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# Path Separability of Graphs

Emilie Diot\*

Cyril Gavoille\*<sup>†</sup>

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## Abstract

In this paper we investigate the structural properties of  $k$ -path separable graphs, that are the graphs that can be separated by a set of  $k$  shortest paths. We identify several graph families having such path separability, and we show that this property is closed under minor taking. In particular we establish a list of forbidden minors for 1-path separable graphs.

## 1 Introduction

“Divide and Conquer” is a common technique in computer science to solve problems. The whole data is separated into different small parts to find a solution in these parts, and then to merge the solutions to obtain the result on the input graph.

A wide theory has been developed for graphs that can be decomposed into small pieces. Such graphs, a.k.a. bounded treewidth graphs, supports polynomial algorithms for many class of problems, whereas no algorithms of complexity better than exponential complexity are known for the general case. This has contributed to new insights into Fixed Parameterized Tractable theory whose consequences for practical algorithms are effective improvements on the running time [AKCF<sup>+</sup>04, KBvH01].

There are however problems that can be efficiently solved even for graphs without small separators (or equivalently of large treewidth). Large separators but “well shaped” reveal very useful for approximation algorithms. For instance, if the separator has a small diameter, or a small dominating set, then distances between vertices can be computed efficiently up to some small additive errors (see [DG07, CDE<sup>+</sup>08, UY09] for works about the treelength of a graph).

An important observation due to Thorup [Tho04] is that separators consisting of a set of shortest paths are also very useful for the design of compact routing scheme, distance and reachability oracles. More precisely, he used the fact that every weighted planar graph with  $n$  vertices has a set of three shortest paths whose deletion split the graph into connected components of at most  $n/2$  vertices (a decomposition into components of at most  $2n/3$  vertices using two shortest paths was early proved in [LT79]). Using a recursive decomposition, and sampling each such shortest path, he showed that distances between any pair of vertices can

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\*Université de Bordeaux, LaBRI, France. E-mail: {diot,gavoille}@labri.fr

<sup>†</sup>The second author is also member of the “Institut Universitaire de France”. Both authors are supported by the ANR-project “ALADDIN”, and the équipe-projet commune LaBRI-INRIA Bordeaux Sud-Ouest “CÉPAGE”.

be approximated up to a factor of  $1 + \varepsilon$  in polylogarithmic time, for every  $\varepsilon > 0$ , with an oracle of size<sup>1</sup>  $O(\varepsilon^{-1}n \log n)$ .

This notion of “shortest path” separator has been extended in [AG06]. Roughly speaking, a  $k$ -path separator is the union of  $k$  shortest paths whose removal halve the graph. The formal definition is actually slightly more complicated and is described in Section 2. The same authors have showed that  $k$ -path separable graphs have efficient solutions for several “Object Location Problems” including compact routing schemes, distance oracles, and small-world navigability. Based on the deep Robertson-Seymour’s decomposition [RS03], they show in particular that every weighted graph excluding a minor  $H$  has a  $(1 + \varepsilon)$ -approximate distance oracle of size  $O(\varepsilon^{-1}kn \log n)$ , where  $k = k(H)$  depending only of  $H$ . Actually, the oracle can be distributed into balanced labels, each of size  $O(\varepsilon^{-1}k \log n)$  such that distance queries can be answered from given the source-destination labels only. The graphs excluding a fixed minor is a huge family of graphs including (and not restricted to) bounded treewidth graphs, planar graphs, and graphs of bounded genus.

## 1.1 Our results

An approximate distance oracle for a graph  $G$  is a data-structure that quickly returns, for any source-destination pair, an approximation on cost of a shortest path connecting them. Such data-structures are obtained by preprocessing  $G$  where each edge has a weight corresponding to the cost of traversing this edge (or length). However, in practice, the number  $n$  of vertices of  $G$  is large whereas the number of interesting vertices for which we want approximate the distance is small (say  $t$ ). Current solutions [AG06, Tho04] provide oracles of size  $O(t \log n)$  whereas a space independent of  $n$  would be preferable.

Such a compression can be achieved by adding weights on the vertices of the input graph. Typically, interesting vertices receive a weight 1 whereas the others receive a weight 0. A  $k$ -path separator on such vertex- and edge-weighted graph is then defined as previously, excepted that the removal of the separator must leave connected components of size at most half the total vertex-weight of the graph. The size of the oracles is improved since  $\log t$  recursion levels suffice instead of  $\log n$  in the initial formulation.

In this paper we extend the classical notion of  $k$ -path separability to edge- and vertex-weighted graphs. In particular, we prove that previous results (e.g., the 3-path separability of planar graphs) still hold in this new framework.

We establish a connection between separators corresponding to the border of a face and  $k$ -path separability, and we identify several families of graphs that are 1-path and 2-path separable. We note that most of our proofs are constructive, and lead to polynomial and even linear algorithms.

More interestingly, we show that the family of graphs that are  $k$ -path separable for any weight function is minor-closed. Combined with the Graph Minor Theory of Robertson and Seymour [RS04], it follows that the  $k$ -path separability can be theoretically tested in cubic time [RS95] for each fixed  $k$ , although no algorithm is currently known. Finally, we provide a first step towards the characterization of 1-path separable graphs.

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<sup>1</sup>The *size* is actually the number of “distance items” stored in the oracle.

## 2 Preliminaries

A *minor* of a graph  $G$  is a subgraph of a graph obtained from  $G$  by edge contraction. We denote by  $K_r$  the complete graph (or clique) on  $r$  vertices, and  $K_{p,q}$  the complete bipartite graph. For convenience, the term *component* is a short for connected component.

A *vertex-weight function* (resp. *edge-weight function*) is a non-negative real function defined on the vertices (resp. edges) of a graph. A non-negative real function applying on both vertices and edges is simply called *weight function*. A weighted graph is graph  $G$  having a weight function  $\omega$ , that we denote also by  $(G, \omega)$ . The *weight* of a subgraph  $H$  of  $G$ , denoted by  $\omega(H)$ , is the sum of the weights over the vertices of  $H$ .

A *half-separator* for a graph  $G$  with vertex-weight function  $\omega$  is a subset of vertices  $S$  such that each component of  $G \setminus S$  has weight at most  $\omega(G)/2$ . Observe that the deletion of a half-separator does not necessarily disconnect the graph.

A *k-path separator* of a weighted graph  $G$  is a subgraph  $P_0 \cup P_1 \cup \dots$  where each  $P_i$  is a subgraph composed of the union of  $k_i$  minimum cost paths in  $G \setminus \bigcup_{j < i} P_j$ , and where  $\sum_i k_i \leq k$ . A *k-path separator* is said *strong* if it consists of  $P_0$  only, i.e., composed of the union of  $k$  minimum cost paths in  $G$ . A weighted graph is (strongly) *k-path separable* if every induced subgraph has a (strong) *k-path separator*.

A *tree-decomposition* of a graph  $G$  is a tree  $T$  whose vertices, called *bags*, are subsets of vertices of  $G$  such that:

1. for every vertex  $u$  of  $G$ , there exists a bag  $X$  of  $T$  such that  $u \in X$ ;
2. for every edge  $\{u, v\}$  of  $G$ , there exists a bag  $X$  of  $T$  such that  $u, v \in X$ ; and
3. for every vertex  $u$  of  $G$ , the set of bags containing  $u$  induces a subtree of  $T$ .

An important property following from the last two points is that every path between  $u \in X$  and  $v \in Y$  in  $G$  has to intersect all the bags on the path from  $X$  and  $Y$  in  $T$ . Therefore, the deletion of every bag  $X$  disconnects  $G$  provided that  $T \setminus X$  is composed of more than one subtree and that no bags  $Y \subseteq X$ .

The *width* of a tree-decomposition  $T$  is  $\max_{X \in T} \{|X| - 1\}$ . A *treewidth- $t$*  graph is a graph having a tree-decomposition of width  $t$ , and the treewidth of  $G$  is the minimum  $t$  such that  $G$  is a treewidth- $t$  graph.

We will use several times the following basic result.

**Lemma 1** *Every tree-decomposition of a vertex-weighted graph has a bag that is a half-separator of the graph. Such a bag is called center of the tree-decomposition.*

**Proof.** Let  $T$  be a tree-decomposition of a graph  $G$  with vertex-weight function  $\omega$ . We choose an arbitrary bag  $R$  to be the root of  $T$ , and we denote by  $\pi(X)$  the parent of the bag  $X$  in  $T$ . For a bag  $X$ , we denote by  $T_X$  the subtree of  $T$  rooted at  $X$ . By abuse of notation,  $T_X$  denotes also the subgraph induced by all vertices contained in the bags of  $T_X$ .

For every bag  $X$  we define the value:

$$\rho(X) = \sum_{v \in T_X \setminus \pi(X)} \omega(v)$$

as the weight of the vertices that are in  $T_X \setminus \pi(X)$ , with the convention that  $\pi(R) = \emptyset$ . Note that  $\rho(R) = \omega(G)$ . Consider a component  $H$  of  $G \setminus X$ . If  $H$  belongs to  $T \setminus T_X$ , then

$\omega(H) \leq \rho(R) - \rho(X) = \omega(G) - \rho(X)$ . If  $H$  belongs to  $T_F$  for some child  $F$  of  $X$ , then  $\omega(H) \leq \rho(F)$ , because the vertices of  $\pi(F) = X$  are not in  $H$ .

The center  $C$  can be found, traversing  $T$  from its root  $R$ , as follows:

1. If  $\rho(F) < \omega(G)/2$  for all the children  $F$  of the current bag  $X$ , then set  $C = X$  and stop.
2. Otherwise, choose a child  $F$  of  $X$  with  $\rho(F) \geq \omega(G)/2$ , set  $X = F$  and go to Step 1.

The above procedure stops after at most the depth of  $T$  steps. Indeed, at any time along the traversal  $\rho(X) \geq \omega(G)/2$ , and  $\rho(X)$  is non-increasing ( $\rho(X) \leq \rho(\pi(X))$ ) provided  $X \neq R$ . Moreover, whenever it stops at  $C$ ,  $\rho(C) \geq \omega(G)/2$  and  $\rho(F) < \omega(G)/2$  for all the children of  $C$ .

Consider a component  $H$  of  $G \setminus C$ . Either  $H$  is contained in  $T \setminus T_C$ , or it is contained in  $T_F$  for some child  $F$  of  $C$ . In the former case,  $\omega(H) \leq \omega(G) - \rho(C) \leq \omega(G)/2$ . In the latter case,  $\omega(H) \leq \rho(F) < \omega(G)/2$ . Hence,  $C$  is the center as claimed.  $\square$

For the sake of presentation, we prove the following folklore results.

### Proposition 1

1. Every weighted treewidth- $t$  graph is strongly  $\lceil (t+1)/2 \rceil$ -path separable.
2. Every weighted planar graph is strongly 3-path separable.
3. Every weighted  $n$ -vertex graph is strongly  $\lceil n/4 \rceil$ -path separable
4. The uniform<sup>2</sup> weighted clique  $K_{4k+1}$  is not  $k$ -path separable.

### Proof.

1. Consider any subgraph  $H$  of a weighted graph  $G$ . The treewidth of  $H$  is at most the treewidth of  $G$ . So  $H$  has a tree-decomposition of width  $\leq t$ , so with bags consisting of at most  $t+1$  vertices. By Lemma 1, the center  $C$  of the tree-decomposition is a half-separator. It can be covered by at most  $\lceil |C|/2 \rceil \leq \lceil (t+1)/2 \rceil$  shortest paths. Therefore,  $H$  has a strong  $\lceil (t+1)/2 \rceil$ -path separator, and thus  $G$  is strongly  $\lceil (t+1)/2 \rceil$ -path separable.

2. It is well-known that every planar graph has a tree-decomposition such that every bag consists of at most three shortest paths. This comes from the well-known fact that every planar graph having a depth- $h$  rooted tree has a tree-decomposition where each bag consists of 3 paths of the tree starting from the root (see [FG06][pp. 305]). By Lemma 1, the center  $C$  of the tree-decomposition is a half-separator. So,  $C$  forms a strong 3-path separator.

3. Consider any subgraph  $H$  of a weighted graph  $(G, \omega)$  with  $n$  vertices. Let  $W$  be the smallest set of vertices in  $H$  such that  $\omega(W) \geq \omega(H)/2$ . Thus, the components of  $H \setminus W$  have weight  $\leq \omega(H)/2$ . It is clear that  $W$  contains at most half the vertices of  $H$ , i.e.,  $|W| \leq \lceil |V(H)|/2 \rceil$ . A set of at most  $\lceil |W|/2 \rceil$  shortest paths suffices to cover  $W$ . Therefore,  $H$  has a strong  $k$ -path separator with  $k = \lceil \lceil |V(H)|/2 \rceil / 2 \rceil \leq \lceil n/4 \rceil$ , completing the proof of Point 3.

4. Let us show that the uniform weighted  $K_{4k+1}$  has no  $k$ -path separator. Indeed, every  $k$ -path separator  $S$  consists of at most  $2k$  vertices since every shortest path in a clique consists of an edge.  $K_{4k+1} \setminus S$  is a clique on at least  $2k+1$  vertices, so  $S$  is not a half-separator.  $\square$

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<sup>2</sup>That is with a unit weight for all vertices and edges.

As remark in [AG06], the  $k$ -path separability of minor-free graphs holds also for vertex-weighted graphs. However the formal proof of this result cannot be considered as folklore, and its self-contained proof is currently more than the page limitation of this paper.

One of the unresolved problems we left open is to know whether they are planar graphs that are not 2-path separable.

### 3 Face-Separable Graphs

As we will see later in Section 4, graphs that are  $k$ -path separable have strong structural properties. In particular, planarity plays an important role, at least for  $k = 1$  in the light of Theorem 3: all 1-path separable graphs are planar but  $K_{3,3}$ . In this section we will see that a half-separator of special “shape” implies a low path separability of the graph. Interestingly, this half-separator is defined independently of the shortest path metric of the graph. It only depends on the vertex-weight function.

A half-separator  $S$  of a weighted graph  $G$  is a *face-separator* if  $G$  has a plane embedding such that  $S$  is the border of a face. A weighted graph is *face-separable* if every induced subgraph has a face-separator.

By definition, outerplanar graphs are face-separable, since the outerface contains all vertices of the graph. We will see that the family of face-separable graphs includes more general graphs, like the series-parallel graphs, the subdivisions of a  $K_4$  (Proposition 2), and even includes some unbounded treewidth planar graphs (Proposition 3).

The main result of this section is:

**Theorem 1** *Every face-separable weighted graph is strongly 2-path separable.*

**Proof.** To prove the result it suffices to show that if a weighted graph has a face-separator, then it has a strong 2-path separator. So, consider any weighted graph  $(G_0, \omega)$  having a face-separator  $B_0$ .

Observe that  $B_0$  is not necessarily a cycle. (See for instance face  $F$  in the graph depicted on Fig. 1(b)). It might even be not connected, e.g. if  $B_0$  is the border of the outerface and  $G_0$  is not connected. For technical reason, we will need later in the proof that the face-separator is biconnected. Note also that it might be possible that all bicomponents of  $G_0$  are face-separable but not  $G_0$ , as suggested by Fig. 1(a). So, face-separability of a graph does not reduce to face-separability of its bicomponents.

However, we will show in Claim 1 that  $B_0$  contains a cycle which is a face-separator of  $G_0$ , as suggested by Fig. 1(b).

**Claim 1** *Separator  $B_0$  either contains a vertex or an edge that is a half-separator of  $G_0$ , or contains a cycle that is a face-separator of  $G_0$ .*

**Proof.** We concentrate our attention on the heaviest component  $G$  of  $G_0$  intersecting  $B_0$ . Denote by  $B = B_0 \cap G$ , the part of the face-separator in component  $G$ . Note that  $B$  is connected. Moreover, all components of  $G_0 \setminus B$  (those of  $G \setminus B$  and of  $G_0 \setminus G$ ) have weight  $\leq \omega(G_0)/2$ .

We consider a plane embedding of  $G$  such that  $B$  is the border of its outerface. Note that every vertex of  $G \setminus B$  is drawn in a region delimited by a cycle of  $B$ . The bicomponents of

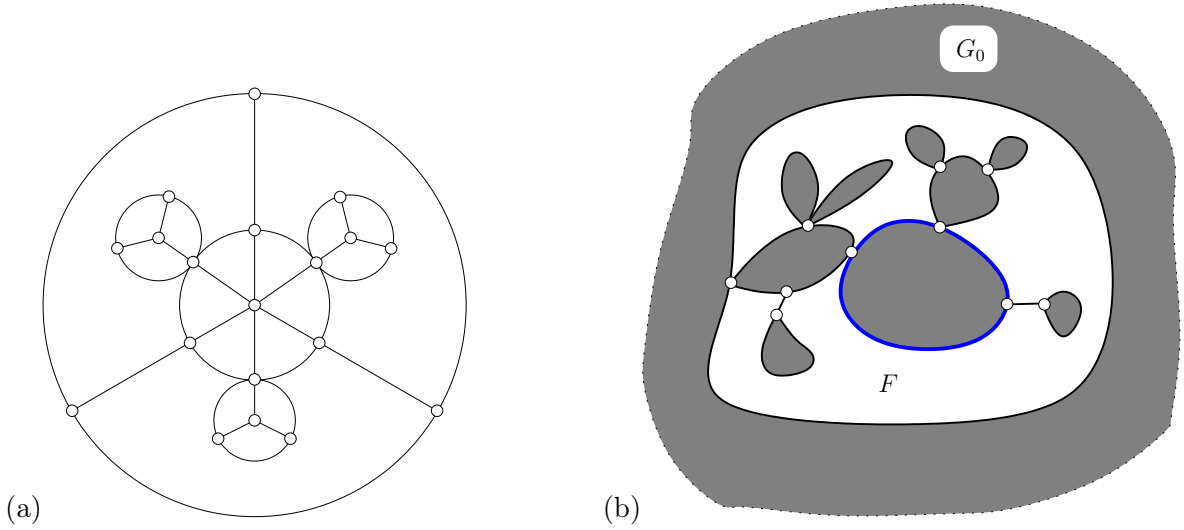


Figure 1: (a) A uniform weighted graph with 19 vertices that is not face-separable whereas all its bicomponents are. (b) Graph  $G_0$  is depicted rayed, and face-separator  $B_0$ , which is not biconnected, is the border of face  $F$ . The face-separator provided by Claim 1 is drawn blue. To make  $B$  the border of a face, the embedding of  $G_0$  has to be changed.

$B$  form a tree of cycles, connected together by cut-vertices or bridges. We consider a tree-decomposition  $T$  of  $B$  where each bag corresponds to a bicomponent or a bridge of  $B$ . One can check that  $T$  is indeed a tree-decomposition of  $B$ . We fix a bag  $R$  as the root of  $T$ . Denote by  $\pi(X)$  the parent bag of  $X$ , and set  $\pi(R) = \emptyset$ . We denote by  $G(X)$  the subgraph induced by the vertices of  $G$  that are either in  $X \setminus \pi(X)$  or inside the region delimited by  $X$ . Note that  $\{G(X)\}_X$  forms a partition of the vertices of  $G$ , whereas the set of bags of  $T$ ,  $\{X\}_X$ , is not a partition of the vertices of  $B$  (because of the cut-vertices).

We now define a vertex-weight function  $\omega_B$  on  $B$  in order to find out a suitable half-separator for  $G$ . Roughly speaking, we uniformly distribute on the vertices of any bag  $X$  the weight (under  $\omega$ ) of the vertices of  $G(X)$ . More precisely, for every bag  $X$  of  $T$ , and every vertex  $u \in X \setminus \pi(X)$ , we set  $\omega_B(u) = \omega(G(X))/|V(G(X))|$ . Clearly,  $\sum_{u \in X \setminus \pi(X)} \omega_B(u) = \omega(G(X))$ , and thus  $\omega_B(B) = \omega(G)$  by partitioning.

From Lemma 1,  $T$  has a center  $C$  that is a half-separator of the weighted graph  $(B, \omega_B)$ . The components of  $G \setminus (C \cup G(C))$  have weights no more than the weights (under  $\omega_B$ ) of the components of  $B \setminus C$ , which is  $\leq \omega_B(B)/2 = \omega(G)/2$ . Therefore,  $C \cup G(C)$  is a half-separator of  $G$ . We observe that the weight of each component of  $G(C) \setminus C$  is  $\leq \omega(G)/2$ , since  $B$  is a half-separator of  $G$  and the vertices of  $G(C) \setminus C$  cannot be adjacent to any vertex outside  $C$ . Thus  $C$  is in fact a half-separator of  $G$ . This also a half-separator of  $G_0$  because we have seen that  $\omega(G) \leq \omega(G_0)$ , and all components of  $G_0 \setminus G$  have weight  $\leq \omega(G_0)/2$ .

Center  $C$  is a bag of the tree-decomposition of  $B$ . So, it is composed either of a vertex (if  $|B| = 1$ ), of an edge, or of a cycle. We are done in the first two cases. So, assume that the center  $C$  in a cycle. We remark that it is possible to embed component  $G$  such that  $C$  is the border of the outerface of  $G$ . This is due to the fact that all bicomponents attached to  $C$  are attached by cut-vertices or bridges. It is possible to redraw each such components inside  $C$

by preserving planarity of the embedding. It follows that  $C$  is a face-separator of  $G$ , and also of  $G_0$ . This concludes the proof of the claim.  $\square$

If  $B_0$  contains a vertex or an edge that is a half-separator of  $G_0$ , then it forms a 1-path separator for  $G_0$  and we are done. So, assume we are in the second case of Claim 1, and let us denote by  $B$  the cycle of  $B_0$  that is a face-separator for  $G_0$ . We denote by  $G$  the component of  $G_0$  containing  $B$ . We now redraw graph  $G$  on the plane such that the face whose cycle  $B$  is the border is not the outerface.

Let  $T$  be any shortest-path spanning tree of  $G$  rooted at some vertex  $r$  of  $B$ . For every vertex  $u$  of  $T$ , we denote by  $T_u$  the path from  $r$  to  $u$  in  $T$ . We will show that there exist two vertices  $u, v$  of  $B$  forming a non-tree edge of  $T$  such that  $T_u \cup T_v$  is a half-separator of  $G_0$ .

Given two vertices  $u, v$  of  $B$ , possibly with  $u = v$ , we denote by  $X_{u,v}$  the heaviest component of  $G \setminus (T_u \cup T_v)$ , and by  $B_{u,v} = B \cap X_{u,v}$ . We check that  $B_{u,v}$  is connected because otherwise by planarity  $X_{u,v}$  would intersect  $T_u$  or  $T_v$ .

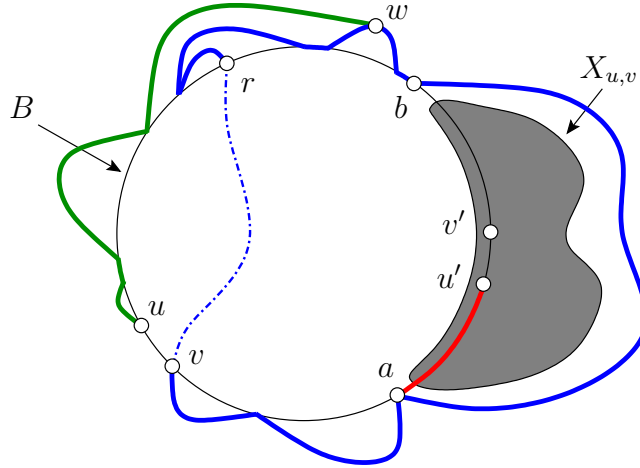


Figure 2: Illustration for the proof of Theorem 1. Path  $T_v$  is in blue, path  $T_u$  becomes green after its blue common prefix with  $T_v$  ( $w$  is the nearest common ancestor of  $u, v$ ). Component  $B_{u,v}$  is the segment from  $a$  to  $b$  when traversing  $B$  counter-clockwise,  $a$  and  $b$  excluded. The red path from  $a$  to  $u'$  is included in  $T$ . Note that the subpath of  $T_v$  from  $a$  to  $b$  may be drawn differently (with  $X_{u,v}$  touching the outerface).

We now assume that  $u, v$  are chosen in  $B$  such that they form a non-tree edge of  $T$  and such that  $|V(B_{u,v})|$  is minimum. If  $|V(B_{u,v})| = 0$ , then  $T_u \cup T_v$  is the willing half-separator for  $G_0$  because the heaviest component  $X_{u,v}$  contains no vertices of  $B$  and  $B$  is a half-separator of  $G_0$ .

So, assume that  $|V(B_{u,v})| \geq 1$ . Let  $x$  be any vertex of  $B_{u,v}$ , and assume that  $r, u, v, x, r$  are encountered in this order when traversing  $B$  counter-clockwise. If not, we consider a reverse drawing of  $G$  (like in a mirror). We have  $v \neq r$  because  $v$  and  $r$  does not belong to  $B_{u,v}$ . An important observation is that:

**Claim 2**  $B_{v,v} = B_{u,v}$ .



In other words path  $T_u$  is useless to minimize  $|V(B_{u,v})|$ .

**Proof.** This is due to the fact that path  $T_v$  “separates”  $u$  from  $X_{u,v}$ , i.e., every path from  $u$  to any vertex of  $X_{u,v}$  must meet  $T_v$ . Indeed, this is trivially true if  $u = r$ , and otherwise,  $u$  and  $x$  falls into different regions delimited by the closed curve defined by  $T_v$  and an extra curve lying on the face from  $r$  to  $v$  (see the dashed curve on Fig. 2). Thus,  $X_{v,v} = X_{u,v}$  and  $B_{v,v} = B_{u,v}$ .  $\square$

According to Claim 2, let us define  $a, b$  to be the two vertices of  $T_v$  delimiting  $B_{v,v}$  (see Fig. 2). In fact, Claim 2 holds for every vertex  $z \in T_v$  that is descendant of  $a$ :  $B_{z,z} = B_{u,v}$ . And, clearly, if  $z$  is a descendant of  $a$  in  $T$  (and not necessary in  $T_v$ ), then  $|V(B_{z,z})| \leq |V(B_{u,v})|$ . That is, the number of vertices of the largest component of  $G \setminus T_z$  is no more than the one of  $G \setminus (T_u \cup T_v)$ .

Let  $u', v'$  be the first non-tree edge of  $T$  that belongs to  $B_{v,v}$  when traversing  $B$  counter-clockwise ( $u'$  traversed before  $v'$ ). Although  $u' = v = a$  and  $v' = b$  are possible,  $u'$  or  $v'$  belongs to  $B_{v,v}$ : if all edges of  $B$  from  $a$  to  $b$  belong to  $T$ , then  $T_v$  would form a cycle in the tree  $T$ . Thus the edges  $uv$  and  $u'v'$  are distinct (they differ by at least one endpoint), and  $u'v'$  is a non-tree edge of  $T$  in  $B$  as well.

We have seen that for every descendant  $z$  of  $a$ ,  $|V(B_{z,z})| \leq |V(B_{u,v})|$ . If  $u' \neq a$ , then  $|V(B_{u',u'})| < |V(B_{u,v})|$  and so  $|V(B_{u',v'})| < |V(B_{u,v})|$ : a contradiction,  $|V(B_{u,v})|$  is minimum. If  $u' = a$ ,  $|V(B_{u',u'})| = |V(B_{u,v})|$ . However, by considering path  $T_{v'}$ ,  $|V(B_{u',v'})| < |V(B_{u,v})|$  since in this case  $v' \in B_{u,v}$  and  $v' \notin T_v$ . This is a contradiction too.

Therefore, we have proved that  $|V(B_{u,v})| = 0$  for some non-tree edge of  $T$  in  $B$ , and that therefore,  $T_u \cup T_v$  is a strong 2-path separator for the input graph  $(G_0, \omega)$ .  $\square$

The bound given by Theorem 1 is best possible because there are face-separable graphs that are not 1-path separable. This can be proved by combining Proposition 2 and the fact there are treewidth-2 graphs and subdivisions of  $K_4$  that are not 1-path separable – see Fig. 6.

**Proposition 2** *Every weighted treewidth-2 graph or weighted subdivision of  $K_4$  is face-separable.*

**Proof.** Let  $(G, \omega)$  be any weighted treewidth-2 graph. It is known that every treewidth-2 graph is a subgraph of a series-parallel graph, and in particular a planar graph. As any subgraph of  $G$  is also a treewidth-2 graph, it is sufficient to prove that  $G$  has a face-separator.

We consider the graph  $H$  obtained from  $G$  by adding as many edges as possible while preserving a treewidth-2 graph. Let  $T$  be a tree-decomposition of  $H$  of width 2, and let  $C$  be the center of  $T$ . Bag  $C$  is composed of a  $K_3$ . We embed  $H$  in the plane such that  $C$  is the border of a face of this embedding. This is possible by moving some subgraph from inside the  $K_3$  to outside. If not,  $H$  would contain a  $K_4$ -minor, contradicting the fact that  $H$  has treewidth 2. We can now remove the edges that have been added to  $H$  in order to obtain  $G$ , and we consider the border  $S$  of the face containing the three vertices of  $C$ . Such a face exists since deleting edges can only enlarge the existing faces of a plane embedding. We have  $S \subseteq C$ , and  $C$  is a half-separator of  $H$  (Lemma 1). Note that  $H$  has the same total weight of  $G$  (we have added only edges). It follows that  $S$  is a half-separator for  $G$ . This completes the first part of the statement of the proposition.

Consider now a subdivision  $G$  of  $K_4$  having a vertex-weight function  $\omega$ , and  $H$  be an induced subgraph of  $G$ . If  $H$  is a proper subgraph of  $G$  (i.e.,  $H \neq G$ ), then  $H$  is outerplanar

and thus has a face-separator. So, assume that  $H = G$ .

We assume given a plane embedding of  $H$ . We denote by  $v_1, \dots, v_4$  the four degree-3 vertices of  $H$ , and by  $P_{i,j}$  the path between  $v_i$  and  $v_j$ , for all  $i, j \in \{1, \dots, 4\}$ . Let  $w_i = \omega(v_i)$ , and let  $p_{i,j} = \omega(P_{i,j} \setminus \{v_i, v_j\})$  be the sum of the weights of vertices on  $P_{i,j}$  excluding its extremities.

Let assume that  $H$  has no face-separator. There are four possible faces  $F_1, \dots, F_4$ , each one bordered by three paths. Faces are ordered such that whenever the border of  $F_i$  is removed, the remaining component is composed of three paths sharing vertex  $v_i$ . The total weight of this component is  $w_i + \sum_{j \neq i} p_{i,j}$ . As the border of  $F_i$  is not a face-separator, we have  $w_i + \sum_{j \neq i} p_{i,j} > \omega(H)/2$ . This holds for each  $i \in \{1, \dots, 4\}$ . Summing these four equations, it turns out (observe that each path  $p_{i,j}$  occurs twice in this sum):

$$\sum_i w_i + 2 \sum_{i \neq j} p_{i,j} > 2\omega(H) = 2 \left( \sum_i w_i + \sum_{i \neq j} p_{i,j} \right).$$

It implies  $0 > \sum_i w_i$ : a contradiction, by definition  $\omega(v) \geq 0$  for each vertex  $v$ . Therefore, one of the face  $F_i$  is a face-separator for  $H$ , that completes the proof.  $\square$

Up to now, the graphs we have proved to be face-separable are all of treewidth  $\leq 3$ . From Proposition 1 (Point 1), they are 2-path separable. It is natural to ask whether all face-separable graphs have such a low treewidth property. We answer negatively to this question.

**Proposition 3** *For every  $n$ , there is a uniform weighted face-separable graph with at most  $n$  vertices whose treewidth is  $\Omega(\log \log n)$ .*

**Proof.** The proof is based on the construction of a graph called  $G_p$ , for integral  $p \geq 1$ . It has treewidth at least  $k = p - O(\log \log p)$  because we can show it contains a  $k \times k$ -grid minor, and the number of vertices of  $G_p$  is  $n < 2^{2^p}$ . In other words, the treewidth of  $G_p$  is at least  $\log \log n - O(\log^{(4)} n)$ .

Graph  $G_p$  is composed of a tree  $T_p$  of depth  $p$  where each vertex of depth  $i < p$  has exactly  $d(i)$  children, for some function  $d$  defined later. Furthermore, for each depth  $i$ , a path linking all depth- $i$  vertices is added to  $T_p$  to form  $G_p$ . Let us denote by  $L(i)$  the number of depth- $i$  vertices in  $T_p$ . The values  $L(i)$  and  $d(i)$  obey to the following induction:  $L(0) = 1$  and  $L(i) = L(i-1) \cdot d(i-1)$ , where  $d(i) = \sum_{j=0}^i L(j)$ . The first values of  $L(i)$  and  $d(i)$  are given in the table hereafter, and  $G_4$  is depicted on Fig. 3.

To prove Proposition 3, we show that every subgraph  $H$  of  $G_p$  contains a face-separator. An important property we use is that in  $G_p$ , the number  $d(i)$  of children for a vertex of depth  $i$  is at least the vertex number of the graph induced by  $T_{i-1}$ . The key point is that  $H$  is either outerplanar, or there must exist a vertex  $v$  of depth  $i$  in  $T_p$  such that all its children belongs to the border of the outerface of  $H$ . In the first case,  $H$  is trivially face-separable. In the second one, and using the property on  $d(i)$ , we derive that at least half the vertices of  $H$  lie on the outerface.  $\square$

$i$	0	1	2	3	4	5	...
$L$	1	1	2	8	96	10368	...
$d$	1	2	4	12	108	10464	...

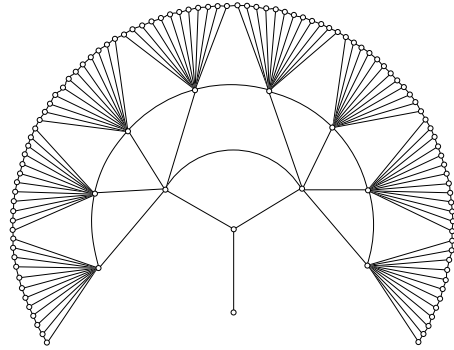


Figure 3: The graph  $G_4$  with 108 vertices.

## 4 The Hierarchy of Separable Graphs

For every integer  $k \geq 1$ , we denote by  $\text{PS}_k$  the family of all the graphs  $G$  that are  $k$ -path separable for every weight function  $\omega$ . More formally,

$$\text{PS}_k = \{G \mid \forall \omega, (G, \omega) \text{ is } k\text{-path separable}\} .$$

We define similarly the family  $\text{SPS}_k$  of all the graphs that are strongly  $k$ -path separable for every weight function.

We have seen that every weighted planar graph is strongly 3-path separable. In other words, planar graphs are in  $\text{SPS}_3$ . In Section 3, we have seen that treewidth-2 graphs are face-separable, and thus strongly 2-path separable. Thus this family is in  $\text{SPS}_2$ . We will show in Proposition 6 that outerplanar graphs are in  $\text{PS}_1$ . Obviously, families  $\text{PS}_1$  and  $\text{SPS}_1$  coincide.

Clearly, for each  $k$ ,  $\text{SPS}_k \subset \text{PS}_k$  since a strongly  $k$ -path separator is a particular  $k$ -path separator. Also, the hierarchies  $\text{PS}_1 \subset \dots \subset \text{PS}_k$  and  $\text{SPS}_1 \subset \dots \subset \text{SPS}_k$  are strict because of the complete graph. By Proposition 1 (points 3 and 4),  $K_{4k+1} \in \text{SPS}_{k+1}$  and  $K_{4k+1} \notin \text{PS}_k$ . The family  $\text{PS}_k$  is however much larger than  $\text{SPS}_k$  as suggested by the next proposition.

**Proposition 4** *For each  $k > 4$ , there is a graph  $A_k$  with  $O(k^2)$  vertices such that  $A_k \notin \text{SPS}_k$ , but  $A_k \in \text{PS}_4$ .*

**Proof.** Consider the graph  $A_k$  composed of a  $(2k+1) \times (2k+1)$ -grid in which a vertex  $v$  is connected to all the vertices of the grid.  $A_k$  has  $O(k^2)$  vertices. Edges incident to  $v$  have weights  $1/2$ , whereas all other vertex or edge weights are unitary. Graph  $A_k$  has no strong  $k$ -path separator. The removal of any set of  $k$  shortest paths deletes at most  $2k+1$  different vertices: all of them go thru  $v$ . However,  $2k+2$  vertices are required to halve the graph because its treewidth is  $2k+1$ . Therefore,  $A_k \notin \text{SPS}_k$ . However, for every weight function  $\omega$ ,  $(A_k, \omega)$  has a 4-path separator. The first path consists of the universal vertex  $v$ , and the three others are defined as in the planar case (since  $A_k \setminus \{v\}$  is planar, and thus 3-path separable). Therefore,  $A_k \in \text{PS}_4$ .  $\square$

For the study of  $\text{PS}_k$  and  $\text{SPS}_k$  graphs families, the next proposition tell us that we can restrict our attention to only  $k$ -path separator of biconnected graphs.

**Proposition 5** *A graph  $G$  belongs to  $\text{PS}_k$  (resp.  $\text{SPS}_k$ ) if and only if every weighted bicomponent of  $G$  has a (resp. strong)  $k$ -path separator.*

**Proof.** Because the proof is similar for  $\text{PS}_k$  and  $\text{SPS}_k$ , we denote by  $P_k \in \{\text{PS}_k, \text{SPS}_k\}$  any of these families. Assume  $G \in P_k$ . It is clear from the definitions of  $k$ -path separability and of  $P_k$  that every induced weighted subgraph of  $G$  must have a  $k$ -path separator. Let us show the other way.

Assume that every weighted bicomponent of  $G$  has a  $k$ -path separator. We will show that  $G \in P_k$ . For that we need to show that any weighted induced subgraph of  $G$ , say  $(H, \omega_H)$ , has a  $k$ -path separator. We restrict our attention to the case where  $H$  is connected (otherwise it suffices to consider the largest component of  $H$ ). Consider the weight function  $\omega_G$  for  $G$  defined by: for every vertex  $x \in H$ , we set  $\omega_G(x) = \omega_H(x)$ , and  $\omega_G(x) = 0$  otherwise. Observe that  $\omega_G(G) = \omega_H(H)$ . For every edge  $e \in H$ , we set  $\omega_G(e) = \omega_H(e)$ , and  $\omega_G(e) = 1 + \sum_{e' \in H} \omega_H(e')$  for all other edges. Note that any shortest path in  $G$  between two vertices of  $H$  can use only edges of  $H$ .

It is not difficult to see that the decomposition in which each bag consists of a bicomponent of  $G$  is a tree-decomposition of  $G$ . As a consequence (Lemma 1), the center  $C$  of this tree-decomposition is a bicomponent of  $G$ , and a half-separator for  $(G, \omega_G)$ . The vertices of  $C \cap H$  are also a half-separator for  $(H, \omega_H)$ .

Let us define the vertex-weight function  $\omega_C$  for  $C$  as follows. For every edge  $e \in C \cap H$ , we set  $\omega_C(e) = \omega_H(e)$ , and we set  $\omega_C(e) = 1 + \sum_{e' \in G} \omega_G(e')$  for all the other edges. Note that any shortest path in  $G$  between two vertices of  $C \cap H$  can use only edges of  $H$ . For each vertex  $x$  of  $C \cap H$ , we set  $\omega_C(x) = \omega_H(x) + \sum_{v \in H(C,x)} \omega_H(v)$  where  $H(C, x)$  is the set of vertices of  $H$  that are connected to  $x$  by a path of  $H$  that does not intersect any vertex of  $C$  (excepted  $x$ ). We set  $\omega_C(x) = 0$  for all other vertices. Note that  $H(C, x) = \emptyset$  if  $x$  is not a cut-vertex of  $G$ . Note also that  $\omega_C(C) = \omega_H(H)$ . Since every weighted bicomponent of  $G$  has a  $k$ -path separator, then in particular,  $(C, \omega_C)$  has a  $k$ -path separator, say  $S$ . We remark that each path composing  $S$  intersects vertices of  $H$  into a sub-path. This is due to the fact that  $H$  is connected and once the path enter a node of  $H$  it cannot leave  $H$  (edges not in  $H$  have too much large weight). So,  $S \cap H$  is composed of  $k$  paths. We set  $S_H = S \cap H$ .

It remains to show that  $S_H$  is a  $k$ -path separator for  $(H, \omega_H)$ . The weights of each component of  $C \setminus S$  that appears in  $H \setminus S_H$  are the same. Hence, these weights are at most  $\omega_H(H)/2$  by construction. However, there are components of  $H \setminus S_H$  that are not attached to any component of  $C \setminus S$ . This occurs each time that such a component of  $H \setminus S_H$ , say  $X$ , is attached by some cut-vertex of  $G$  belonging to  $S$ . Since  $C$  is the center of  $(G, \omega_G)$ ,  $\omega_G(X) \leq \omega_G(G)/2$ . However, we have seen that  $\omega_G(X) = \omega_H(X)$  and  $\omega_G(G) = \omega_H(H)$ . Hence  $\omega_H(X) \leq \omega_H(H)/2$ . Therefore,  $S_H$  is a half-separator for  $(H, \omega_H)$  as required, and thus a  $k$ -path separator for  $(H, \omega_H)$ .  $\square$

#### 4.1 Closed under minor taking

The remarkable property of  $\text{PS}_k$  and  $\text{SPS}_k$  families is they are closed under minor taking. From the Graph Minor Theorem, such families can be characterized by a finite list of forbidden minors, and membership of a given graph in one of these families can be done in time  $O(n^3)$  for fixed  $k$ .

**Theorem 2** For each integer  $k \geq 1$ , the families  $\text{PS}_k$  and  $\text{SPS}_k$  are minor-closed.

**Proof.** We give the proof for the family  $\text{SPS}_k$ , the proof for  $\text{PS}_k$  is similar. Let  $H$  be any minor of a graph  $G$ . We will prove that if  $G$  is  $k$ -path separable, then  $H$  is  $k$ -path separable too. To prove that  $H$  is  $k$ -path separable, we need to prove that  $H$  has a  $k$ -path separator for every induced subgraph of  $H$ . However, since every subgraph of  $H$  is also a minor of  $G$ , we simply show that  $H$  has a  $k$ -path separator.

It is not difficult to see that if  $H$  is a minor of  $G$ , then with each vertex  $u$  of  $H$  we can associate a connected subgraph of  $G$ , called *super-node* of  $u$ , such that if  $(u, v)$  is an edge of  $H$ , then there exists an edge of  $G$ , called *super-edge* of  $(u, v)$ , connecting a vertex of the super-node of  $u$  and a vertex of the super-node of  $v$ . (If there are several such edges we select only one.) The super-nodes must be pairwise disjoint (see Fig. 4).

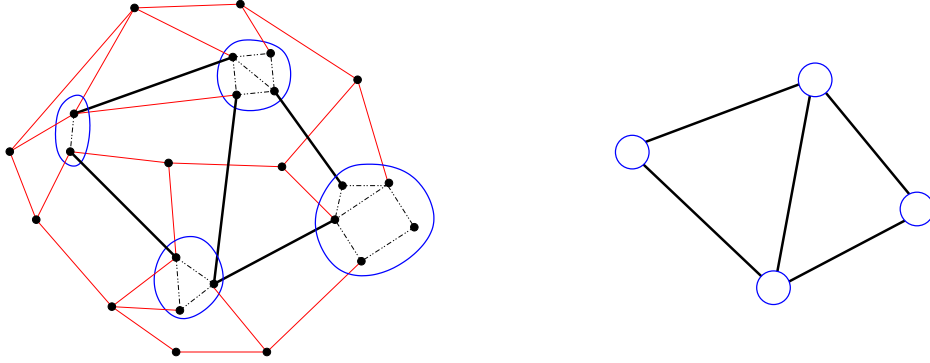


Figure 4: A graph  $G$  and a minor  $H$ .

Let  $\omega_H$  be any weight function on  $H$ . From  $\omega_H$ , we construct a weight function  $\omega_G$  on  $G$  as follows. For every edge  $(x, y)$  of  $G$  that is a super-edge of  $(u, v)$  (colored black on Fig. 4), we set  $\omega_G(x, y) = \omega_H(u, v)$ . For every edge  $(x, y)$  of  $G$  such that  $x$  and  $y$  both belongs to the same super-node (called *internal-edge* and dashed on Fig. 4), we set  $\omega_G(x, y) = 0$ . And, for all other edges  $(x, y)$  of  $G$  (called *external-edge* and colored red on Fig. 4), we set  $\omega_G(x, y) = 1 + \sum_{e \in E(H)} \omega_H(e)$ , so that the cost of a path in  $G$  using any such edge is strictly larger than the cost of any simple path in  $H$ . The weight of a vertex  $x$  that belongs to the super-node of  $u$  is  $\omega_G(x) = \omega_H(u)/t_u$ , where  $t_u$  is the number of vertices of the super-node of  $u$ . Note that the sum of weights of the vertices of the super-node of  $u$  is precisely  $\omega_H(u)$ . The weight of all other vertices is 0. Observe that  $\omega_G(G) = \omega_H(H)$ .

Since  $G \in \text{SPS}_k$ , the weighted graph  $(G, \omega_G)$  has a  $k$ -path separator  $S_G$  consisting of  $k$  shortest paths in  $G$ . Let  $H_0, H_1, \dots$  be the components of  $H$ , and assume that  $\omega_H(H_0)$  is maximum. With each path  $P$  of  $S_G$  that intersects a super-node of a vertex of  $H_0$ , we associate a path  $Q$  in  $H_0$  as follows. Let  $(U_0, \dots, U_t)$  be the ordered sequence of all the super-nodes of vertices of  $H_0$  traversed by  $P$ . We denote by  $u_i$  the vertex of  $H_0$  such  $U_i$  is the super-node of  $u_i$ . Path  $Q$  is obtained by adding an edge between  $u_{i-1}$  to  $u_i$ , for each  $i \in \{1, \dots, t\}$ . We claim that the set composed of each path  $Q$  constructed from  $P$  as above, and denote by  $S_H$ , is a half-separator of  $H$ .

First, let us show that  $Q$  is a shortest path in  $H_0$  (and thus in  $H$ ). Path  $P$  between the last vertex of  $U_{i-1}$  and the first vertex of  $U_i$  consists of the super-edge of  $(u_{i-1}, u_i)$ , because  $H_0$

is connected and the weight of this super-edge is less than the weight of any external-edges. Thus  $Q$  is a path in  $H_0$ . Now, assume that there exists a path  $Q'$  in  $H_0$ , from  $u_0$  to  $u_t$ , that is shorter than  $Q$ . Then, from  $Q'$  we can construct a shorter path in  $G$  (shorter than  $P$ ) from the last vertex of  $U_0$  to the first vertex of  $U_t$ . This is due to the fact that each super-node is connected and internal-edges have weight 0. This contradicts that  $P$  is a shortest path, hence  $Q$  is a shortest path in  $H_0$ .

It remains to show that  $S_H$  is a half-separator of  $H$ . Observe that for  $i \neq 0$ ,  $\omega_H(H_i) \leq \omega_H(H)/2$  because  $\omega(H_0)$  is maximum. Let  $X_{H_0}$  be the set of vertices of any component in  $H_0 \setminus S_H$ . Then, there must exist a component  $X_G$  in  $G \setminus S_G$  wholly containing all the super-nodes of the vertices of  $X_{H_0}$ . Let  $v$  be a vertex of  $X_{H_0}$  whose its super-node belongs to none component of  $G \setminus S_G$ . Then, there exists a vertex of this super-node that is in  $S_G$ . From our construction,  $v$  belongs to  $S_H$  (vertices of  $Q$  and super-nodes of  $P$  correspond): contradiction. Therefore,  $\omega_H(X_{H_0}) \leq \omega_G(X_G)$ . Moreover,  $\omega_G(X_G) \leq \omega_G(G)/2 = \omega_H(H)/2$ . Thus,  $S_H$  is a half-separator of  $H$ , completing the proof.  $\square$

## 4.2 One-path separable graphs

In this part, we concentrate our attention to the graphs that belong to  $\text{PS}_1$ . From Proposition 2, outerplanar graphs and subdivisions of outerplanar graphs are face-separable (they are treewidth-2 graphs), and thus belong to  $\text{PS}_2$  (and even to  $\text{SPS}_2$ ). Actually, outerplanar graphs are in  $\text{PS}_1$ :

**Proposition 6** *Every weighted outerplanar graph is 1-path separable.*

**Proof.** Let  $(G, \omega)$  be any weighted outerplanar graph. Since outerplanar graphs are hereditary family, i.e., closed under induced subgraphs taking, it suffices to show that  $(G, \omega)$  has a 1-path separator.

Let  $r$  be any vertex of  $G$ . We assume that a outerplane embedding of  $G$  is given. From  $r$ , we traverse all the vertices by following the border of the outerface, and we denote by  $v_i$  the  $i$ -th vertices encountered. We have  $v_1 = r$ . Let  $i_0$  be the smallest integer such that  $\sum_{i=1}^{i_0} \omega(v_i) \geq \omega(G)/2$ . We consider a shortest path  $P$  between  $r$  and  $v_{i_0}$ .

We shall prove that  $P$  is a 1-path separator. Let  $A$  be  $\{v_i \mid i < i_0\}$  and  $B$  be  $\{v_i \mid i > i_0\}$ . Note that  $\omega(A)$  and  $\omega(B) \geq \omega(G)/2$ . So, we just need to prove that there is no edge between vertices of  $A$  and vertices of  $B$ . We will prove this by contradiction.

Let  $a$  be a vertex of  $A$  and  $b$  a vertex of  $B$  such that  $(a, b)$  is an edge. The edge  $(a, b)$  cannot cross  $P$  by planarity. Thus, the edge belongs to the border of the outerface, which implies that either  $v_{i_0}$ , or  $r$  is not on the border of the outerface: contradiction.  $\square$

Unfortunately, Proposition 6 does not generalize to treewidth-2 graphs. As depicted on Fig. 6, there are simple series-parallel graphs and subdivisions of  $K_4$  that are not in  $\text{PS}_1$ .

The family  $\text{PS}_1$  does not reduce to outerplanar graphs, as shown in the next proposition (Proposition 7). A *globe graph* is a subdivision of  $K_{2,r}$ , for some  $r$ , in which the two degree- $r$  vertices may be adjacent.

**Proposition 7** *Every weighted globe graph is 1-path separable.*



**Proof.** Let  $(G, \omega)$  be any weighted globe graph. According to Proposition 5, it suffices to show that  $(G, \omega)$  has a 1-path separator to prove that  $G \in \text{PS}_1$ .

Let  $P_1, \dots, P_r$  be the  $r$  paths composing  $G$ , and let us denote by  $x, y$  the common extremities of all the  $P_i$ 's. We assume that  $\omega(P_1)$  is maximum over all  $P_i$ 's. Consider a shortest path spanning tree  $T$  rooted at  $x$ .

If all the edges of  $P_1$  are in  $T$ , then we select  $P_1$ . It is a shortest path and a half-separator since the components of  $G \setminus P_1$  consists of all the  $P_i \setminus \{x, y\}$  for  $i > 1$ , each one having a weight  $\leq \omega(G)/2$  (otherwise this would contradict that  $\omega(P_1)$  is maximum).

Otherwise, let  $(a, b)$  be an the edge of  $P_1$  that is not in  $T$ . There is only one such edge in  $P_1$ , otherwise  $T$  would be not connected. When going from  $x$  to  $y$  on  $P_1$  we assume that  $a$  is traversed before  $b$ . Denote by  $A$  the path in  $T$  from  $x$  to  $a$ , and by  $B$  the path in  $T$  from  $x$  to  $b$ . Note that  $A, B$  are shortest paths in  $G$ .

If  $\omega(A) \geq \omega(G)/2$ , then we select  $A$ . Any component of  $G \setminus A$  have weight at most  $\omega(G) - \omega(A) \leq \omega(G)/2$ . Finally, if  $\omega(A) < \omega(G)/2$ , then we select  $B$ . The components of  $G \setminus B$  are  $A \setminus \{x\}$  and  $P_i \setminus \{x, y\}$  for all  $i > 1$ , and thus all of them have weight  $\leq \omega(G)/2$ . In all cases, we have selected a half-separator for  $G$ .  $\square$

A first attempt to characterize  $\text{PS}_1$  is given by Theorem 3.

**Theorem 3** *Every biconnected graph of  $\text{PS}_1$  is either isomorphic to  $K_{3,3}$ , or planar and excludes the list of minors depicted on Fig. 6.*

**Proof.** Let  $(G, \omega)$  be a biconnected weighted graph with  $G \in \text{PS}_1$ . First assume that  $G$  is non planar. From Kuratowski's criteria,  $G$  contains a subdivision of  $K_5$  or  $K_{3,3}$ .

The complete graph  $K_5$  is not 1-path separable graph from Proposition 1 (cf. Point 4 with  $k = 2$ ). From Theorem 2, it follows that  $G$  cannot contain a subdivision of  $K_5$ , so it must contain a subdivision of  $K_{3,3}$ .

We shall proof that the only non planar graph in  $\text{PS}_1$  is  $K_{3,3}$ . We show that any proper subdivision of  $K_{3,3}$  or super-graph of  $K_{3,3}$  is not in  $\text{PS}_1$ .

Denote by  $M$  a  $K_{3,3}$  in which only one edge is subdivided into two edges. We set unitary all the weights so that the total vertex-weight is 7. The diameter of  $M$  is two. The deletion of any shortest path deletes at most three vertices. Moreover, such a deletion cannot disconnect  $M$ , and thus leaves a component with at least  $4 > 7/2$  vertices.  $M$  is not 1-path separable.

Denote by  $M'$  a super-graph of  $K_{3,3}$  composed of  $K_{3,3}$  with only one more edge. We denote by  $\{x_1, x_2, x_3\}$  and  $\{y_1, y_2, y_3\}$  the vertex set of each part of  $K_{3,3}$ . The weight function  $\omega$  for  $M'$  is set as follows (cf. Fig. 5):  $\omega(x_i) = 2$ ,  $\omega(y_2) = \omega(y_3) = 3$ ,  $\omega(x_1, y_2) = \omega(x_2, y_3) = 2$ , all the other weights are unitary. The total vertex-weight of  $M'$  is  $\omega(M') = 13$ .

Consider any path  $P$  in  $M'$ . There are three cases:

1.  $P$  connects  $x_i$  to  $x_j$ : it goes thru  $y_1$  and has length 2, and leaves a component of weight  $2 + 3 + 3 = 8 > \omega(M')/2 = 6.5$ .
2.  $P$  connects two adjacent vertices: it has length 1, of weight at most 6, thus it leaves a component of four vertices of weight  $> 7 > \omega(M')/2$ .
3.  $P$  connects  $y_1$  to  $y_2$  or  $y_3$ : it has length 2, and leaves a component of weight  $3 + 2 + 2 = 7 > \omega(M')/2$ .

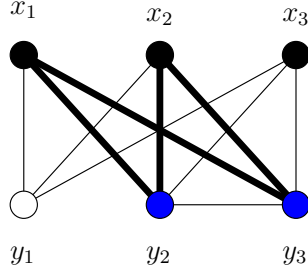


Figure 5: The weighted graph  $M'$ : a  $K_{3,3}$  plus one edge. Black vertices and bold edges have weight 2, blue vertices have weight 3, other vertices and edges have weight 1.

In all cases,  $M' \setminus P$  has a component of weight  $> \omega(M')/2$ . Thus  $M'$  has no 1-path separator. Therefore, if  $G$  is not planar, then  $G$  can only be isomorphic to  $K_{3,3}$ .

We prove now that  $K_{3,3} \in \text{PS}_1$ . According to Proposition 5, it suffices to show that every weighted  $K_{3,3}$  has a 1-path separator. W.l.o.g. assume that  $\omega(x_1) \geq \omega(x_2) \geq \omega(x_3)$ .

Define  $P_1$  be a shortest path from  $x_1$  to  $x_2$ , and assume  $P_1$  contains  $y_{i_1}$ . Define  $P_2$  be a shortest path from  $y_{i_2}$  to  $y_{i_3}$  (with  $i_1, i_2, i_3$  pairwise different indices), and assume it contains  $x_{j_1}$  (denote by  $j_2, j_3$  the two other  $x$ 's indices). We show that  $P_1$  or  $P_2$  is a 1-path separator. By contradiction, if  $P_1$  is not a half-separator, then (and similarly for  $P_2$ )  $\omega(P_1) < \omega(K_{3,3})/2$  and  $\omega(G \setminus P_1) > \omega(K_{3,3})/2$ . As  $\omega(P_1)$  is lower bounded by  $\omega(x_1) + \omega(x_2) + \omega(y_{i_1})$  and  $\omega(G \setminus P_1)$  upper bounded by  $\omega(x_3) + \omega(y_{i_2}) + \omega(y_{i_3})$  (and similarly for  $P_2$ ), it follows that:

$$\omega(x_1) + \omega(x_2) + \omega(y_{i_1}) < \omega(x_3) + \omega(y_{i_2}) + \omega(y_{i_3}) \quad (1)$$

$$\omega(x_{j_1}) + \omega(y_{i_2}) + \omega(y_{i_3}) < \omega(x_{j_2}) + \omega(x_{j_3}) + \omega(y_{i_1}) \quad (2)$$

By summing these equations, we obtain:

$$\begin{aligned} \omega(x_1) + \omega(x_2) + \omega(x_{j_1}) &< \omega(x_3) + \omega(x_{j_2}) + \omega(x_{j_3}) \\ \Rightarrow \omega(x_1) + \omega(x_2) + \omega(x_3) &< \omega(x_3) + \omega(x_{j_2}) + \omega(x_{j_3}) \leq \omega(x_3) + \omega(x_2) + \omega(x_1) \end{aligned}$$

since, by assumption,  $\omega(x_3) \leq \omega(x_{j_1})$  and  $\omega(x_{j_2}) + \omega(x_{j_3}) \leq \omega(x_2) + \omega(x_1)$ . This leads to a contradiction. Thus,  $P_1$  or  $P_2$  is a 1-path separator for  $K_{3,3}$ .

For planar graphs, we manage to find forbidden minors represented in Fig. 6. To prove that each minor  $M$  of this list is indeed excluded, we exhibit a particular weight function  $\omega$  for  $M$ . Actually, each vertex and edge has weight 1 or 2 as depicted on Fig. 6. We then exhaustively check that, for each pair  $u, v$  of vertices of  $M$ , the deletion of any shortest path from  $u$  to  $v$  leaves a component of weight  $> \omega(M)/2$ .

To illustrate this, consider for instance the “wheel graph”, composed of a cycle of length 5 and a degree-5 vertex, called hereafter center, connected of all vertices of the cycle. The total weight of the graph is 7, the center has weight 2. Any shortest path from the center to a non-center vertex consists of one edge. Therefore its deletion leaves a path of 4 vertices, so of weight 4. Any shortest path between two non-center vertices consists of 2 edges at most, so leaving a component with the center and two (or more) non-center vertices, thus of weight at least 4. In both cases, the weight is  $> 7/2$ . This graph has no half-separator composed of a shortest path, and thus is not in  $\text{PS}_1$ .  $\square$



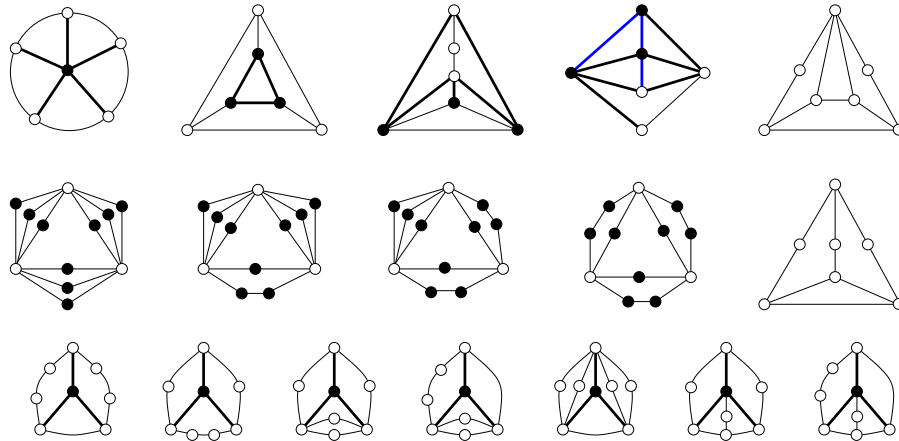


Figure 6: Forbidden minors for planar graphs in  $PS_1$ . Black vertices and bold edges have weight 2, blue edges have weight 3, other vertices and edges have weight 1. There are exactly three non-planar forbidden minors for  $PS_1$  (non depicted in the figure):  $K_5$ ,  $K_{3,3}$  plus one edge, and  $K_{3,3}$  whose one edge is subdivided into two edges.

## 5 Conclusion

In this paper we have investigated the family of graphs that are  $k$ -path separable. Graph Minor Theory implies that such a family can be characterized by a finite set of forbidden minors that we have started to list for  $k = 1$ .

We propose here a list of further researches.

1. Determine the full list of forbidden minors for 1-path separable graphs.
2. Find an explicit polynomial time algorithm to determine if a graph is  $k$ -path separable, for fixed  $k$ .
3. Prove or disprove that planar graphs are 2-path separable.
4. Prove NP-completeness for the problem of determining whether a given *weighted* graph has a  $k$ -path separator.
5. Extend the study to more general isometric separators, not only shortest paths.

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