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# On hitting times of the winding processes of planar Brownian motion and of Ornstein-Uhlenbeck processes, via Bougerol's identity.

S. VAKEROUDIS<sup>1 2</sup>

## Abstract

We develop some identities in law in terms of planar complex valued Ornstein-Uhlenbeck processes ( $Z_t = X_t + iY_t, t \geq 0$ ) including planar Brownian motion, which are equivalent to the well known Bougerol identity for linear Brownian motion ( $\beta_t, t \geq 0$ ): for any fixed  $u > 0$ :

$$\sinh(\beta_u) \stackrel{(law)}{=} \hat{\beta}_{(\int_0^u ds \exp(2\beta_s))}$$

These identities in law for 2-dimensional processes allow us to study the distribution of hitting times  $T_c^\theta \equiv \inf\{t : \theta_t = c\}$ , ( $c > 0$ ),  $T_{-d,c}^\theta \equiv \inf\{t : \theta_t \notin (-d, c)\}$ , ( $c, d > 0$ ) and more specifically of  $T_{-c,c}^\theta \equiv \inf\{t : \theta_t \notin (-c, c)\}$ , ( $c > 0$ ) of the continuous winding processes  $\theta_t = \text{Im}(\int_0^t \frac{dZ_s}{Z_s})$ ,  $t \geq 0$  of complex Ornstein-Uhlenbeck processes.

**Key words:** Planar Brownian motion, Ornstein-Uhlenbeck process, winding process, Bougerol's identity, exit time from a cone.

## 1 Introduction

The conformal invariance of planar Brownian motion has deep consequences as to the structure of its trajectories (see, e.g., Le Gall [LeG90]). In particular, a number of articles have been devoted to the study of its continuous winding process ( $\theta_t, t \geq 0$ ): Spitzer [Spi58], Williams [Wil74], Durrett

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[Dur82], Messulam-Yor [MeY82], Pitman-Yor [PiY86], Le Gall-Yor [LeGY87], Bertoin-Werner [BeW94], Yor [Yor97], Pap-Yor [PaY00], Bentkus-Pap-Yor [BPY03]. In this paper, we take up again the study of the first hitting times:

$$T_{-d,c}^\theta \equiv \inf\{t : \theta_t \notin (-d, c)\}, \quad (c, d > 0)$$

this time in relation with Bougerol's well-known identity: for fixed  $u > 0$ :

$$\sinh(\beta_u) \stackrel{(law)}{=} \hat{\beta}_{(\int_0^u ds \exp(2\beta_s))},$$

where  $(\hat{\beta}_t, t \geq 0)$  is a Brownian motion<sup>†</sup>, independent of  $\beta$ .

In particular, it turns out that: for fixed  $c > 0$ :

$$\theta_{T_c^{\hat{\beta}}} \stackrel{(law)}{=} C_{a(c)} \quad (\star)$$

where  $\hat{\beta}$  is a BM<sup>†</sup> independent of  $(\theta_u, u \geq 0)$ ,  $T_c^{\hat{\beta}} = \inf\{t : \hat{\beta}_t = c\}$ ,  $(C_t, t \geq 0)$  is a standard Cauchy process and  $a(c) = \arg \sinh(c) \equiv \log(c + \sqrt{1 + c^2})$ ,  $c \in \mathbb{R}$ .

The identity  $(\star)$  yields yet another proof of the celebrated Spitzer theorem:

$$\frac{2}{\log t} \theta_t \xrightarrow[t \rightarrow \infty]{(law)} C_1,$$

with the help of Williams' "pinching method" (see Williams [Wil74] and Messulam-Yor [MeY82]).

Moreover, we study the distributions of  $T_{-\infty,c}^\theta$  and  $T_{-c,c}^\theta$ . In particular, we give an explicit formula for the density function of  $T_{-c,c}^\theta$  and for the first moment of  $\ln(T_{-c,c}^\theta)$ .

The last section of the paper is devoted to developing similar results when planar Brownian motion is replaced by a complex valued Ornstein-Uhlenbeck process. We note that Bertoin-Werner [BeW94] already made discussions of windings for planar Brownian motion using arguments related to Ornstein-Uhlenbeck processes.

Firstly, we obtain some analogue of  $(\star)$  when  $T_c^{\hat{\beta}}$  is replaced by  $T_c^{(\lambda)} = T_c^\theta = \inf\{t : |\theta_t^Z| = c\}$ , the corresponding time for an Ornstein-Uhlenbeck process with parameter  $\lambda$ . Secondly, we exhibit the distribution of  $T_c^{(\lambda)}$ . More specifically, we derive the asymptotics of  $E[T_c^{(\lambda)}]$  for  $\lambda$  large and for  $\lambda$  small.

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<sup>†</sup>When we simply write: Brownian motion, we always mean real-valued Brownian motion, starting from 0. For 2-dimensional Brownian motion, we indicate planar or complex BM.

## 2 The Brownian motion case

### 2.1 A reminder on planar Brownian motion

Let  $(Z_t = X_t + iY_t, t \geq 0)$  denote a standard planar Brownian motion, starting from  $x_0 + i0, x_0 > 0$ , where  $(X_t, t \geq 0)$  and  $(Y_t, t \geq 0)$  are two independent linear Brownian motions, starting respectively from  $x_0$  and 0.

As is well known (see e.g. Itô-McKean [ItMK65]), since  $x_0 \neq 0$ ,  $(Z_t, t \geq 0)$  does not visit a.s. the point 0 but keeps winding around 0 infinitely often. In particular, the continuous winding process  $\theta_t = \text{Im}(\int_0^t \frac{dZ_s}{Z_s}), t \geq 0$  is well defined.

Furthermore, there is the skew product representation:

$$\log |Z_t| + i\theta_t \equiv \int_0^t \frac{dZ_s}{Z_s} = (\beta_u + i\gamma_u) \Big|_{u=H_t}^t = \int_0^t \frac{ds}{|Z_s|^2} \quad (1)$$

where  $(\beta_u + i\gamma_u, u \geq 0)$  is another planar Brownian motion starting from  $\log x_0 + i0$ .

Rewriting (1) as:

$$\log |Z_t| = \beta_{H_t}; \quad \theta_t = \gamma_{H_t} \quad (2)$$

we easily obtain that the  $\sigma$ -fields  $\sigma\{|Z_t|, t \geq 0\}$  and  $\sigma\{\beta_u, u \geq 0\}$  are identical, whereas  $(\gamma_u, u \geq 0)$  is independent from  $(|Z_t|, t \geq 0)$ .

A number of studies of the properties of the first hitting time (see Figure 2)

$$T_{-d,c}^\theta \equiv \inf\{t : \theta_t \notin (-d, c)\}, \quad (c, d > 0)$$

have been developed, going back to Spitzer [Spi58].

In particular, it is well known (Spitzer [Spi58], Burkholder [Bur77], Revuz-Yor [ReY99] Ex. 2.21/page 196) that:

$$E \left[ (T_{-d,c}^\theta)^p \right] < \infty \quad \text{if and only if} \quad p < \frac{\pi}{2(c+d)}. \quad (3)$$

Moreover, Spitzer's asymptotic theorem (see e.g. Spitzer [Spi58]) states that:

$$\frac{2\theta_t}{\log t} \xrightarrow[t \rightarrow \infty]{(law)} C_1 \stackrel{(law)}{=} \gamma_{T_1^\beta}, \quad (4)$$

where  $C_1$  is a standard Cauchy variable.

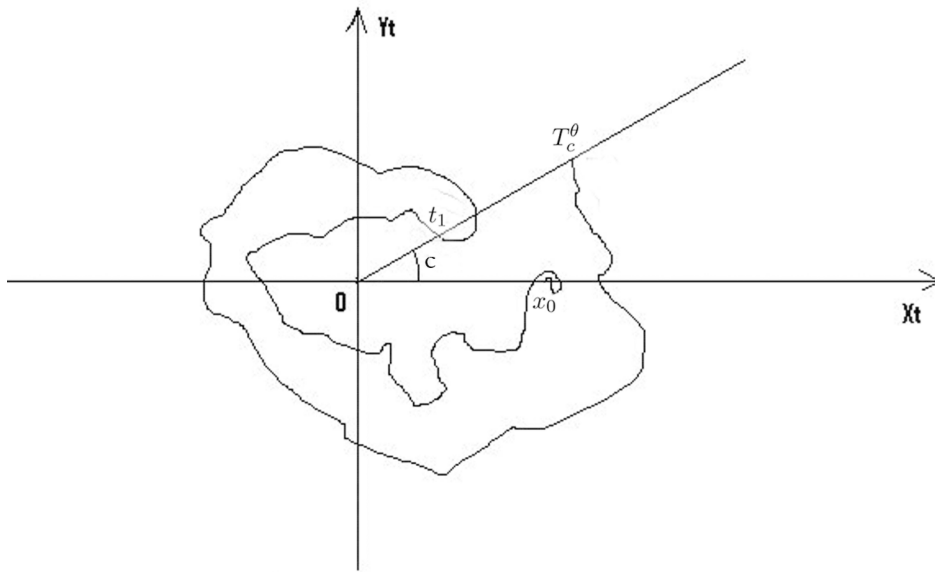


Figure 1: Diagram for  $T_c^\theta$ . The planar BM starts from  $x_0 + i0$ .  
 $t_1$  does not count because the angle is negative.

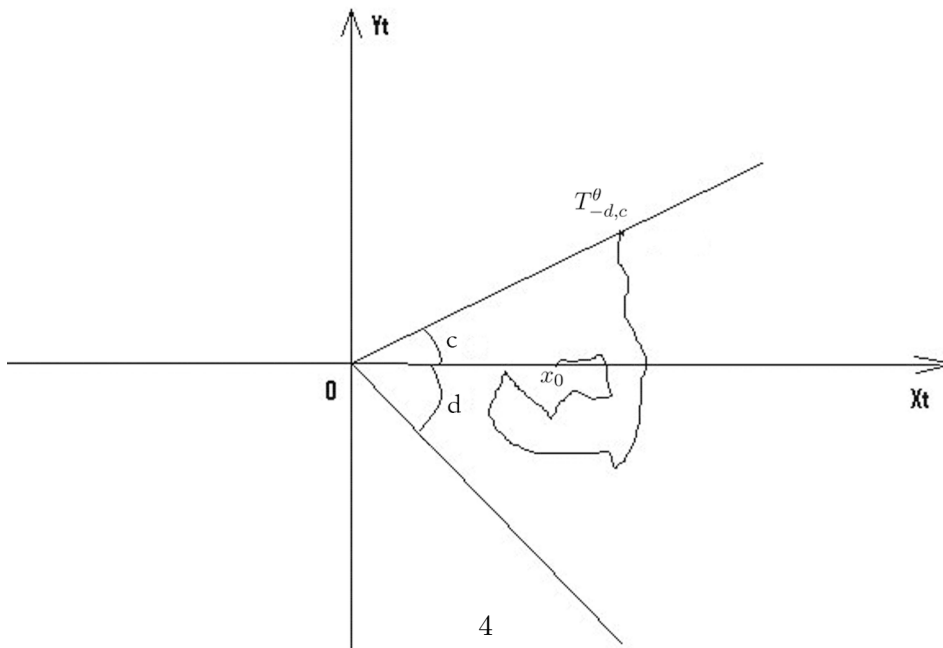


Figure 2: Diagram for  $T_{-d,c}^\theta$ . The planar BM starts from  $x_0 + i0$ .

## 2.2 On the Laplace transform of the distribution of the hitting time $T_c^\theta \equiv T_{-\infty, c}^\theta$

Now, we use the representation (2) to access the distribution of  $T_c^\theta$  (see Figure 1). We define  $T_c^\gamma \equiv \inf\{t : \gamma_t \notin (-\infty, c)\}$  the hitting time associated to the Brownian motion  $(\gamma_t, t \geq 0)$ . Note that:

from (2),  $H_{T_c^\theta} = T_c^\gamma$ , hence  $T_c^\theta = H_u^{-1} \Big|_{u=T_c^\gamma}$ , where

$$H_u^{-1} \equiv \inf\{t : H_t > u\} = \int_0^u ds \exp(2\beta_s) := A_u \quad (5)$$

from (2). Thus, we have obtained:

$$T_c^\theta = A_{T_c^\gamma} \quad (6)$$

where  $(A_u, u \geq 0)$  and  $T_c^\gamma$  are independent, since  $\beta$  and  $\gamma$  are independent. We can write:  $\beta_s = (\log x_0) + \beta_s^{(0)}$ , with  $(\beta_s^{(0)}, s \geq 0)$  a standard one-dimensional Brownian motion starting from 0. Then, we deduce from (6) that:

$$T_c^\theta = x_0^2 \left( \int_0^{T_c^\gamma} ds \exp(2\beta_s^{(0)}) \right). \quad (7)$$

From now on, for simplicity, we shall take  $x_0 = 1$ , but this is really no restriction, as the dependency in  $x_0$ , which is exhibited in (7), is very simple. We shall also make use of Bougerol's identity (Bougerol [Bou83], see also Alili-Dufresne-Yor [ADY97] and Yor [Yor01], p. 200) which is very useful to study the distribution of  $A_u$ . For any fixed  $u > 0$ :

$$\sinh(\beta_u) \stackrel{(law)}{=} \hat{\beta}_{A_u} = \hat{\beta}_{(\int_0^u ds \exp(2\beta_s))} \quad (8)$$

where on the right hand side,  $(\hat{\beta}_t, t \geq 0)$  is a Brownian motion, independent of  $A_u \equiv \int_0^u ds \exp(2\beta_s)$ .

Thus, from (8) and (6), and as it is well known (see e.g. Revuz-Yor [ReY99]) that the law of  $\beta_{T_c^\gamma}$  is the Cauchy law with parameter  $c$ , i.e., with density:

$$h_c(y) = \frac{c}{\pi(c^2 + y^2)}.$$

we deduce that:

**Proposition 2.1** *For fixed  $c > 0$ , there is the following identity in law:*

$$\sinh(C_c) \stackrel{(law)}{=} \hat{\beta}_{(T_c^\theta)}. \quad (9)$$

where, on the left hand side,  $(C_c, c \geq 0)$  denotes a standard Cauchy process and on the right hand side,  $(\beta_u, u \geq 0)$  is a one-dimensional BM, independent from  $T_c^\theta$ .

We may now identify the densities of the variables found on both sides of (9), i.e.:

on the left hand side:  $\frac{1}{\sqrt{1+x^2}} h_c(\arg \sinh x) = \frac{1}{\sqrt{1+x^2}} h_c(a(x))$  ;

on the right hand side:  $E \left[ \frac{1}{\sqrt{2\pi T_c^\theta}} \exp \left( -\frac{x^2}{2T_c^\theta} \right) \right]$

where  $a(x) = \arg \sinh(x)$

Thus, we have obtained the following:

**Proposition 2.2** *The distribution of  $T_c^\theta$  is characterized by:*

$$E \left[ \frac{1}{\sqrt{2\pi T_c^\theta}} \exp \left( -\frac{x}{2T_c^\theta} \right) \right] = \frac{1}{\sqrt{1+x}} \frac{c}{\pi(c^2 + \log^2(\sqrt{x} + \sqrt{1+x}))}, \quad x \geq 0. \quad (10)$$

The proof of Proposition 2.2 follows from:  $a(y) = \arg \sinh(y) \equiv \log(y + \sqrt{1+y^2})$  and by making the change of variable  $y^2 = x$ . Let us now define the probability:

$$Q_c = \sqrt{\frac{\pi}{2T_c^\theta}} c \cdot P$$

The fact that  $Q_c$  is a probability follows from (10) by taking  $x = 0$ . Thus we obtain that  $c E[\sqrt{\pi/2T_c^\theta}] = 1$ , and we may write:

$$E_{Q_c} \left[ \exp \left( -\frac{x^2}{2T_c^\theta} \right) \right] = \frac{1}{\sqrt{1+x^2}} \frac{1}{1 + \frac{1}{c^2} \log^2(x + \sqrt{1+x^2})}, \quad \forall x \geq 0 \quad (11)$$

which yields the Laplace transform of  $1/T_c^\theta$  under  $Q_c$ .

Let us now take a look at what happens if we make  $c \rightarrow \infty$ . If we denote by  $T_1^\beta \equiv \inf\{t : \beta_t = 1\}$  the first hitting time of level 1 for a standard BM  $\beta$  and by  $N$  a standard Gaussian variable  $\mathcal{N}(0, 1)$ , from equation (11), we obtain:

$$\lim_{c \rightarrow \infty} E_{Q_c} \left[ e^{-x/2T_c^\theta} \right] = E \left( e^{-xN^2/2} \right) = E \left( e^{-x/2T_1^\beta} \right) \quad (12)$$

which means that :  $T_c^\theta \xrightarrow[c \rightarrow \infty]{(law)} T_1^\beta$ . (At this point, one may wonder whether there is some kind of convergence in law involving  $(\theta_u, u \geq 0)$ , under  $Q_c$ , as  $c \rightarrow \infty$ , but, we shall not touch this point).

From Proposition 2.2 we deduce the following:

**Corollary 2.3** Let  $\varphi(x)$  denote the Laplace transform of Proposition 2.2 i.e. the expression on the right hand side of (10), which is the Laplace transform of  $1/2T_c^\theta$  under  $Q_c$ . Then, the Laplace transform of  $1/2T_c^\theta$  under  $P$  is:

$$E \left[ \exp \left( -\frac{x}{2T_c^\theta} \right) \right] = \int_x^\infty \frac{dw}{\sqrt{w-x}} \varphi(w) \quad (13)$$

**Proof of Corollary 2.3** From Fubini's theorem, we deduce from (10) that:

$$\begin{aligned} E \left[ \exp \left( -\frac{x}{2T_c^\theta} \right) \right] &= \int_0^\infty \frac{dy}{\sqrt{y}} E \left[ \frac{1}{\sqrt{2\pi T_c^\theta}} \exp \left( -\frac{x+y}{2T_c^\theta} \right) \right] \\ &= \int_0^\infty \frac{dy}{\sqrt{y}} \varphi(x+y), \quad y = xt \\ &= \sqrt{x} \int_0^\infty \frac{dt}{\sqrt{t}} \varphi(x(1+t)), \quad v = 1+t \\ &= \sqrt{x} \int_1^\infty \frac{dv}{\sqrt{v-1}} \varphi(xv), \quad xv = w \end{aligned}$$

which finally yields formula (13). □

### 2.3 Some related identities in law

This subsection is strongly related to Dufresne-Yor [DuY10].

A slightly different look at the combination of Bougerol's identity (8) and the skew-product representation (1) leads to the following striking identities in law:

**Proposition 2.4** Let  $(\delta_u, u \geq 0)$  be a 1-dimensional Brownian motion independent of the planar Brownian motion  $(Z_u, u \geq 0)$ , starting from 1. Then, the following identities in law hold:

$$(i) H_{T_b^\delta} \stackrel{(law)}{=} T_{a(b)}^\beta \quad (ii) \theta_{T_b^\delta} \stackrel{(law)}{=} C_{a(b)} \quad (iii) \bar{\theta}_{T_b^\delta} \stackrel{(law)}{=} |C_{a(b)}|$$

where  $C_A$  is a Cauchy variable with parameter  $A$  and  $\bar{\theta}_u = \sup_{s \leq u} \theta_s$ .

**Proof of Proposition 2.4** From the symmetry principle (see the original Note: André [And87] and e.g.: Gallardo [Gal08] for a detailed discussion), Bougerol's identity may be equivalently stated as:

$$\sinh(\bar{\beta}_u) \stackrel{(law)}{=} \bar{\delta}_{A_u(\beta)} \quad (14)$$



Consequently, the laws of the first hitting times of a fixed level  $b$  by the processes on each side of (14) are identical, that is:

$$T_{a(b)}^\beta \stackrel{(law)}{=} H_{T_b^\delta}$$

which is (i).

(ii) follows from (i) since:

$$\theta_u \stackrel{(law)}{=} \gamma_{H_u}$$

with  $(\gamma_s, s \geq 0)$  a Brownian motion independent of  $(H_u, u \geq 0)$  and  $(C_u, u \geq 0)$  may be represented as  $(\gamma_{T_u^\beta}, u \geq 0)$ .

(iii) follows from (ii), again with the help of the symmetry principle. □

We now remark that Proposition 2.2 may be derived from (iii) in Proposition 2.4. Indeed, for  $c > 0$ , we have from (iii) (let  $N \sim \mathcal{N}(0, 1)$  with  $N$  independent from  $T_c^\theta$ ):

$$\begin{aligned} P\left(\bar{\theta}_{T_b^\delta} < c\right) &= P\left(T_b^\delta < T_c^\theta\right) = P\left(b < \bar{\delta}_{T_c^\theta}\right) \\ &= P\left(b < \sqrt{T_c^\theta}|N|\right) \\ &= P\left(\frac{b}{\sqrt{T_c^\theta}} < |N|\right) \\ &= \sqrt{\frac{2}{\pi}}E\left[\int_{\frac{b}{\sqrt{T_c^\theta}}}^{\infty} dy e^{-y^2/2}\right] \end{aligned}$$

and by changing variables :  $y^2 = x/T_c^\theta$ , we obtain Proposition 2.2.

## 2.4 Recovering Spitzer's theorem

The identity (ii) in Proposition 2.4 is reminiscent of Williams' remark (see Williams [Wil74] and Messulam-Yor [MeY82]), that:

$$H_{T_r^R} \stackrel{(law)}{=} T_{\log r}^\delta \tag{15}$$

where here  $R$  starts from 1 and  $\delta$  starts from 0 (in fact, this is a consequence of (2) ). For a number of variants of (15), see Yor [Yor85] and Mansuy-Yor

[MaY08]. This was D. Williams' starting point for a non-computational proof of Spitzer's result (4). We note that in (ii),  $T_b^\delta$  is independent of the process  $(\theta_u, u \geq 0)$  while in (15)  $T_r^R$  depends on  $(\theta_u, u \geq 0)$ . Actually, we can mimic Williams' "pinching method" to derive Spitzer's theorem (4) from (ii) in Proposition 2.4.

**Proposition 2.5 (A new proof of Spitzer's theorem)**

As  $t \rightarrow \infty$ ,  $\theta_{T_{\sqrt{t}}^\delta} - \theta_t$  converges in law, which implies that:

$$\frac{1}{\log t} \left( \theta_{T_{\sqrt{t}}^\delta} - \theta_t \right) \xrightarrow[t \rightarrow \infty]{(P)} 0 \quad (16)$$

which, in turn, implies Spitzer's theorem (see formula (4)):

$$\frac{2}{\log t} \theta_t \xrightarrow[t \rightarrow \infty]{(law)} C_1$$

**Proof of Proposition 2.5** From equation (ii) of Proposition 2.4 we note:

$$\frac{1}{\log b} \theta_{T_b^\delta} \stackrel{(law)}{=} \frac{C_{a(b)}}{\log b} \xrightarrow[b \rightarrow \infty]{(law)} C_1$$

So, for  $b = \sqrt{t}$  we have:

$$\frac{2}{\log t} \theta_{T_{\sqrt{t}}^\delta} \xrightarrow[b \rightarrow \infty]{(law)} C_1$$

On the other hand, following Williams' "pinching method", we note that:

$$\frac{1}{\log t} \left( \theta_{T_{\sqrt{t}}^\delta} - \theta_t \right) \xrightarrow[t \rightarrow \infty]{(law)} 0$$

since  $Z_u = x_0 + Z_u^{(0)}$  and also, as we change variables  $u = tv$  and we use the scaling property we obtain:

$$\theta_{T_{\sqrt{t}}^\delta} - \theta_t \equiv \text{Im} \left( \int_t^{T_{\sqrt{t}}^\delta} \frac{dZ_u}{Z_u} \right) \xrightarrow{(law)} \text{Im} \left( \int_1^{T_1^\delta} \frac{dZ_v^{(0)}}{Z_v^{(0)}} \right)$$

Here, the limit variable is -in our opinion- of no other interest than its existence which implies (16), hence (4).

□

## 2.5 On the distributions of $T_c^\theta \equiv T_{-\infty,c}^\theta$ and $T_{-c,c}^\theta$

**Proposition 2.6** *The asymptotic equivalence:*

$$(\log t) P(T_c^\theta > t) \xrightarrow{t \rightarrow \infty} (4c)/\pi \quad (17)$$

holds.

As a consequence, for  $\eta > 0$ ,  $E[(\log T_c^\theta)_+]^\eta < \infty$  if and only if  $\eta < 1$  (where  $(\cdot)_+$  denotes the positive part).

**Proof of Proposition 2.6** We shall rely upon the asymptotic distribution of  $H_t \equiv \int_0^t \frac{ds}{|Z_s|^2}$  which is given by (see e.g. Revuz-Yor [ReY99]):

$$\frac{4H_t}{(\log t)^2} \xrightarrow[t \rightarrow \infty]{(law)} T_1^\beta \equiv \inf\{t : \beta_t = 1\} \quad (18)$$

or equivalently:

$$\frac{\log t}{2\sqrt{H_t}} \xrightarrow[t \rightarrow \infty]{(law)} |N| \quad (19)$$

where  $N$  is a standard Gaussian variable  $\mathcal{N}(0, 1)$ .

We note that, from representation (2) of  $\theta_t$ , the result (18) is equivalent to Spitzer's theorem (see e.g. Spitzer [Spi58]):

$$\frac{2\theta_t}{\log t} \xrightarrow[t \rightarrow \infty]{(law)} C_1 \stackrel{(law)}{=} \gamma_{T_1^\beta}, \quad (20)$$

where  $C_1$  is a standard Cauchy variable.

We shall now use this, in order to deduce Proposition 2.6. We denote  $S_t^\theta \equiv \sup_{s \leq t} \theta_s \equiv S_{H_t}^\gamma$  and we note that (from scaling):

$$P(T_c^\theta \geq t) = P(S_{H_t}^\gamma \leq c) = P(\sqrt{H_t} S_1^\gamma \leq c) \quad (21)$$

since  $\gamma$  and  $H$  are independent. Thus, we have (since  $S_1^\gamma \stackrel{(law)}{=} |N|$  and by making the change of variable  $x = \frac{ay}{\sqrt{H_t}}$ ):

$$\begin{aligned} P(T_c^\theta \geq t) &= \sqrt{\frac{2}{\pi}} E \left[ \int_0^{c/\sqrt{H_t}} dx e^{-\frac{x^2}{2}} \right] \\ &= \sqrt{\frac{2}{\pi}} c E \left[ \int_0^1 \frac{dy}{\sqrt{H_t}} \exp\left(-\frac{c^2 y^2}{2H_t}\right) \right]. \end{aligned} \quad (22)$$

Thus, we now deduce from (19) that:

$$\frac{\log t}{2} P(T_c^\theta \geq t) \xrightarrow{t \rightarrow \infty} \sqrt{\frac{2}{\pi}} c E[|N|] = \frac{2}{\pi} c. \quad (23)$$

which is precisely (17).

It is now elementary to deduce from (23) that: for  $\eta > 0$ :

$$E[(\log T_c^\theta)_+^\eta] < \infty \Leftrightarrow 0 < \eta < 1$$

since (23) is equivalent to:

$$u P(\log T_c^\theta > u) \xrightarrow{u \rightarrow \infty} \left(\frac{4c}{\pi}\right). \quad (24)$$

Consequently, Fubini's theorem yields:

$$E[(\log T_c^\theta)_+^\eta] = \int_0^\infty du \eta u^{\eta-1} P(\log T_c^\theta > u)$$

and from (24) this is finite if and only if:

$$\int_0^\infty du u^{\eta-2} < \infty \Leftrightarrow \eta < 1$$

So,  $E[(\log T_c^\theta)_+^\eta] < \infty \Leftrightarrow 0 < \eta < 1$ .

□

Now we give several examples of random times  $T$  which may be studied quite similarly to  $T_c^\theta$ . We make the following hypothesis:

let us call  $T : C(\mathbb{R}_+, \mathbb{R}) \rightarrow \mathbb{R}_+$  an *intrinsic time* if

$$T(f_{g(\cdot)}) = g^{-1}(T(f)) \quad (25)$$

for any  $f \in C(\mathbb{R}_+, \mathbb{R})$  and  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  continuous and strictly increasing. Now, if  $T(\cdot)$  is an intrinsic time, then it follows that:  $H_{T(\theta)} = T(\gamma)$ , hence  $T(\theta) = A_{T(\gamma)}$ , defined with respect to the law of  $Z$ , issued from  $x_0 \neq 0$ . Using Bougerol's identity, we obtain:

$$\sinh(\beta_{T(\gamma)}) \stackrel{(law)}{=} \hat{\beta}_{A_{T(\gamma)}} = \hat{\beta}_{T(\theta)}. \quad (26)$$

where  $(\hat{\beta}_u, u \geq 0)$  is a 1-dimensional Brownian motion independent of  $(\beta, \gamma)$  (or equivalently, of  $Z$ ). Consequently, denoting by  $h_T$  the density of  $\beta_{T(\gamma)}$ , we deduce from (26) that:

$$E \left[ \frac{1}{\sqrt{2\pi T(\theta)}} \exp\left(-\frac{x^2}{2T(\theta)}\right) \right] = \frac{1}{\sqrt{1+x^2}} h_T(\log(x + \sqrt{1+x^2})) \quad (27)$$

or equivalently, changing  $x$  in  $\sqrt{x}$ , we obtain:

$$E \left[ \frac{1}{\sqrt{2\pi T(\theta)}} \exp\left(-\frac{x}{2T(\theta)}\right) \right] = \frac{1}{\sqrt{1+x}} h_T(\log(\sqrt{x} + \sqrt{1+x})). \quad (28)$$

In a number of cases,  $h_T$  is known explicitly, for example:

(i)

$$T(\gamma) = T_{-d,c}^\gamma \Leftrightarrow T(\theta) = \int_0^{T_{-d,c}^\gamma} ds \exp(2\beta_s) = T_{-d,c}^\theta$$

So:

$$E \left[ \frac{1}{\sqrt{2\pi T_{-d,c}^\theta}} \exp\left(-\frac{x}{2T_{-d,c}^\theta}\right) \right] = \frac{1}{\sqrt{1+x}} h_{-d,c}(\log(\sqrt{x} + \sqrt{1+x})) \quad (29)$$

where  $h_{-d,c}$  is the density of the variable  $\beta_{T_{-d,c}^\gamma}$ . The law of  $\beta_{T_{-d,c}^\gamma}$  may be obtained from its characteristic function which is given by (see Revuz-Yor [ReY99], page 73):

$$\begin{aligned} E \left[ \exp(i\lambda\beta_{T_{-d,c}^\gamma}) \right] &= E \left[ \exp\left(-\frac{\lambda^2}{2}T_{-d,c}^\gamma\right) \right] \\ &= \frac{\cosh\left(\frac{\lambda}{2}(c-d)\right)}{\cosh\left(\frac{\lambda}{2}(c+d)\right)}. \end{aligned}$$

In particular, for  $c = d$ , we recover the very classical formula:

$$E \left[ \exp(i\lambda\beta_{T_{-c,c}^\gamma}) \right] = \frac{1}{\cosh(\lambda c)}.$$

It is well known that (see e.g. Biane-Yor [BiY87], P. Lévy [Lev80]):

$$\begin{aligned} E \left[ \exp(i\lambda\beta_{T_{-c,c}^\gamma}) \right] &= \frac{1}{\cosh(\lambda c)} = \frac{1}{\cosh(\pi\lambda\frac{c}{\pi})} \\ &= \int_{-\infty}^{\infty} e^{i(\frac{\lambda c}{\pi})x} \frac{1}{2\pi} \frac{1}{\cosh(\frac{x}{2})} dx \\ &\stackrel{y=\frac{cx}{\pi}}{=} \int_{-\infty}^{\infty} e^{i\lambda y} \frac{1}{2\pi} \frac{\frac{\pi}{c}}{\cosh(\frac{y\pi}{2c})} dy \\ &= \int_{-\infty}^{\infty} e^{i\lambda y} \frac{1}{2c} \frac{1}{\cosh(\frac{y\pi}{2c})} dy. \end{aligned} \quad (30)$$

So, the density of  $\beta_{T_{-c,c}^\gamma}$  is:

$$h_{-c,c}(x) = \left(\frac{1}{2c}\right) \frac{1}{\cosh\left(\frac{x\pi}{2c}\right)} = \left(\frac{1}{c}\right) \frac{1}{e^{\frac{x\pi}{2c}} + e^{-\frac{x\pi}{2c}}}$$

and

$$h_{-c,c}(\log(\sqrt{x} + \sqrt{1+x})) = \left(\frac{1}{c}\right) \frac{1}{(\sqrt{x} + \sqrt{1+x})^\zeta + (\sqrt{x} + \sqrt{1+x})^{-\zeta}},$$

where  $\zeta = \frac{\pi}{2c}$ . However using:

$$(\sqrt{x} + \sqrt{1+x})^{-\zeta} = (\sqrt{1+x} - \sqrt{x})^\zeta \quad (31)$$

we obtain:

$$h_{-c,c}(\log(\sqrt{x} + \sqrt{1+x})) = \left(\frac{1}{c}\right) \frac{1}{(\sqrt{x} + \sqrt{1+x})^\zeta + (\sqrt{1+x} - \sqrt{x})^\zeta} \quad (32)$$

So we deduce that (for  $c = d$ ):

$$\begin{aligned} & E \left[ \frac{1}{\sqrt{2\pi T_{-c,c}^\theta}} \exp\left(-\frac{x}{2T_{-c,c}^\theta}\right) \right] \\ &= \left(\frac{1}{c}\right) \left(\frac{1}{\sqrt{1+x}}\right) \frac{1}{(\sqrt{x} + \sqrt{1+x})^\zeta + (\sqrt{1+x} - \sqrt{x})^\zeta}. \end{aligned} \quad (33)$$

□

(ii) As a second example of an intrinsic time, let us consider the time introduced in exercise 6.2 in Chaumont-Yor [ChY03], p. 178 (we use a slightly different notation). Let  $(\beta_t, t \geq 0)$  be a real valued Brownian motion and define  $S_t = \sup_{s \leq t} \beta_s$ ,  $I_t = \inf_{s \leq t} \beta_s$ . Let  $\gamma = \rho$  and  $\theta = \hat{\theta}$  in (1) and for  $c > 0$ :

$$T(\gamma) = T_c^\rho = \inf \{t : S_t - I_t = c\}.$$

Thus:

$$T(\theta) = \int_0^{T_c^\rho} ds \exp(2\beta_s) := T_c^{\hat{\theta}}.$$

So:

$$E \left[ \frac{1}{\sqrt{2\pi T_c^{\hat{\theta}}}} \exp\left(-\frac{x}{2T_c^{\hat{\theta}}}\right) \right] = \frac{1}{\sqrt{1+x}} h_c(\log(\sqrt{x} + \sqrt{1+x})) \quad (34)$$

where  $h_c$  is the density of the variable  $\beta_{T_c^\rho}$ . The law of  $\beta_{T_c^\rho}$  may be obtained from its characteristic function which is given by (see Biane-Yor [BiY87], and Chaumont-Yor [ChY03]):

$$\begin{aligned}
E [\exp(i\lambda\beta_{T_c^\rho})] &= E \left[ \exp\left(-\frac{\lambda^2}{2}T_c^\rho\right) \right] = \frac{1}{(\cosh(\lambda\frac{c}{2}))^2} = \frac{1}{(\cosh(\pi\lambda\frac{c}{2\pi}))^2} \\
&= \int_{-\infty}^{\infty} e^{i(\frac{\lambda c}{2\pi})x} \frac{1}{2\pi} \frac{x}{\sinh(\frac{x}{2})} dx \\
&\stackrel{y=\frac{cx}{2\pi}}{=} \int_{-\infty}^{\infty} e^{i\lambda y} \frac{1}{2\pi} \frac{\frac{2\pi y}{c}}{\sinh(\frac{\pi y}{c})} \frac{2\pi}{c} dy \\
&= \int_{-\infty}^{\infty} e^{i\lambda y} \frac{2\pi}{c^2} \frac{y}{\sinh(\frac{\pi y}{c})} dy. \tag{35}
\end{aligned}$$

So, the density of  $\beta_{T_c^\rho}$  is:

$$h_c(x) = \left(\frac{2\pi x}{c^2}\right) \frac{1}{\sinh(\frac{\pi x}{c})} = \frac{2\pi}{c^2} \frac{x}{e^{\frac{\pi x}{c}} - e^{-\frac{\pi x}{c}}}$$

and

$$h_c(\log(\sqrt{x} + \sqrt{1+x})) = \frac{2\pi}{c^2} \frac{x}{(\sqrt{x} + \sqrt{1+x})^{\hat{\zeta}} - (\sqrt{x} + \sqrt{1+x})^{-\hat{\zeta}}},$$

where  $\hat{\zeta} = \frac{\pi}{c}$ . Thus:

$$\begin{aligned}
&E \left[ \frac{1}{\sqrt{2\pi T_c^\rho}} \exp\left(-\frac{x}{2T_c^\rho}\right) \right] \\
&= \frac{2\pi}{c^2} \frac{x}{\sqrt{1+x}} \frac{1}{(\sqrt{x} + \sqrt{1+x})^{\hat{\zeta}} - (\sqrt{1+x} - \sqrt{x})^{\hat{\zeta}}}. \tag{36}
\end{aligned}$$

We note that this study may be related to Pitman-Yor [PiY03]; and more precisely  $\beta_{T_c^\rho}$  and  $T_c^\rho$  correspond to the variables  $C_2$  and  $\hat{C}_2$  respectively (see e.g. Table 6 in p. 312).

□

Let us now return to the case of  $T_{-c,c}^\theta$  (example (i)). More specifically, we shall obtain its density function  $f(t)$ .

**Proposition 2.7** *The density function  $f$  of  $T_{-c,c}^\theta$  is given by:*

$$f(t) = \frac{1}{\sqrt{2c}} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\nu_k)}{\Gamma(2\nu_k)} \frac{1}{\sqrt{t}} e^{-\frac{1}{4t}} M_{\frac{1}{2}, \nu_k} \left(\frac{1}{2t}\right) \tag{37}$$

where  $M_{a,b}(\cdot)$  is the Whittaker function with parameters  $a, b$ . Equivalently:

$$f(t) = \frac{\sqrt{2}}{c} \sum_{k=0}^{\infty} (-1)^k \frac{1}{\sqrt{t}} e^{-\frac{1}{2t}} \left(\frac{1}{2t}\right)^{\nu_k + \frac{1}{2}} \nu_k \sum_{n=0}^{\infty} \frac{\Gamma(\nu_k + n)}{\Gamma(2\nu_k + n + 1)} \frac{1}{n!} \left(\frac{1}{2t}\right)^n \quad (38)$$

where  $\nu_k = \frac{\pi}{4c}(2k + 1)$ .

**Proof of Proposition 2.7** The following calculation relies upon a private note by A. Comtet [Com06]. We denote:

$$\varphi_{\zeta}(x) = (\sqrt{x} + \sqrt{1+x})^{\zeta} + (\sqrt{1+x} - \sqrt{x})^{\zeta}.$$

Noting:

$$\sqrt{1+x} = \cosh \frac{y}{2} \iff y = 2 \arg \cosh(\sqrt{1+x}). \quad (39)$$

we get:

$$\begin{aligned} \varphi_{\zeta}(x) &= \left(\sinh \frac{y}{2} + \cosh \frac{y}{2}\right)^{\zeta} + \left(\cosh \frac{y}{2} - \sinh \frac{y}{2}\right)^{\zeta} \\ &= 2 \cosh \frac{y\zeta}{2} \end{aligned}$$

So, from (33), we have:

$$II := E \left[ \frac{1}{\sqrt{2\pi T_{-c,c}^{\theta}}} \exp\left(-\frac{x}{2T_{-c,c}^{\theta}}\right) \right] = \frac{1}{\psi} \frac{1}{\cosh \frac{y}{2}} \frac{1}{\cosh \frac{\pi y}{2\psi}} \quad (40)$$

where  $\psi = 2c$ . However, expanding  $\cosh \frac{\pi y}{2\psi}$ , we get:

$$\frac{1}{\cosh \frac{\pi y}{2\psi}} = 2 \frac{e^{-\frac{\pi y}{2\psi}}}{1 + e^{-\frac{\pi y}{\psi}}} = 2 \sum_{k=0}^{\infty} \left(-e^{-\frac{\pi y}{\psi}}\right)^k e^{-\frac{\pi y}{2\psi}}$$

and from (39), we deduce that:

$$\begin{aligned} II &= \sum_{k=0}^{\infty} \frac{2(-1)^k}{\psi \cosh \frac{y}{2}} e^{-\frac{\pi}{2\psi}(2k+1)y} \\ &= \sum_{k=0}^{\infty} \frac{4(-1)^k}{\psi \sqrt{2} \sqrt{2 \sinh \frac{y}{2} \cosh \frac{y}{2}}} \sqrt{\frac{\sinh \frac{y}{2}}{\cosh \frac{y}{2}}} e^{-\nu_k y} \\ &= \sum_{k=0}^{\infty} \frac{4(-1)^k}{\psi \sqrt{2} \sqrt{2 \sinh \frac{y}{2} \cosh \frac{y}{2}}} \sqrt{\tanh \frac{y}{2}} e^{-\nu_k y}, \end{aligned}$$



where  $\nu_k = \frac{\pi}{2\psi}(2k+1)$ .

From (33), we have  $1+x = \cosh^2 \frac{y}{2} \iff x = \sinh^2 \frac{y}{2}$ , thus:

$$\left(\tanh \frac{y}{2}\right)^{1/2} = \sqrt{\frac{\sinh \frac{y}{2}}{\cosh \frac{y}{2}}} = \left(\frac{\sqrt{x}}{\sqrt{1+x}}\right)^{1/2} = \left(\frac{x}{1+x}\right)^{1/4}.$$

Moreover, we know that (see e.g. Abramowitz-Stegun [AbSt70], equation 8.6.10, or Lebedev [Leb63]):

$$i\sqrt{\frac{\pi}{2\sinh y}}e^{-\nu_k y} = Q_{\nu_k-1/2}^{1/2}(\cosh y)$$

where  $\{Q_b^a(\cdot)\}$  is the family of Legendre functions and  $\cosh y = 2x+1$ . So, we deduce:

$$II = \sum_{k=0}^{\infty} \frac{4(-i)}{\psi\sqrt{\pi}}(-1)^k \left(\frac{x}{1+x}\right)^{1/4} Q_{\nu_k-1/2}^{1/2}(2x+1). \quad (41)$$

By using formula (7.621.9, page 861) from Gradshteyn-Ryzhik [GrR65]:

$$\int_0^{\infty} e^{-sw} M_{l,\nu_k}(w) \frac{dw}{w} = \frac{2\Gamma(1+2\nu_k)}{\Gamma(\frac{1}{2}+\nu_k+l)} \left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^{l/2} Q_{\nu_k-1/2}^l(2s) \quad (42)$$

with:  $l = \frac{1}{2}$ ,  $\nu_k = \frac{\pi}{2\psi}(2k+1)$ ,  $s = x + \frac{1}{2}$  and  $M_{\cdot,\cdot}(\cdot)$  denoting the Whittaker function, which is defined as:

$$M_{a,b}(w) = w^{b+\frac{1}{2}} e^{-\frac{1}{2}w} \frac{\Gamma(2b+1)}{\Gamma(\frac{1}{2}+b-a)} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2}+b-a+n)}{\Gamma(2b+1+n)} \frac{w^n}{n!}.$$

we have:

$$-2i \frac{\Gamma(1+2\nu_k)}{\Gamma(1+\nu_k)} \left(\frac{x}{1+x}\right)^{1/4} Q_{\nu_k-1/2}^{1/2}(2x+1) = \int_0^{\infty} e^{-sw} M_{1/2,\nu_k}(w) \frac{dw}{w} \quad (43)$$

From (41) and by changing the variable  $w = \frac{1}{2t}$ , we deduce:

$$\begin{aligned} II &= \sum_{k=0}^{\infty} \frac{2}{\psi\sqrt{\pi}}(-1)^k \frac{\Gamma(\nu_k+1)}{\Gamma(2\nu_k+1)} \int_0^{\infty} \frac{dw}{w} \exp\left(-w\left(x+\frac{1}{2}\right)\right) M_{1/2,\nu_k}(w) \\ &= \sum_{k=0}^{\infty} \int_0^{\infty} \frac{dt}{t} \frac{2}{\psi\sqrt{\pi}}(-1)^k \frac{\Gamma(\nu_k+1)}{\Gamma(2\nu_k+1)} \exp\left(-\frac{1}{4t} - \frac{x}{2t}\right) M_{1/2,\nu_k}\left(\frac{1}{2t}\right). \end{aligned} \quad (44)$$

By using the equations (40) and (44), we conclude:

$$\begin{aligned}
& E \left[ \frac{1}{\sqrt{2\pi T_{-c,c}^\theta}} \exp \left( -\frac{x}{2T_{-c,c}^\theta} \right) \right] \\
&= \sum_{k=0}^{\infty} \int_0^{\infty} \frac{dt}{t} \frac{2}{\psi\sqrt{\pi}} (-1)^k \frac{\Gamma(\nu_k + 1)}{\Gamma(2\nu_k + 1)} \exp \left( -\frac{1}{4t} - \frac{x}{2t} \right) M_{\frac{1}{2}, \nu_k} \left( \frac{1}{2t} \right) \\
&= \sum_{k=0}^{\infty} \int_0^{\infty} \frac{dt}{t} \frac{2}{\psi\sqrt{\pi}} (-1)^k \frac{\Gamma(\frac{\pi}{4a}(2k+1) + 1)}{\Gamma(2\frac{\pi}{4a}(2k+1) + 1)} \exp \left( -\frac{1}{4t} - \frac{x}{2t} \right) M_{\frac{1}{2}, \frac{\pi}{4a}(2k+1)} \left( \frac{1}{2t} \right).
\end{aligned} \tag{45}$$

This is the Laplace transform of  $1/2T_{-c,c}^\theta$ , and so, the density function  $f$  of  $T_{-c,c}^\theta$  is given by:

$$f(t) = \frac{2\sqrt{2}}{\psi} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\nu_k + 1)}{\Gamma(2\nu_k + 1)} \frac{1}{\sqrt{t}} e^{-\frac{1}{4t}} M_{\frac{1}{2}, \nu_k} \left( \frac{1}{2t} \right) \tag{46}$$

$$= \frac{\sqrt{2}}{c} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\frac{\pi}{4a}(2k+1) + 1)}{\Gamma(2\frac{\pi}{4a}(2k+1) + 1)} \frac{1}{\sqrt{t}} e^{-\frac{1}{4t}} M_{\frac{1}{2}, \frac{\pi}{4a}(2k+1)} \left( \frac{1}{2t} \right) \tag{47}$$

$$= \frac{\sqrt{2}}{c} \sum_{k=0}^{\infty} (-1)^k \frac{\nu_k \Gamma(\nu_k)}{2\nu_k \Gamma(2\nu_k)} \frac{1}{\sqrt{t}} e^{-\frac{1}{4t}} M_{\frac{1}{2}, \nu_k} \left( \frac{1}{2t} \right) \tag{48}$$

where the Whittaker function  $M_{\frac{1}{2}, \nu_k} \left( \frac{1}{2t} \right)$  is:

$$\begin{aligned}
& M_{\frac{1}{2}, \frac{\pi}{4c}(2k+1)} \left( \frac{1}{2t} \right) \\
&= \left( \frac{1}{2t} \right)^{\frac{\pi}{4c}(2k+1) + \frac{1}{2}} e^{-\frac{1}{4t}} \frac{\Gamma(\frac{\pi}{2c}(2k+1) + 1)}{\Gamma(\frac{\pi}{4c}(2k+1))} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{\pi}{4c}(2k+1) + n)}{\Gamma(\frac{\pi}{2c}(2k+1) + 1 + n)} \frac{1}{n!} \left( \frac{1}{2t} \right)^n \\
&= \left( \frac{1}{2t} \right)^{\nu_k + \frac{1}{2}} e^{-\frac{1}{4t}} \frac{\Gamma(2\nu_k + 1)}{\Gamma(\nu_k)} \sum_{n=0}^{\infty} \frac{\Gamma(\nu_k + n)}{\Gamma(2\nu_k + 1 + n)} \frac{1}{n!} \left( \frac{1}{2t} \right)^n \\
&= \left( \frac{1}{2t} \right)^{\nu_k + \frac{1}{2}} e^{-\frac{1}{4t}} (2\nu_k) \frac{\Gamma(2\nu_k)}{\Gamma(\nu_k)} \sum_{n=0}^{\infty} \frac{\Gamma(\nu_k + n)}{(2\nu_k + n)\Gamma(2\nu_k + n)} \frac{1}{n!} \left( \frac{1}{2t} \right)^n.
\end{aligned} \tag{49}$$

Thus, from (48) and (49), we deduce (38).

□

Figure 3: The diagram of  $f(t)$  with respect to the time  $t$ .

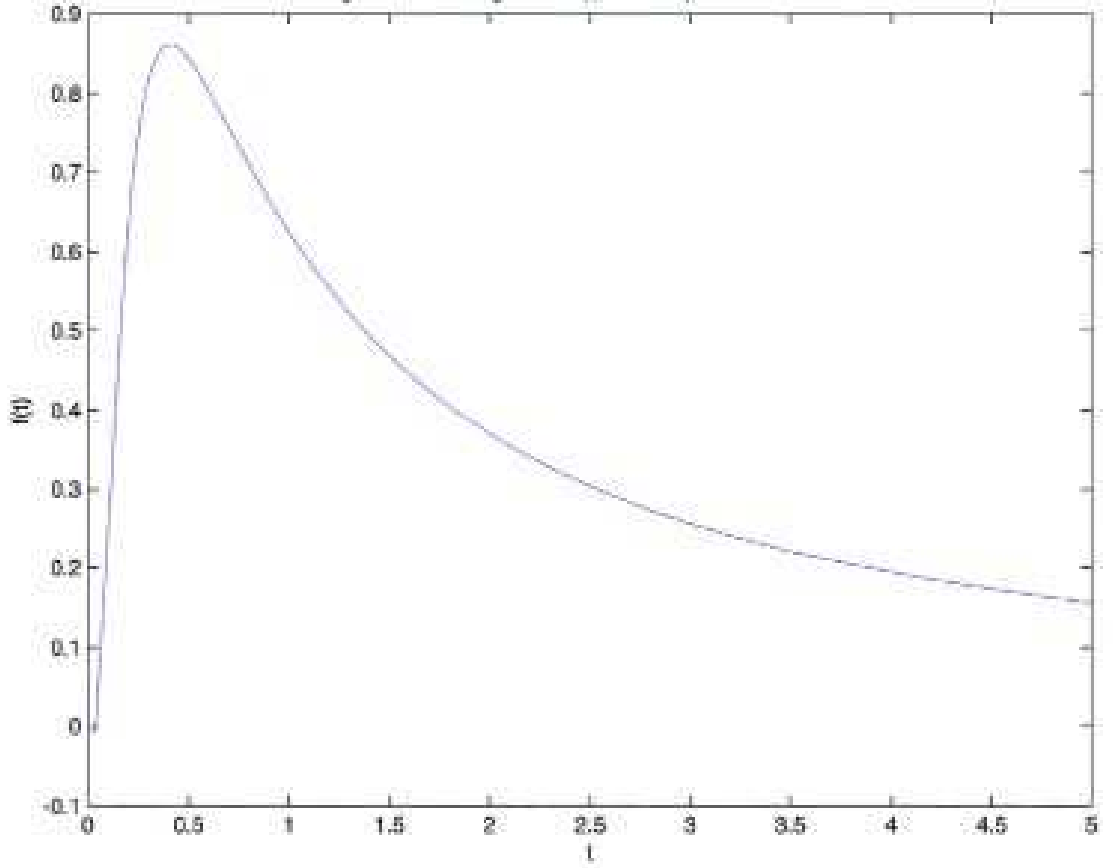
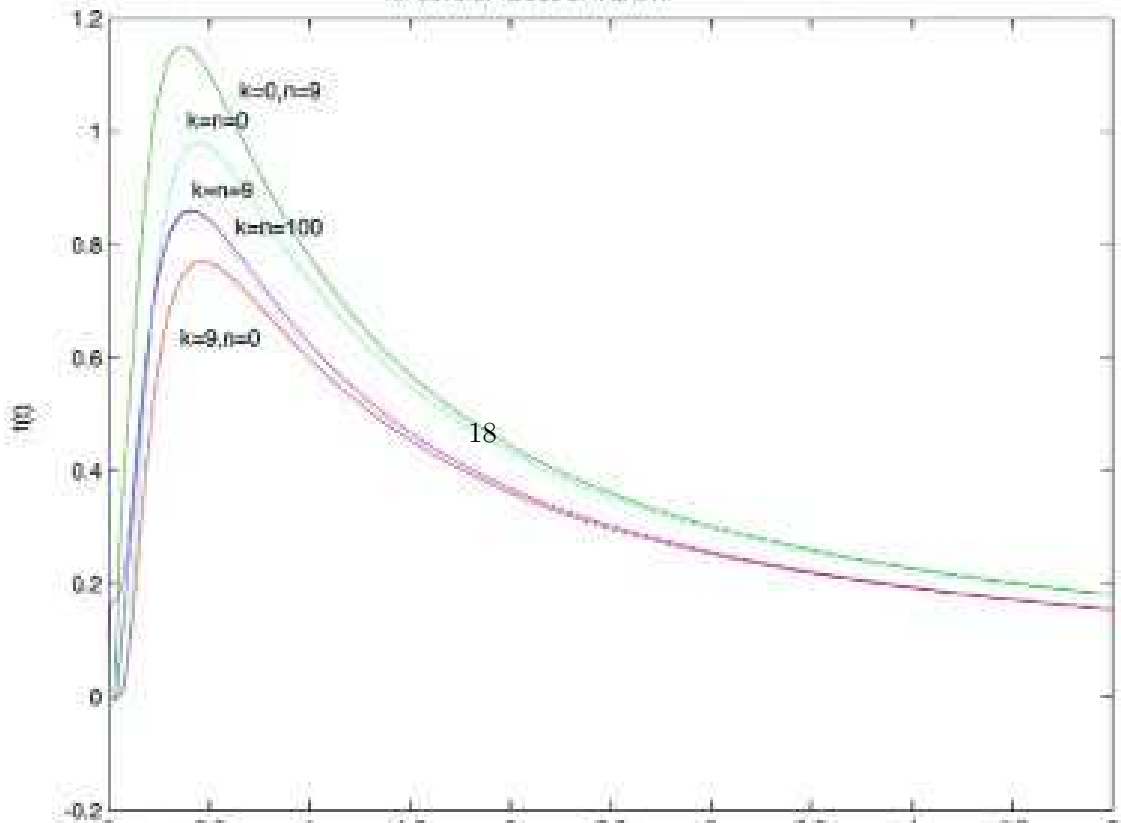


Figure 4: The diagram of  $f(t)$  with respect to the time  $t$  for several values of  $k$  and  $n$ .



**Remark 2.8** • *Figure 3 represents the approximation of the density function  $f$  with respect to the time  $t$  (for  $k = n = 9$ ) whereas figure 4 represents the approximation of  $f$  with respect to the time  $t$  for several values of  $k$  and  $n$ .*

- *From Figure 4, we may remark that the approximation  $k = n = 9$  is sufficiently good (comparing to the one for  $k = n = 100$ ).*
- *For the case  $k = n = 9$  it seems that locally, in a small area around 0,  $f(t) < 0$  which is not right. This is due to the first negative ( $k = 1$ ) term of the sum and due to the fact that we have omitted many terms. However, this is not a problem because it appears only locally. Similar irregularities have already been observed in previous articles (see e.g. Ishiyama p.275 [Ish05]).*

## 2.6 On the first moment of $\ln(T_{-c,c}^\theta)$

**Proposition 2.9** *The first moment of  $\ln(T_{-c,c}^\theta)$  has the following integral representation:*

$$E \left[ \ln \left( T_{-c,c}^\theta \right) \right] = 2 \int_0^\infty \frac{dz}{\cosh \left( \frac{\pi z}{2} \right)} \ln (\sinh (cz)) + \ln (2) + C_E \quad (50)$$

where  $C_E = -\Gamma'(1)$  is the Euler-Mascheroni constant (also called Euler's constant).

**Proof of Proposition 2.9** Let us return to equations (2) and (6). So, for  $t = T_{-c,c}^\theta$ , we have:

$$\theta_{T_{-c,c}^\theta} = \gamma_{H_{T_{-c,c}^\theta}} \iff H_{T_{-c,c}^\theta} = T_{-c,c}^\gamma \iff T_{-c,c}^\theta = A_{T_{-c,c}^\gamma} \quad (51)$$

Thus, for  $\varepsilon > 0$ :

$$E \left[ \left( T_{-c,c}^\theta \right)^\varepsilon \right] = E \left[ \left( A_{T_{-c,c}^\gamma} \right)^\varepsilon \right]$$

Consider  $(\delta_t, t \geq 0)$  a Brownian motion, independent of  $A_t$ . Then, Bougerol's identity and the scaling property yield ( $\mathcal{G}_a$  denotes a gamma variable with parameter  $a$ , and  $N^2 \stackrel{(law)}{=} 2\mathcal{G}_{1/2}$ ).

$$\begin{aligned} E \left[ |\sinh (B_t)|^{2\varepsilon} \right] &= E \left[ |\delta_{A_t}|^{2\varepsilon} \right] = E \left[ (A_t^\varepsilon) |\delta_1|^{2\varepsilon} \right] \\ &= E \left[ A_t^\varepsilon \right] E \left[ (2\mathcal{G}_{1/2})^\varepsilon \right] \\ &= E \left[ A_t^\varepsilon \right] (2^\varepsilon) \frac{\Gamma \left( \frac{1}{2} + \varepsilon \right)}{\Gamma \left( \frac{1}{2} \right)} \end{aligned}$$

Thus, for  $t = T_{-c,c}^\gamma$ , we have:

$$E \left[ \left| \sinh \left( B_{T_{-c,c}^\gamma} \right) \right|^{2\varepsilon} \right] = E \left[ A_{T_{-c,c}^\gamma}^\varepsilon \right] (2^\varepsilon) \frac{\Gamma\left(\frac{1}{2} + \varepsilon\right)}{\Gamma\left(\frac{1}{2}\right)} \quad (52)$$

Recall that (see e.g. Biane-Yor [BiY87], P. Lévy [Lev80]):

$$E \left[ \exp(i\lambda B_{T_{-c,c}^\gamma}) \right] = E \left[ \exp\left(-\frac{\lambda^2}{2} T_{-c,c}^\gamma\right) \right] = \frac{1}{\cosh(\lambda c)}$$

and the density of  $\beta_{T_{-c,c}^\gamma}$  is:

$$h_{-c,c}(y) = \left(\frac{1}{2c}\right) \frac{1}{\cosh\left(\frac{y\pi}{2c}\right)} = \left(\frac{1}{c}\right) \frac{1}{e^{\frac{y\pi}{2c}} + e^{-\frac{y\pi}{2c}}}$$

Thus, on the left hand side of (52), we have:

$$\begin{aligned} E \left[ \left| \sinh \left( B_{T_{-c,c}^\gamma} \right) \right|^{2\varepsilon} \right] &= \int_{-\infty}^{\infty} \frac{dy}{2c} \frac{1}{\cosh\left(\frac{\pi y}{2c}\right)} \left| \sinh(y) \right|^{2\varepsilon} \\ &= \int_0^{\infty} \frac{dy}{c} \frac{1}{\cosh\left(\frac{\pi y}{2c}\right)} (\sinh y)^{2\varepsilon} \\ &= \int_0^{\infty} dz \frac{1}{\cosh\left(\frac{\pi z}{2}\right)} (\sinh(cz))^{2\varepsilon}. \end{aligned}$$

where we have made the change of variable  $z = \frac{y}{c}$ . So, from (52), by writing:

$$E \left[ A_{T_{-c,c}^\gamma}^\varepsilon \right] = E \left[ \left( T_{-c,c}^\theta \right)^\varepsilon \right] = E \left[ e^{\varepsilon \ln(T_{-c,c}^\theta)} \right]$$

we deduce:

$$\frac{\Gamma\left(\frac{1}{2} + \varepsilon\right)}{\Gamma\left(\frac{1}{2}\right)} E \left[ e^{\varepsilon \ln(T_{-c,c}^\theta)} \right] = \frac{1}{2^\varepsilon} \int_0^{\infty} \frac{dz}{\cosh\left(\frac{\pi z}{2}\right)} (\sinh(cz))^{2\varepsilon}$$

and by removing 1 from both sides, we obtain:

$$\frac{\Gamma\left(\frac{1}{2} + \varepsilon\right)}{\Gamma\left(\frac{1}{2}\right)} E \left[ e^{\varepsilon \ln(T_{-c,c}^\theta)} \right] - 1 = \int_0^{\infty} \frac{dz}{\cosh\left(\frac{\pi z}{2}\right)} \left( \frac{(\sinh(cz))^{2\varepsilon}}{2^\varepsilon} - 1 \right) \quad (53)$$

On the left hand side, we apply the trivial identity  $ab - 1 = a(b - 1) + a - 1$  with  $a = \frac{\Gamma\left(\frac{1}{2} + \varepsilon\right)}{\Gamma\left(\frac{1}{2}\right)}$  and  $b = E \left[ e^{\varepsilon \ln(T_{-c,c}^\theta)} \right]$ , we divide by  $\varepsilon$  and we take the limit for  $\varepsilon \rightarrow 0$ . Thus:

$$\begin{aligned} \frac{a(b-1)}{\varepsilon} &= \frac{\Gamma\left(\frac{1}{2} + \varepsilon\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{E \left[ e^{\varepsilon \ln(T_{-c,c}^\theta)} \right] - 1}{\varepsilon} \\ &\xrightarrow{\varepsilon \rightarrow 0} E \left[ \ln \left( T_{-c,c}^\theta \right) \right] \end{aligned}$$

and:

$$\begin{aligned} \frac{a-1}{\varepsilon} &= \frac{1}{\varepsilon} \left( \frac{\Gamma\left(\frac{1}{2} + \varepsilon\right)}{\Gamma\left(\frac{1}{2}\right)} - 1 \right) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \left( \frac{\Gamma\left(\frac{1}{2} + \varepsilon\right) - \Gamma\left(\frac{1}{2}\right)}{\varepsilon} \right) \\ &\xrightarrow{\varepsilon \rightarrow 0} \frac{1}{\sqrt{\pi}} \Gamma'\left(\frac{1}{2}\right) = \frac{1}{\sqrt{\pi}} (-\sqrt{\pi}) (C_E + 2 \ln 2) = -(C_E + 2 \ln 2) \end{aligned}$$

On the right hand side of (53), we have:

$$\frac{1}{\varepsilon} \left[ \left( \frac{(\sinh(cz))^2}{2} \right)^\varepsilon - 1 \right] = \frac{1}{\varepsilon} \left[ \exp \left( \varepsilon \ln \left( \frac{(\sinh(cz))^2}{2} \right) \right) - 1 \right]$$

hence:

$$\begin{aligned} &\frac{1}{\varepsilon} \int_0^\infty \frac{dz}{\cosh\left(\frac{\pi z}{2}\right)} \left( \frac{(\sinh(cz))^{2\varepsilon}}{2^\varepsilon} - 1 \right) \\ &\xrightarrow{\varepsilon \rightarrow 0} \int_0^\infty \frac{dz}{\cosh\left(\frac{\pi z}{2}\right)} \ln \left( \frac{(\sinh(cz))^2}{2} \right) \\ &= 2 \int_0^\infty \frac{dz}{\cosh\left(\frac{\pi z}{2}\right)} (\ln(\sinh(cz))) - \ln(2) \end{aligned}$$

which finishes the proof. □

### 3 The Ornstein-Uhlenbeck case

#### 3.1 An identity in law for Ornstein-Uhlenbeck processes, which is connected to Bougerol's identity

Consider the complex valued Ornstein-Uhlenbeck (OU) process:

$$Z_t = z_0 + \tilde{Z}_t - \lambda \int_0^t Z_s ds \quad (54)$$

where  $\tilde{Z}_t$  is a complex valued Brownian motion (BM),  $z_0 \in \mathbb{C}$  and  $\lambda \geq 0$  and  $T_c^{(\lambda)} \equiv T_c^{\theta^Z} \equiv \inf \{t \geq 0 : |\theta_t^Z| = c\}$  ( $\theta_t^Z$  is the continuous winding process associated to  $Z$ ) denoting the first hitting time of the symmetric conic boundary of angle  $c$  for  $Z$ . It is well known that (see e.g. Revuz-Yor [ReY99]):

$$\begin{aligned} Z_t &= e^{-\lambda t} \left( z_0 + \int_0^t e^{\lambda s} d\tilde{Z}_s \right) \\ &= e^{-\lambda t} (\mathbb{B}_{\alpha_t}) \end{aligned} \quad (55)$$

where, in the second equation, with the help of Dambis-Dubins-Schwarz Theorem,  $(\mathbb{B}_t, t \geq 0)$  is a complex valued Brownian motion starting from  $z_0$  and

$$\alpha_t = \int_0^t e^{2\lambda s} ds = \frac{e^{2\lambda t} - 1}{2\lambda} .$$

We are interested in the study of the continuous winding process  $\theta_t^Z = \text{Im}(\int_0^t \frac{dZ_s}{Z_s}), t \geq 0$ . By applying Itô's formula to (55), we have:

$$dZ_s = e^{-\lambda s} (-\lambda) \mathbb{B}_{\alpha_s} ds + e^{-\lambda s} d(\mathbb{B}_{\alpha_s})$$

We divide by  $Z_s$  and we obtain:

$$\frac{dZ_s}{Z_s} = (-\lambda) ds + \frac{d\mathbb{B}_{\alpha_s}}{\mathbb{B}_{\alpha_s}}$$

and so:

$$\text{Im} \left( \frac{dZ_s}{Z_s} \right) = \text{Im} \left( \frac{d\mathbb{B}_{\alpha_s}}{\mathbb{B}_{\alpha_s}} \right)$$

which means that:

$$\theta_t^Z = \theta_{\alpha_t}^{\mathbb{B}} .$$

Thus, the following holds:

**Proposition 3.1** *Using the notation which we introduced before, we have:*

$$\theta_t^Z = \theta_{\alpha_t}^{\mathbb{B}} \quad (56)$$

and:

$$T_c^{(\lambda)} = \frac{1}{2\lambda} \ln \left( 1 + 2\lambda T_{-c,c}^{\theta^{\mathbb{B}}} \right) \quad (57)$$

where  $T_{-c,c}^{\theta}$  is the exit time from a cone of angle  $c$  for the complex valued BM  $\mathbb{B}$ .

**Proof of Proposition 3.1** We define

$$\begin{aligned} T_c^{(\lambda)} &\equiv T_c^{\theta^Z} \equiv \inf \{ t \geq 0 : |\theta_t^Z| = c \} \\ &= \inf \{ t \geq 0 : |\theta_{\alpha_t}^{\mathbb{B}}| = c \} \end{aligned} \quad (58)$$

Thus, we deduce that  $\alpha_{T_c^{(\lambda)}} = T_c^{\theta^{\mathbb{B}}} \equiv T_{-c,c}^{\theta}$ . However,  $T_{-c,c}^{\theta}$  (the exit time from a cone for the BM) has already been studied in the previous chapter and we know the explicit formula of its density function (Proposition 2.7). So:

$$T_c^{(\lambda)} = \alpha^{-1} \left( T_c^{\theta^{\mathbb{B}}} \right) = \alpha^{-1} \left( T_{-c,c}^{\theta} \right) \quad (59)$$

where  $\alpha^{-1}(t) = \frac{1}{2\lambda} \ln(1 + 2\lambda t)$ . Consequently:

$$T_c^{(\lambda)} = \frac{1}{2\lambda} \ln \left( 1 + 2\lambda T_{-c,c}^{\theta} \right)$$

and

$$E \left[ T_c^{(\lambda)} \right] = \frac{1}{2\lambda} E \left[ \ln \left( 1 + 2\lambda T_{-c,c}^{\theta} \right) \right] \quad (60)$$

which finishes the proof. □

From now on, for simplicity, we shall take  $z_0 = 1$  (but this is really no restriction, as the dependency in  $z_0$ , which is exhibited in (7), is very simple). The following Proposition may be considered as an extension of the identity in law (ii) in Proposition 2.4, which yields from Bougerol's identity.

**Proposition 3.2** *Consider  $(Z_t^\lambda, t \geq 0)$  and  $(U_t^\lambda, t \geq 0)$  two independent Ornstein-Uhlenbeck processes and call  $T_b^{(\lambda)}(U^\lambda) = \inf \{ t \geq 0 : e^{\lambda t} U_t^\lambda = b \}$ . Then, an Ornstein-Uhlenbeck extension of identity in law (ii) in Proposition 2.4 is the following:*

$$\theta_{T_b^{(\lambda)}(U^\lambda)}^{Z^\lambda} \stackrel{(law)}{=} C_{a(b)} \quad (61)$$

where  $a(x) = \arg \sinh(x)$ .



**Proof of Proposition 3.2** Let us consider a second Ornstein-Uhlenbeck process  $(U_t^\lambda, t \geq 0)$  independent of the first one. Then, taking equation (55) for  $U_t^\lambda$ , we have:

$$e^{\lambda t} U_t^\lambda = \delta_{\left(\frac{e^{2\lambda t} - 1}{2\lambda}\right)} \quad (62)$$

where  $(\delta_t, t \geq 0)$  is a complex valued Brownian motion starting from  $z_0 = 1$ . So:

$$T_b^{(\lambda)}(U^\lambda) = \frac{1}{2\lambda} \ln \left( 1 + 2\lambda T_b^\delta \right) \quad (63)$$

Equation (56) for  $t = \frac{1}{2\lambda} \ln \left( 1 + 2\lambda T_b^\delta \right)$ , equivalently:  $\alpha(t) = T_b^\delta$  becomes (we suppose that  $z_0 = 1$ ):

$$\theta_{T_b^{(\lambda)}(U^\lambda)}^{Z^\lambda} = \theta_{\frac{1}{2\lambda} \ln(1+2\lambda T_b^\delta)}^{Z^\lambda} = \theta_{u=T_b^\delta}^{\mathbb{B}} \stackrel{(law)}{=} C_{a(b)}$$

□

### 3.2 On the distribution of $T_{-c,c}^\theta$ for an Ornstein-Uhlenbeck process

Now we turn to the study of the density function of

$$T_c^{(\lambda)} \equiv T_c^{\theta^Z} \equiv \inf \{ t \geq 0 : |\theta_t^Z| = c \}$$

and its first moment.

**Proposition 3.3** *Asymptotically for  $\lambda$  large, for  $z_0 = 1$ , we have:*

$$2\lambda E \left[ T_c^{(\lambda)} \right] - \ln(2\lambda) \xrightarrow{\lambda \rightarrow \infty} E \left[ \ln \left( T_{-c,c}^\theta \right) \right] \quad (64)$$

and:

$$E \left[ \ln \left( T_{-c,c}^\theta \right) \right] = 2 \int_0^\infty \frac{dz}{\cosh\left(\frac{\pi z}{2}\right)} \ln(\sinh(cz)) + \ln(2) + C_E \quad (65)$$

where  $C_E$  is Euler's constant.

For  $c < \frac{\pi}{8}$ , we have the asymptotic equivalence:

$$\frac{1}{\lambda} \left( E \left[ T_c^{(\lambda)} \right] - E \left[ |\sinh \left( B_{T_{-c,c}^\gamma} \right)|^2 \right] \right) \xrightarrow{\lambda \rightarrow 0} -\frac{1}{3} E \left[ |\sinh \left( B_{T_{-c,c}^\gamma} \right)|^4 \right]. \quad (66)$$

Equivalently:

$$\frac{d}{d\lambda} \Big|_{\lambda=0} E \left[ T_c^{(\lambda)} \right] = \lim_{\lambda \rightarrow 0} \left[ \frac{1}{\lambda} \left( E \left[ T_c^{(\lambda)} \right] - E \left[ T_c^{(0)} \right] \right) \right] = -\frac{1}{3} E \left[ |\sinh \left( B_{T_{-c,c}^\gamma} \right)|^4 \right]. \quad (67)$$

Moreover:

$$E \left[ \left| \sinh \left( B_{T_{-c,c}^\gamma} \right) \right|^4 \right] = \int_0^\infty \frac{dz}{\cosh \left( \frac{\pi z}{2} \right)} (\sinh(cz))^4. \quad (68)$$

More precisely, for  $c < \frac{\pi}{8}$ :

$$E \left[ \left| \sinh \left( B_{T_{-c,c}^\gamma} \right) \right|^4 \right] = \frac{1}{8} \left( \frac{1}{\cos(4c)} - 4 \frac{1}{\cos(2c)} + 3 \right) \quad (69)$$

and asymptotically:

$$E \left[ \left| \sinh \left( B_{T_{-c,c}^\gamma} \right) \right|^4 \right] \underset{c \rightarrow 0}{\simeq} 5c^4. \quad (70)$$

### Proof of Proposition 3.3

#### $\lambda$ large

Let us return to equation (60). For  $\lambda \rightarrow \infty$ , we have:

$$\begin{aligned} E \left[ T_c^{(\lambda)} \right] &= \frac{1}{2\lambda} E \left[ \ln \left( 1 + 2\lambda T_{-c,c}^\theta \right) \right] \\ &= \frac{1}{2\lambda} E \left[ \ln \left( 2\lambda \left( T_{-c,c}^\theta + \frac{1}{2\lambda} \right) \right) \right] \\ &= \frac{\ln(2\lambda)}{2\lambda} + \frac{1}{2\lambda} E \left[ \ln \left( T_{-c,c}^\theta + \frac{1}{2\lambda} \right) \right] \end{aligned}$$

Thus:

$$2\lambda E \left[ T_c^{(\lambda)} \right] - \ln(2\lambda) \xrightarrow{\lambda \rightarrow \infty} E \left[ \ln \left( T_{-c,c}^\theta \right) \right]$$

which is precisely (64). Moreover, by the integral representation (50) for  $E \left[ \ln \left( T_{-c,c}^\theta \right) \right]$ , we deduce (65).

#### $\lambda$ small

For  $\lambda \rightarrow 0$ , we may approximate:

$$\begin{aligned} T_c^{(\lambda)} &= \frac{1}{2\lambda} \ln \left( 1 + 2\lambda T_{-c,c}^\theta \right) \\ &= \frac{1}{2\lambda} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2\lambda T_{-c,c}^\theta)^n}{n} \\ &= T_{-c,c}^\theta - \lambda (T_{-c,c}^\theta)^2 + O(\lambda^2) + \dots \\ &= A_{T_{-c,c}^\gamma} - \lambda (A_{T_{-c,c}^\gamma})^2 + O(\lambda^2) + \dots \end{aligned}$$

because:

$$A_{T_{-c,c}^\gamma} = T_{-c,c}^\theta.$$

For  $c < \frac{\pi}{8}$ , all the positive moments of  $A_{T_{-c,c}^\gamma}$  are finite:  $E \left[ \left( A_{T_{-c,c}^\gamma} \right)^p \right] < \infty$  ( $p = 1, 2, \dots$ ), thus:

$$\begin{aligned} E \left[ T_c^{(\lambda)} \right] &= E \left[ A_{T_{-c,c}^\gamma} \right] - \lambda E \left[ \left( A_{T_{-c,c}^\gamma} \right)^2 \right] + O(\lambda^2) + \dots \\ \Rightarrow \frac{1}{\lambda} \left( E \left[ T_c^{(\lambda)} \right] - E \left[ A_{T_{-c,c}^\gamma} \right] \right) &= -E \left[ \left( A_{T_{-c,c}^\gamma} \right)^2 \right] + O(\lambda) + \dots \\ &\xrightarrow{\lambda \rightarrow 0} -E \left[ \left( A_{T_{-c,c}^\gamma} \right)^2 \right] \end{aligned}$$

Following the proof of Proposition 2.9, Bougerol's identity and the scaling property yield:

$$\begin{aligned} E \left[ (\sinh(B_t))^2 \right] &= E \left[ (\delta_{A_t})^2 \right] = E \left[ (A_t) (\delta_1)^2 \right] = E[A_t] E \left[ (\delta_1)^2 \right] \\ &= E[A_t]. \end{aligned}$$

Thus, for  $t = T_{-c,c}^\gamma$ , we have:

$$E \left[ A_{T_{-c,c}^\gamma} \right] = E \left[ \left| \sinh \left( B_{T_{-c,c}^\gamma} \right) \right|^2 \right]$$

Similarly:

$$\begin{aligned} E \left[ (\sinh(B_t))^4 \right] &= E \left[ (\delta_{A_t})^4 \right] = E \left[ (A_t)^2 (\delta_1)^4 \right] = E \left[ (A_t)^2 \right] E \left[ (\delta_1)^4 \right] \\ &= 3E \left[ (A_t)^2 \right]. \end{aligned}$$

Thus, for  $t = T_{-c,c}^\gamma$ , we have:

$$E \left[ \left( A_{T_{-c,c}^\gamma} \right)^2 \right] = \frac{1}{3} E \left[ \left| \sinh \left( B_{T_{-c,c}^\gamma} \right) \right|^4 \right]$$

So, we deduce (66). In order to prove (67), it suffices to remark that:

$$E \left[ T_c^{(0)} \right] = E \left[ T_{-c,c}^\theta \right] = E \left[ A_{T_{-c,c}^\gamma} \right] = E \left[ \left| \sinh \left( B_{T_{-c,c}^\gamma} \right) \right|^2 \right].$$

On the one hand, by using the density of  $B_{T_{-c,c}^\gamma}$ :

$$\begin{aligned} E \left[ \left| \sinh \left( B_{T_{-c,c}^\gamma} \right) \right|^4 \right] &= \int_{-\infty}^{\infty} \frac{dy}{2c} \frac{1}{\cosh\left(\frac{\pi y}{2c}\right)} \left| \sinh(y) \right|^4 \\ &= \int_0^{\infty} \frac{dy}{c} \frac{1}{\cosh\left(\frac{\pi y}{2c}\right)} (\sinh y)^4 \\ &\stackrel{z=\frac{y}{c}}{=} \int_0^{\infty} dz \frac{1}{\cosh\left(\frac{\pi z}{2}\right)} (\sinh(cz))^4. \end{aligned}$$

which is finite if and only if  $c < \frac{\pi}{8}$ . In order to prove this, it suffices to use the standard expressions:  $\sinh(x) = \frac{e^x - e^{-x}}{2}$  and  $\cosh(x) = \frac{e^x + e^{-x}}{2}$ . On the other hand (note  $T \equiv T_{-c,c}^\gamma$ ), we remark that  $-B_T \stackrel{(law)}{=} B_T$  and (see e.g. Revuz-Yor [ReY99], ex.3.10)  $E[e^{kB_T}] = E\left[e^{\frac{k^2}{2}T}\right] = \frac{1}{\cos(kc)}$ , for  $0 \leq k < \pi(2c)^{-1}$ , thus:

$$\begin{aligned} E\left[|\sinh(B_T)|^4\right] &= \frac{1}{2^4} E\left[(e^{B_T} - e^{-B_T})^4\right] \\ &= \frac{1}{2^4} E\left[e^{4B_T} - 4e^{3B_T - B_T} + 6e^{2B_T - 2B_T} - 4e^{B_T - 3B_T} + e^{-4B_T}\right] \\ &= \frac{1}{2^4} (2E[e^{4B_T}] - 8E[e^{2B_T}] + 6) \\ &= \frac{1}{2^3} \left(\frac{1}{\cos(4c)} - 4\frac{1}{\cos(2c)} + 3\right) \end{aligned}$$

which is precisely (69) and this is finite if and only if  $c < \frac{\pi}{8}$ . Moreover, asymptotically for  $c \rightarrow 0$ , by using the scaling property, we have:

$$\begin{aligned} E\left[|\sinh(B_{T_{-c,c}^\gamma})|^4\right] &= E\left[\left(\sinh(cB_{T_{-1,1}^\gamma})\right)^4\right] \stackrel{c \rightarrow 0}{\simeq} c^4 E\left[\left(B_{T_{-1,1}^\gamma}\right)^4\right] \\ &= \underbrace{c^4 3 E\left[\left(T_{-1,1}^\gamma\right)^2\right]}_{5/3} = 5c^4 \end{aligned}$$

where  $E\left[\left(T_{-1,1}^\gamma\right)^2\right] = 5/3$  comes from Pitman-Yor [PiY03] (by using the notation of this paper, Table 3:  $E[X_t^2] = \frac{t(2+3t)}{3}$  for  $X_t = C_1$  and  $t = 1$ ). This asymptotics may also be obtained by (69) by developing  $\cos(4c)$  and  $\cos(2c)$  into series up to the second order term and keeping the terms of the order  $c^4$ . □

**Remark 3.4** *If we slightly modify the above study for the Ornstein-Uhlenbeck process by inserting a diffusion coefficient  $D$ :*

$$Z_t = z_0 + \sqrt{2D}\tilde{Z}_t - \lambda \int_0^t Z_s ds,$$

we obtain:

$$\begin{aligned} Z_t &= e^{-\lambda t} \left( z_0 + \sqrt{2D} \int_0^t e^{\lambda s} d\tilde{Z}_s \right) \\ &= e^{-\lambda t} (\mathbb{B}_{\alpha t}) \end{aligned} \tag{71}$$

where in the second equation we used Dambis-Dubins-Schwarz Theorem with

$$\begin{aligned}\alpha_t &= 2D \int_0^t e^{2\lambda s} ds = D \frac{e^{2\lambda t} - 1}{\lambda} \\ \Rightarrow \alpha_t^{-1} &= \frac{1}{2\lambda} \ln \left( 1 + \frac{\lambda}{D} t \right)\end{aligned}$$

So:

$$2\lambda E \left[ T_c^{(\lambda)} \right] - \ln \left( \frac{\lambda}{D} \right) \xrightarrow{\lambda \rightarrow \infty} E \left[ \ln \left( T_{-c,c}^\theta \right) \right] \quad (72)$$

because:

$$\begin{aligned}E \left[ T_c^{(\lambda)} \right] &= \frac{1}{2\lambda} E \left[ \ln \left( 1 + \frac{\lambda}{D} T_{-c,c}^\theta \right) \right] \\ &= \frac{1}{2\lambda} E \left[ \ln \left( \frac{\lambda}{D} \left( T_{-c,c}^\theta + \frac{D}{\lambda} \right) \right) \right] \\ &= \frac{\ln \left( \frac{\lambda}{D} \right)}{2\lambda} + \frac{1}{2\lambda} E \left[ \ln \left( T_{-c,c}^\theta + \frac{D}{\lambda} \right) \right].\end{aligned}$$

Moreover:

$$\begin{aligned}E \left[ \ln \left( T_{-c,c}^\theta \right) \right] &= 2 \ln(z_0) + E \left[ \ln \left( T_{-c,c}^{\theta(1)} \right) \right] \\ &= 2 \ln(z_0) + \int_0^\infty \frac{dz}{\cosh \left( \frac{\pi z}{2} \right)} \ln(\sinh(cz)) + \ln(2) + O(\frac{1}{8})\end{aligned}$$

where  $T_{-c,c}^{\theta(1)}$  denotes the first hitting time of the symmetric conic boundary of angle  $c$  for a Brownian motion  $Z$  starting from 1.

For  $\lambda$  small, we replace  $2T_{-c,c}^\theta$  by  $\frac{z_0^2}{D} T_{-c,c}^\theta$  in the proof of Proposition 3.3 ( $\lambda$  small case) and we deduce:

$$\begin{aligned}T_c^{(\lambda)} &= \frac{1}{2\lambda} \ln \left( 1 + \lambda \frac{z_0^2}{D} T_{-c,c}^\theta \right) \\ &= \frac{1}{2\lambda} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\lambda \frac{z_0^2}{D} T_{-c,c}^\theta)^n}{n} \\ &= \frac{z_0^2}{2D} T_{-c,c}^\theta - \lambda \left( \frac{z_0^2}{2D} \right)^2 (T_{-c,c}^\theta)^2 + O(\lambda^2) + \dots\end{aligned}$$

By repeating the previous calculation, for  $c < \frac{\pi}{8}$ , we obtain:

$$\frac{1}{\lambda} \left( E \left[ T_c^{(\lambda)} \right] - \frac{z_0^2}{2D} E \left[ |\sinh(B_{T_{-c,c}^\theta})|^2 \right] \right) \xrightarrow{\lambda \rightarrow 0} -\frac{1}{3} \left( \frac{z_0^2}{2D} \right)^2 E \left[ |\sinh(B_{T_{-c,c}^\theta})|^4 \right]$$

and  $E \left[ \left| \sinh \left( B_{T_{-c,c}^\gamma} \right) \right|^4 \right]$  is given by (68), (69) and asymptotically, for  $c \rightarrow 0$  by (70).

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