{int}D_T$ .

**Proposition 5.5.** *Let  $(t_0, x_0) \in \bar{D}_Y$ , with  $t_0 = T$ , and  $\varphi$  be a smooth function such that*

$$(\text{strict}) \max_{\bar{D}_Y} (v^* - \varphi) = (v^* - \varphi)(t_0, x_0) = 0. \quad (5.29)$$

*Assume that  $(T, x_0, v^*(T, x_0)) \in \text{int}D_T$ . Then,  $\min \{\varphi(T, x_0) - w^*(x_0), M_*\varphi(T, x_0)\} \leq 0$ .*

**Proof.** Assume to the contrary that

$$\min \{ \varphi(T, x_0) - w^*(x_0) , M_* \varphi(T, x_0) \} =: 2\eta > 0 .$$

Let  $\tilde{\varphi}$  be defined by  $\tilde{\varphi}(t, x) := \varphi(t, x) + \sqrt{T-t+\varepsilon} - \sqrt{\varepsilon}$  for  $\varepsilon > 0$  small. Clearly,  $(T, x_0)$  achieves a strict maximum of  $v^* - \tilde{\varphi}$ , and it follows from the identity  $v(T, \cdot) = w(x_0)$ , the fact that  $\varphi(T, x_0) - w^*(x_0) > 0$  and (5.29) that

$$\max_{(\{T\} \times \bar{B}_\varepsilon(x_0)) \cup ([T-\varepsilon, T] \times \partial B_\varepsilon(x_0))} (v^* - \tilde{\varphi}) := -\zeta < 0 .$$

Also observe that, the fact that  $M_* \varphi(T, x_0) > 0$ , means that

$$\max \{ R_* \tilde{\varphi}(T, x_0) , G \tilde{\varphi}(T, x_0) \} > 0 .$$

Since  $\partial_t \tilde{\varphi} \rightarrow -\infty$  as  $t \rightarrow T$  and  $\varepsilon \rightarrow 0$ , we can find  $\varepsilon > 0$  small enough such that

$$\max \{ F_* \tilde{\varphi}(T, x_0) , G \tilde{\varphi}(T, x_0) \} > 0 .$$

Moreover, the assumption  $(T, x_0, \tilde{\varphi}(T, x_0)) = (T, x_0, v^*(T, x_0)) \in \text{int} D_T$ , recall (3.11)-(3.12), implies that, for  $\varepsilon > 0$  small enough,

$$\left\{ (t, x, y) \in [0, T] \times \mathbb{R}^{d+1} \text{ s.t. } (t, x) \in B_\varepsilon(T, x_0), |y - \tilde{\varphi}(t, x)| \leq \varepsilon \right\} \subset \text{int} D .$$

Following line by line the arguments of Proposition 5.2 then leads to a contradiction to (GDP2) of Corollary 2.4.  $\square$

We finally consider the case  $(T, x_0, v^*(T, x_0)) \in \partial D_T$ .

**Proposition 5.6.** *Let  $(t_0, x_0) \in \bar{D}_Y$ , with  $t_0 = T$ , and  $\varphi$  be a smooth function such that*

$$(\text{strict}) \max_{\bar{D}_Y} (v^* - \varphi) = (v^* - \varphi)(t_0, x_0) = 0 . \quad (5.30)$$

*Assume that  $(T, x_0, v^*(T, x_0)) \in \partial D_T$ . Then,  $\min \{ \varphi(T, x_0) - w^*(x_0) , M_*^{\text{in}} \varphi(T, x_0) \} \leq 0$ .*

**Proof.** The result follows from an obvious combination of the arguments used in the proofs of Proposition 5.5 and Proposition 5.3 above.  $\square$

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