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# EA-Matrix integrals of associative algebras and equivariant localization.

Serguei Barannikov

To R.K.Gordin on the occasion of his 70th birthday

**Keywords** Mirror symmetry, Gromov-Witten invariants, Noncommutative varieties, Batalin-Vilkovisky formalism

**Abstract** The EA-matrix integrals, introduced in [B1], are studied in the case of graded associative algebras with odd or even scalar product. I prove that the EA-matrix integrals for associative algebras with scalar product are integrals of equivariantly closed differential forms with respect to Lie algebra  $gl_N(A)$ .

## Introduction

The theory of periods of noncommutative varieties, depending on commutative parameters, was introduced in [B5]. The analogue of top-degree holomorphic form in this setting was shown in *loc.cit.* to be certain element of semi-infinite subspace of negative cyclic homology. The integrals of this element satisfy the second order equation with respect to the parameters of deformations of the varieties. It was proven in *loc.cit.* that the generating function of genus zero Gromov-Witten invariants of complete intersection in  $\mathbb{CP}^d$  with trivial canonical class coincides with the coefficient of this second order equation for the family of mirror varieties. This approach had singled out the  $A_\infty$ -algebras/categories, satisfying cyclic homology analogue of degeneration of Hodge to de Rham spectral sequence, as the proper definition of (smooth and compact) noncommutative varieties.

The EA-matrix integrals were introduced in [B1] as a set of periods of associative, more generally  $A_\infty$ -algebras or noncommutative varieties, depending on *noncommutative* parameters:

$$\mathcal{F}(Y) = \int_{\Gamma} \exp(\text{Tr}\langle Y, X \rangle + \frac{1}{3!} m_{\tilde{A}}(X, \frac{\partial}{\partial X})) \prod_{\alpha, i, j} dX_i^{\alpha, j}$$

$\tilde{A} = A \otimes q_N$  /  $\tilde{A} = A \otimes gl_N$  in even/odd scalar product case, here  $q_N$  is the odd matrix algebra, see *loc.cit.* It was shown in theorem 3 in *loc.cit.* that the matrix Airy integral from [K] corresponds in this way to the simplest associative algebra of one dimension  $A = \{e | e^2 = e\}$ .

The usual varieties correspond here to  $A_\infty$ -algebras of endomorphisms of generators of their  $D^b(\text{Coh})$ -categories.

The asymptotic expansion of EA-matrix integrals via BV formalism was shown in [B1, B3] to define, as a sum over generalized ribbon graphs, a generating function for series of cohomology classes of compactified moduli spaces of curves of all genus. A particular example is the formula for cohomology-valued generating function for products of  $\psi$ -classes,  $\psi_i = c_1(T_{p_i}^*)$ , in the cohomology  $H^*(\bar{\mathcal{M}}_{g,n})$  calculated by the stable ribbon graph complex ([B3]):

$$\sum_{\sum d_i = d} \psi_1^{d_1} \dots \psi_n^{d_n} \prod_{i=1}^n \frac{(2d_i - 1)!!}{\lambda_i^{(2d_i + 1)}} = \left[ \sum_{G \in \Gamma_{g,n}^{dec, odd}} G \frac{2^{-\chi(G)}}{|\text{Aut}(G)|} \prod_{e \in \text{Edge}(G)} \frac{1}{\lambda_{i(e)} + \lambda_{j(e)}} \right] \quad (0.1)$$

where the sum on the right is over *stable ribbon* oriented graphs of genus  $g$  with  $n$  numbered punctures, with  $2d + n$  edges, and such that at each vertex the cyclically ordered subsets of edges have arbitrary *odd* cardinality.

In this paper it is proven that the EA-matrix integrals for associative algebras with scalar product are integrals of equivariantly closed differential forms with respect to the Lie algebra  $gl_N(A)$ . This generalizes and clarifies the similar result with respect to the Lie algebra  $gl_N$  from [B2]. The localization formula for the  $gl_N(A)$ -action then leads in [B4] to calculation of these EA matrix integrals via determinants and  $\tau$ -functions of integrable hierarchies.

The paper is dedicated to Raphael Kalmanovich Gordin, with gratitude.

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## Notations

For a  $\mathbb{Z}/2\mathbb{Z}$ -graded vector space  $A = A_0 \oplus A_1$  denote via  $\Pi A$  the parity inverted vector space,  $(\Pi A)_0 = A_1$ ,  $(\Pi A)_1 = A_0$ . For an element  $a$  from  $\mathbb{Z}/2\mathbb{Z}$ -graded vector space  $A$  denote by  $\pi a \in \Pi A$  the same element considered with parity reversed.

## 1 Equivariantly closed De Rham differential form.

Let  $A = A_0 \oplus A_1$  denotes a  $\mathbb{Z}/2\mathbb{Z}$ -graded associative algebra,  $\dim_k A_0 = r < \infty$ ,  $\text{char}(k) = 0$ , with multiplication denoted by  $m_2 : A^{\otimes 2} \rightarrow A$ . Let  $A$  be endowed with *odd* invariant non-degenerate scalar product  $\langle \cdot, \cdot \rangle : A_0 \otimes A_1 \rightarrow k$ . The multiplication tensor can be viewed then as the  $\mathbb{Z}/3\mathbb{Z}$ -cyclically invariant linear function on  $(\Pi A)^{\otimes 3}$

$$m_A : (\pi a_1, \pi a_2, \pi a_3) \mapsto (-1)^{\bar{a}_2+1} \langle m_2(a_1, a_2), a_3 \rangle, \quad m_A \in (\text{Hom}((\Pi A)^{\otimes 3}, k))^{\mathbb{Z}/3\mathbb{Z}}.$$

The odd symmetric scalar product on  $A$  corresponds to the odd anti-symmetric product  $\langle \cdot, \cdot \rangle^\pi$  on  $\Pi A$ :

$$\langle \pi a_1, \pi a_2 \rangle^\pi = (-1)^{\bar{a}_1+1} \langle a_1, a_2 \rangle$$

The tensor product with the matrix algebra  $gl_N$  is again naturally a  $\mathbb{Z}/2\mathbb{Z}$ -graded associative algebra with the odd scalar product  $A \otimes gl_N$ . The cyclic tensor

$$m_{A \otimes gl_N} \in (\text{Hom}((\Pi A \otimes gl_N)^{\otimes 3}, k))^{\mathbb{Z}/3\mathbb{Z}} \quad (1.1)$$

restricted to the diagonal  $\Pi A \otimes gl_N \subset (\Pi A \otimes gl_N)^{\otimes 3}$  is  $GL(N)$ -invariant cubic polynomial, denoted by  $m_{A \otimes gl_N}(Z)$ ,  $Z \in \Pi A \otimes gl_N$ . The associativity of the algebra  $A$  translates into the equation

$$\{m_{A \otimes gl_N}(Z), m_{A \otimes gl_N}(Z)\} = 0, \quad (1.2)$$

where  $\{\cdot, \cdot\}$  is the odd Poisson bracket corresponding to the odd anti-symmetric product  $Tr|_{gl_N^{\otimes 2}} \otimes \langle \cdot, \cdot \rangle^\pi$  on  $\Pi A \otimes gl_N$ .

**Proposition 1** *The algebra of functions on  $\Pi A \otimes gl_N$  is identified naturally, preserving the odd Poisson bracket, with the algebra of polyvectors on the even affine space  $\Pi A_1 \otimes gl_N$ .  $\square$*

This is analogous to algebra of functions on symplectic space being identified naturally, preserving Poisson bracket, with algebra of functions on cotangent bundle of given lagrangian subspace.

Denote by  $X^\alpha \in gl_N$ ,  $P_\alpha \in \Pi gl_N$  the matrices of coordinates on  $\Pi A \otimes gl_N$  corresponding to a choice of a dual pair of bases  $\{e^\alpha\}$ ,  $\{\xi_\alpha\}$  on  $A_0$  and  $A_1$  so that

$$Z = \sum_{\alpha} \pi \xi_{\alpha} \otimes X^{\alpha} + \pi e^{\alpha} \otimes P_{\alpha} \quad (1.3)$$

Then  $(P_{\alpha})_j^i$  corresponds to the vector field  $\frac{\partial}{\partial (X^{\alpha})_i^j}$  on  $\Pi A_1 \otimes gl_N$ . The cubic polynomial  $\frac{1}{3!} m_{A \otimes gl_N}(Z)$  corresponds to the sum of the function and the bivector,

$$\frac{1}{3!} \sum_{\alpha, \beta, \gamma} (m_A)_{\alpha \beta \gamma} Tr(X^{\alpha} X^{\beta} X^{\gamma}) + \frac{1}{2} \sum_{\alpha, \beta, \gamma} (m_A)^{\beta \gamma}_{\alpha} Tr(X^{\alpha} P_{\beta} P_{\gamma}). \quad (1.4)$$

The odd Poisson bracket is generated by the odd second order Batalin-Vilkovisky differential  $\Delta$  acting on the algebra of functions on  $\Pi A \otimes gl_N$

$$\{f_1, f_2\} = (-1)^{\bar{f}_1} (\Delta(f_1) f_2 - \Delta(f_2) f_1) + (-1)^{\bar{f}_1} f_1 \Delta(f_2) \quad (1.5)$$

$$\Delta = \sum_{\alpha, i, j} \frac{\partial^2}{\partial X_i^{\alpha, j} \partial P_{\alpha, j}^i}$$

### 1.1 Divergence-free condition

Let us assume from now on that the Lie algebra  $A_0$  is unimodular.

**Condition 2** (*unimodularity of  $A_0$* ) For any  $a \in A_0$

$$tr([a, \cdot]_{A_0}) = 0 \quad (1.6)$$

**Proposition 3** *The unimodularity of  $A_0$  (1.6) implies*

$$\Delta m_{A \otimes gl_N}(Z) = 0. \quad (1.7)$$

□

Next proposition is the standard corollary of the equations (1.2), (1.7) and the relation (1.5).

**Proposition 4** *The exponent of the sum (1.4) is closed under the Batalin-Vilkovisky differential*

$$\Delta \exp \left( \frac{1}{3!} m_{A \otimes gl_N}(Z) \right) = 0.$$

□

### 1.2 Closed De Rham differential form

The affine space  $\Pi A_1 \otimes gl_N$  has a holomorphic volume element, defined canonically up to a multiplication by a constant

$$\varpi = \lambda \prod_{\alpha, i, j} dX_i^{\alpha, j}.$$

It identifies the polyvectorfields on  $\Pi A_1 \otimes gl_N$  with the de Rham differential forms  $\Omega_{\Pi A_1 \otimes gl_N}$  on the same affine space via

$$\gamma \mapsto \gamma \vdash \varpi$$

The Batalin-Vilkovisky differential  $\Delta$  corresponds then to the De Rham differential  $d_{DR}$  acting on the differential forms. By the proposition 4 the polyvector  $\exp \frac{1}{3!} (m_{A \otimes gl_N}(Z))$  defines the closed differential form

$$\Psi(X) = \exp \left( \frac{1}{3!} \sum_{\alpha, \beta, \gamma} (m_A)_{\alpha\beta\gamma} Tr(X^\alpha X^\beta X^\gamma) + \frac{1}{2} \sum_{\alpha, \beta, \gamma} (m_A)_{\alpha}^{\beta\gamma} Tr(X^\alpha \frac{\partial}{\partial X^\beta} \wedge \frac{\partial}{\partial X^\gamma}) \right) \vdash \lambda \prod_{\alpha, i, j} dX_i^{\alpha, j} \quad (1.8)$$

$$d_{DR}\Psi(X) = 0$$

It is a sum of the closed differential forms of degrees  $rN^2$ ,  $rN^2 - 2$ , ....

### 1.3 Equivariantly closed differential form

The unimodularity (1.6) implies the invariance of  $\varpi$  under the co-adjoint action of the Lie algebra  $A_0 \otimes gl_N$

$$X \mapsto [Y, X],$$

$Y \in A_0 \otimes gl_N$ . Consider the  $A_0 \otimes gl_N$ -equivariant differential forms on  $\Pi A_1 \otimes gl_N$ :

$$\Omega_{\Pi A_1 \otimes gl_N}^{A_0 \otimes gl_N} = (\Omega_{\Pi A_1 \otimes gl_N} \otimes \mathcal{O}_{A_0 \otimes gl_N})^{A_0 \otimes gl_N}.$$

The  $A_0 \otimes gl_N$ -equivariant differential is given by

$$d_{A_0 \otimes gl_N} \Phi(Y) = d_{DR} \Phi - \sum_{\alpha, l, j} Y_{\alpha, j}^l (i_{[E_i^j \otimes e^\alpha, \cdot]} \Phi)$$

$\Phi \in \Omega_{\Pi A_1 \otimes gl_N}^{A_0 \otimes gl_N}$ , where  $i_\gamma$  denotes the contraction operator with respect to the vector field  $\gamma$ , see e.g. [DKV]. This differential corresponds, when passing to functions on  $\Pi A \otimes gl_N$ , to the sum

$$\begin{aligned} \Delta_{A_0 \otimes gl_N} : f(Z, Y) &\mapsto \Delta f - \frac{1}{2} Tr \langle [Y, Z], Z \rangle^\pi f, \\ f(Z, Y) &\in (\mathcal{O}_{\Pi A \otimes gl_N} \otimes \mathcal{O}_{A_0 \otimes gl_N})^{A_0 \otimes gl_N} \end{aligned}$$

of the Batalin-Vilkovisky differential and the operator of multiplication by the odd quadratic function

$$\frac{1}{2} Tr \langle [Y, Z], Z \rangle^\pi = m_{A \otimes gl_N}(Y \otimes Z \otimes Z). \quad (1.9)$$

The function depends on the equivariant parameters  $Y \in A_0 \otimes gl_N$ .

**Theorem 1** *The product of the closed de Rham differential form  $\Psi(X)$  (1.8) with the function  $\exp Tr \langle Y, X \rangle$ ,  $Y \in A_0 \otimes gl_N$ ,  $X \in \Pi A_1 \otimes gl_N$ , is  $A_0 \otimes gl_N$ -equivariantly closed differential form:*

$$d_{A_0 \otimes gl_N} (\exp(Tr \langle Y, X \rangle + \frac{1}{3!} m_{A \otimes gl_N}(X, \frac{\partial}{\partial X})) \vdash \lambda \prod_{\alpha, i, j} dX_i^{\alpha, j}) = 0$$

*Proof* Denote by  $i_{m(X \frac{\partial}{\partial X} \frac{\partial}{\partial X})}$  the operator of contraction with the bivector field  $\frac{1}{2} \sum_{\alpha, \beta, \gamma} (m_A)_{\alpha}^{\beta\gamma} Tr(X^\alpha \frac{\partial}{\partial X^\beta} \wedge \frac{\partial}{\partial X^\gamma})$  and by  $R_{Tr(Y dX)}$  the operator of exterior multiplication by the 1-form  $Tr \langle Y, dX \rangle$  acting on differential forms,

$$R_{Tr(Y dX)} = [d_{DR}, i_{Tr \langle Y, X \rangle}]$$

where  $i_{Tr \langle Y, X \rangle}$  is the multiplication by the linear function  $Tr \langle Y, X \rangle$ . Then

$$[i_{m(X \frac{\partial}{\partial X} \frac{\partial}{\partial X})}, R_{Tr(Y dX)}] = i_{[Y, Y]}$$

This is simply a particular case of the standard relation

$$[i_{\gamma_1}, Lie_{\gamma_2}] = i_{[\gamma_1, \gamma_2]}$$

for the action of polyvector fields. Notice that

$$d_{DR}e^{Tr\langle Y, X \rangle} = e^{Tr\langle Y, X \rangle} (d_{DR} + R_{Tr(YdX)})$$

and that

$$R_{Tr(YdX)} \exp \left( i_{m(X \frac{\partial}{\partial X} \frac{\partial}{\partial X})} \right) = \exp \left( i_{m(X \frac{\partial}{\partial X} \frac{\partial}{\partial X})} \right) (R_{Tr(YdX)} + i_{[,Y]}) \quad (1.10)$$

Since  $d_{DR}\Psi(X) = 0$ , and  $R_{Tr(YdX)} \prod_{\alpha, i, j} dX_i^{\alpha, j} = 0$ , therefore

$$d_{DR}(e^{Tr\langle Y, X \rangle} \Psi(X)) = i_{[,Y]} e^{Tr\langle Y, X \rangle} \Psi(X). \quad \square$$

## 2 The integral.

The closed differential form  $\Psi(X)$  is integrated over the cycles, which are standard in the theory of exponential integrals  $\int_{\Gamma} \exp f$ , see ([AVG] and references therein):

$$\Gamma \in H_*(M, \text{Re}(f) \rightarrow -\infty), \quad M = \Pi A_1 \otimes gl_N(\mathbb{C}). \quad (2.1)$$

Here  $f$  is the first term in (1.4), which is the restriction of the cubic polynomial  $\frac{1}{3!}m_{A \otimes gl_N}(Z)$  on  $M$ .

The relative homology are the same for such  $f$ ,  $f \neq 0$ , and for  $f + Tr\langle Y, X \rangle$  since linear term is dominated by the cubic term when  $|X| \rightarrow +\infty$ . Choosing a real form of  $A_0 \otimes gl_N(\mathbb{C})$  and taking the cycles in  $H_*(M, \text{Re}(f) \rightarrow -\infty)$  invariant with respect to this Lie algebra gives natural cycles for integration of the equivariantly closed differential form  $e^{Tr\langle Y, X \rangle} \Psi(X)$

$$\mathcal{F}(Y) = \int_{\Gamma} \exp(Tr\langle Y, X \rangle + \frac{1}{3!}m_{A \otimes gl_N}(X, \frac{\partial}{\partial X})) \vdash \prod_{\alpha, i, j} dX_i^{\alpha, j}$$

In general the integration cycles are the elements of the equivariant homology

$$H_{*, A_0 \otimes gl_N}(M, \text{Re}(\frac{1}{3!} \sum_{\alpha, \beta, \gamma} (m_A)_{\alpha\beta\gamma} Tr(X^\alpha X^\beta X^\gamma)) \rightarrow -\infty)$$

One can consider also the normalized integral

$$\widehat{\mathcal{F}}(Y) = \int_{\Gamma} \exp(Tr\langle Y, X \rangle + \frac{1}{3!}m_{A \otimes gl_N}(X, \frac{\partial}{\partial X})) \vdash \prod_{\alpha, i, j} dX_i^{\alpha, j} / \mathcal{F}_{[2]}(Y) \quad (2.2)$$

where  $\mathcal{F}_{[2]}(Y)$  is the corresponding Gaussian integral of the quadratic part of  $f + Tr\langle Y, X \rangle$  at a critical point  $(-Y)^{\frac{1}{2}}$ .

Let the associative algebra  $A_0$  has an *anti-involution*  $a \rightarrow a^\dagger$

$$(ab)^\dagger = b^\dagger a^\dagger, (ca)^\dagger = \bar{c} a^\dagger, \text{tr}(a^\dagger) = \overline{\text{tr}(a)}, (a^\dagger)^\dagger = a$$

The anti-involution defines the natural cycle for the equivariant integration. This anti-involution extends naturally to  $A_0 \otimes gl_N(\mathbb{C})$ . Then the Lie subalgebra of anti-hermitian elements in  $A_0 \otimes gl_N(\mathbb{C})$

$$u_N(A_0) = \{Y^\dagger = -Y \mid Y \in A_0 \otimes gl_N(\mathbb{C})\}$$

is a real form of  $A_0 \otimes gl_N(\mathbb{C})$ . And the space of hermitian elements in the dual space

$$\Gamma = \{X^\dagger = X \mid X \in A_0^\vee \otimes gl_N(\mathbb{C})\} \quad (2.3)$$

is invariant under the action of  $u_N(A_0)$ . Then the “real-slice”  $\Gamma$  is the natural choice of the cycle for the equivariant integration.

The localization formula for equivariant cohomology reduces the integral of the equivariantly closed form  $\Omega$  over  $\Gamma$  to the integral over the fixed locus  $F$ ,

$$\int_{\Gamma} \Omega = \int_F \frac{\Omega}{eu(N_F)} \quad (2.4)$$

where  $eu(N_F)$  is the euler class of the normal bundle of  $F$  in  $\Gamma$ , see [AB], [DKV]. Calculating the integral using the equivariant localization leads to generalized Vandermonde determinants and  $\tau$ -functions.

Let for simplicity the algebra  $A$  with odd scalar product is the tensor product  $A = A_0 \otimes q_1$  of the even associative algebra  $A_0$  with scalar product, denoted  $\eta(y_1, y_2)$ , and the algebra  $q_1 = \{1, \xi \mid \xi^2 = 1\}$  with the odd scalar product  $\langle 1, \xi \rangle = 1$ .

Assume that the natural scalar product on the Lie algebra of anti-hermitian elements in  $A_0$  is positive definite

$$-\eta(y, y) = \eta(y, y^\dagger) > 0.$$

Otherwise one can apply to  $\Gamma$  a partial Wick rotation.

Then the localization formula (2.4) after some calculations leads to the following result ([B4]):

**Proposition 5** *The integral (2.2), written in variables  $t \in HC^*(A)$ , is a  $\tau$ -function of KP-type hierarchy and, in particular, satisfies the Hirota quadratic equations.*

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