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# A note on upper estimates for Pickands constants

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## Abstract

*Pickands constants*  $\mathcal{H}_{B_\alpha}$  play a significant role in the extreme value theory of Gaussian processes. Recall that

$$\mathcal{H}_{B_\alpha} := \lim_{T \rightarrow \infty} \frac{\mathbb{E} \exp \left( \sup_{t \in [0, T]} (\sqrt{2} B_\alpha(t) - t^\alpha) \right)}{T},$$

where  $\{B_\alpha(t), t \geq 0\}$  is a fractional Brownian motion with Hurst parameter  $\alpha/2$  and  $\alpha \in (0, 2]$ .

In this note we derive new upper bounds for  $\mathcal{H}_{B_\alpha}$  and  $\alpha \in (1, 2]$ . The obtained results improve bounds given by Shao (1996).

**Key words:** fractional Brownian motion, Gaussian process, Pickands constants, scaled Brownian motion.

**AMS 2000 Subject Classification:** Primary 60G15, Secondary 60G70, 68M20.

## 1 Introduction

Let  $\{B_\alpha(t), t \geq 0\}$  be a fractional Brownian motion with Hurst parameter  $\frac{\alpha}{2}$ , i.e., a centered Gaussian process with covariance function  $\text{Cov}(B_\alpha(t), B_\alpha(s)) = \frac{1}{2} (t^\alpha + s^\alpha - |t - s|^\alpha)$ , where  $0 < \alpha \leq 2$ . In this note we focus on estimates of

$$\mathcal{H}_{B_\alpha} := \lim_{T \rightarrow \infty} \frac{\mathcal{H}_{B_\alpha}(T)}{T}$$

where  $\mathcal{H}_{B_\alpha}(T) = \mathbb{E} \exp \left( \sup_{t \in [0, T]} (\sqrt{2} B_\alpha(t) - t^\alpha) \right)$ .

Constants  $\mathcal{H}_{B_\alpha}$  play a significant role in the extreme value theory of Gaussian processes and in the literature are called *Pickands constants*. We refer to, e.g., Pickands (1969), Shao (1996) or the monograph of Piterbarg (1996) for the description of theories where Pickands constants appear.

Both in view of theoretical interest and recent efforts to find numerical approximations of  $\mathcal{H}_{B_\alpha}$  (see Burnecki et al., 2002), there is a need in analyzing properties of Pickands constants. In particular, despite a vast literature that deals with properties of  $\mathcal{H}_{B_\alpha}$  (e.g. Shao (1996), Dębicki et al. (2003), Dębicki (2006)), the exact value of Pickands constants is known only for  $\mathcal{H}_{B_1} = 1$  and  $\mathcal{H}_{B_2} = \frac{1}{\sqrt{\pi}}$  (see, e.g., Bräker (1993), Piterbarg (1996)). Complementary, in Shao (1996) it was shown that

$$(\alpha/4)^{1/\alpha} \left(1 - e^{-1/\alpha}(1 + 1/\alpha)\right) \leq \mathcal{H}_{B_\alpha} \leq \left(\sqrt{\alpha} \left(0.77\sqrt{\alpha} + 2.41(8.8 - \alpha \ln(0.4 + 2.5/\alpha))^{1/2}\right)\right)^{2/\alpha}$$

for  $\alpha \in (0, 1)$  and

$$5.2^{-1/\alpha} 0.625 \leq \mathcal{H}_{B_\alpha} \leq (\alpha e / \sqrt{\pi})^{2/\alpha} \quad (1)$$

for  $\alpha \in (1, 2)$ . Other lower bounds were found in Dębicki et al. (2003):

$$\frac{\alpha/2}{2^{2+\frac{2}{\alpha}} \Gamma\left(\frac{1}{\alpha}\right)} \leq \mathcal{H}_{B_\alpha}$$

for  $0 < \alpha \leq 2$ .

In this note we derive new upper bounds for  $\mathcal{H}_{B_\alpha}$  when  $\alpha \in (1, 2)$ , which improve (1). The proofs rely on an application of properties of *generalized Pickands constants* introduced in Dębicki (2002).

Throughout the paper  $\Phi(\cdot)$  denotes distribution function of the standard normal random variable. Let  $\Psi(\cdot) = 1 - \Phi(\cdot)$ .

## 2 Generalized Pickands constants and $\mathcal{H}_{B_\alpha}(T)$

The idea of getting upper bounds of  $\mathcal{H}_{B_\alpha}$  is based on an appropriate use of the results on comparison of *generalized Pickands constants* developed in Dębicki (2002). Below we recall the definition and some useful properties of generalized Pickands constants.

Following Dębicki (2002), let  $\eta(\cdot)$  be a centered Gaussian process with stationary increments, a.s. continuous sample paths,  $\eta(0) = 0$  and with variance function  $\sigma_\eta^2(t) = \text{Var}(\eta(t))$  that satisfies

**C1**  $\sigma_\eta^2(t) \in C^1([0, \infty))$  is strictly increasing and there exist  $\varepsilon > 0$  such that

$$\limsup_{t \rightarrow \infty} \frac{t\sigma_\eta^2(t)}{\sigma_\eta^2(t)} \leq \varepsilon;$$

**C2**  $\sigma_\eta^2(t)$  is regularly varying at 0 with index  $\alpha_0 \in (0, 2]$  and  $\sigma_\eta^2(t)$  is regularly varying at  $\infty$  with index  $\alpha_\infty \in (0, 2)$ .

Let

$$\mathcal{H}_\eta := \lim_{T \rightarrow \infty} \frac{\mathcal{H}_\eta(T)}{T},$$

where  $\mathcal{H}_\eta(T) = \mathbb{E} \exp \left( \sup_{t \in [0, T]} (\sqrt{2}\eta(t) - \sigma_\eta^2(t)) \right)$ . The existence and properties of  $\mathcal{H}_\eta$  are analyzed in Dębicki (2002) and Dębicki (2006). Following Dębicki (2002) we call  $\mathcal{H}_\eta$  *generalized Pickands constant*. Recall some useful properties of generalized Pickands constants (for the proofs we refer to Dębicki, 2002):

**Lemma 2.1** *Let  $\eta_1(t), \eta_2(t)$  be centered Gaussian processes with stationary increments that satisfy C1-C2.*

- 1) *Function  $\mathcal{H}_{\eta_1}(\cdot)$  is subadditive;*
- 2) *If  $\sigma_{\eta_1}^2(t) \leq \sigma_{\eta_2}^2(t)$  for all  $t \geq 0$ , then  $\mathcal{H}_{\eta_1} \leq \mathcal{H}_{\eta_2}$ .*

The following expressions for  $\mathcal{H}_{aB_1}(T)$  and  $\mathcal{H}_{aB_2}(T)$  play an important role in further analysis.

**Lemma 2.2** *For each  $T > 0$  and  $a > 0$*

$$\mathcal{H}_{aB_1}(T) = 2\Phi \left( a\sqrt{\frac{T}{2}} \right) + \sqrt{\frac{a^2 T}{\pi}} \exp \left( -\frac{a^2 T}{4} \right) + a^2 T \Phi \left( \sqrt{a^2 T/2} \right).$$

**Proof** Using self-similarity of Brownian motion, we have

$$\begin{aligned} \mathcal{H}_{aB_1}(T) &= \mathbb{E} \exp \left( \sup_{t \in [0, T]} \left( \sqrt{2}aB_1(t) - a^2 t \right) \right) = \\ &= \mathbb{E} \exp \left( \sup_{t \in [0, 2a^2 T]} \left( B_1(t) - t/2 \right) \right), \end{aligned}$$

which combined with the fact that (see, e.g., Baxter & Donsker, 1957)

$$\begin{aligned} \rho(x) &:= \frac{d}{dx} \mathbb{P} \left( \sup_{t \in (0, 2a^2 T)} \{B_1(t) - t/2\} \leq x \right) \\ &= \sqrt{\frac{1}{\pi a^2 T}} \exp \left( -\frac{(x + a^2 T)^2}{4a^2 T} \right) + e^{-x} \Psi \left( \frac{a^2 T - x}{\sqrt{2a^2 T}} \right), \end{aligned}$$

leads to the following chain of equations

$$\begin{aligned}
& \mathbb{E} \exp \left( \sup_{t \in [0, 2a^2T]} \left( B_1(t) - \frac{1}{2}t \right) \right) = \\
&= \int_0^\infty \sqrt{\frac{1}{\pi a^2 T}} \exp \left( -\frac{(x - a^2 T)^2}{4a^2 T} \right) + \Psi \left( \frac{a^2 T - x}{\sqrt{2a^2 T}} \right) dx = \\
&= 2\Phi \left( a\sqrt{\frac{T}{2}} \right) + \int_0^\infty \int_{-\infty}^{\frac{a^2 T - x}{\sqrt{2a^2 T}}} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{y^2}{2} \right) dy dx = \\
&= 2\Phi \left( a\sqrt{\frac{T}{2}} \right) + \int_{-\infty}^{a\sqrt{\frac{T}{2}}} \int_0^{-y\sqrt{2a^2 T} + a^2 T} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{y^2}{2} \right) dx dy = \\
&= 2\Phi \left( a\sqrt{\frac{T}{2}} \right) + \sqrt{\frac{a^2 T}{\pi}} \exp \left( -\frac{a^2 T}{4} \right) + a^2 T \Phi \left( \sqrt{a^2 T/2} \right).
\end{aligned}$$

This completes the proof.  $\square$

**Lemma 2.3** For each  $T > 0$  and  $a > 0$

$$\mathcal{H}_{aB_2}(T) = 1 + T \frac{a}{\sqrt{\pi}}.$$

**Proof** Since  $B_2(t) = t\mathcal{N}$ , where  $\mathcal{N}$  is a standard normal random variable, then

$$\begin{aligned}
\mathcal{H}_{aB_2}(T) &= \mathbb{E} \exp \left( \sup_{t \in [0, \sqrt{2}aT]} \left( \mathcal{N}t - \frac{1}{2}t^2 \right) \right) = \\
&= \int_{-\infty}^0 \exp(0) \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right) dx + \int_0^{\sqrt{2}aT} \exp \left( \frac{x^2}{2} \right) \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right) dx \\
&\quad + \int_{\sqrt{2}aT}^\infty \exp \left( T\sqrt{2}ax - a^2 T^2 \right) \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right) dx \\
&= 1 + T \frac{a}{\sqrt{\pi}}.
\end{aligned}$$

This completes the proof.  $\square$

### 3 Main results

In this section we give two upper bounds for  $\mathcal{H}_{B_\alpha}$  (Theorems 3.1 and 3.2) and compare them with bounds obtained by Shao (1996); (1). The estimate given in Theorem 3.1 uniformly improves (1) and gives right values for  $\alpha = 1$  and  $\alpha = 2$ . Theorem 3.2 provides sharper bounds for  $\alpha$  in the neighborhood of 1 (see Picture 1).

**Theorem 3.1** *Let  $\alpha \in (1, 2)$ . Then*

$$\mathcal{H}_{B_\alpha} \leq \inf_{T>0} \frac{\left(1 + T\sqrt{\frac{\alpha-1}{\pi}}\right) \left(2\Phi\left(\sqrt{\frac{(2-\alpha)T}{2}}\right) + \sqrt{\frac{(2-\alpha)T}{\pi}}e^{-\frac{(2-\alpha)T}{4}} + (2-\alpha)T\Phi\left(\sqrt{\frac{(2-\alpha)T}{2}}\right)\right)}{T}.$$

**Proof** Let  $\alpha \in (1, 2)$  and  $\eta(t) = \sqrt{2-\alpha}B_1(t) + \sqrt{\alpha-1}B_2(t)$ , where  $B_1(\cdot), B_2(\cdot)$  are independent fractional Brownian motions.

The proof consists on two steps.

Step 1. We show that

$$\mathcal{H}_{B_\alpha} \leq \mathcal{H}_\eta. \quad (2)$$

Observe that for each  $t \geq 0$  we have

$$\sigma_{B_\alpha}^2(t) \leq \sigma_\eta^2(t).$$

Indeed, the above is equivalent to the fact that  $f(t) = (2-\alpha) + (\alpha-1)t - t^{\alpha-1} \geq 0$  for each  $t \geq 0$  which follows from the observation that  $f(\cdot)$  attains its minimum (over  $t \geq 0$ ) at  $t^* = 1$  and  $f(1) = 0$ . Hence from Lemma 2.1 we get (2).

Step 2. From subadditivity of generalized Pickands constants (Lemma 2.1) we have

$$\mathcal{H}_\eta \leq \inf_{T>0} \mathcal{H}_\eta(T)/T. \quad (3)$$

Additionally

$$\begin{aligned} \mathcal{H}_\eta(T) &= \\ &= \mathbb{E} \exp \left( \sup_{t \in [0, T]} \left( \sqrt{2} \left( \sqrt{2-\alpha}B_1(t) + \sqrt{\alpha-1}B_2(t) \right) - (2-\alpha)t - (\alpha-1)t^2 \right) \right) \\ &\leq \mathbb{E} \exp \left( \sup_{t \in [0, T]} \sqrt{4-2\alpha}B_1(t) - (2-\alpha)t \right) \exp \left( \sup_{t \in [0, T]} \sqrt{2\alpha-2}B_2(t) - (\alpha-1)t^2 \right) \\ &= \mathcal{H}_{\sqrt{2-\alpha}B_1}(T) \mathcal{H}_{\sqrt{\alpha-1}B_2}(T). \end{aligned}$$

We complete the proof combining this with Lemmas 2.2, 2.3.  $\square$

As an corollary to Theorem 3.1 we can get the following, more explicit bound for  $\mathcal{H}_{B_\alpha}$ .

**Corollary 3.1** *Let  $\alpha \in (1, 2)$ . Then*

$$\mathcal{H}_{B_\alpha} \leq \frac{\left(1 + \sqrt{\frac{2 + \sqrt{\frac{2}{\pi e}} \sqrt{\alpha-1}}{(2-\alpha)\sqrt{\pi}}}\right) \left(2 + \sqrt{\frac{2}{\pi e}} + \sqrt{\frac{2\sqrt{\pi} + \sqrt{\frac{2}{e}}}{\sqrt{\alpha-1}}} \sqrt{2-\alpha}\right)}{\sqrt{\frac{2\sqrt{\pi} + \sqrt{\frac{2}{e}}}{(2-\alpha)\sqrt{\alpha-1}}}.$$

**Proof** Due to Theorem 3.1

$$\begin{aligned} \mathcal{H}_{B_\alpha} &\leq \inf_{T>0} \left( 1 + T\sqrt{\frac{\alpha-1}{\pi}} \right) \times \\ &\quad \times \frac{2\Phi\left(\sqrt{\frac{(2-\alpha)T}{2}}\right) + \sqrt{\frac{(2-\alpha)T}{\pi}} \exp\left(-\frac{(2-\alpha)T}{4}\right) + (2-\alpha)T\Phi\left(\sqrt{(2-\alpha)T/2}\right)}{T}. \end{aligned}$$

Using that  $\Phi\left(\sqrt{\frac{(2-\alpha)T}{2}}\right) \leq 1$  and  $\sqrt{\frac{(2-\alpha)T}{\pi}} \exp\left(-\frac{(2-\alpha)T}{4}\right) \leq \sqrt{\frac{2}{\pi e}}$ , we obtain

$$\begin{aligned} \mathcal{H}_{B_\alpha} &\leq \inf_{T>0} \frac{\left(1 + T\sqrt{\frac{\alpha-1}{\pi}}\right) \left(2 + \sqrt{\frac{2}{\pi e}} + T(2-\alpha)\right)}{T} \\ &= \frac{\left(1 + T^*\sqrt{\frac{\alpha-1}{\pi}}\right) \left(2 + \sqrt{\frac{2}{\pi e}} + T^*(2-\alpha)\right)}{T^*}, \end{aligned}$$

where

$$T^* = \sqrt{\frac{2\sqrt{\pi} + \sqrt{\frac{2}{e}}}{(2-\alpha)\sqrt{\alpha-1}}}.$$

This completes the proof.  $\square$

**Theorem 3.2** *Let  $\alpha \in (1, 2)$ . Then*

$$\mathcal{H}_{B_\alpha} \leq \alpha \left( \frac{2 + \sqrt{\frac{2}{\pi e}}}{\alpha - 1} \right)^{1 - \frac{1}{\alpha}}.$$

**Proof** From subadditivity of  $\mathcal{H}_{B_\alpha}(T)$  (Lemma 2.1) we have

$$\mathcal{H}_{B_\alpha} \leq \inf_{T \geq 0} \frac{\mathcal{H}_{B_\alpha}(T)}{T}. \quad (4)$$

Let  $S_\alpha(t) := B_1(t^\alpha)$  for  $t \geq 0$ . Observe that for  $\alpha \in (1, 2)$  and each  $s, t \geq 0$

$$\text{Cov}(S_\alpha(t), S_\alpha(s)) = s^\alpha \wedge t^\alpha \leq \frac{1}{2}(s^\alpha + t^\alpha - |t - s|^\alpha) = \text{Cov}(B_\alpha(t), B_\alpha(s)),$$

while  $\text{Var}(S_\alpha(t)) = \text{Var}(B_\alpha(t))$ . Thus due to Slepian inequality (see, e.g., Theorem C.1 in Piterbarg, 1996) for each  $u \geq 0$

$$\mathbb{P}\left(\sup_{t \in [0, T]} \left(\sqrt{2}B_\alpha(t) - t^\alpha\right) > u\right) \leq \mathbb{P}\left(\sup_{t \in [0, T]} \left(\sqrt{2}S_\alpha(t) - t^\alpha\right) > u\right).$$



The above implies that

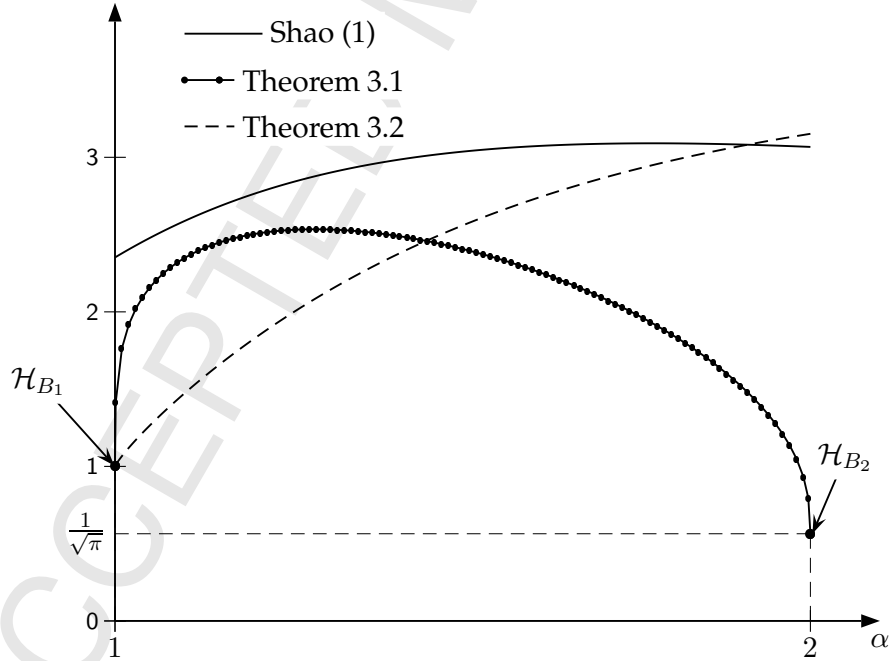
$$\begin{aligned}
\mathcal{H}_{B_\alpha}(T) &= \mathbb{E} \exp \left( \sup_{t \in [0, T]} \left( \sqrt{2} B_\alpha(t) - t^\alpha \right) \right) = \\
&\leq \mathbb{E} \exp \left( \sup_{t \in [0, T]} \left( \sqrt{2} B_1(t^\alpha) - t^\alpha \right) \right) \\
&= \mathbb{E} \exp \left( \sup_{t \in [0, T^\alpha]} \left( \sqrt{2} B_1(t) - t \right) \right) \leq 2 + \sqrt{\frac{2}{\pi e}} + T^\alpha,
\end{aligned} \tag{5}$$

where (5) follows from Lemma 2.2. Combination of (5) with (4) and the fact that

$$\arg \inf_{t \geq 0} \frac{2 + \sqrt{\frac{2}{\pi e}} + T^\alpha}{T} = \left( \frac{2 + \sqrt{\frac{2}{\pi e}}}{\alpha - 1} \right)^{\frac{1}{\alpha}}$$

completes the proof.  $\square$

In Picture 1 we compare upper bound (1) with estimates given in Theorem 3.1 and 3.2.



Picture 1

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