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Higher dimensional matrix Airy functions and equivariant cohomology.

SERGUEI BARANNIKOV

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ABSTRACT. I consider the simplest example of my equivariantly closed matrix integral from [B06], starting from super associative algebra with an odd trace.

1. EQUIVARIANTLY CLOSED MATRIX DE RHAM DIFFERENTIAL FORM FROM ASSOCIATIVE ALGEBRAS WITH ODD TRACE.

Let $A = A_0 \oplus A_1$ denotes a $\mathbb{Z}/2\mathbb{Z}$ -graded associative algebra, $\dim_k A = r < \infty$, with multiplication denoted by $m_2 : A^{\otimes 2} \rightarrow A$ and odd invariant scalar product $\langle \cdot, \cdot \rangle : A^{\otimes 2} \rightarrow \Pi k$. The multiplication tensor can be written as the $\mathbb{Z}/3\mathbb{Z}$ -cyclically invariant linear function on $(\Pi A)^{\otimes 3}$ which I denote by m_A

$$m_A : (\pi a_1, \pi a_2, \pi a_3) \rightarrow (-1)^{\overline{a_2}+1} \langle m_2(a_1, a_2), a_3 \rangle, \quad m_A \in (\text{Hom}(\Pi A, k)^{\otimes 3})^{\mathbb{Z}/3\mathbb{Z}}.$$

The odd scalar product on A corresponds to the odd anti-symmetric product $\langle \cdot, \cdot \rangle^\pi$ on ΠA :

$$\langle \pi a_1, \pi a_2 \rangle^\pi = (-1)^{\overline{a_1}+1} \langle a_1, a_2 \rangle$$

The cyclic tensor m_A extends to the $GL(N)$ -invariant cubique polynomial on $\mathbb{Z}/2\mathbb{Z}$ -graded vector space $gl(N) \otimes \Pi A$

$$Tr(m_A) : Z \rightarrow Tr(m_A(Z^{\otimes 3})), \quad Z \in gl(N) \otimes \Pi A$$

where Tr denotes the trace on tensor powers

$$gl(N)^{\otimes r} \rightarrow k, \quad Z_1 \otimes \dots \otimes Z_r \rightarrow Tr(Z_1 \cdot \dots \cdot Z_r)$$

The associativity of the algebra A translates into the equation

$$\{Tr(m_A), Tr(m_A)\} = 0 \tag{1}$$

where $\{\cdot, \cdot\}$ is the odd Poisson bracket corresponding to the odd anti-symmetric product $Tr|_{gl(N)^{\otimes 2} \otimes \langle \cdot, \cdot \rangle^\pi}$. The space of polynomial, respectfully analytic, functions on $gl(N) \otimes \Pi A$ is identified naturally, preserving the odd Poisson bracket, with polyvectors on $gl(N) \otimes \Pi A_1$ with polynomial, respectfully analytic, coefficients. If I denote by $X^\alpha \in gl(N)$, $P_\alpha \in \Pi gl(N)$ the coordinates on $gl(N) \otimes \Pi A$ corresponding to a choice of a dual pair of bases $\{e^\alpha\}$, $\{\xi_\alpha\}$ on A_0 and A_1 , so that $Z = \sum_\alpha X^\alpha \otimes \pi \xi_\alpha + P_\alpha \otimes \pi e^\alpha$, then $(P_\alpha)_j^i$ corresponds to the vector field $\frac{\partial}{\partial (X^\alpha)_i^j}$ on $gl(N) \otimes \Pi A_1$. The cyclic polynomial $Tr(m_A)$ corresponds to the sum of the function and the bivector,

$$\frac{1}{3!} \sum_{\alpha, \beta, \gamma} (m_A)_{\alpha\beta\gamma} Tr(X^\alpha X^\beta X^\gamma) + \frac{1}{2} \sum_{\alpha, \beta, \gamma} (m_A)_\alpha^{\beta\gamma} Tr(X^\alpha P_\beta P_\gamma) \tag{2}$$

which I'll denote by the same symbol $Tr(m_A)$ when it does not seem to lead to a confusion.

The odd Poisson bracket is generated by the odd second order Batalin-Vilkovisky differential acting on functions on $gl(N) \otimes \Pi A$

$$\{f_1, f_2\} = (-1)^{\bar{f}_1} (\Delta(f_1 f_2) - \Delta(f_1) f_2 + (-1)^{\bar{f}_1} f_1 \Delta(f_2)) \quad (3)$$

$$\Delta = \sum_{\alpha, i, j} \frac{\partial^2}{\partial X_i^{\alpha, j} \partial P_{\alpha, j}^i}$$

1.1. Divergence-free (unimodularity) condition.. I'll assume from now on that the Lie algebra A_0 is unimodular.

Condition 1. (*unimodularity of A_0*) For any $a \in A_0$

$$tr([a, \cdot]_{A_0}) = 0 \quad (4)$$

Proposition 2. *The unimodularity of A_0 (4) is equivalent to*

$$\Delta Tr(m_A) = 0 \quad (5)$$

where $N > 2$. \square

Next proposition is a standard corollary of the equations (1), (5) and the relation (3).

Proposition 3. *The exponent of the sum (2) is closed under the Batalin-Vilkovisky differential*

$$\Delta(\exp Tr(m_A)) = 0.$$

\square

1.2. Closed De Rham differential form.. The affine space $gl(N) \otimes \Pi A_1$ has a holomorphic volume element, defined canonically up to a multiplication by a constant

$$\varpi = \lambda \prod_{\alpha, i, j} dX_i^{\alpha, j}.$$

It identifies the polyvectorfields on $gl(N) \otimes \Pi A_1$ with de Rham differential forms $\Omega_{gl(N) \otimes \Pi A_1}$ on the same affine space via

$$\gamma \rightarrow \gamma \lrcorner \varpi$$

The Batalin-Vilkovisky differential Δ corresponds then to the De Rham differential d_{DR} . By proposition 3 the polyvector $\exp Tr(m_A)$ defines the closed differential form

$$\Psi(X) = \exp^{\frac{1}{\hbar}} \left(\frac{1}{3!} \sum_{\alpha, \beta, \gamma} (m_A)_{\alpha\beta\gamma} Tr(X^\alpha X^\beta X^\gamma) + \frac{1}{2} \sum_{\alpha, \beta, \gamma} (m_A)_{\alpha}^{\beta\gamma} Tr(X^\alpha \frac{\partial}{\partial X^\beta} \wedge \frac{\partial}{\partial X^\gamma}) \right) \lrcorner \lambda \prod_{\alpha, i, j} dX_i^{\alpha, j} \quad (6)$$

$$d_{DR} \Psi(X) = 0$$

which is a sum of closed forms of degrees $rN^2, rN^2 - 2, \dots$

1.3. Equivariantly closed differential form.. The unimodularity (4) implies the invariance of ϖ under adjoint action of the Lie algebra $gl(N) \otimes A_0$

$$X \rightarrow [Y, X]$$

and it is equivalent to the invariance of ϖ if $N > 2$.

Consider the $gl(N) \otimes A_0$ -equivariant differential forms on $gl(N) \otimes \Pi A_1$:

$$\Omega_{gl(N) \otimes \Pi A_1}^{gl(N) \otimes A_0} = (\Omega_{gl(N) \otimes \Pi A_1} \otimes \mathcal{O}_{gl(N) \otimes A_0})^{gl(N) \otimes A_0}.$$

The $gl(N) \otimes A_0$ -equivariant differential

$$d_{gl(N) \otimes A_0} \Phi = d_{DR} \Phi + \sum_{\alpha, l, j} Y_{\alpha, j}^l (i_{[E_i^j \otimes e^\alpha, \cdot]} \Phi)$$

$\Phi \in \Omega_{gl(N) \otimes \Pi A_1}^{gl(N) \otimes A_0}$, where i_γ denotes the contraction operator, corresponds when passing back to $gl(N) \otimes \Pi A$ to the sum

$$\begin{aligned} \Delta_{gl(N) \otimes A_0} & : f(Z, Y) \rightarrow \Delta f + \frac{1}{2} Tr \langle [Y, Z], Z \rangle^\pi f, \\ f(Z, Y) & \in (\mathcal{O}_{gl(N) \otimes \Pi A} \otimes \mathcal{O}_{gl(N) \otimes A_0})^{gl(N) \otimes A_0} \end{aligned}$$

of Batalin-Vilkovisky differential and the operator of multiplication by the odd quadratic function $\frac{1}{2} Tr \langle [Y, Z], Z \rangle^\pi = Tr(m_A)(Y \otimes Z \otimes Z)$. The function depends on the equivariant parameters $Y \in gl(N) \otimes A_0$. Notice that the odd product $\langle \cdot, \cdot \rangle^\pi : (\Pi A)^{\otimes 2} \rightarrow \Pi k$ can be viewed as the even pairing

$$\langle \cdot, \cdot \rangle^\pi : A_0 \otimes \Pi A_1 \rightarrow k$$

Theorem 4. *The product of the de Rham closed differential form $\Psi(X)$ (6) with the function $\exp Tr \langle Y, X \rangle^\pi$, $Y \in gl(N) \otimes A_0$ is $gl(N) \otimes A_0$ -equivariantly closed differential form:*

$$d_{gl(N) \otimes A_0} \exp(Tr \langle Y, X \rangle^\pi + \frac{1}{3!} Tr(m_A)(X, \frac{\partial}{\partial X})) \vdash \lambda \prod_{\alpha, i, j} dX_i^{\alpha, j} = 0$$

Proof. Denote by $i_{Tr(X \frac{\partial}{\partial X} \frac{\partial}{\partial X})}$ the operator of contraction with the bivector field $\frac{1}{2} \sum_{\alpha, \beta, \gamma} (m_A)_\alpha^{\beta\gamma} Tr(X^\alpha \frac{\partial}{\partial X^\beta} \wedge \frac{\partial}{\partial X^\gamma})$ and by $R_{Tr(Y dX)}$ the operator of exterior multiplication by the 1-form $Tr \langle Y, dX \rangle^\pi$. Then

$$[i_{Tr(X \frac{\partial}{\partial X} \frac{\partial}{\partial X})}, R_{Tr(Y dX)}] = i_{[\cdot, Y]}$$

This is simply a particular case of the standard relation

$$[i_{\gamma_1}, Lie_{\gamma_2}] = i_{[\gamma_1, \gamma_2]}$$

for the action of polyvectorfields. Therefore

$$d_{DR}(e^{Tr \langle Y, X \rangle^\pi} \Psi(X)) = i_{[\cdot, Y]} e^{\frac{1}{h} Tr \langle Y, X \rangle^\pi} \Psi(X)$$

■

2. THE INTEGRAL

My closed differential form $\Psi(X)$ can now be integrated over the cycles, which are standard in the theory of exponential integrals $\int_{\Gamma} \exp f$, see ([AVG] and references therein): $\Gamma \in H_*(M, \text{Re}(f) \rightarrow -\infty)$. Here f is the first term in (2) which is the restriction of the cyclic polynomial $\text{Tr}(m_A)$ to $\mathfrak{gl}(N) \otimes \Pi A_1 = M$,

$$\mathcal{F}_0 = \int_{\Gamma} \exp\left(\frac{1}{3!} \text{Tr}(m_A)\left(X, \frac{\partial}{\partial X}\right)\right) \vdash \lambda \prod_{\alpha, i, j} dX_i^{\alpha, j}$$

In generic situation the cohomology of such f and $f + \text{Tr}\langle Y, X \rangle^{\pi}$ are the same, since linear term is dominated when $|X| \rightarrow +\infty$. Choosing a real form of $\mathfrak{gl}(N) \otimes A_0$ and taking the cycles in $H_*(M, \text{Re}(f) \rightarrow -\infty)$ invariant under it, this gives the natural cycles for integration of the equivariantly closed differential form $e^{\text{Tr}\langle Y, X \rangle^{\pi}} \Psi(X)$

$$\mathcal{F}(Y) = \int_{\Gamma} \exp\left(\text{Tr}\langle Y, X \rangle^{\pi} + \frac{1}{3!} \text{Tr}(m_A)\left(X, \frac{\partial}{\partial X}\right)\right) \vdash \lambda \prod_{\alpha, i, j} dX_i^{\alpha, j}$$

For example if the algebra is defined over R , then I can take the real slice, i.e. the cycle of antihermitian matrices as such a cycle. Localizing the integral then leads to some generalisation of Vandermond determinants and τ -functions.

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