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# Continuum Approximation for Congestion Dynamics Along Freeway Corridors

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In this paper, congestion dynamics along crowded freeway corridors are modeled as a conservation law with a source term that is continuous in space. The source term represents the net inflow from ramps, postulated here as a location-dependent function of the demand for entering and exiting the corridor. Demands are assumed time-independent, which is appropriate for understanding the onset of congestion. Numerical and analytical results reveal the existence of four well-defined regions in time-space, two of which are transient. The conditions for the existence of congestion both in the freeway and in the on-ramps are identified, as well as the set of on-ramps that are most likely to become active bottlenecks. The results in this paper help explain the stochastic nature of bottleneck activation, and can be applied to devise effective system-wide ramp metering strategies that would prevent excessively long on-ramp queues.

*Key words:* Continuum Approximation; kinematic wave model; traffic dynamics; congestion

## 1. Introduction.

The main drawback of current traffic flow network models based on the kinematic wave model (Lighthill and Whitham 1955, Richards 1956) is the treatment of boundary conditions at merges. Invariably, it is assumed full priority for the entering flows, which is convenient for mathematical tractability but only reasonable under very light traffic (see e.g. Coclite and Piccoli 2002, Bayen et al. 2004, Gugat et al. 2005, Coclite et al. 2005, Bastin et al. 2007, Jin and Zhang 2003, 2004). Under congested conditions, however, empirical evidence reported by Cassidy and Ahn (2005) indicates that the available capacity on the freeway is allocated to competing streams according to a “merge ratio”, as per Daganzo’s model (Daganzo 1996), which is a constant independent of flows.

To circumvent this problem, this paper uses a result in Laval and Leclercq (2008) who obtained a continuum model for lane-changing rates in congested traffic. In the case of merges, this means that freeway inflows are expressed as a function of the flow on the freeway and the demand for entering the freeway. The resulting formulation allowed us to better understand the onset of morning and evening commute congestion along crowded freeway corridors, and its relationship with the demand for entering and exiting along the corridor. In order to obtain analytical solutions and better understand system dynamics, a continuum approximation was used to represent inflows and outflows to the corridor.

This paper is organized as follows. Section 2 formulates the problem as a conservation law in both the freeway and the on-ramps, as well as the appropriate boundary conditions. The numerical solution of this problem is carried out in section 3, which reveals considerable insight. Based on this insight, section 4 shows the analytical solution for three important cases. Finally, section 5 presents a discussion and outlook.

## 2. Problem Formulation.

Consider a long  $n$ -lane freeway corridor of length  $L$  with entrances and exits evenly spaced  $\delta$  distance units apart. The number of entrances and exits is large, and therefore freeway inflow and outflow rates can be treated as continuous variables in time  $t$  and location  $x$ ,  $\phi^+(t, x)$  and  $\phi^-(t, x)$ , in units of veh/time-distance. There exists a homogeneous fundamental diagram  $q(k)$  that gives the flow  $q(t, x)$  as a function of the local density  $k(t, x)$ ; both quantities defined as totals across all of freeway lanes. Therefore, the traffic conservation law can be expressed as

$$k_t + s(k)k_x = \phi^+ - \phi^-, \quad (1)$$

where  $s(k) = dq(k)/dk$  is the speed of characteristics, and variables in subscript represent partial derivatives. We will use the superscript “ $x$ ” to denote variables belonging to the on-ramp that merges at location  $x$  in the freeway. With this notation, traffic dynamics at on-ramps are given by

$$k_t^x + s^x(k^x)k_y^x = 0, \quad (2)$$

where  $k^x(t, y)$  gives the density at location  $y$  along the on-ramp. The length of all on-ramps is  $d$  and are assumed identical to a single freeway lane.

It is reasonable to assume that exit flows are Markovian; i.e.,

$$\phi^- = \beta q, \quad (3)$$

where  $\beta(t, x)$  is the proportion per unit distance of the freeway flow that exits at  $(t, x)$ . The determination of  $\phi^+(t, x)$  is more elaborate because one has to capture driver merging behavior, which depends on both the traffic state in the freeway and in the on-ramp. To this end, let  $\alpha(t, x)$  be the *demand* rate for entering the freeway at  $(t, x)$ , in units of veh/time-distance. Notice that  $\alpha$  gives the number of vehicles willing to enter the freeway and not the number of vehicles that actually enter the freeway. To obtain the latter, in this paper we modify eqn. (6) in Laval and Leclercq (2008) which pertains to discretionary lane-changing rates in multilane freeways. Accordingly, the actual freeway inflow rate may be expressed as

$$\phi^+(t, x) = \min\left\{1, \frac{\mu(k(t, x))}{\lambda(k(t, x))}\right\} \lambda^x(k^x(t, d)) / \delta, \quad (4)$$

where  $\mu$  and  $\lambda$  are the receiving and sending function of kinematic wave theory. For a triangular fundamental diagram with free-flow speed  $u$ , wave speed  $w$ , and jam density  $\kappa$ , we have

$$\mu(k) = \min\{(n\kappa - k)w, nQ\}, \quad (5a)$$

$$\lambda(k) = \min\{uk, nQ\}, \quad (5b)$$

$$\mu^x(k^x) = \min\{(\kappa - k^x)w, Q\}, \quad (5c)$$

$$\lambda^x(k^x) = \min\{uk^x, Q\}. \quad (5d)$$

where  $Q = uw\kappa/(u + w)$  is capacity of one lane. Notice that  $q(k) = \min\{\lambda(k), \mu(k)\}$  and  $q^x(k^x) = \min\{\lambda^x(k^x), \mu^x(k^x)\}$ .

Notice that (4) is the continuum limit (as the cell size goes zero) of the model introduced in Laval and Daganzo (2006) for the flow of lane changes, which allocates the available capacity to each approach according to its sending function. In the case of discrete merges, this approach was first introduced in Jin and Zhang (2003). The appendix shows that this type of models are equivalent to Daganzo’s model (Daganzo 1996), which has been validated empirically (Cassidy and Ahn 2005). In particular, they produce constant merging priorities independent of the (congested) flow in the competing approaches. Without loss of generality, (5b) and (5d) imply equal priority between the

76 on-ramp and the freeway shoulder lane. The appendix also shows how one can modify (5d) to  
 77 account for a different priorities.

78 To model on ramps as a continuum, the term  $1/\delta$  in (4) “diffuses” the demand of a discrete  
 79 on-ramp (given by (5d)) within a vicinity of length  $\delta$ . Similarly, we use the following initial and  
 80 boundary conditions for on-ramps:

$$81 \quad k^x(0, y) = \alpha(0, x)\delta/u, \quad (6a)$$

$$82 \quad k^x(t, 0) = \alpha(t, x)\delta/u, \quad (6b)$$

$$83 \quad q^x(t, d) = \phi^+(t, x)\delta. \quad (6c)$$

85 Initial condition (6a) assumes that all on-ramps are filled with the demand at  $t = 0$ . Boundary  
 86 conditions (6b) and (6c) imply that the demand at time  $t$  enters the on-ramp at  $y = 0$ , and that  
 87 the flow that actually enters the freeway (i.e., that exits the on-ramp at  $y = d$ ) is given by (4).

88 We also assume that the freeway is empty at  $t = 0$  and that there is no inflow from the upstream  
 89 end of the freeway; the only inflow is due to on-ramps in  $0 \leq x \leq L$ . Notice that the latter is without  
 90 loss of generality because one can always increase  $L$  to account for higher freeway flows at a given  
 91 location. Thus, in addition to (3) and (4), the following are the initial and boundary conditions for  
 92 the freeway:

$$93 \quad k(0, x) = 0, \quad (7a)$$

$$94 \quad k(t, 0) = 0. \quad (7b)$$

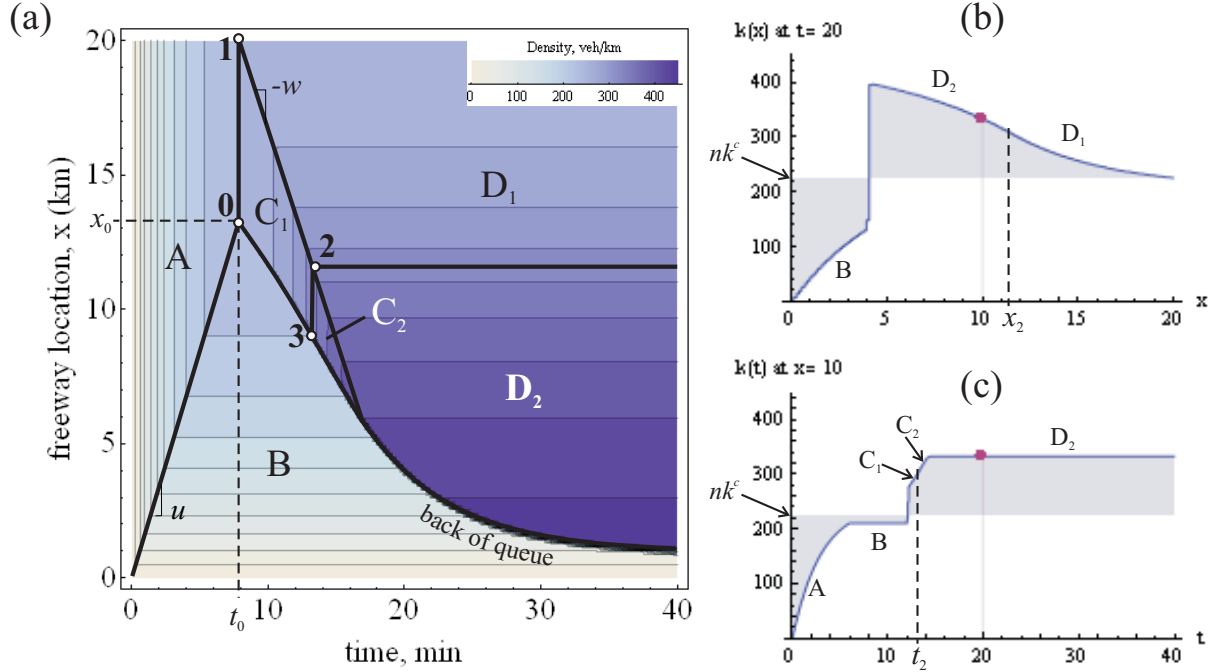
96 As formulated, the problem is general enough to capture any demand pattern whatsoever. The  
 97 numerical solutions for such problems would be straightforward, but simulation is a lousy tool for  
 98 understanding the problem. Therefore, in the sequel we restrict our attention to time-independent  
 99 and location-monotonic demand patterns, i.e.,  $\alpha = \alpha(x), \beta = \beta(x), d^2\alpha/dx^2 = d^2\beta/dx^2 = 0$ . Depend-  
 100 ing on the sign of the slope of  $\alpha(x)$  and  $\beta(x)$ , this should provide an adequate approximation  
 101 for the morning and evening commute problems. Additionally, this simplification will allow us to  
 102 obtain analytical solutions that reveal important insights which should be true in general.

103 The next section shows the numerical solution of the problem formulated in this section. The  
 104 numerical solution reveals important properties of the problem, which are useful for the analytical  
 105 solution presented in section 4.

### 106 3. Numerical Solution.

107 In this section we apply Godunov’s scheme to the problem (1)-(7) solving the Riemann problems  
 108 using the sending and receiving functions (5) as described in Daganzo (1996). Without loss of  
 109 generality, we have chosen  $w = u$  in these experiments in order to obtain exact solutions, i.e., free  
 110 of numerical errors (the reader is referred to Leclercq et al. 2007, for a demonstration). For the  
 111 numerical examples in this section we have used  $n=3$  lanes,  $w = u = 100$  km/hr and  $\kappa=150$  veh/km  
 112 (and therefore  $Q=7500$  veh/hr).

113 The main insight revealed by the numerical solutions is that there exists four distinct regions in  
 114 the time-space plane where density obeys a distinctive pattern: two of these regions correspond to  
 115 free-flow states (labeled regions A and B in the sequel) and two are congested (C and D). Regions  
 116 A and C are transient while B and D give the steady state of the system (until demands drop at  
 117 the end of the rush hour). Of course, the evolution of density inside each region will depend on the  
 118 functions  $\alpha(x)$  and  $\beta(x)$ , but as we will see in the sequel the shape of the regions is rather general  
 119 for typical rush-hour patterns.



**Figure 1** Numerical solution for the constant demand case:  $a = 4,850$  veh/hr/km,  $b = 0.2$  km<sup>-1</sup> and  $L = 20$  km and  $\delta = 1$  km. (a) Time-space density contour map for the freeway. Bold lines correspond to interfaces between the different traffic states, while thin lines are isodensity contours. (b) and (c): evolution of density at  $t = 20$  min and  $x = 10$  km, respectively.

### 3.1. Constant demands

We start our discussion with the simplest case where  $\alpha(x) = a$  and  $\beta(x) = b$  are constant in  $0 < x < L$  and zero elsewhere. Figure 1a shows a density contour map obtained numerically. Region A corresponds to the “filling up” of the freeway, where, as time passes, the density increases. As can be seen in figure the isodensity contours are vertical, meaning that at a given time the density in the freeway is constant for  $x \geq ut$ . Inside region B densities have reached a free-flow equilibrium, where isodensity contours are horizontal. Point “0” in the figure marks the beginning of congestion. At this point the density reaches the critical density and therefore a shock waves propagates upstream, which corresponds to the back of the queue. At the same time, a wave is emanated from point “1” in the figure, which is not a result of a restriction from downstream of  $x = L$  but of the absence of lateral inflow and outflow for  $x > L$ . This wave eventually meets the back of the queue, and marks the boundary between congested regions C and D. The difference between these regions is that in the former the isodensity contours are vertical and in the latter, horizontal. These congested regions can be broken down into two subregions: regions C<sub>1</sub> and D<sub>1</sub> where on-ramps are in free-flow conditions, and regions C<sub>2</sub> and D<sub>2</sub> where on-ramps are congested. The boundary between the two sub regions are defined by points “2” and “3” in the figure. Point “2” corresponds to the most downstream location where on-ramp congestion is first observed, while point “3” is the analogous for the most upstream location.

Figures Fig. 1b and 1c show the evolution of density at  $t = 20$  min and  $x = 10$  km, respectively. As expected, there is a sharp shock between region B and the congested regions. It can be seen in part b of the figure that point “2” corresponds to an inflection point in the spatial evolution of density inside region D.

A density map at selected time instants is presented in Fig. 2 for both the freeway and the on-ramps. Consistently with Fig. 1a, it can be seen that congestion at on-ramps starts around  $t = 14$  min simultaneously between locations  $x_2$  and  $x_3$ . On-ramp queues propagate upstream thereafter,

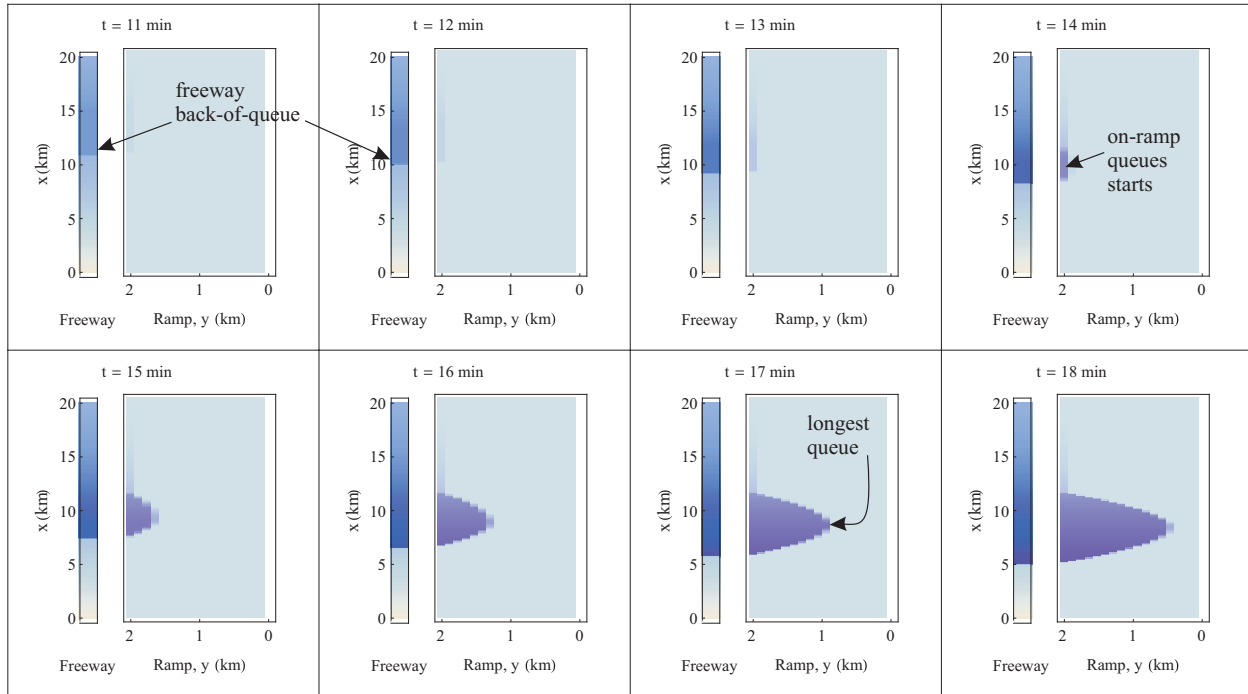


Figure 2 A density map at selected time instants for both the freeway and the on-ramps.

145 eventually reaching the beginning of the on-ramp at  $y = 0$ . It is interesting to note that at a given  
 146 time instant larger queues are observed roughly in the midpoint between  $x_2$  and the back of the  
 147 freeway queue.

### 148 3.2. Monotonic demand patterns.

149 Here we extend our previous analysis to include a first-order dependency between the demand  
 150 and freeway location. To this end, we consider all combinations, the following specifications for  
 151 on-ramps:

$$152 \quad \alpha(x) = a, \quad \text{(constant demand)} \quad (8a)$$

$$153 \quad \alpha(x) = a(1 - x/L), \quad \text{(decreasing demand)} \quad (8b)$$

$$154 \quad \alpha(x) = ax/L, \quad \text{(increasing demand)} \quad (8c)$$

156 and the for exits:

$$157 \quad \beta(x) = b, \quad \text{(constant demand)} \quad (9a)$$

$$158 \quad \beta(x) = b(1 - x/L), \quad \text{(decreasing demand)} \quad (9b)$$

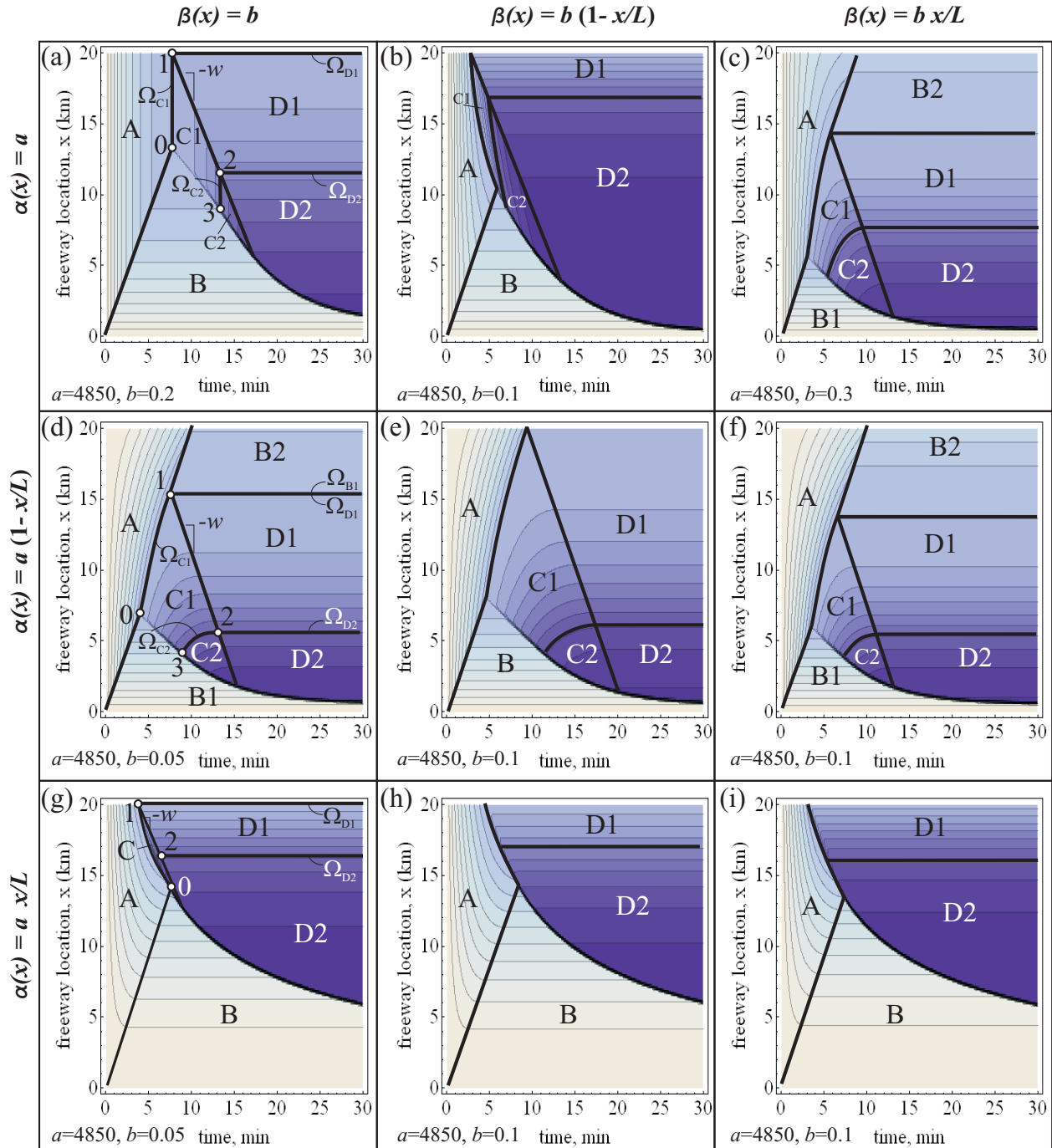
$$159 \quad \beta(x) = bx/L, \quad \text{(increasing demand)} \quad (9c)$$

161 Notice that specifications (8) and (9) are valid in  $0 \leq x \leq L$ ; elsewhere all functions are identically  
 162 zero.

163 The numerical solution for all combinations between (8) and (9) are shown in Fig. 3. It can be  
 164 seen that the main qualitative difference with respect to the constant demand case of Fig. 1 are  
 165 that

166 1. the isodensity contours in regions A and C are no longer vertical, and are determined by  
 167 on-ramp demand. This can be verified by comparing the second and third row of the figure.

168 2. there is a new free-flow region  $B_2$  that appears downstream of  $D_1$  in three cases (parts c, d  
 169 and f in the figure). These regions appear because of the small on-ramp demand and-or a large  
 170 exit demand near the end of the freeway segment.



**Figure 3** Numerical solution for all combinations between (8) and (9). Bold lines correspond to interfaces between the different traffic states, while thin lines are isodensity contours.

171 3. region C tends to disappear (but still exists) in the case of increasing on-ramp demand; see  
172 last row in the figure.

173 4. the interface between regions C and A are either forward moving (parts c, d, e and f of the  
174 figure) or backward moving (b, g, h and i).

175 Notice how pictures d and g in the figure are similar to all cases within their respective row. This  
176 suggests that, at least qualitatively, the behavior of the system is not very sensitive to the exit  
177 demand pattern, and that the constant exit demand case should provide good approximations for

178 more complicated patterns. A similar argument can be made for the constant on-ramp demand.  
 179 It is apparent how pictures b and c in the figure are very similar to the pictures in the third  
 180 and second row, respectively. This result is intuitive since, in terms of net inflow, an increasing  
 181 (decreasing) exit demand and a decreasing (increasing) on demand are interchangeable.

182 As expected, congestion is more severe farther away from the downtown area, in all cases. This  
 183 is evident from the figure since higher densities are observed closer to the beginning of the freeway  
 184 segment.

## 185 4. Analytical Solution

186 In this section we derive the analytical solution for the three cases in the first column of Fig. (3),  
 187 i.e.  $\beta(x) = b$ . Unfortunately, these are the only cases where one can obtain fully algebraic solutions;  
 188 all other cases involve terms that can only be evaluated numerically. However, it was founded in  
 189 section 3 that the particular form of  $\beta(x)$  does not affect significantly the results compared to the  
 190 constant- $\beta$  case, at least qualitatively. Therefore, it is expected that the insights obtained for the  
 191 three cases analyzed in this section are valid in general.

192 The solution method proposed here recognizes that inside each region identified in the previous  
 193 section, the density obeys a simplified form of (1) supplemented with “ad-hoc” boundary conditions.  
 194 In particular, since each region is either congested or uncongested, the speed of characteristics  $s(k)$   
 195 is either  $-w$  or  $u$ , respectively, independent of the density, which facilitates matters considerably.  
 196 The idea then is to solve each region independently choosing the appropriate boundary conditions.

197 In general, a boundary condition specifies (exogenous) values for the density,  $k_0(t, x)$ , along a  
 198 trajectory in the time-space plane,  $(t, x) \in \Omega$ ; i.e.:

$$199 \quad k(t, x) = k_0(t, x), \quad \forall (t, x) \in \Omega. \quad (10)$$

200 where  $\Omega$  is the set of points defining the trajectory. For the examples of the previous section, part  
 201 a of Fig. 3 shows the trajectory relevant for each region,  $\Omega_A, \Omega_B \dots \Omega_{D_2}$ .

202 It is important to note that the proposed region-by-region solution method does not require the  
 203 explicit solution of the kinematic wave model for on-ramps (2). This is true because inside each  
 204 region the inflow rate  $\phi^+$  can be determined exogenously. In particular,  $\phi^+$  takes only two possible  
 205 values depending on the traffic states prevailing in the on-ramp just upstream of the merge; i.e.:

$$206 \quad \phi^+(t, x) = \begin{cases} \alpha(x), & \text{(uncongested on-ramp),} \\ (n\kappa - k(t, x))w/(n\delta), & \text{(congested on-ramp).} \end{cases} \quad (11)$$

207 It is clear that in free-flow  $\phi^+$  corresponds to the on-ramp demand  $\alpha(x)$ . To obtain the congested  
 208 portion of (11), we note that on-ramp congestion takes place only when the freeway is congested,  
 209 in which case (4) can be written as  $\phi^+ = \mu\lambda^x/(\lambda\delta)$ . Since both the freeway and the on-ramp are  
 210 congested then  $\mu = (n\kappa - k)w, \lambda = nQ$  and  $\lambda^x = Q$ , and therefore  $\phi^+ = (n\kappa - k)w/(n\delta)$  as sought.

211 Finally, Table 1 shows a summary of all the information needed to solve the kinematic wave  
 212 model (1) inside each region, for the case of constant demands. All other cases are solved similarly,  
 213 the only difference being the definition of the boundary conditions. Note that in the table  $k^c$  refers  
 214 to the critical density in a single lane.

### 215 4.1. Constant demands

216 In this section we consider demands that are constant along the freeway corridor; as per (8b) and  
 217 (9a). Next, we use Table 1 to obtain the analytical expressions for the density inside regions A



**Table 1** Summary of specifications for solving the kinematic wave model (1)-(7) inside each region

region	freeway	on-ramps	$s(k)$	$\phi^+ - \phi^-$	$\Omega$	$k_0(t, x)$
A	free-flow	free-flow	$u$	$\alpha - \beta uk$	$t = 0$	0
B	free-flow	free-flow	$u$	$\alpha - \beta uk$	$x = 0$	0
C1	congested	free-flow	$-w$	$\alpha - \beta(n\kappa - k)w$	$t = t_0$	$nk^c$
C2	congested	congested	$-w$	$(1/(n\delta) - \beta)(n\kappa - k)w$	$t = t_2$	$k_{C_1}(t_2, \Omega_{C_2})$
D1	congested	free-flow	$-w$	$\alpha - \beta(n\kappa - k)w$	$x = x_1$	$nk^c$
D2	congested	congested	$-w$	$(1/(n\delta) - \beta)(n\kappa - k)w$	$x = x_2$	$k_{D_1}(t_2, \Omega_{D_2})$

through  $D_2$ , which are labeled  $k_A, k_B, \dots$ . This can be accomplished using the method of characteristics (see e.g. LeVeque 1993), which recognizes that along characteristic lines the density obeys an ordinary differential equation that can be solved straightforwardly. It can be shown that:

$$k_A(t) = (1 - \exp(-but))a/(ub), \quad (12a)$$

$$k_B(x) = (1 - \exp(-bx))a/(ub), \quad (12b)$$

$$k_{C_1}(t) = n\kappa - \frac{1}{bw} \left( a(1 - c_1^{\frac{w}{u}+1}) \exp(bwt) \right), \quad (12c)$$

$$k_{C_2}(t) = n\kappa - \frac{n\delta a}{w} \left( \frac{c_0}{c_1} \right)^{\frac{1}{b\delta n} - 1} \left( \frac{1}{c_1} \right)^{\frac{c_0 w}{b\delta n u}} \exp\left(-\frac{wc_0}{\delta n} t\right), \quad (12d)$$

$$k_{D_1}(x) = n\kappa - \frac{1}{bw} (a(1 + c_1) \exp(b(L - x))), \quad (12e)$$

$$k_{D_2}(x) = n\kappa - \frac{n\delta a}{w} \left( \frac{c_0}{c_1} \right)^{\frac{1}{b\delta n} - 1} \exp\left(-\frac{c_0}{\delta n} (L - x)\right). \quad (12f)$$

where we have defined the (dimensionless) constants

$$c_0 = 1 - bn\delta, \quad (13a)$$

$$c_1 = 1 - bnQ/a. \quad (13b)$$

The above solutions are valid only in the relevant regions A... $D_2$  defined in Fig. 1. To formalize the definition of these regions we will next find the coordinates  $(t_i, x_i)$  of points  $i = 0, 1, 2, 3$  in the figure.

Point “0” is the point closest to the origin where the density equals the critical density. Therefore, it can be obtained by solving  $k_B(x) = nk^c$  for  $x$ , which gives

$$x_0 = \frac{1}{b} \log\left(\frac{1}{c_1}\right), \quad (14a)$$

$$t_0 = x_0/u. \quad (14b)$$

In the numerical example of Fig. 1 this gives  $x_0 = 13.14$  km and  $t_0 = 7.88$  min, as can be verified in the figure. Clearly,

$$x_1 = L, \quad (15a)$$

$$t_1 = t_0. \quad (15b)$$

The location of point “2”,  $x_2$ , marks the location of the most downstream on-ramp that first suffers congestion. At this point the demand on the on-ramp  $a\delta$  equals the capacity allocated to the on-ramp according to the merge model,  $q(k_{D_1}(x))/n$ . Accordingly, solving  $a\delta = (n\kappa - k_{D_1}(x))w/n$  for  $x$  gives

$$x_2 = L - \frac{1}{b} \log\left(\frac{c_0}{c_1}\right), \quad \text{and} \quad (16a)$$

$$t_2 = (L - x_2)/w + t_0, \quad (16b)$$

252 which gives  $x_2 = 11.44$  and  $t_2 = 13.02$  min for Fig. 1.

253 The formulae for points “0” and “2” above can be used to obtain the conditions for the existence  
 254 of congestion both in the freeway and the on-ramps. In particular, no freeway congestion means that  
 255 point “0” appears beyond the length of our freeway segment; i.e.,  $x_0 > L$ . Similarly no congestion  
 256 on the on-ramps means  $x_2 < 0$ . Using (14a) and (16a) gives

$$257 \quad a \leq \frac{bnQ}{1 - e^{-bL}}, \quad (\text{no freeway congestion}) \quad (17a)$$

$$258 \quad a \leq \frac{bnQ}{1 - c_0 e^{-Lb}}. \quad (\text{no on-ramp congestion}) \quad (17b)$$

260 In the numerical example of Fig. 1 we obtained the conditions  $a \leq 4,533.2$  and  $a \leq 4,584$ , respec-  
 261 tively. Notice that for very long freeway segments the exponential terms in (17) vanish, so that a  
 262 common no-congestion condition simplifies to  $a \leq bnQ$ .

263 As was mentioned in section 3, the highest densities are observed near the beginning of the  
 264 freeway segment, i.e., in region  $D_2$ . But this high density is always smaller than the jam density.  
 265 In fact, the limit of (12f) tends to the jam density only for very long freeways; i.e., as  $L \rightarrow \infty$ . For  
 266 example, in the case of Fig. 1 we have that for  $L = 20$  km  $k_{D_2}(0) = 418 < n\kappa = 450$  veh/km.

267 It is possible to obtain the equation of the back of the queue; see Fig. 1a. This can be done using  
 268 the classic “shock condition” which asserts that the speed of the shock is given by the ratio between  
 269 the difference in flow and the difference density between the neighboring traffic states, respectively.  
 270 In our case, one would obtain three formulae for the shock speed, one for each pair of neighboring  
 271 traffic states B- $C_1$ , B- $C_2$  and B- $D_2$ . Unfortunately, these equations are very complicated and not  
 272 particularly insightful and therefore are not included here.

#### 273 4.2. Decreasing on-ramp demand.

274 In this section we assume demands obeying (8b) and (9a); i.e., on-ramp demand decreases lin-  
 275 early along the freeway. This could represent a first-order approximation for morning (evening)  
 276 commute in cities where people tend to live in the suburbs (downtown area). The relevant bound-  
 277 ary conditions are shown in Fig. 3d. Notice that these are similar to the constant demand case  
 278 in Table 1 except for state C and the new state  $B_2$ . In the latter case the boundary condition  
 279 is simply  $k(t, x_1) = nk^c$ , but unfortunately for  $C_1$  and  $C_2$  the boundary conditions are far more  
 280 complicated. In fact, as can be seen in the figure,  $\Omega_{C_1}(t)$  and  $\Omega_{C_2}(t)$  correspond to the isodensity  
 281 curves  $k_A(t, x) = nk^c$  and  $k_{C_1}(t, x) = k_{D_1}(x_2)$ , respectively, which turn the problem mathematically  
 282 intractable. This has no profound impact in our analysis since these regions are only transient. For  
 283 the remaining regions it can be shown that

$$284 \quad k_A(t, x) = c_4 \left( 1 + b(L - x) - e^{-btu} (b(L + tu - x) - 1) \right), \quad (18a)$$

$$285 \quad k_{B_1}(x) = c_4 \left( 1 + b(L - x) - e^{-bx} (bL + 1) \right), \quad (18b)$$

$$286 \quad k_{B_2}(x) = c_4 \left( 1 + b(L - x) - e^{-b(c_2 - L + x)} \right), \quad (18c)$$

$$287 \quad k_{D_1}(x) = n\kappa - c_4 \left( 1 + b(L - x) - e^{-b(c_2 - L + x)} \right), \quad (18d)$$

$$288 \quad k_{D_2}(x) = n\kappa - c_4 e^{-bc_2 - \frac{L - xc_0 - c_3}{\delta n} - \frac{c_3}{b}} \left( e^{b(c_2 + L) - c_3} (1 + c_3) - e^{bL} \right), \quad (18e)$$

290 where we have defined the (positive) constants

$$291 \quad c_2 = nQLb/a, \quad (19a)$$

$$292 \quad c_3 = -\mathcal{W} \left( -\frac{1}{c_0} \exp(-bc_2 - \frac{1}{c_0}) \right) - \frac{1}{c_0}, \quad (19b)$$

$$293 \quad c_4 = \frac{a}{b^2 Lu}. \quad (19c)$$

294

295 In (19b) the term  $\mathcal{W}(z)$  represent the (real-valued) Lambert W-function, which gives the solution  
296 of  $z = we^w$ . The coordinates for the different points in Fig. 3d can be obtained similarly to the  
297 constant demand case. Solving  $k_{B_1}(x) = nk^c$  for  $x$  gives

$$x_0 = L - c_2 + \frac{1}{b} (1 + \mathcal{W}(-b(L + 1) \exp(-b(L - c_2) - 1))), \quad (20a)$$

$$t_0 = x_0/u. \quad (20b)$$

301 For the example in Fig. 3d we have that  $x_0 = 7.48$  km and  $t_0 = x_0/u = 4.48$  min. To obtain point “1”  
302 we note that it is the most downstream point in A where the density equals  $nk^c$ . It is straightforward  
303 to show that

$$x_1 = L - (e^{bt_1} - 1)/b + ut_1, \quad (21a)$$

$$t_1 = -c_2/u - (1 + \mathcal{W}(-\exp(-bc_2 - 1)))/(bu), \quad (21b)$$

307 which in the case of Fig. 3d gives  $t_1 = 7.59$  min and  $x_1 = 15$  km. Solving  $\delta\alpha(x) = w(n\kappa - k_{D_1}(x))/n$   
308 for  $x$  gives

$$x_2 = L - c_3/b, \quad (22a)$$

$$t_2 = (x_1 - x_2)/w + t_1. \quad (22b)$$

312 In the case of Fig. 3d this gives  $t_2 = 13.42$  min and  $x_2 = 5.28$  km. With all, the congestion conditions  
313 become

$$a \leq \frac{b^2 nQL}{1 - (bL + 1) \exp(-bL)}, \quad (\text{no freeway congestion}) \quad (23a)$$

$$a \leq \frac{b^2 nQL}{bL - \log(bLc_0 + 1)}. \quad (\text{no on-ramp congestion}) \quad (23b)$$

317 which for Fig. 3d give  $a \leq 4, 257.5$  and  $a \leq 2, 923.5$ , respectively. Interestingly, for very long freeways  
318 one obtains the same non-congestion condition  $a \leq bnQ$  as in the constant demand case. The reader  
319 can verify this by taking the limit  $L \rightarrow \infty$  in (23).

320 Similarly to the constant on-ramp demand case the limit of (18e) tends to the jam density as  
321  $L \rightarrow \infty$ . For Fig. 3d we have  $k_{D_2}(0) = 431 < n\kappa = 450$  veh/km.

### 322 4.3. Increasing on-ramp demand

323 Here demands are given by (8c) and (9a); i.e., on-ramp demand decreases linearly along the freeway,  
324 as an approximation for the morning (evening) commute cities where people tend to live in the  
325 downtown area (suburbs). Proceeding similarly as in previous sections, a summary of results follow.

$$k_A(t, x) = c_4 (but + (1 - bx)(1 - e^{bt_1})), \quad (24a)$$

$$k_B(x) = c_4 (bx - 1 + e^{-bx}), \quad (24b)$$

$$k_{D_1}(x) = n\kappa - c_4 (1 + b(L - x) - e^{-b(c_2 - L + x)}), \quad (24c)$$

$$k_{D_2}(x) = n\kappa - c_4 \left( e^{bL - \frac{c_5}{b\delta n} + \frac{xc_0}{\delta n}} (1 - b(L - c_2)) - e^{-\frac{(c_5 - bx)c_0 + 1}{b\delta n}} (1 - c_5) \right), \quad (24d)$$

332 where  $c_5 = \mathcal{W}\left(-\frac{1}{c_0} e^{bL - \frac{1}{c_0}} (1 - b(L - c_2))\right) + \frac{1}{c_0}$  is a dimensionless constant. As opposed to the  
333 previous two sections, the back of the queue starts in point “1” in Fig. 3g, which is located at  
334 the downstream end of the freeway segment. The time when congestion starts,  $t_1$ , is such that  
335  $k_A(t_1, L) = nk^c$ ; i.e.:

$$x_1 = L, \quad (25a)$$

$$t_1 = \frac{1}{bu} (bL - 1 - \mathcal{W}(e^{bL-1} (bL - 1 - c_2))) \quad (25b)$$

339 which in the case of Fig. 3g gives  $t_1 = 3.83$  min. Point “2” is given by

340 
$$x_2 = c_5/b, \tag{26a}$$

341 
$$t_2 = (L - x_2)/w + t_1. \tag{26b}$$

343 Unfortunately, point “0” is located in the back of the queue, and it becomes mathematically  
 344 intractable to obtain its coordinates. However, we can still find the congested conditions. In the  
 345 case of the freeway, this condition is equivalent to having  $t_1 \geq L/u$ , since  $L/u$  is the latest time for  
 346 congestion to form (after this time the freeway is in steady state). In the case of the on-ramps, we  
 347 can use the same argument as in previous sections, i.e.,  $x_2 < 0$ . Interestingly, these two conditions  
 348 are mathematically equivalent; i.e.:

349 
$$a \leq \frac{b^2 n Q L}{bL - 1 + \exp(-bL)}, \tag{27}$$
 (no freeway and on-ramp congestion)

350 which also tends to  $a \leq bnQ$  for very long freeways.

## 351 5. Discussion.

352 The continuum approximation proposed in this paper reveals considerable insights on the opera-  
 353 tion of freeway corridors, and yet it requires only two parameters,  $a$  and  $b$ . As in any continuum  
 354 approximation, one has to interpret the results with care when comparing with real-world situ-  
 355 ations. For example, one should interpret vehicular density as an indication of the likelihood to  
 356 encounter congestion at a given time-space point. It is clear that in the real world demands are far  
 357 more complicated than idealized here, yet our results give a first order approximation. And even  
 358 if real-world demands behaved as postulated here, the accuracy of the predictions would decrease  
 359 with the actual distance between ramps,  $\delta$ .

360 Of particular interest is region C, which is a transient and marks the beginning of congestion.  
 361 The on-ramps covered by this region (e.g., the ones located, roughly, in  $x_0 \leq x \leq x_1$ ) should be  
 362 interpreted as the *most likely on-ramps* to become active bottlenecks. In fact, it is commonly  
 363 observed in the field that the location of the active bottleneck in a freeway corridor varies from day  
 364 to day, but always within a relatively short segment. This can be seen in Fig. 4, which presents a  
 365 collection of density maps for 12 days from a real freeway corridor. It can be seen how the active  
 366 bottleneck changes location within the same rush hour following a rather reproducible pattern in  
 367 time-space. This is consistent with the shape of the interface between regions C and A shown in  
 368 Fig. 3, which can be either forward moving (parts c, d, e and f of the figure), backward moving  
 369 (b, g, h and i) or vertical (a). In each case one would expect that the active bottleneck would shift  
 370 location according to the shape of this interface.

371 The findings in this paper may be used to explore a new kind of ramp metering strategy. It was  
 372 found that on-ramp queues form in a well-defined portion of the freeway segment, i.e., in  $x \leq x_2$ .  
 373 Since on-ramp queues can be very deleterious (capacity drop and spill-back to city streets) one  
 374 can devise a ramp metering strategy that would spread on-ramp queues more evenly across the  
 375 entire freeway corridor. This could be done by metering the on-ramps downstream of  $x_2$  in such  
 376 a way that the freeway flow in  $x \leq x_2$  enables higher on-ramp discharge. Notice that this result is  
 377 independent of the demand profile. While other authors have pointed out that for minimizing total  
 378 cost the closer to the CBD the more restrictive metering rates should be, our finding is different:  
 379 (i) there may be a multitude of strategies (but always implemented downstream of  $x_2$ ) that may  
 380 achieve similar queue lengths distribution; and (ii) Figs. 3c and 3f show that there are instances  
 381 where there is no reason for metering near the CBD as the freeway there is in free-flow conditions.  
 382 Research in this realm continues.

## Appendix A: The Merge Model.

This appendix shows that (4) is equivalent to Daganzo’s model (Daganzo 1996), and that one needs a single additional parameter,  $\zeta$ , to modify (5d) in order to obtain experimentally observed merge ratios (Cassidy and Ahn 2005). To simplify the exposition we consider a “discrete” merge for the demonstration in the first subsection, and show how to obtain the parameter  $\zeta$  in the continuum approximation in the second subsection.

### A.1. Discrete on-ramps.

Consider Fig. 5 which shows a discrete on-ramp and its “merge diagram” describing the actual flows through the merge,  $q_i$  in approach  $i=1$  (freeway), 2 (on-ramp), given a demand vector  $(q_1^0, q_2^0)$ . A demand vector is feasible if each component is less than the approach capacity  $Q_i$ , and if it is below the total capacity line in the figure; i.e., if the total demand  $q_1 + q_2$  is less than the available capacity downstream of the merge,  $\mu_1$ . The merge ratio  $p$  is such that when both approaches are congested, each one discharges at a “minimum capacity”  $q_i^*$ ,  $i = 1, 2$ ; i.e.,  $p = q_2^*/q_1^*$  and  $q_1^* = \mu_1/(1+p)$ ,  $q_2^* = \mu_1 p/(1+p)$ . An approach is said to be congested if  $q_i^0 > q_i^*$ . Using this notation (4) can be expressed as

$$q_2 = \min\left\{1, \frac{\mu_1}{\lambda_1}\right\} \lambda_2, \quad \text{while} \quad (28a)$$

$$q_1 = \mu_1 - q_2, \quad (28b)$$

where  $\lambda_i$  is the sending functions for approach  $i = 1, 2$ . Using  $Q_1 = nQ$  and  $Q_2 = \zeta Q$  these demands can be expressed as

$$\lambda_1 = \min\{q_1^0, nQ\}, \quad (29a)$$

$$\lambda_2 = \min\{q_2^0, \zeta Q\}. \quad (29b)$$

where we have introduced the parameter  $\zeta$  for allowing calibrating the model against empirical merge ratios.

To prove that (4) is equivalent to Daganzo’s model, we note that a merge can be in one of three states: both approach in free-flow, both approaches in congestion, and one approach congested and the other one in free-flow. Next we examine each case individually.

When all approaches are in free-flow  $\mu_1 \geq \lambda_1$  and  $\lambda_i = q_i^0$ ,  $i = 1, 2$  so that (28)-(29) give  $q_2 = q_2^0$ , as expected. When all approaches are congested  $\mu_1 \leq \lambda_1 = nQ$  and  $\lambda_2 = \zeta Q$  and therefore  $q_2 = \mu_1 \zeta/n$ . Since in this case  $q_2$  should equal the prediction of Daganzo’s model  $q_2^* = \mu_1 p/(1+p)$ , it is clear that one should choose  $\zeta$  as

$$\zeta = \frac{np}{1+p}. \quad (30)$$

Now we examine the case where the freeway is congested and the on-ramp is uncongested. In this case one gets  $q_2 = \mu_1 q_2^0/(nQ)$ , which corresponds to point “2” in Fig. 5a. But this point is not stable because on-ramp demand is greater than the allocated capacity, i.e.  $q_2^0 \geq \mu_1 q_2^0/(nQ)$ . Therefore, a queue will grow in the on-ramp propagating upstream at a speed  $s_2$ ; see part b of the figure. However, as soon as the queue appears in the on-ramp, the sending function  $\lambda_2$  jumps to its maximum value  $\zeta Q$ , bringing total demand to point “3” in Fig. 5a. At this point, (28)-(29) give  $q_2 = \mu_1 \zeta/n$ ; i.e., point “4” in the figure. Again, this point is not stable because  $q_2^0 \leq \mu_1 \zeta/n$ , which means that the on-ramp back-of-queue will eventually recede at a speed  $s_4$  in Fig. 5b. As soon as this queue clears, total demand will be given by point “0” and the cycle starts over. This cycle produces a “flip-flop” between points “2” and “4” in the figure. It turns out that the long-run average between these points is point “1”, which establishes the sought equivalence. This is true because the on-ramp queue does not grow indefinitely, but goes back and forth at speeds  $s_2$  and  $s_4$ , respectively. This means that the long-run average flow discharging from the on-ramp equals its input flow,  $q_2 = q_2^0$ . Clearly,  $q_1 = \mu_1 - q_2$ . This concludes the demonstration.

### A.2. Continuous on-ramps.

Here we show how to choose  $\zeta$  in the continuum approximation. Consider a single on-ramp at  $x = 0$  and let  $x = \delta$  be the beginning of the next upstream on-ramp. Using the notation in the main text, it is now clear that when both the freeway and the on-ramp are congested then  $\phi^+(x) = q(x)\zeta/(n\delta)$ ,  $0 \leq x \leq \delta$ . In steady-state conditions the conservation equation can be expressed as the ODE:

$$\frac{dq(x)}{dx} = q(x) \frac{\zeta}{n\delta}, \quad (31a)$$

$$q(0) = \mu_1, \quad (31b)$$

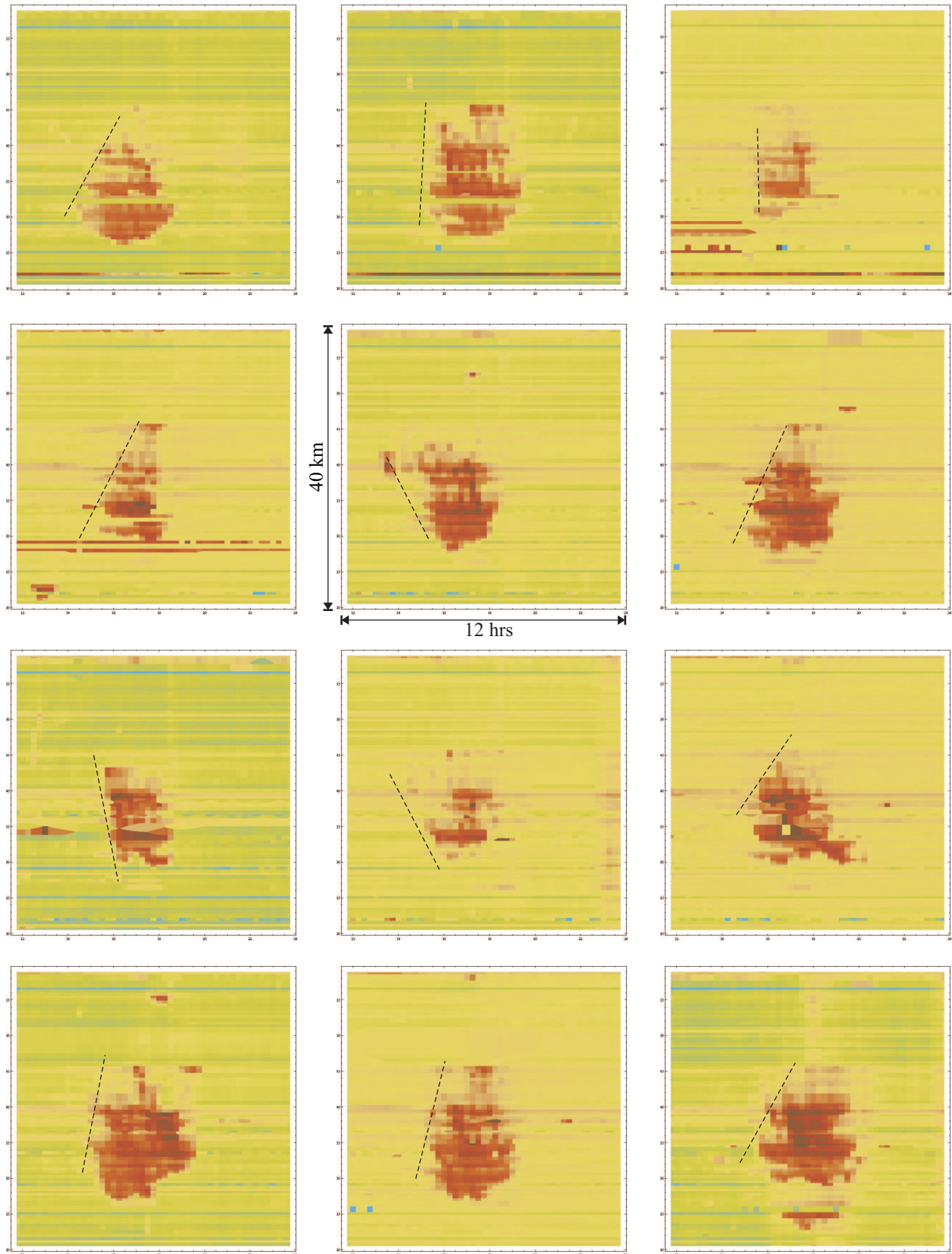
434 whose solution is  $q(x) = \mu_1 \exp(-x\zeta/(n\delta))$ . Evaluating this solution at  $x = \delta$  and combining it with Daganzo's  
435 model prediction, i.e.  $q(\delta) = \mu_1/(1+p)$ , gives the appropriate value for  $\zeta$ ,

436 
$$\zeta = n \ln(1+p). \tag{32}$$

437 Finally, it is worth noting that for simulation purposes, it is straightforward to incorporate explicitly the  
438 lengths,  $\Delta$ , of the insertion section (i.e., the freeway section where vehicles can change lanes from the on-  
439 ramp to the freeway). The only changes are (i) replacing  $\delta$  with  $\Delta$ , and (ii) set  $\phi^+ = 0$  outside the insertion  
440 sections.

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**Figure 4** Density map from a 40 km corridor on the I-285 freeway in Atlanta, Georgia. This data corresponds to the period noon-midnight, all lanes combined, eastbound direction for selected days in January 2008. The dashed lines indicate the direction of the free-flow to congestion transitions A-C in the text.

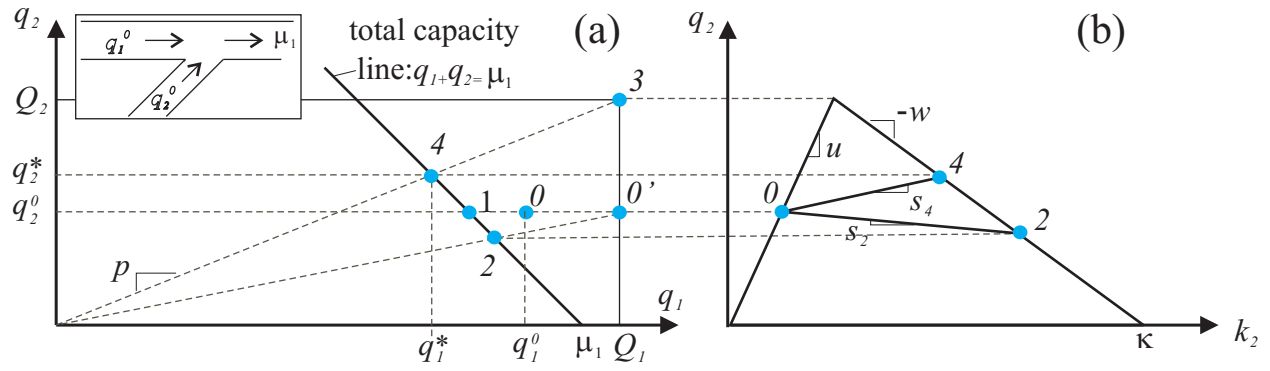


Figure 5 (a) Daganzo's merge diagram; (b) fundamental diagram on approach 2.