Isotropic Contact Forces in Arbitrary Representation: Heterogeneous Few-Body Problems and Low Dimensions

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Ludovic Pricoupenko
Laboratoire de Physique Théorique de la Matière Condensée,
Université Pierre et Marie Curie and CNRS, 4 place Jussieu, 75252 Paris, France.
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The Bethe-Peierls asymptotic approach which models pairwise short-range forces by contact conditions is introduced in arbitrary representation for spatial dimensions less than or equal to three. The formalism is applied in various situations and emphasis is put on the momentum representation. In presence of a transverse harmonic confinement, dimensional reduction toward 2D or 1D physics is derived within this formalism. The energy theorem relating the mean energy of an interacting system to the asymptotic behavior of the one-particle density matrix illustrates the method in its second quantized form. Integral equations that encapsulate the Bethe-Peierls contact condition for few-body systems are derived. In 3D, for three-body systems supporting Efimov states, a nodal condition is introduced in order to obtain universal results from the Skorniakov Ter-Martirosian equation and the Thomas collapse is avoided. Four-body bound states eigenequations are derived and the 2D '3+1' bosonic ground state is computed as a function of the mass ratio.

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The non-perturbative zero-range limit of a pairwise interaction was first introduced by Bethe and Peierls in the context of Nuclear Physics for the description of the deuteron [1–3]. In this modeling, the finite range pairwise interacting potential was replaced by a zero range potential and a peculiar asymptotic behavior was imposed on the wave function as the two interacting particles approach towards each other. More than sixty years after this pioneering work, the Bethe-Peierls approach appeared to be especially relevant for modeling pairwise interaction in few- and many-body systems of ultra-cold atoms where details of the interaction are useless for understanding a large class of low energy process [4–17]. This remarkable simplicity follows essentially from two features: -first, the temperature in these systems is sufficiently low that s-wave scattering is the dominant process in two-body collisions [18]; -second, the 3D s-wave scattering length (denoted hereafter by a_3) in the two-body scattering can be tuned by use of a magnetic Feshbach Resonance (FR) toward an arbitrary large value with respect to the range of the interatomic forces, denoted in what follows by b_3 [19, 20] and which is of the order of the van der Waals radius:

$$b_3 \sim \left(\frac{\mu C_6}{\hbar^2}\right)^{1/4}, \qquad (1)$$

where C_6 is the van der Waals coefficient of the interatomic potential and μ is the reduced mass of the two interacting particles. In the vicinity of a broad FR the two-body scattering cross-section $4\pi a_3^2$ is very large with respect to the non-resonant cross section ($\sim 4\pi b_3^2$) and this justifies the use of the 3D Bethe-Peierls approach parameterized by the 3D scattering length only.

Ultra-cold atomic systems can be trapped in very anisotropic harmonic potentials leading to a D-dimensional behavior where $D = 1$ (quasi-1D systems) or $D = 2$ (quasi-2D systems) [21]. In an ideal D-dimensional atomic wave guide, non interacting atoms move freely along D direction(s) while they remain frozen in the lowest state of the transverse zero-point motion characterized by a length denoted hereafter by $a_{\rm trans}$. The associated energy $E_{\rm trans} = \hbar^2/(2\mu a_{\rm trans}^2)$, where μ is the reduced mass of the interacting pair, defines the limit of energy beyond which higher transverse states are populated and a 3D-like behavior is progressively recovered for increasing energies. In the actual experimental state-of-the-art, atomic waveguides have a transverse atomic length which is large as compared to the 3D potential radius (b_3). Therefore for low energy process *i.e.* for collisional energies E such that $|E| \ll \hbar^2/(\mu b_3^2)$, the short-range pairwise interacting potential can be described through the 3D Bethe-Peierls approach. For collisional energies much smaller than $E_{\rm trans}$ the transverse excited states of the trap are populated only *via* virtual process while for large interatomic separation r such that $r \gg a_{\rm trans}$, only the ground transverse state is occupied. In this regime of collisional energies and in the free D-dimensional subspace, scattering process can be deduced from a D-dimensional effective pairwise interaction characterized by a finite range hereafter denoted by b_D, which is of the order of the transverse length $a_{\rm trans}$. Hence for low energy D-dimensional process where $|E| \ll \hbar^2/(\mu b_D^2)$ the effective low-dimensional interaction can be analogously to the 3D case, replaced by a zero-range force. Therefore in each dimension D less or equal to three, a D-dimensional Bethe-Peierls approach can be relevant and useful as a tool for exploring few- and many-body shallow states properties. The dimensional reduction of the effective interaction from 3D to 1D has been first achieved in Ref. [4] and from 3D to 2D in Ref. [5].

One of the purpose of this paper is to show that the zero-range potential approach can be handle in a very simple and unified way in any dimension $D \leq 3$. The key tool used along these lines is introduced in section-I

of the paper. It consists in the general expression of the Bethe-Peierls asymptotic condition which can be used in any representation for all dimension D. Substitution of the δ-source terms of the zero range approach by a family of well behaved functions of vanishing but finite support, allows one to avoid technical problems concerning double limit calculations (*i.e.* evaluation of infinite series in the zero range limit). While used usually in the configuration space, the zero-range approach can thus be implemented directly in the momentum representation for translation invariant systems. The link between this formulation of the Bethe-Peierls approach and the Λ-potential introduced in Ref. [22] is analyzed. The method is illustrated in section-II with the example of the dimensional reduction issue from 3D towards 1D and 2D. The Bethe-Peierls contact condition is directly expressed in the functional basis which diagonalizes the free Hamiltonian and known results are obtained in a simple manner. For N-body systems, integral equations which encapsulate the Bethe-Peierls contact condition are obtained in section-III. It is also shown how to express this zero-range force approach in a second quantized representation. As an example, energy theorems [23–26] are derived by using this last formulation. Section-IV of the paper is devoted to the three- and four-body shallow bound state issue. For three identical bosons in 3D, the original Skorniakov Ter-Martirosian (STM) equation [27] appears as a straightforward application of the results of the preceding section. A nodal condition is imposed on the three-body wave function in order to avoid the Thomas collapse [28] while keeping the simplicity of the zero-range approach [29]. Known results of universal theory [30] are recovered from the regularized STM equation. For two-mass component fermionic systems in 3D, critical mass ratios for the Efimov threshold are given in each partial wave by using the STM equation. In 2D, the eigenenergy equation for two-mass component tetramers is studied numerically. Energy of the ground s-wave of '3+1' bosonic tetramers in the zero-range limit is computed as a function of the mass ratio.

In this paper the norm of any vector $\mathbf{v}$ is denoted by v. For a uni-dimensional system ($D = 1$), $\mathbf{v}$ is algebraic and $v = |\mathbf{v}|$ is the absolute value of $\mathbf{v}$.

I. ISOTROPIC CONTACT FORCES

A. Two-body transition matrix and scattering amplitude

This section reviews basic notions of two-body scattering in D-dimensional spaces which are useful in the formulation of the Bethe-Peierls approach.

Two colliding particles are described in their center of mass frame by the relative particle of reduced mass μ and relative momentum $\mathbf{k}$. They are supposed to interact through a short range potential V, and the Hamiltonian for the relative particle is $H = H_0 + V$, where H_0 is the free Hamiltonian. In what follows, as a consequence of the small collisional energy, scattering is supposed to occur in the s-wave channel of the relative particle only. For an incoming atomic wave of momentum $\mathbf{k}_0$ and of collisional energy $E = \hbar^2 k_0^2/2\mu$ (measured from the continuum threshold), the scattering state $|\Psi_{\mathbf{k}_0}\rangle$ of the relative particle verifies the Lippmann-Schwinger equation:

$$|\Psi_{\mathbf{k}_0}\rangle = |\mathbf{k}_0\rangle + \frac{1}{E + i0^+ - H_0} V |\Psi_{\mathbf{k}_0}\rangle. \tag{2}$$

We now turn to the $\mathbf{k}$-representation with the convention $\langle \mathbf{r}|\mathbf{k}\rangle = \exp(i\mathbf{k}\cdot\mathbf{r})$:

$$\langle \mathbf{r}|\Psi\rangle = \int \frac{d^D\mathbf{k}}{(2\pi)^D} \langle \mathbf{k}|\Psi\rangle \exp(i\mathbf{k}\cdot\mathbf{r}). \tag{3}$$

In the $\mathbf{k}$-representation, the scattering states in Eq. (2) can be written as a function of the half on-shell transition matrix (or t-matrix) defined by:

$$\langle \mathbf{k}|T(E + i0^+)|\mathbf{k}_0\rangle = \langle \mathbf{k}|V|\Psi_{\mathbf{k}_0}\rangle, \tag{4}$$

so that:

$$\langle \mathbf{k}|\Psi_{\mathbf{k_0}}\rangle = (2\pi)^D \delta(\mathbf{k} - \mathbf{k}_0) + \frac{\langle \mathbf{k}|T(E + i0^+)|\mathbf{k}_0\rangle}{E + i0^+ - \frac{\hbar^2 k^2}{2\mu}}. \tag{5}$$

The pairwise interaction is supposed to be short range with a typical radius denoted by b_D. For a small energy $|E| \ll \hbar^2/\mu b_D^2$ and momentum $|\mathbf{k}|b_D \ll 1$, the half on-shell t-matrix depends on the collisional energy E only and coincides with the low energy on-shell t-matrix. In what follows this function is denoted by T_D:

$$\langle \mathbf{k}|T(E + i0^+)|\mathbf{k}_0\rangle \simeq T_D(E + i0^+). \tag{6}$$

At the lowest order in energy (implying that $|k_0|b \ll 1$), the expression of T_D can be parameterized by the D-dimensional scattering length a_D only. It can be written as [31, 32]:

$$T_D(E + i0^+) = \frac{\Omega_D \hbar^2}{2\mu} \times \begin{cases} \dfrac{a_3}{(1 + ia_3k_0)}; & (D = 3) \\ \dfrac{-1}{\ln(-ia_2k_0e^{\gamma}/2)}; & (D = 2) \\ \dfrac{-ik_0}{(1 + ia_1k_0)}; & (D = 1) \end{cases}, \tag{7}$$

where Ω_D is the full D-dimensional space angle:

$$\Omega_3 = 4\pi,\ \Omega_2 = 2\pi,\ \text{and}\ \Omega_1 = 2. \tag{8}$$

In Eq. (7) for the two-dimensional case, γ is the Euler's constant, and the 2D scattering length a_2 is always positive. The transition matrices in Eq. (7) are the basic objects for describing low energy scattering process in D-dimensional few- and many-body systems. Scattering

properties can be equivalently described through the D-dimensional scattering amplitudes defined by:

$$f_3(k_0) = -a_3/(1+ik_0a_3); \tag{9}$$
$$f_2(k_0) = 1/\ln(-ik_0a_2e^{\gamma}/2); \tag{10}$$
$$f_1(k_0) = -1/(1+ik_0a_1). \tag{11}$$

The notion of scattering amplitude is often used in configuration space where the scattering states for large relative coordinates ($r \gg b_D$) are deduced from Eq. (7) with:

$$\langle \mathbf{r}|\Psi_{\mathbf{k_0}}\rangle = e^{i\mathbf{k}_0\cdot\mathbf{r}} + f_D(k_0) \times \begin{cases} \frac{e^{ik_0r}}{r}; & (D=3) \\ \frac{i\pi}{2}H_0^{(1)}(k_0r); & (D=2) \\ e^{ik_0r}; & (D=1) \end{cases} \cdot \tag{12}$$

In Eq. (12), for $D=2$ the outgoing scattering wave function $H_0^{(1)}$ is the Hankel's function of order zero [33]; for $D=1$, f_1 is also called the even scattering amplitude [4].

B. Bethe-Peierls asymptotic approach in arbitrary representation

Surprisingly the zero-range approximation has been essentially used in the configuration space while other representations can yield substantial simplifications. For example, as a consequence of translation invariance, the momentum representation is very well suited for solving the few-body problem in homogeneous space. In this section, it is shown how the zero-range approximation can be formulated in arbitrary representation.

While for finite range forces Eq. (6) is valid for a small relative momentum k only (*i.e.* for $kb_D \ll 1$), in the zero-range potential approximation this equality is extended for arbitrary large values of k. That way, the short range pairwise potential V is replaced by a formal zero-range potential such that the scattering states in this approximation coincide with Eq. (5) in the small relative momentum limit ($kb_D \ll 1$) or with Eq. (12) for large interparticle distances ($r \gg b_D$). The simplest way to implementing the zero-range approximation is to follow the Bethe-Peierls method [1], where for any state $|\Psi\rangle$ described by the Hamiltonian H_0+V, the pairwise short range potential V is replaced by a delta-source term $|\delta^D\rangle$ with an amplitude S_Ψ:

$$V|\Psi\rangle \longrightarrow S_\Psi|\delta^D\rangle, \tag{13}$$

and S_Ψ is deduced from an asymptotic condition on the state $|\Psi\rangle$. In Eq. (13), $|\delta^D\rangle$ is such that the usual δ-distribution is obtained in the configuration space:

$$\langle \mathbf{r}|\delta^D\rangle = \delta(\mathbf{r}), \tag{14}$$

and $|\delta^D\rangle$ is also denoted by the formal ket of zero relative coordinates $|0_{\mathbf{r}}\rangle$. From Eq. (5) the scattering state $|\Psi_{\mathbf{k_0}}\rangle$ is written in the zero-range potential approach as:

$$|\Psi_{\mathbf{k_0}}\rangle = |\mathbf{k}_0\rangle + \frac{T_D(E+i0^+)}{E+i0^+-H_0}|\delta^D\rangle. \tag{15}$$

Injecting Eq. (13) in the Lippmann-Schwinger equation (2) gives:

$$|\Psi_{\mathbf{k_0}}\rangle = |\mathbf{k}_0\rangle + \frac{S_{\Psi_{\mathbf{k_0}}}}{E+i0^+-H_0}|\delta^D\rangle. \tag{16}$$

Identifying Eq. (15) and Eq. (16) shows that the source amplitude associated with a scattering state $|\Psi_{\mathbf{k_0}}\rangle$ is nothing but the on-shell transition matrix: $S_{\Psi_{\mathbf{k_0}}} = T_D(E+i0^+)$. In configuration space, scattering states in Eq. (15) or in Eq. (12) are singular for vanishingly small interparticle distances. As shown in what follows, this singular behavior defines the Bethe-Peierls asymptotic condition. From Eq. (15) one finds that all the scattering states for a given dimension D have the same singular behavior which reduces to:

$$\langle \mathbf{r}|\Psi\rangle \underset{\mathbf{r}\to 0}{=} A \times \begin{cases} \left(\frac{1}{a_3}-\frac{1}{r}\right)+O(r); & (D=3) \\ \ln\left(\frac{r}{a_2}\right)+O(r); & (D=2) \\ (r-a_1)+O(r); & (D=1) \end{cases} \cdot \tag{17}$$

In Eq. (17) A is a function of the energy E but does not depend on r. It is important to note that Eq. (17) is obtained in the center of mass frame, meaning that the limit $r \to 0$ is taken at fixed value of the center of mass of the two colliding particles. Conversely, for a given dimension D, Eq. (17) is the sufficient asymptotic condition which permits to obtain the source amplitude $S_{\Psi_{\mathbf{k}_0}}$ of a scattering state $|\Psi_{\mathbf{k}_0}\rangle$ at energy E in Eq. (16). The asymptotic condition in Eq. (17) referred also as the 'contact condition' or the Bethe-Peierls condition, remains unchanged for any linear combination of scattering states and can be thus used to found any eigenstate in the zero-range approximation for a system where an external potential is included within the free Hamiltonian H_0. For a positive energy E, the Lippmann Schwinger equation is

$$|\Psi\rangle = |\Psi^{(0)}\rangle + \frac{S_\Psi}{E+i0^+-H_0}|\delta^D\rangle, \tag{18}$$

where $|\Psi^{(0)}\rangle$ is the complementary solution *i.e* it is an eigenstate of the free Hamiltonian H_0 at energy E (for a negative energy $|\Psi^{(0)}\rangle = 0$ and the $+i0^+$ prescription in Eq. (18) is not useful). The relation between the number A in Eq. (17) and the source amplitude S_Ψ in Eq. (18) is given by:

$$A = \frac{2\mu \mathrm{S}_\Psi}{\Omega_D\hbar^2}. \tag{19}$$

The crucial point of this section is to show that the asymptotic condition in Eq. (17) can be expressed in arbitrary representation. In what follows for convenience,

the delta distribution is represented by the $\epsilon \to 0$ limit of a Gaussian weight:

$$\langle \mathbf{r}|\delta_\epsilon^D\rangle = \frac{1}{(2\pi\epsilon^2)^{D/2}} \exp\left(-\frac{r^2}{\epsilon^2}\right), \tag{20}$$

so that:

$$\delta^D(\mathbf{r}) = \lim_{\epsilon\to 0}\langle \mathbf{r}|\delta_\epsilon^D\rangle. \tag{21}$$

With this particular choice, in the momentum space the representation of the delta term has the same expression [denoted by $\chi_\epsilon(k)$] for all dimensions:

$$\langle \mathbf{k}|\delta_\epsilon^D\rangle = \chi_\epsilon(k) = \exp\left(-\frac{k^2\epsilon^2}{4}\right). \tag{22}$$

Matrix elements and states without index ϵ are considered in their zero-range limit which corresponds in this formalism to the limit $\epsilon \to 0$:

$$\lim_{\epsilon\to 0}|\Psi_\epsilon\rangle = |\Psi\rangle \tag{23}$$

For the formulation of the zero-range approximation it is also useful to introduce the 'reference state' denoted by $|\phi_\epsilon^\Lambda\rangle$:

$$\langle \mathbf{k}|\phi_\epsilon^\Lambda\rangle = -\frac{2\mu}{\hbar^2}\times\frac{\chi_\epsilon(k)}{k^2+\Lambda^2}. \tag{24}$$

This state results from the action of the 2-body Green's function G_{E_Λ} in free space on $|\delta_\epsilon^D\rangle$ at the negative energy E_Λ:

$$|\phi_\epsilon^\Lambda\rangle = G_{E_\Lambda}|\delta_\epsilon^D\rangle, \tag{25}$$

where:

$$G_{E_\Lambda} = \frac{1}{E_\Lambda - H_0} \quad \text{and} \quad E_\Lambda = -\frac{\hbar^2\Lambda^2}{2\mu} < 0. \tag{26}$$

In Eq. (26) the parameter Λ is chosen positive ($\Lambda \in \mathbb{R}^+$) according to the usual prescription in scattering theory. In the zero-range limit where $\epsilon \to 0$, in the configuration space the reference state has the same type of singularity as in Eq. (17):

$$\langle \mathbf{r}|\phi^\Lambda\rangle = \frac{2\mu}{\Omega_D\hbar^2}\begin{cases} \left(-\dfrac{1}{r}+\Lambda\right)+O(r); & (D=3) \\ \ln\left(\dfrac{\Lambda r e^\gamma}{2}\right)+O(r); & (D=2) \\ \left(r-\dfrac{1}{\Lambda}\right)+O(r); & (D=1) \end{cases}. \tag{27}$$

The contact condition in Eq. (17) can then be written in terms of the reference state as:

$$\lim_{r\to 0}\lim_{\epsilon\to 0}\langle \mathbf{r}|\Psi_\epsilon - \mathrm{S}_\Psi\phi_\epsilon^\Lambda\rangle = \frac{\mathrm{S}_\Psi}{T_D(E_\Lambda)}. \tag{28}$$

In the configuration space, the state $|\Psi_\epsilon - \mathrm{S}_\Psi\phi_\epsilon^\Lambda\rangle$ is a smooth function for (r,ϵ) close to $(0,0)$. It is thus possible to commute the $r\to 0$ and $\epsilon\to 0$ limits in Eq. (28):

$$\lim_{r\to 0}\lim_{\epsilon\to 0}\langle \mathbf{r}|\Psi_\epsilon - \mathrm{S}_\Psi\phi_\epsilon^\Lambda\rangle = \lim_{\epsilon\to 0}\lim_{r\to 0}\langle \mathbf{r}|\Psi_\epsilon - \mathrm{S}_\Psi\phi_\epsilon^\Lambda\rangle. \tag{29}$$

For particles moving in presence of an external potential, the source amplitude in configuration space depends on the center of mass coordinates of the interacting pair. Hence, in general situations the source amplitude is replaced by a state associated with the center of mass of the interacting pair:

$$S_\Psi \longrightarrow |\mathrm{S}_\Psi\rangle. \tag{30}$$

Finally, the Bethe-Peierls condition can be written without specifying any representation as:

$$\lim_{\epsilon\to 0}\langle 0_{\mathbf{r}}|\Psi_\epsilon - \mathrm{S}_\Psi\phi_\epsilon^\Lambda\rangle = \frac{|\mathrm{S}_\Psi\rangle}{T_D(E_\Lambda)}. \tag{31}$$

Eq. (31) is by construction, invariant in a change of $\Lambda \in \mathbb{R}^+$: as it will be shown in the next section, one recovers the *so-called* Λ-freedom of the Λ-potential [22]. This freedom permits to simplify exact calculations without introducing any approximation, or also to improve approximate schemes [22, 35, 36]. Moreover, the $\epsilon\to 0$ limit can be taken equivalently as follows:

$$\lim_{\epsilon\to 0}\langle \delta_\epsilon^D|\Psi_\epsilon - \mathrm{S}_\Psi\phi_\epsilon^\Lambda\rangle = \frac{|\mathrm{S}_\Psi\rangle}{T_D(E_\Lambda)} \tag{32}$$

or also by substituting the ket in the left hand side of Eq. (32) by its $\epsilon\to 0$ limit:

$$\lim_{\epsilon\to 0}\langle \delta_\epsilon^D|\Psi - \mathrm{S}_\Psi\phi^\Lambda\rangle = \frac{|\mathrm{S}_\Psi\rangle}{T_D(E_\Lambda)}. \tag{33}$$

The contact conditions in Eqs. (31,32,33) can be expressed in any desired representation by inserting a closure relation in the scalar product concerning the relative particle. For example in the momentum representation, one can insert at the right of the bra $\langle 0_{\mathbf{r}}|$ in Eq. (32) the closure relation:

$$\int \frac{d^D\mathbf{k}}{(2\pi)^D}|\mathbf{k}\rangle\langle\mathbf{k}| = \mathbb{I}, \tag{34}$$

and this gives:

$$\lim_{\epsilon\to 0}\left(\int \frac{d^D\mathbf{k}}{(2\pi)^D}\chi_\epsilon(k)\langle\mathbf{k}|\Psi - \mathrm{S}_\Psi\phi^\Lambda\rangle\right) = \frac{|\mathrm{S}_\Psi\rangle}{T_D(E_\Lambda)}. \tag{35}$$

In the next sections, it will be shown that Eqs. (31,32,33) allow one to obtain in a simple way the standard integral equations of few-body problems in the zero range limit and also to compute the induced scattering resonances and related scattering problems in presence of an harmonic transverse confinement. As a conclusion of this part, it is interesting (and useful) to note that for $D = 3$

or $D=1$, it is possible to eliminate the presence of the reference function in the contact condition by performing a specific limit on Λ. For $D=3$, in the $\Lambda \to 0$ limit the contact condition can be written as:

$$\lim_{\epsilon\to 0} \partial_\epsilon \left(\epsilon \langle \delta_\epsilon^{D=3}|\Psi\rangle\right) = \frac{\mu |\mathrm{S}_\Psi\rangle}{2\pi\hbar^2 a_3}. \tag{36}$$

Eq. (36) can be written in the alternative following form:

$$\operatorname*{Reg}_{\epsilon\to 0} \langle \delta_\epsilon^{D=3}|\Psi\rangle = \frac{\mu |\mathrm{S}_\Psi\rangle}{2\pi\hbar^2 a_3}, \tag{37}$$

where $\operatorname*{Reg}_{\epsilon\to 0}$ extracts the regular part of $\langle \delta_\epsilon^{D=3}|\Psi\rangle$ in the limit where $\epsilon \to 0$. For $D=1$, elimination of the reference function occurs in the limit where Λ tends to ∞. One obtains:

$$\lim_{\epsilon\to 0} \langle \delta_\epsilon^{D=1}|\Psi\rangle = -\frac{\mu a_1}{\hbar^2}|S_\Psi\rangle. \tag{38}$$

C. Link with the *s*-wave Λ-potential

The zero-range s-wave pseudo-potential was introduced by Fermi [37] in order to perform calculations in the first order Born approximation. The Fermi pseudo-potential was not thus constructed for non perturbative approach (for large values of a_3 for example). The modern formulation of the zero-range pseudo-potential for $D=3$ is due to Breit [2]. In this approach, the Bethe-Peierls condition is included in the Schrödinger equation *via* the zero-range pseudo-potential. It has been shown in the configuration space that there exists in each dimension D a family of zero-range potentials: the *so-called* Λ-potentials [22]. This section, links the Bethe-Peierls approach and the Λ-potential in arbitrary representation. For this purpose, a family of operators $\mathrm{R}_\epsilon^\Lambda$ is introduced. They act on a state verifying the asymptotic Bethe-Peierls condition in Eq. (17) as:

$$\mathrm{R}_\epsilon^\Lambda |\Psi_\epsilon\rangle = |\delta_\epsilon^D\rangle \lim_{\epsilon\to 0} \langle \delta_\epsilon^D|\Psi_\epsilon - \mathrm{S}_\Psi \phi_\epsilon^\Lambda\rangle. \tag{39}$$

These operators have the two obvious properties:

$$\mathrm{R}_\epsilon^{\Lambda=\frac{1}{a_D}} |\Psi_\epsilon\rangle = 0 \quad \text{and} \quad \mathrm{R}_\epsilon^\Lambda |\phi_\epsilon^\Lambda\rangle = 0. \tag{40}$$

Explicit expressions for $\mathrm{R}_\epsilon^\Lambda$ can be found without specifying any representation as follows:

$$\mathrm{R}_\epsilon^\Lambda |\Psi_\epsilon\rangle = |\delta_\epsilon^D\rangle \lim_{\epsilon\to 0} r_\epsilon^\Lambda \left[\langle \delta_\epsilon^D|\Psi_\epsilon\rangle\right], \tag{41}$$

where r_ϵ^Λ is an operator defined in each dimension as:

$$r_\epsilon^\Lambda[\,\cdot\,] = \begin{cases} \left[\left(\partial_\epsilon + \sqrt{\frac{\pi}{2}}\Lambda\right)\epsilon\,\cdot\right]; & (D=3) \\ \left[\left(1+\frac{\epsilon}{2}\ln(e^\gamma\Lambda^2\epsilon^2/2)\partial_\epsilon\right)\cdot\right]; & (D=2) \\ \left[\left(1+\sqrt{\frac{\pi}{2}}\frac{1}{\Lambda}\partial_\epsilon\right)\cdot\right]; & (D=1) \end{cases} \tag{42}$$

Expressions of r_ϵ^Λ in Eq. (42) depend on the choice made for the short range functions $\langle \mathbf{r}|\delta_\epsilon^D\rangle$, given here by Eq. (20). The source term in the stationary Schrödinger equation can be expressed in terms of $\mathrm{R}_\epsilon^\Lambda$ by using Eqs. (31,39):

$$(H_0 - E)\,|\Psi_\epsilon\rangle + T_D(E_\Lambda)\mathrm{R}_\epsilon^\Lambda |\Psi_\epsilon\rangle = 0. \tag{43}$$

In the limit $\epsilon \to 0$, the pseudo-potential in this equation coincides exactly with the Λ-potential introduced in Ref. [22], where the coupling constant g_Λ is nothing but the transition matrix at energy E_Λ:

$$g_\Lambda = T_D(E_\Lambda). \tag{44}$$

D. Context of resonant scattering

The zero-range approximation is especially interesting in the regime of parameters where the pairwise potential leads to a resonant scattering at low collisional energy. In this regime, the cross-section proportional to $|f_D|^2$ has a large value at small relative momentum ($k_0 b_D \ll 1$) and expressions in Eqs. (9,10,11,12) are good approximations. For $D=3$, this corresponds to the regime where the scattering length a_3 is large in absolute value as compared to the potential radius b_3 and the maximum is at $k_0 = 0$. This regime can be achieved by using the Feshbach Resonance (FR) mechanism (see Ref. [20] for a general review of FR in ultracold atoms). The FR involves the coupling between atoms (in the 'open channel') and a molecular state (in the 'closed channel') characterized by a size of the order of the potential range b_3. This two-channel description provides the expression of the scattering length (a_3) as a function of the external magnetic field $\mathcal{B}$ in the vicinity of a given resonance located at $\mathcal{B} = \mathcal{B}_0$ which can be parameterized by the formula [38]:

$$a_3 = a_{\mathrm{bg}}\left(1 - \frac{\Delta\mathcal{B}}{\mathcal{B}-\mathcal{B}_0}\right). \tag{45}$$

In Eq. (45), $\Delta\mathcal{B}$ is the width of the magnetic FR and a_{bg} is the 'background' scattering length *i.e.* the scattering length away from the FR. The magnetic width ($\Delta\mathcal{B}$) can be also characterized by the 'width radius' $R^\star$ defined by

$$R^\star = \frac{\hbar^2}{m a_{\mathrm{bg}} \delta\mu \Delta\mathcal{B}}, \tag{46}$$

where $\delta\mu$ is the difference of magnetic moment for an atomic pair in the open- and the closed-channel [20, 39]. In the vicinity of a broad resonance, $R^\star$ is of the order of magnitude or smaller than the 3D potential radius b_3 and the 3D scattering amplitude can be approximated by Eq. (9) in a large interval of momentum where $|k_0|b_3 \ll 1$.

As shown in the next section, the resonant behavior in low dimensional atomic wave guide can be reached by tuning the length a_{trans} of the transverse confinement and/or the 3D scattering length. For $D=1$ the resonant regime occurs in the limit where the 1D-scattering

length a_1 is small with respect to the transverse length (*i.e.* $|a_1| \ll b_1$) and also for vanishing relative momentum $k_0 = 0$. In 2D, the maximum in $|f_2|^2$ occurs at the momentum $k_0 = 2/(e^\gamma a_2)$ (where $|f_2| = 2/\pi$) and the resonant regime is thus reached at low energy if and only if the 2D-scattering length is large as compared to the transverse length (*i.e.* $a_2 \gg b_2$).

In the resonant regimes defined above, the probability of scattering for two colliding particles is large so that the pairwise short range potential affects the form of the wave function at interparticle distances which are large with respect to the potential radius b_D. More precisely, the scattering states in Eq. (5) can be decomposed in an incoming part $|\phi_{\rm inc}\rangle = |\mathbf{k}_0\rangle$ and a scattered part $|\phi_{\rm scatt}\rangle$. At large distance and in the resonant regime:

$$\left|\frac{\langle \mathbf{r}|\phi_{\rm scatt}\rangle}{\langle \mathbf{r}|\phi_{\rm inc}\rangle}\right| \underset{r\gg b_D}{=} \begin{cases} |a_3|/r; & (D=3) \\ \sqrt{e^\gamma a_2}/\sqrt{\pi r}; & (D=2) \\ 1; & (D=1) \end{cases}. \qquad (47)$$

For $D = 3$ or $D = 2$ this ratio is greater than or of the order of unity for $b_D \ll r < |a_D|$, where $|a_D|$ is arbitrarily large and is equal to unity for arbitrary large distance in one-dimensional systems ($D = 1$). The zero-range potential approach is a formalism which permits to evaluate accurately the wave function in configurations where particles are outside the potential radius while configuration where two or more particles are inside the potential radius are not reliably described. Consequently, the Bethe-Peierls asymptotic approach is very well suited for studying systems in the resonant regime where the wavefunction is modified by the interaction at interparticle distances which are large as compared to the potential radius.

In the three-dimensional space ($D = 3$), for a positive scattering length a_3 the on-shell transition matrix T_3 in Eq. (7) has a real pole at negative energy $E_2 = -\hbar^2/(2\mu a_3^2)$. In the resonant regime, this pole is associated with the existence of a shallow dimer which is thus very well described in the zero range approach (the probability that the relative pair has a radius greater than the potential radius b_3 is $\exp(-2b_3/a_3) \sim 1$). For $D = 2$, the pole is at $E_2 = -2\hbar^2/(\mu a_2^2 \exp(2\gamma))$, and in the resonant regime it is also the signature of the existence of a shallow dimer. For $D = 1$, in the resonant regime ($a_1 \to 0$) the approximation of the transition matrix in Eq. (7) doesn't possess a negative low energy pole and next order terms in the energy expansion are needed to find the pole corresponding to the lowest bound state [40].

II. DIMENSIONAL REDUCTION

In this section, the dimensional reduction issue is solved by using the zero-range approximation with the Bethe-Peierls condition expressed through Eq. (36). This section gives alternative derivations of the results given in Refs. [4–7]. The problematic of the 3D→1D (respectively 3D→2D) dimensional reduction is as follows: two particles move freely in 1D (resp. 2D) while they are confined in the transverse direction by an atomic wave guide built from a 2D (resp. 1D) trapping potential. The collisional energy is such that at large interatomic distances, the particles are confined in the ground state of the trapping potential. The problem is to find the low-energy transition matrix of this low dimensional quasi-2D (resp quasi-1D) scattering process as a function of the wave guide parameters and of the three dimensional scattering length a_3. In what follows only the case of harmonic trapping where the center of mass and relative motions decouple are considered. In the center of mass frame the source amplitude S_Ψ is constant and the scattering state can be written as:

$$|\Psi\rangle = |\Psi^{(0)}\rangle + |\Psi^{\rm int}\rangle, \qquad (48)$$

where $|\Psi^{(0)}\rangle$ is the incident wave, the 'interacting part' is

$$|\Psi^{\rm int}\rangle = \frac{\mathrm{S}_\Psi}{E + i0^+ - H_0}|0_{\mathbf{r}}\rangle, \qquad (49)$$

and the source amplitude is found from the Bethe-Peierls condition by solving the equation:

$$\langle 0_{\mathbf{r}}|\Psi^{(0)}\rangle + \underset{\epsilon\to 0}{\rm Reg}\langle \delta_\epsilon^{\mathrm{D}=3}|\Psi^{\rm int}\rangle = \frac{\mu \mathrm{S}_\Psi}{2\pi\hbar^2 a_3}. \qquad (50)$$

These quasi-1D and quasi-2D scattering problems have been initially solved by using the Bethe-Peierls method in the configuration space in Ref. [4] and Refs. [5, 6]. In what follows, an alternative derivation is given in order to illustrate the formalism introduced in the previous section.

A. Linear Atomic Wave Guide

The 3D→1D reduction problem, is considered here in the case where the two colliding atoms are confined in an isotropic two-dimensional harmonic trap while they move freely along the third direction (z). In this case, the problem can be solved in the center of mass frame where the non-interacting Hamiltonian is

$$H_0 = -\frac{\hbar^2}{2\mu}\partial_z^2 + \mathcal{H}_{\rm 2D}, \qquad (51)$$

and $\mathcal{H}_{\rm 2D}$ is the Hamiltonian for the transverse motion:

$$\mathcal{H}_{\rm 2D} = -\frac{\hbar^2}{2\mu}(\partial_x^2 + \partial_y^2) + \frac{1}{2}\mu\omega_\perp^2(x^2 + y^2) - \hbar\omega_\perp. \qquad (52)$$

The linear atomic wave guide is characterized by the transverse length:

$$a_\perp = \sqrt{\frac{\hbar}{\mu\omega_\perp}}, \qquad (53)$$

and the zero-range approximation of the $D=3$ pairwise potential is justified in the limit where $a_\perp \gg b_3$. In Eq. (52), the zero-point energy has been subtracted. That way, the energy E of the scattering states is measured with respect to the continuum threshold. In this quasi-1D scattering problem at energy $E=\hbar^2k_0^2/2\mu$, the incoming state for the relative particle ($|\Psi^{(0)}\rangle$) in Eq. (48) has a momentum $\mathbf{k}_0$ along z and is in the ground-state of the transverse 2D-oscillator:

$$|\Psi^{(0)}\rangle = |n_x=0\rangle|n_y=0\rangle|k_z=k_0\rangle, \tag{54}$$

where n_x and $|n_x\rangle$ (resp. n_y and $|n_y\rangle$) are the quantum number and eigenstate of the 1D-harmonic oscillator of atomic frequency $\omega_\perp$ and mass μ along x (resp. y). The system is quasi-1D for collisional energies such that:

$$E < E^{\rm 1D}_{\rm trans}, \tag{55}$$

where $E^{1D}_{\rm trans}=2\hbar\omega_\perp$. Eq. (55) defines the monomode regime of the atomic waveguide such that outgoing particles are trapped at large relative distances in the ground state of the transverse trap. The contribution of the incoming wave in Eq. (50) is:

$$\langle 0_{\mathbf{r}}|\Psi^{(0)}\rangle = \phi_0(0)^2 = \frac{1}{\sqrt{\pi a_\perp}}, \tag{56}$$

where $\phi_n(x)=\langle x|n\rangle$ is given by:

$$\phi_n(x) = \frac{\exp\left(\frac{-x^2}{2a_\perp^2}\right)}{\pi^{1/4}\sqrt{a_\perp}} H_n\left(\frac{x}{a_\perp}\right). \tag{57}$$

In the calculation of $\langle\delta_\epsilon^{\rm D=3}|\Psi^{\rm int}\rangle$, the quantum numbers of the non-interacting Hamiltonian are introduced directly by inserting the closure relation:

$$\int_{-\infty}^{+\infty}\frac{dk_z}{2\pi}|k_z\rangle\langle k_z|\sum_{n_x=0}^{\infty}|n_x\rangle\langle n_x|\sum_{n_y=0}^{\infty}|n_y\rangle\langle n_y| = \mathbb{I}. \tag{58}$$

Only the even values of n_x and n_y gives a non vanishing contribution, and one obtains

$$\langle\delta_\epsilon^{\rm D=3}|\Psi^{\rm int}\rangle = \frac{\mathrm{S}_\Psi}{\hbar\omega_\perp}\int_{-\infty}^{+\infty}\frac{dk_z}{2\pi}\sum_{p,q=0}^{\infty}\exp\left(-\frac{k_z^2\epsilon^2}{4}\right)\times \frac{\langle\delta_\epsilon^{\rm D=1}|2p\rangle\langle\delta_\epsilon^{\rm D=1}|2q\rangle\phi_{2p}^*(0)\phi_{2q}^*(0)}{\tau-\frac{k_z^2a_\perp^2}{4}-p-q}, \tag{59}$$

where the dimensionless energy variable τ is defined by:

$$\tau = \frac{E}{2\hbar\omega_\perp}+i0^+. \tag{60}$$

The behavior of $\langle\delta_\epsilon^{\rm D=3}|\Psi^{\rm int}\rangle$ in the limit where ϵ vanishes is more easily obtained in the domain of negative collisional energies ($\tau<0$) than in the domain of positive collisional energies $E>0$ and are related to each other by analyticity. For $\tau<0$ one can use the identity

$$\frac{1}{\frac{k_z^2a_\perp^2}{4}+n+p-\tau} = \int_0^\infty du\, e^{-u\left(\frac{k_z^2a_\perp^2}{4}+n+p-\tau\right)}, \tag{61}$$

which allows one to decouple the discrete summations from the integration over k_z in Eq. (59). From Eq. (57), one can deduce the following limit:

$$\lim_{\epsilon\to0}\sum_{p=0}^{\infty}\langle\delta_\epsilon^{\rm D=1}|2p\rangle\phi_{2p}^*(0)e^{-pu} = \frac{|\phi_0(0)|^2}{\sqrt{1-e^{-u}}}. \tag{62}$$

Using Eq. (62) and integrating over k_z in Eq. (59) gives:

$$\langle\delta_\epsilon^{\rm D=3}|\Psi^{\rm int}\rangle \underset{\epsilon\to0}{=} \frac{-\mu\mathrm{S}_\Psi}{2\pi^{3/2}\hbar^2a_\perp}\int_0^\infty\frac{du}{\sqrt{u+\frac{\epsilon^2}{a_\perp^2}}}\frac{e^{u\tau}}{1-e^{-u}}. \tag{63}$$

In the limit where $\epsilon\to0$, this expression diverges as $1/\epsilon$ and its regular part coincides with a 'Partie finie de Hadamard' [41, 42]:

$$\operatorname*{Reg}_{\epsilon\to0}\langle\delta_\epsilon^{D=3}|\Psi^{\rm int}\rangle = \frac{\mu\mathrm{S}_\Psi}{2\pi^{3/2}\hbar^2a_\perp}\,\mathrm{P.f.}\int_0^\infty\frac{du}{\sqrt{u}}\frac{e^{u\tau}}{1-e^{-u}}. \tag{64}$$

One recognizes in Eq. (64) an expression similar to the integral representation of the Hurwitz zeta function $\zeta(s,z)$ defined for $\Re(s)>1$ and $0<z<1$ by [43]:

$$\zeta(s,z) = \frac{1}{\Gamma(s)}\int_0^\infty dt\frac{t^{s-1}e^{-tz}}{1-e^{-t}}. \tag{65}$$

However, in Eq. (65) the integral diverges for $s=1/2$. The Partie finie de Hadamard in Eq. (64) permits to achieve a meromorphic continuation in the variable s of Eq. (65) and to identify the function $\zeta(1/2,z)$ [44]. Hence:

$$\mathrm{S}_\Psi = \frac{2\pi^{1/2}\hbar^2}{\mu\left[\zeta\left(\frac12,-\tau\right)+\frac{a_\perp}{a_3}\right]}. \tag{66}$$

For positive energies and in the monomode regime ($0<E<2\hbar\omega_\perp$), the scattering state written in the configuration representation in Eq. (48) has a non evanescent contribution at large relative length $z\gg a_\perp$, in the subspace of the transverse ground-state only. This property allows one to identify a quasi-1D scattering process with:

$$\langle z, n_x=0, n_y=0|\Psi\rangle \underset{|z|\gg a_\perp}{=} \exp(ik_0z) - \frac{i\mu\mathrm{S}_\Psi}{\sqrt{\pi}\hbar^2a_\perp k_0}\exp(ik_0|z|), \tag{67}$$

and $a_\perp$ plays the role of a 1D-potential radius (*i.e.* $b_1\sim a_\perp$). The low-energy scattering amplitude $f^{\rm 3D\to1D}$ of this quasi-1D system is thus:

$$f^{\rm 3D\to1D}(k_0) = -\frac{i\mu\mathrm{S}_\Psi}{\sqrt{\pi}\hbar^2a_\perp k_0}. \tag{68}$$

In the $|k_z\rangle$-representation, the scattering states read:

$$\langle k_z, n_x = 0, n_y = 0|\Psi\rangle = (2\pi)\delta(k_z - k_0) + \frac{\mathrm{S}_\Psi}{\sqrt{\pi}a_\perp\left(E + i0^+ - \frac{\hbar^2 k_z^2}{2\mu}\right)} \quad (69)$$

so that, the quasi-1D scattering transition matrix $T^{3\mathrm{D}\to 1\mathrm{D}}$ can be expressed as:

$$T^{3\mathrm{D}\to 1\mathrm{D}} = \frac{\mathrm{S}_\Psi}{\sqrt{\pi}a_\perp}. \quad (70)$$

Finally, from Eqs. (66,70) one obtains:

$$T^{3\mathrm{D}\to 1\mathrm{D}} = \frac{2\hbar^2}{\mu a_\perp} \times \frac{1}{\zeta(\frac{1}{2}, -\tau) + \frac{a_\perp}{a_3}}. \quad (71)$$

In the low energy limit ($k_0 a_\perp \ll 1$):

$$\zeta\left(\frac{1}{2}, -\tau\right) \underset{\tau\to 0}{\simeq} \frac{1}{\sqrt{-\tau}} + \zeta\left(\frac{1}{2}\right) \quad (72)$$

and for positive energies $\sqrt{-\tau} = -ik_0 a_\perp/2$. From Eq. (7), one can then identify a 1D-scattering length which is a function of the transverse length $a_\perp$ and of the 3D scattering length a_3 [40]:

$$a_1 = -\frac{a_\perp}{2}\left[\frac{a_\perp}{a_3} + \zeta\left(\frac{1}{2}\right)\right]. \quad (73)$$

B. Planar atomic wave guide

In the 3D→2D reduction problem, the two colliding particles are confined in a planar harmonic wave guide of frequency ω_z along the z-direction while they move freely in the two other directions. The non-interacting Hamiltonian in the center of mass frame reads:

$$H_0 = -\frac{\hbar^2}{2\mu}(\partial_x^2 + \partial_y^2) + \mathcal{H}_z, \quad (74)$$

with, the 1D-harmonic trap along z:

$$\mathcal{H}_z = -\frac{\hbar^2}{2\mu}\partial_z^2 + \frac{\mu}{2}\omega_z^2 z^2 - \frac{\hbar\omega_z}{2}. \quad (75)$$

In Eq. (75) the zero point energy of the transverse trap has been subtracted, so that similarly to the previous 3D→1D reduction problem, the energy of an eigenstate is measured with respect to the continuum threshold and coincides with the collisional energy. In what follows, $|n_z\rangle$ is the eigenstate of quantum number n_z for the 1D-harmonic oscillator of frequency ω_z and mass μ. The subsequent derivation supposes that the 3D zero-range approximation is justified, *i.e.* that the atomic wave guide is such that:

$$a_z \gg b_3, \quad (76)$$

where a_z is the characteristic length of the atomic wave guide:

$$a_z = \sqrt{\frac{\hbar}{\mu\omega_z}}. \quad (77)$$

In the quasi-2D scattering problem the incoming state is:

$$|\Psi^{(0)}\rangle = |\mathbf{k}_{2\mathrm{D}} = \mathbf{k}_0\rangle|n_z = 0\rangle, \quad (78)$$

where $\mathbf{k}_0$ is the 2D-wave relative wavevector of the incoming wave and $\mathbf{k}_{2\mathrm{D}} = k_x\hat{\mathbf{e}}_x + k_y\hat{\mathbf{e}}_y$ is the 2D collisional relative momentum. The collisional energy E is $E = \hbar^2 k_0^2/2\mu$ and the monomode regime condition is given by:

$$E < E_{\mathrm{trans}}^{2\mathrm{D}} \quad (79)$$

where, $E_{\mathrm{trans}}^{2\mathrm{D}} = \hbar\omega_z$. Eq. (79) ensures that the system is quasi-2D, *i.e.* the colliding particles are asymptotically trapped in the ground state of the transverse trap. The behavior of $\langle\delta_\epsilon^{\mathrm{D}=3}|\Psi^{\mathrm{int}}\rangle$ in the limit where $\epsilon \to 0$ is obtained by using the same technics as for the 3D→1D reduction problem. The quantum numbers of the free Hamiltonian are introduced by insertion of the closure relation:

$$\int \frac{d^2\mathbf{k}_{2\mathrm{D}}}{(2\pi)^2}|\mathbf{k}_{2\mathrm{D}}\rangle\langle\mathbf{k}_{2\mathrm{D}}| \sum_{n_z=0}^{\infty}|n_z\rangle\langle n_z| = \mathbb{I}. \quad (80)$$

In the 3D Bethe-Peierls asymptotic condition, only even values of n_z contributes and one obtains:

$$\langle\delta_\epsilon^{\mathrm{D}=3}|\Psi^{\mathrm{int}}\rangle = \frac{\mu\mathrm{S}_\Psi a_z^2}{4\pi\hbar^2}\int_0^\infty k_{2\mathrm{D}}dk_{2\mathrm{D}} \times \sum_{p=0}^{\infty}\exp\left(-\frac{k_{2\mathrm{D}}^2\epsilon^2}{4}\right)\frac{\langle\delta_\epsilon^{\mathrm{D}=1}|2p\rangle\phi_{2p}^*(0)}{\tau - \frac{k_{2\mathrm{D}}^2 a_z^2}{4} - p}, \quad (81)$$

where the dimensionless energy variable τ is defined by:

$$\tau = \frac{E}{2\hbar\omega_z} + i0^+. \quad (82)$$

The discrete summation in Eq. (81) is performed in the domain $\tau < 0$ by using the transformation of Eq. (61) together with the identity of Eq. (62). After integration over $k_{2\mathrm{D}}$, the regular part of $\langle\delta_\epsilon^3|\Psi^{\mathrm{int}}\rangle$ can be expressed in the limit where $\epsilon \to 0$ as:

$$\underset{\epsilon\to 0}{\mathrm{Reg}}\langle\delta_\epsilon^3|\Psi^{\mathrm{int}}\rangle = -\frac{\mu\mathrm{S}_\Psi}{2\pi\hbar^2}|\phi_0(0)|^2 J(\tau), \quad (83)$$

where the function $J(\tau)$ is defined in the domain $\tau < 0$ by:

$$J(\tau) = \mathrm{P.f.}\int_0^\infty \frac{du}{u}\frac{\exp(\tau u)}{\sqrt{1 - \exp(-u)}}. \quad (84)$$

From Eq. (50), one finally obtains:

$$\mathrm{S}_\Psi = \frac{2\pi\hbar^2\phi_0(0)}{\mu\left[\frac{1}{a_3} + |\phi_0(0)|^2 J(\tau)\right]}. \quad (85)$$

Projection of the wavefunction on the ground-state of the 1D transverse harmonic oscillator $\langle \mathbf{k}_{2\mathrm{D}}, n_z = 0|\Psi\rangle$ gives the quasi-2D transition matrix:

$$T^{3\mathrm{D}\to 2\mathrm{D}} = -\frac{\pi\hbar^2}{\mu\left[\frac{\sqrt{\pi}a_z}{2a_3} + \frac{J(\tau)}{2}\right]}. \quad (86)$$

In the domain of positive energy Eq. (82) is continued analytically from the identity

$$J(\tau) = \ln\left(-\frac{B}{2\pi\tau}\right) + \sum_{n=1}^{\infty} \ln\left(\frac{n}{n-\tau}\right)\frac{(2n-1)!!}{(2n)!!}, \quad (87)$$

where $B \simeq 0.9049$ is defined by [34]:

$$\ln\left(\frac{Be^\gamma}{2\pi}\right) = \int_0^\infty du\left(\frac{u^{-1}}{\sqrt{1-e^{-u}}} - \frac{1}{u^{3/2}} - \frac{1}{1+u}\right), \quad (88)$$

and $\gamma = 0.5772\ldots$ is the Euler's constant. The 2D low energy condition is given by:

$$|E| \ll \hbar\omega_z, \quad (89)$$

and a_z plays the role of a 2D-potential radius, *i.e.* $b_2 \sim a_z$. Only the first logarithmic term in Eq. (87) contributes at the lowest order in the limit of Eq. (89), and one can identify the resulting expression of the quasi-2D transition matrix $T^{3\mathrm{D}\to 2\mathrm{D}}$ with the 2D low energy transition matrix T_2 in Eq. (7). The 2D scattering length a_2 of this quasi-2D scattering problem can thus be expressed in terms of a_z and of the 3D scattering length a_3 [6, 7, 36] as:

$$a_2 = a_z e^{-\gamma}\sqrt{\frac{2\pi}{B}}\exp\left(-\frac{a_z\sqrt{\pi}}{2a_3}\right). \quad (90)$$

It is interesting to discuss a little bit further the condition for having a low energy resonant behavior in this quasi-2D system. The 2D cross-'section' (this is a length) is proportional to $|T^{3\mathrm{D}\to 2\mathrm{D}}(k_0)|^2/k_0$ and a maximum occurs in the low-energy regime Eq. (89) for:

$$\frac{a_z}{a_3} = \frac{1}{\sqrt{\pi}}\ln\left(\frac{\pi E}{B\hbar\omega_z}\right). \quad (91)$$

The right hand side of Eq. (91) is large and negative which shows that the resonance occurs for sufficiently large values of $|a_3|$ with respect to a_z in the domain of negative scattering lengths. At fixed value of the collisional energy E and of the scattering length $a_3 < 0$, the resonance can be reached by tuning the harmonic frequency of the atomic waveguide: this is the *so-called* quasi-2D Confined Induced Resonance first found in Ref. [5].

III. FORMALISM FOR N-BODY SYSTEMS

A. k-representation

In this section the Bethe-Peierls method is applied to few- and many-body systems in the **k**-representation. The system is composed of N-particles of respective masses $m_1, m_2, \ldots m_N$ and of momenta $\mathbf{k}_1, \mathbf{k}_2, \ldots \mathbf{k}_N$. In what follows, for convenience the set of momenta is denoted by the short-hand notation $\{\mathbf{k}\}$. The relative and total momenta of a given pair of particles (ij), are denoted by

$$\boldsymbol{k}_{ij} = \frac{m_j\mathbf{k}_i - m_i\mathbf{k}_j}{m_i + m_j}\ ,\ \boldsymbol{K}_{ij} = \mathbf{k}_i + \mathbf{k}_j, \quad (92)$$

and the reduced mass of the pair (ij) is denoted by:

$$\mu_{i,j} = \frac{m_i m_j}{m_i + m_j} \quad (93)$$

It is also convenient to introduce the set of momenta denoted by $\{\boldsymbol{\xi}^{ij}\}$ with $\{\boldsymbol{\xi}^{ij}\} = \boldsymbol{\xi}_1^{ij}, \boldsymbol{\xi}_2^{ij} \ldots \boldsymbol{\xi}_N^{ij}$ where:

$$\boldsymbol{\xi}_n^{ij} = \begin{cases} \boldsymbol{K}_{ij}\sqrt{\dfrac{m_i}{2(m_i+m_j)}} & \text{for } n = i \\ \boldsymbol{K}_{ij}\sqrt{\dfrac{m_j}{2(m_i+m_j)}} & \text{for } n = j \\ \mathbf{k}_n & \text{otherwise} \end{cases}. \quad (94)$$

This last notation permits to isolate in the kinetic energy the contribution of the relative particle (ij) from the $N-1$ other momentum:

$$\sum_{n=1}^{N}\frac{\mathbf{k}_n^2}{m_n} = \frac{\boldsymbol{k}_{ij}^2}{\mu_{i,j}} + \sum_{n=1}^{N}\frac{\left(\xi_n^{ij}\right)^2}{m_n}. \quad (95)$$

A non vanishing source amplitude $|\mathrm{S}_\Psi^{i=j}\rangle$ is associated with any pair of particles interacting via the short range pairwise potential in the many-body state $|\Psi_\epsilon\rangle$. In the case where the system is composed of particles with spin, in what follows $|\Psi_\epsilon\rangle$ denotes the projection over a given spin configuration of the many-body state including the symmetry imposed by the quantum statistics. The free Hamiltonian of the system (which may includes an external potential) is denoted by H_0, and the N-body stationary states at energy E verifies:

$$(H_0 - E)|\Psi_\epsilon\rangle = -\sum_{i<j}|(ij):\delta_\epsilon^D\rangle|\mathrm{S}_\Psi^{i=j}\rangle. \quad (96)$$

In Eq. (96), $|(ij):\delta_\epsilon^D\rangle$ is the state $|\delta_\epsilon^D\rangle$ for the relative particle (ij) of momentum $\boldsymbol{k}_{ij}$, and the source amplitude $|\mathrm{S}_\Psi^{i=j}\rangle$ is a state for the $N-1$ other particles of momenta $\{\boldsymbol{\xi}^{ij}\}$. If the pair (ij) does not interact then $|\mathrm{S}_\Psi^{i=j}\rangle = 0$, otherwise it satisfies the Bethe-Peierls contact condition:

$$\lim_{\epsilon\to 0}\langle(ij):\delta_\epsilon^D|\left(|\Psi\rangle - |(ij):\phi^\Lambda\rangle|\mathrm{S}_\Psi^{i=j}\rangle\right) = \frac{|\mathrm{S}_\Psi^{i=j}\rangle}{T_D(E_\Lambda)}, \quad (97)$$

where the reference state $|(ij):\phi_\epsilon^\Lambda\rangle$ is defined in the **k**-representation as:

$$\langle \boldsymbol{k}_{ij}|(ij):\phi_\epsilon^\Lambda\rangle = -\frac{2\mu_{i,j}}{\hbar^2}\times\frac{\chi_\epsilon(k_{ij})}{k_{ij}^2+\Lambda^2} \tag{98}$$

In what follows Eqs. (96,97) are used in the case where there is no external potential and $|\Psi^{(0)}\rangle$ denotes the complementary solution of Eq. (96) which satisfies the boundary conditions of the problem considered (for example an incoming plane-wave for a scattering problem). The many-body wavefunction can then be written as:

$$\langle\{\mathbf{k}\}|\Psi_\epsilon\rangle = \langle\{\mathbf{k}\}|\Psi^{(0)}\rangle + \mathcal{G}_E(\{\mathbf{k}\})\sum_{i<j}\chi_\epsilon(k_{ij})\langle\{\boldsymbol{\xi}^{ij}\}|\mathrm{S}_\Psi^{i\leftrightharpoons j}\rangle \tag{99}$$

where $\mathcal{G}_E(\{\mathbf{k}\})$ is the N-body Green's function in the **k**-representation:

$$\mathcal{G}_E(\{\mathbf{k}\}) = \frac{1}{E+i0^+-\sum_{n=1}^N\frac{\hbar^2k_n^2}{2m_n}} \tag{100}$$

For each interacting pair, the contact condition Eq. (97) gives in the **k**-representation an integral equation in terms of the source terms and of the complementary solution $|\Psi^{(0)}\rangle$. Interestingly, one can extract the contribution of the interacting pair (ij) in the contact condition Eq. (97) without any integration by using the fact that this equation is invariant in a change of $\Lambda\in\mathbb{R}^+$ (*i.e.* by using the Λ-freedom). To this end, one expresses the Green's function in Eq. (99) as:

$$\mathcal{G}_E(\{\mathbf{k}\}) = \frac{1}{E_{\mathrm{col}}^{(ij)}+i0^+-\frac{\hbar^2k_{ij}^2}{2\mu_{i,j}}}, \tag{101}$$

where $E_{\mathrm{col}}^{(ij)}$ is the collisional energy of the pair (ij) defined by:

$$E_{\mathrm{col}}^{(\mathrm{ij})} = E-\sum_{n=1}^N\frac{\hbar^2\left(\xi_n^{ij}\right)^2}{2m_n}. \tag{102}$$

It is the energy of the pair (ij) in its own center of mass frame while the other particles do not interact for a given set of momenta $\{\mathbf{k}\}$ and a total energy E.

For a negative energy ($E<0$) in which case $|\Psi^{(0)}\rangle=0$, *without loss of generality* one can make the particular choice $\Lambda^2=-2\mu_{i,j}E_{\mathrm{col}}^{(\mathrm{ij})}/\hbar^2$ in Eq. (97). That way, the term involving the reference state ϕ^Λ exactly cancels with the term associated with the source amplitude of the pair (ij) ($|\mathrm{S}_\Psi^{i\leftrightharpoons j}\rangle$). By analytical continuation, the same integral equation holds for $E>0$ (in which case $|\Psi^{(0)}\rangle\neq0$). Finally, assuming that the $\epsilon\to0$ limit is well defined, for each interacting pair one obtains the following integral equation:

$$\sum_{\substack{\mathrm{n<p}\\(n,p)\neq(i,j)}}\int\frac{d^D\boldsymbol{k}_{ij}}{(2\pi)^D}\mathcal{G}_E(\{\mathbf{k}\})\langle\{\boldsymbol{\xi}^{np}\}|\mathrm{S}_\Psi^{n\leftrightharpoons p}\rangle = \frac{\langle\{\boldsymbol{\xi}^{ij}\}|\mathrm{S}_\Psi^{i\leftrightharpoons j}\rangle}{T_D(E_{\mathrm{col}}^{(\mathrm{ij})}+i0^+)} - \int\frac{d^D\boldsymbol{k}_{ij}}{(2\pi)^D}\langle\{\mathbf{k}\}|\Psi^{(0)}\rangle. \tag{103}$$

One has to notice that the integrals in the first line of Eq. (103) are performed with the constraint that $\{\boldsymbol{\xi}^{ij}\}$ is held fixed (but $\{\boldsymbol{\xi}^{np}\}$ is not !).

B. Second Quantization

In this section only fermionic particles of same mass m and two spin-components are considered. Generalization to other systems gives similar equations. The creation and annihilation operator of an atomic wave of momentum $\mathbf{k}$ for a spin σ are denoted by $a_{\mathbf{k},\sigma}^\dagger$ and $a_{\mathbf{k},\sigma}$ with the anti-commutation rule:

$$\{a_{\mathbf{k},\sigma},a_{\mathbf{k},\sigma'}^\dagger\} = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')\delta_{\sigma\sigma'}. \tag{104}$$

The general expression of the Λ-potential is a simple way to write the Hamiltonian in its second quantized form. From the definition of the Λ-potential in Eq. (43) together with the expression of regularizing operator r_ϵ^Λ introduced in Eq. (42) the Hamiltonian can be expressed as:

$$H_\epsilon = \int\frac{d^D\mathbf{k}}{(2\pi)^D}\sum_\sigma\frac{\hbar^2k^2}{2m}a_{\mathbf{k},\sigma}^\dagger a_{\mathbf{k},\sigma} + \frac{T_D(E_\Lambda)}{2}\int\frac{d^D\mathbf{K}}{(2\pi)^D} \times\int\frac{d^D\mathbf{k}'}{(2\pi)^D}\chi_\epsilon(k')a_{\frac12\mathbf{K}-\mathbf{k}',\uparrow}^\dagger a_{\frac12\mathbf{K}+\mathbf{k}',\downarrow}^\dagger \times\lim_{\epsilon\to0}r_\epsilon^\Lambda\left[\int\frac{d^D\mathbf{k}}{(2\pi)^D}\chi_\epsilon(k)a_{\frac12\mathbf{K}-\mathbf{k},\downarrow}a_{\frac12\mathbf{K}+\mathbf{k},\uparrow}\cdot\right], \tag{105}$$

where the dot $(\cdot)$ after the annihilation operator $a_{\frac12\mathbf{K}+\mathbf{k},\uparrow}$ reminds that the limit $\epsilon\to0$ depends on the many-body state on which H_ϵ applies. By making the hypothesis that the few- or many-body state is well defined in the zero-range limit by only using the Bethe-Peierls contact condition, the energy theorem [23–26] can be obtain from the identity:

$$\lim_{\epsilon\to0}r_\epsilon^{\Lambda=\frac{1}{a_D}}\left[\langle H_\epsilon\rangle_\epsilon\right] = \lim_{\epsilon\to0}\langle H_\epsilon\rangle_\epsilon = \langle H\rangle. \tag{106}$$

The mean value of the interacting term is exactly zero in this limit and:

$$\langle H\rangle = \lim_{\epsilon\to0}r_\epsilon^{\Lambda=\frac{1}{a_D}}\left[\int\frac{d^D\mathbf{k}}{(2\pi)^D}\sum_\sigma\frac{\hbar^2k^2}{2m}\langle a_{\mathbf{k},\sigma}^\dagger a_{\mathbf{k},\sigma}\rangle_\epsilon\right]. \tag{107}$$

The energy of the system can be thus expressed in terms of the one-body density:

$$n_{\mathbf{k},\sigma} = \lim_{\epsilon\to 0}\langle a^\dagger_{\mathbf{k},\sigma}a_{\mathbf{k},\sigma}\rangle_\epsilon. \tag{108}$$

In the limit where $\epsilon \to 0$, one obtains from Eq. (99) the high momentum behavior of $n_{\mathbf{k},\sigma}$:

$$n_{\mathbf{k},\sigma} \underset{k\to\infty}{=} \frac{\mathrm{S}}{k^4} + O\left(\frac{1}{k^6}\right), \tag{109}$$

where S is spin independent since the interaction occurs between different spin-particles only. Hence, the integrand in Eq. (107) behaves as $\frac{1}{k^2}$ at high momentum and action of the regularizing operator gives:

$$\langle H\rangle = \int \frac{d^D\mathbf{k}}{(2\pi)^D}\sum_\sigma\left[\frac{\hbar^2k^2}{2m}n_{\mathbf{k},\sigma} - \frac{\hbar^2\mathrm{S}a_D^2}{2m(1+k^2a_D^2)}\right] \tag{110}$$

IV. FEW-BODY BOUND STATES

A. Efimov states

In this section, the generic case where Efimov effect occurs is considered: three identical bosons of mass m interact in 3D with a pairwise interaction of vanishing range with respect to the 3D scattering length a_3 [45, 46]. These states have been observed for the first time in ultra-cold atoms [47–50]. In what follows, some universal properties of Refs. [30, 45, 46] are recovered from the STM equation by using the nodal condition introduced in Ref. [29]. However, it is worth pointing out that deviations from universal predictions which are observed in experiments can be taken into account through finite range models [51–54]. The energy of a trimer $E = E_3$ in its center of mass frame is negative (and less than the dimer's energy E_2 if it exists) so that the complementary solution in Eq. (103) is zero ($|\Psi^{(0)}\rangle = 0$). The binding wavenumber $q > 0$ of a trimer is defined from:

$$E_3 = -\frac{\hbar^2q^2}{m}. \tag{111}$$

The Bose statistics imposes that the source amplitudes for each pair of particle coincide with the same function:

$$\langle\{\boldsymbol{\xi}^{12}\}|\mathrm{S}_\Psi^{1\rightleftharpoons 2}\rangle = \langle\{\boldsymbol{\xi}^{13}\}|\mathrm{S}_\Psi^{1\rightleftharpoons 3}\rangle = \langle\{\boldsymbol{\xi}^{23}\}|\mathrm{S}_\Psi^{2\rightleftharpoons 3}\rangle. \tag{112}$$

As a consequence of translation invariance, the source amplitudes in the center of mass frame can be written as:

$$\langle\{\boldsymbol{\xi}^{12}\}|\mathrm{S}_\Psi^{1\leftrightharpoons 2}\rangle = (2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\,\mathrm{F}(\mathbf{k}_3). \tag{113}$$

The integral eigenequation Eq. (103) can be written as:

$$\frac{\mathrm{F}(\mathbf{k})}{f_3\left(i\sqrt{q^2+\frac{3k^2}{4}}\right)} = 8\pi\int\frac{d^3\mathbf{u}}{(2\pi)^3}\,\frac{\mathrm{F}(\mathbf{u})}{u^2+k^2+\mathbf{k}.\mathbf{u}+q^2}. \tag{114}$$

Eq. (114) is the so-called Skorniakov Ter-Martirosian (STM) equation [27]. This equation is rotationaly invariant and can thus be studied in each momentum sector. In Ref. [55], Danilov showed that as it stands Eq. (114) is ill-defined: it supports a continuum of negative energy solutions in the s-wave sector of $F(\mathbf{k})$. Hence, the Bethe-Peierls asymptotic method which is at first sight adapted for modeling the three-boson resonant problem does not permit to derive a well defined eigenequation. The s-wave component of $\mathrm{F}(\mathbf{k})$ of the source amplitude is denoted by $\phi(k)$:

$$\int\frac{d\Omega}{4\pi}\mathrm{F}(\mathbf{k}) = \phi(k), \tag{115}$$

and $\phi(k)$ verifies the following integral equation:

$$\frac{\phi(k)}{f_3\left(i\sqrt{q^2+\frac{3k^2}{4}}\right)} = \frac{2}{\pi}\int_0^\infty du\,\phi(u)\mathcal{K}_q(k,u), \tag{116}$$

where the s-wave kernel of the STM equation is given by:

$$\mathcal{K}_q(k,u) = \frac{u}{k}\ln\left(\frac{u^2+k^2+q^2+ku}{u^2+k^2+q^2-ku}\right). \tag{117}$$

The fact that there exists a continuum of negative energy solutions means that the Bethe-Peierls model is not self-adjoint for the three-boson problem. Danilov found a way to restore the self-adjointness by introducing a supplementary condition on the high momentum asymptotic behavior of the function $\phi(k)$ [55]. The method of Danilov is based on the fact that for all values of q and $a \neq 0$, Eq. (116) supports solutions with the asymptotic behavior:

$$\phi(k) \underset{k\to\infty}{\sim} Ak^{is_0-2} + Bk^{-is_0-2}, \tag{118}$$

where A and B are two constants and s_0 solves the equation $\sin(\pi s_0/6) = \sqrt{3}s_0\cos(\pi s_0/2)/8$ ($s_0 \simeq 1.00624\ldots$). The zero-range approach is made self-adjoint if one fixes the asymptotic phase shift between the two conjugate behavior $k^{\pm is_0}$ for all values of a and q. However, Minlos and Fadeev showed that even with this supplementary phase-shift condition the spectrum is not bounded from below [56]: this is the so-called Thomas collapse which is characteristic of zero-range forces [28]. In 1970, Efimov has solved the three-boson problem in the resonant regime by introducing the notion of three-body parameter which is also in the zero-range limit, a way to fix the asymptotic phase-shift in Eq. (118). In the $\mathbf{k}$-representation it is referred in what follows as $\kappa^\star$ and defined as:

$$\phi(k) \underset{k\to\infty}{\propto} \frac{1}{k^2}\sin\left[s_0\ln\left(\frac{k\sqrt{3}}{\kappa^\star}\right)\right]. \tag{119}$$

By construction the three-body parameter is not unique [Eq. (119) is invariant in a change $\kappa^\star \to \kappa^\star\exp(\pi/s_0)$].

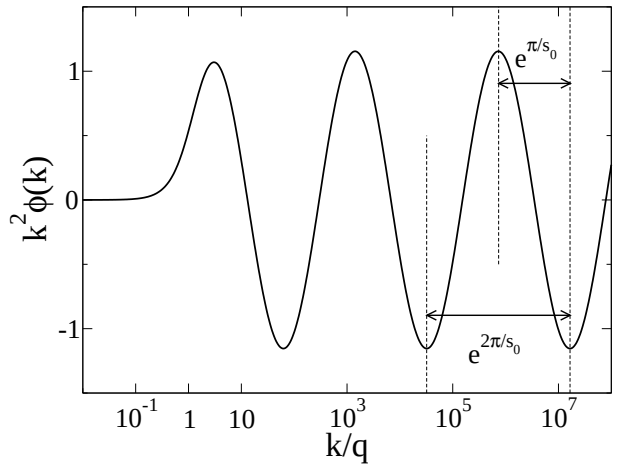

FIG. 1: Shape of the expession of $k^2\phi(k)$ in Eq. (121) plotted in a semi-log plot as a function of k/q.

At unitarity, (*i.e.* $|a| \to \infty$ and $b_3 \to 0$), Efimov showed that the spectrum of trimers is characterized by an accumulation point at zero energy: binding wavenumbers of the trimers are related the each to the others by a scaling factor:

$$q_n = q_p e^{-(n-p)\pi/s_0}, \tag{120}$$

where $(n,p) \in \mathbb{Z}^2$. Interestingly, in Ref. [59] the analytical expression for the source amplitude of the trimers has been found at unitarity:

$$\phi(k) = \frac{1}{k\sqrt{q^2+\frac{3k^2}{4}}} \sin\left[s_0 \,\mathrm{arcsinh}\left(\frac{k\sqrt{3}}{2q}\right)\right]. \tag{121}$$

In particular, this result shows that the choice made for the definition of $\kappa^\star$ in Eq. (119) is such that the spectrum of the zero range theory at unitarity is $q_n = \kappa^\star \exp(n\pi/s_0)$ with $n \in \mathbb{Z}$. The shape of the function $k^2\phi(k)$ of Eq. (121) is given in Fig. (1). For increasing values of k starting from $k=0$, the first zero of $\phi(k)$ is located at the momentum $k = \frac{2q}{\sqrt{3}}\sinh(\pi/s_0)$. Using, this property, it is possible to filter a set of solutions of Eq. (116) satisfying Eqs. (119,120) for sufficiently high quantum numbers (shallow states) and with a spectrum bounded from below. This is done by imposing a nodal condition on the eigenstates:

$$\phi(k_p^{\mathrm{reg}}) = 0, \tag{122}$$

where the node k_p^{reg} is chosen among the set:

$$k_p^{\mathrm{reg}} = \frac{\kappa^\star}{\sqrt{3}} e^{p\pi/s_0} \quad p \in \mathbb{Z}. \tag{123}$$

The position of the node in Eq. (123) fixes the minimum energy of the spectrum. For example if one chooses the nodal condition for $p=2$, using the fact that $\exp(\pi/s_0) \simeq 2\sinh(\pi/s_0)$, the minimum energy E_0 is almost equals to $-\hbar^2{\kappa^\star}^2\exp(2\pi/s_0)/m$ (with a relative error $2\exp(-2\pi/s_0) \simeq 3.8\times10^{-3}$) and the spectrum at unitarity is asymptotically (*i.e.* for large values of n) given by:

$$E_n = -\frac{\hbar^2{\kappa^\star}^2}{m}\exp\left[\frac{-2(n-1)\pi}{s_0}\right] \quad , n \in \mathbb{N}. \tag{124}$$

For a finite scattering length a_3 and for $k|a_3| \gg 1$, eigenfunctions $\phi(k)$ of Eq. (116) have the same behavior as the unitary solutions of Eq. (121) so that Eq. (122) can be also used as a filtering condition and permits to recover the universal spectrum of the zero-range theory for energies much larger than E_0. In principle, the zero-range theory also called 'universal theory' in the literature is recovered by imposing the filtering condition at an arbitrary large node k_p^{reg}. To summarize this discussion, the nodal condition in Eq. (122) has two roles: first, it imposes the Danilov's asymptotic phase shift for states of sufficiently high quantum number and second, it imposes a minimum energy to the spectrum. For a realistic finite range force, the possible values of the binding wavenumbers obtained from the zero range theory in Eq. (120) are such that $q_n b_3 \ll 1$, the ground state energy in Eq. (124) has thus to be chosen higher than $-\hbar^2/(mb_3^2)$.

The nodal condition in Eq. (122) applied on Eq. (116) can be transformed into the integral condition:

$$0 = -\frac{2}{\pi}\int_0^\infty du\,\phi(u)\mathcal{K}_q(k_p^{\mathrm{reg}},u). \tag{125}$$

Subtracting Eq. (125) from Eq. (116) gives the regularized STM equation [29]

$$\frac{\phi(k)}{f_3\left(i\sqrt{q^2+\frac{3k^2}{4}}\right)} = \frac{2}{\pi}\int_0^\infty du\,\left[\mathcal{K}_q(k,u)-\mathcal{K}_q(k_p^{\mathrm{reg}},u)\right]\phi(u), \tag{126}$$

which encapsulates the nodal condition. Numerical solutions of Eq. (126) are obtained by introducing an ultraviolet cut-off Q in the integral. One can verify that results are insensitive to the choice made on the cut-off for $Q \gg k_p^{\mathrm{reg}}$. For instance considering the unitary limit, taking the nodal condition at k_1^{reg} and an UV cut-off at $5\times10^2\times\kappa^\star$, one finds the relative error for the ground state equals to $3.8\times10^{-3} \simeq 2\exp(-2\pi/s_0)$ and for the first excited states ($n \leq 8$), a relative error lesser than 10^{-4}. In practice, the value of the scaling factor $\exp(2\pi/s_0) \simeq 515$ is relatively large so that for the nodal condition at k_2^{reg}, the spectrum is very close to the universal spectrum beginning from the second branch ($n \geq 1$) and for an inverse scattering length $1/|a_3|$ smaller or of the order of $\kappa^\star$.

In Fig. (2), the second Efimov branch of the regularized STM Eq. (126) with the nodal condition at k_2^{reg} has

been plotted as a function of $(1/a_3)$. As in Ref. [30], the thresholds for the apparition of the trimers are denoted by a^* at the atom-dimer continuum limit and by a'^* at the three-atom continuum limit. As a consequence of the choice of the nodal condition at $k_p^{\rm reg}$ for a finite p, results sligthly differ from the zero-range theory in Ref. [30]. The atom-dimer scattering problem (regime

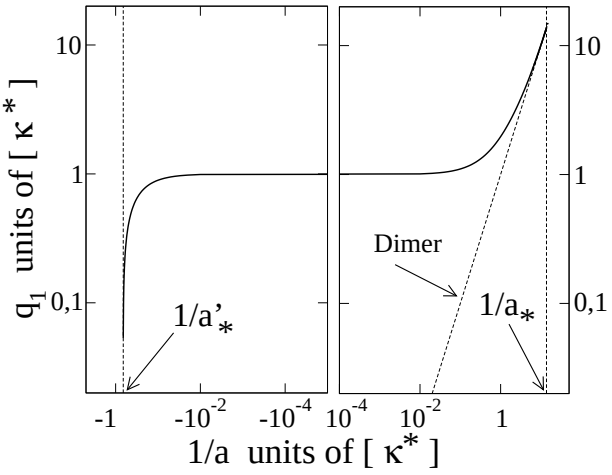

FIG. 2: Solid line: second branch of the trimer's spectrum obtained with the nodal condition at $k_2^{\rm reg}$. Vertical dashed line: trimer's apparition thresholds. Oblique dashed line: atom-dimer continuum limit. The value of the scattering length at the trimer's apparition threshold $a^* \simeq -1.51$ and $a'^* \simeq .0713$ are close to the results of the 'universal theory' in Ref. [30, 59] $a^* \simeq -1.507$ and $a'^* \simeq .0707$. The difference is due to the fact that the zero-range theory is recovered only for a nodal condition at an arbitrarily large $k_p^{\rm reg}$.

where $a_3 > 0$) can be also solved by using the same filtering technique. In what follows, the atom-dimer scattering length is computed as a function of the atomic scattering length a_3 for a given value of the three-body parameter $\kappa^\star$. To this end, the atom-dimer collisional energy is set to zero ($q = 1/a_3$) and the ansatz for the source amplitude is [27]:

$$\phi(k) = 2\pi^2 \frac{\delta(\mathbf{k})}{k^2} + 4\pi \frac{g(k)}{k^2}. \tag{127}$$

Using the regularized STM equation one obtains:

$$\frac{3g(k)a_3}{8\left[1+\sqrt{1+\frac{3}{4}(ka_3)^2}\right]} = \frac{a_3^2}{1+(ka_3)^2} - \frac{a_3^2}{1+(k_p^{\rm reg}a_3)^2} + \int_0^\infty \frac{du}{\pi u^2}\left[\mathcal{K}_{\frac{1}{a_3}}(k,u) - \mathcal{K}_{\frac{1}{a_3}}(k_p^{\rm reg},u)\right] g(u). \tag{128}$$

The atom-dimer scattering length is obtained in the zero momentum limit as

$$a_{\rm ad} = \lim_{k\to 0} g(k), \tag{129}$$

and is plotted in Fig. (3) for the nodal condition taken at $k_2^{\rm reg}$. Fig. (3) is limited to atomic scattering lengths

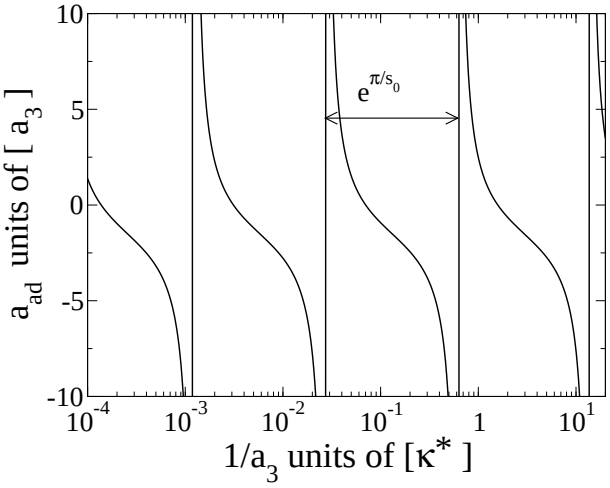

FIG. 3: Atom-dimer scattering length $a_{\rm ad}$ computed with the nodal condition at $k_2^{\rm reg}$ and plotted in a semi-log scale as a function of $\kappa^\star/a_3$ for $a_3 k_2^{\rm reg} \ll 1$.

a_3 much smaller than $1/k_2^{\rm reg}$ *i.e.* in a regime where the nodal condition permits to recover the universal theory with high accuracy. The scattering length $a_{\rm ad}$ diverges at the threshold of apparition of an Efimov's trimer [$a_3 \simeq a^* \exp(n\pi/s_0)$, $n \in \mathbb{N}$] and thus exhibits the log-periodicity which is a characteristics of the Efimov physics.

In Ref. [60] another regularizing technique of the STM equation was derived in the framework of the Effective Field Theory. In this last formulation, the role of the integral counter-term explicitly depends on the ultra-violet integral cut-off of the integral (Q) in such a way that the three-body parameter has a fixed value for all Q. In Refs. [61, 62] a subtraction technique has been also introduced in order to regularize the atom-dimer scattering problem. In these last references, the subtraction is made at zero momentum and therefore imposes the exact value of the atom-dimer scattering length. Thus, this regularization scheme does not correspond to the nodal condition of Eq. (122). It is interesting to write down the equation for the atom-dimer scattering amplitude $g(k)$ obtained within this scheme:

$$\frac{3g(k)a_3}{8\left[1+\sqrt{1+\frac{3}{4}(ka_3)^2}\right]} - \frac{3a_{\rm ad}a_3}{16} = \frac{a_3^2}{1+(ka_3)^2} - a_3^2 + \int_0^\infty \frac{du}{\pi u^2}\left[\mathcal{K}_{\frac{1}{a_3}}(k,u) - \frac{2u^2a_3^2}{1+u^2a_3^2}\right] g(u), \tag{130}$$

and to compare with Eq. (128).

B. Heterogeneous Trimers

As a consequence of interesting predictions [8, 16, 17, 63–68], heteronuclear systems play an important role in ultra-cold physics. In this section, the eigenequation

for heteronuclear trimers is considered. Particles (labeled by i) can be either bosons or fermions and have a mass m_i which can take two possible values: $m_i = m$ or $m_i = M$. Bosons (respectively fermions) of mass M are denoted by B (respectively by F) while bosons (respectively fermions) of mass m are denoted by b (respectively f). Fermionic particles are supposed to have two possible internal states denoted by $\uparrow$ and $\downarrow$. The configurations studied here are thus constructed from the set of particles $\{b, B, f_\uparrow, f_\downarrow, F_\uparrow, F_\downarrow\}$. The present study is also restricted to '2+1' few-body systems where the two-body interaction is non-vanishing for heterogeneous pairs of particles only. As an example in the case of $(bF_\uparrow)$ interacting pairs, other pairs like $(bF_\downarrow)$ or (bB) are not interacting. For fermions neglecting $(F_\uparrow F_\uparrow)$ or $(f_\uparrow f_\uparrow)$ interactions is an exact asumption which is a consequence of Pauli principle. However, neglecting (BB) or (bb) interactions is not an exact hypothesis and only means that these interactions are negligible with respect to an heteronuclear interaction. Thus the true inter-atomic forces of the heteronuclear interacting pairs are in the vicinity of a s-wave resonance while interaction for pairs of identical bosons is neglected. For convenience, the following notations are introduced for the different combinations of particles masses m_i: :

$$M_{(ij)} = m_i + m_j \tag{131}$$

$$\mu_{i,(jk)} = \frac{m_i M_{(jk)}}{m_i + M_{(jk)}} \tag{132}$$

$$\mu_{(ij),(kl)} = \frac{M_{(ij)} M_{(kl)}}{M_{(ij)} + M_{(kl)}}. \tag{133}$$

In this section, two identical atoms of mass M (particles 1 and 2) interact with another one of mass m (particle 3). All the possible bound states can be thus deduced from the two configurations: (BBb) and $(F_\uparrow F_\uparrow b)$. Eigenequation for trimers is obtained for a negative energy $E = E_3$:

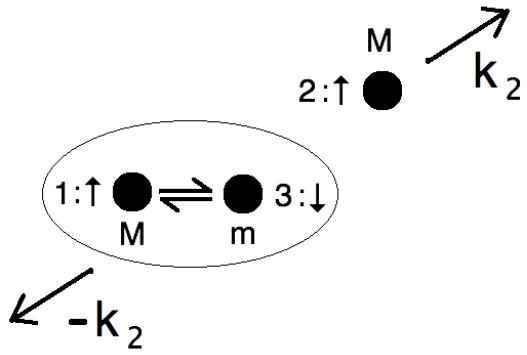

FIG. 4: Schematic representation of the momentum coordinates used for the source amplitude associated with the contact condition between particles 1 and 3.

$$E_3 = -\frac{\hbar^2 q^2}{2\mu_{2,3}} < 0, \tag{134}$$

and if a dimer of binding energy E_2 exists then $E_3 < E_2$. In the center of mass frame, the source amplitude associated with the pair (23) is

$$\langle\{\boldsymbol{\xi}^{23}\}|\mathrm{S}_\Psi^{2\rightleftharpoons 3}\rangle = (2\pi)^D \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)\,\mathrm{F}(\mathbf{k}_1). \tag{135}$$

The other source amplitude is deduced from Eq. (135) by using the exchange symmetry between particle 1 and particle 2:

$$\langle\{\boldsymbol{\xi}^{13}\}|\mathrm{S}_\Psi^{1\rightleftharpoons 3}\rangle = s_{13}(2\pi)^D \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)\,\mathrm{F}(\mathbf{k}_2), \tag{136}$$

where $s_{13} = 1$ when particles (1) and (2) are bosons and $s_{13} = -1$ if they are fermions. For simplifying the notations, it is useful to introduce the mass ratio y defined by:

$$y = \frac{\mu_{2,3}}{m_3} = \frac{M}{M+m}. \tag{137}$$

Assuming that the $\epsilon \to 0$ limit is well defined, the eigenequation is obtained from the contact condition for the pair $(2:M; 3:m)$ in Eq. (103) where $|\Psi^{(0)}\rangle = 0$:

$$\frac{\hbar^2 \mathrm{F}(\mathbf{k})}{2\mu_{2,3} T_D(E_{\mathbf{k}}^{\mathrm{col}})} = -s_{13} \int \frac{d^D\mathbf{u}}{(2\pi)^D} \frac{\mathrm{F}(\mathbf{u})}{u^2 + k^2 + 2y\mathbf{k}.\mathbf{u} + q^2}. \tag{138}$$

In Eq. (138) the identity $d^D\boldsymbol{k}_{23} = d^D\mathbf{k}_2$ has been used (the contact condition is performed at a fixed value of $\mathbf{K}_{23}$ and $E_{\mathbf{k}}^{\mathrm{col}}$ is the collisional energy:

$$E_{\mathbf{k}}^{\mathrm{col}} = E_3 - \frac{\hbar^2 k^2}{2\mu_{1,(23)}} < 0. \tag{139}$$

A detailed study of the three-body bound states in 2D has been performed in Ref. [64], and in this section the discussion is centered on the 3D case. By using the rotational symmetry of the kernel in Eq. (138), one can fix an arbitrary direction: $\hat{e}_{\mathbf{q}}$ and expand the source amplitude $\mathrm{F}(\mathbf{k})$ in terms of partial waves as:

$$\mathrm{F}(\mathbf{k}) = \sum_{l=0}^{\infty} P_l(\hat{e}_{\mathbf{k}} \cdot \hat{e}_{\mathbf{q}}) F_l(k). \tag{140}$$

Each component $F_l(k)$ verifies the following integral equation:

$$\frac{\hbar^2 F_l(k)}{2\mu_{1,3} T_3(E_{\mathbf{k}}^{\mathrm{col}})} = -\frac{s_{13}(-1)^l}{y\pi k} \int_0^\infty du\, u F_l(u) Q_l\left(\frac{u^2 + k^2 + q^2}{2yku}\right), \tag{141}$$

where Q_l is a Legendre function of second kind. One can notice that the standard STM Eq. (114) is obtained by setting formally $s_{13} = 2$ and $y = 1/2$ in Eq. (141). For a sufficiently large value of the mass ratio, the Efimov effect appear in non zero partial waves [57, 58]. In this regime as in the three-boson case, Eq. (141) supports pair of solutions such that $F_l(k) \sim k^{-2\pm is}$ $(s \in \mathbb{R})$ at large momentum k. The regularizing technique can be achieved by fixing the asymptotic phase shift which is a function of the three-body parameter in the partial wave l and denoted here by $\kappa_l^\star$:

$$F_l(k) \underset{k\to\infty}{\propto} \frac{1}{k^2} \sin\left[s \ln\left(\frac{k\sqrt{3}}{\kappa_l^\star} \right) \right]. \tag{142}$$

A set of solutions satisfying Eq. (142) can be filtered from Eq. (141) by imposing the nodal condition:

$$F_l(k_{p,l}^{\rm reg}) = 0 \tag{143}$$

where the node $k_{p,l}^{\rm reg}$ is defined by:

$$k_{p,l}^{\rm reg} = \frac{\kappa_l^\star}{\sqrt{3}} e^{p\pi/s}, p \in \mathbb{N}. \tag{144}$$

Exact zero range theory is obtained in the limit where $p \to \infty$. Universal results are recovered if the node $k_p^{\rm reg}$ is chosen at a large value as compared to all the low energy scales ($k_{p,l}^{\rm reg} \gg 1/|a|, q$). Using the subtracting scheme of the three-boson problem, Eq. (143) can be also incorporated into the integral equation Eq. (141):

$$\frac{\hbar^2 F_l(k)}{2\mu_{1,3} T_3(E_{\mathbf{k}}^{\rm col})} = -\frac{s_{13}(-1)^l}{y\pi} \int_0^\infty du\, u F_l(u) \left[\frac{1}{k} Q_l\left(\frac{u^2+k^2+q^2}{2yku} \right) - \frac{1}{k_l^{\rm reg}} Q_l\left(\frac{u^2+k_l^{\rm reg\,2}+q^2}{2yk_l^{\rm reg}u} \right) \right]. \tag{145}$$

It is interesting to find at resonance the regimes where an Efimov effect occurs from Eq. (141). To this end, zero energy solutions at unitarity $|a_3| = \infty$ are considered (solutions for finite energy have the same high momentum behavior). In this regime, Eq. (141) is scale invariant and this allows one to search for power law solutions: $F_l(k) = k^{\nu+l-2}$, where ν is a function of the mass ratio. For convenience, the following dimensionless parameter t is introduced:

$$t = \arcsin(y) = \arcsin\left(\frac{M}{M+m} \right). \tag{146}$$

Thus, the scattering amplitude is $f_3(\kappa_{\mathbf{k}}) = 1/(k\cos t)$ and Eq. (141) gives for each partial wave l an eigenvalue equation: $\lambda(\nu, l, t) = 0$ where,

$$\lambda(\nu, l, t) = \cos t - \frac{s_{13}(-1)^l}{\pi \sin t} \int_0^\infty dz z^{\nu+l-1} Q_l\left(\frac{1+z^2}{2z\sin t} \right). \tag{147}$$

The integrand in Eq. (147) is positive, hence for a Bose (resp. Fermi) statistics only even (resp. odd) values of l can support a solution. For $x > 1$, $Q_l(x) = \frac{1}{2} P_l(x) \ln(\frac{x+1}{x-1}) + W_l(x)$ where $W_l(x)$ is a polynomial of order l. The polynomial W_l does not contribute in the integral of Eq. (147) and $\lambda(\nu, l, t)$ can be thus expressed in terms of a sum of functions of the form:

$$I(\gamma, t) = \int_0^\infty dz\, z^{\gamma-1} \ln\left(\frac{z^2 + 2z\sin t + 1}{z^2 - 2z\sin t + 1} \right), \tag{148}$$

with $0 \le t \le \frac{\pi}{2}$ and $|\Re(\gamma)| < 1$. In Eq. (148), $I(\gamma, t)$ can be calculated explicitly as:

$$I(\gamma, t) = \frac{2\pi \sin(\gamma t)}{\gamma \cos(\frac{\pi\gamma}{2})}. \tag{149}$$

In the l-wave, Efimov states are characterized by a power law such that $\nu = -l + is$ where s is real. For example, for two identical bosons interacting with another particle ($s_{13} = 1$), in the s-wave sector:

$$\lambda(\nu, l = 0, t) = \cos t - \frac{\sin(\nu t)}{\nu \sin(t) \cos\left(\frac{\pi\nu}{2}\right)}. \tag{150}$$

Eq. (150) admits solution of the form $\nu = is$ for all values of the mass ratio and $s \to 0$ for $t \to 0$. For higher l-waves, Efimov states appear above a critical value of the mass ratio. The threshold is obtained by searching the value of the parameter $t = t_l^{\rm crit}$ such that:

$$\lim_{\nu\to -l} \lambda\left(\nu, l, t_l^{\rm crit}\right) = 0. \tag{151}$$

In the p-wave sector which concerns the case of two identical fermions interacting with another particle ($s_{13} = -1$) one gets

$$\lambda(\nu, l = 1, t) = \cos t - \frac{(\nu+2)\sin(\nu t) - \nu \sin[(\nu+2)t]}{2\nu(\nu+2)\sin^2 t \cos t \cos\left(\frac{\pi\nu}{2}\right)}, \tag{152}$$

and the threshold is obtained from the equation:

$$t_1^{\rm crit} = \tan(t_1^{\rm crit}) - \frac{\pi}{2} \sin^2(t_1^{\rm crit}). \tag{153}$$

Eq. (153) gives the critical mass ratio $\left(\frac{M}{m}\right)_{l=1}^{\rm crit} \simeq 13.6$ found in Refs. [8, 58]. For higher partial waves, the critical values of the mass ratio $\left(\frac{M}{m}\right)_l^{\rm crit}$ found by this method coincide with the ones computed in Ref. [17] where the hyper-spherical method were used. Results are reported in Fig. (5).

V. HETEROGENEOUS TETRAMERS

The four-body problem is the subject of recent theoretical [68, 69, 72–74] and experimental [70, 71] studies in

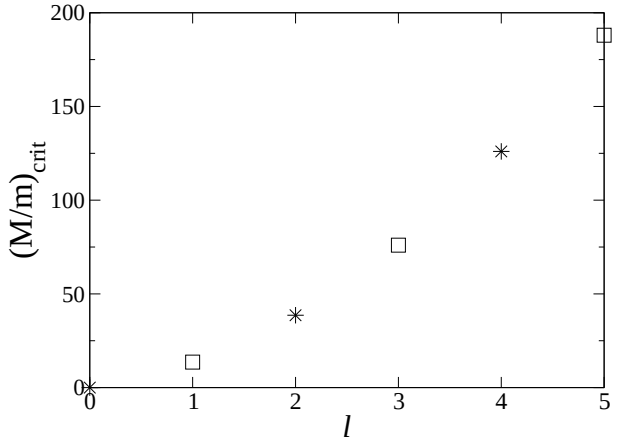

FIG. 5: Critical mass ratio $\left(\frac{M}{m}\right)_l^{\rm crit}$ for the threshold of apparition of an Efimov spectrum in each partial wave l of Eq. (141).

ultra-cold atoms. In this section, zero-range eigenequation for four-particle bound states or 'tetramers' is considered. The binding energy is denoted by E_4 and:

$$E_4 = -\frac{\hbar^2 q^2}{2\mu}, \tag{154}$$

where $E_4 < E_3$ and/or $E_4 < 2E_2$ if a trimer and/or dimer exists. A particle (i) is also characterized by a momentum $\mathbf{k}_i$ and a mass m_i equals to m or M. In the center of mass frame, the source amplitude associated with the pair (12) can be written as:

$$\langle\{\boldsymbol{\xi}^{12}\}|\mathrm{S}_\Psi^{1\leftrightharpoons 2}\rangle = (2\pi)^D\delta(\sum_{n=1}^{4}\mathbf{k}_n)\mathrm{F}(\boldsymbol{K}_{12},\boldsymbol{k}_{34}). \tag{155}$$

The integral equation satisfied by the function F is obtained from Eq. (103) (here $(ij) = (1,2)$). A specific configuration is represented schematically in Fig. (6). The summation at the left hand side of the integral equation is composed of source terms which are deduced from Eq. (155) by using the permutation symmetry and the statistics of the particles. In order to have a general equation for the different possible configurations, a statistical factor denoted by s_{np} is introduced for each pair of particles. Depending on the system, for two interacting particles n and p, the statistical factor is $s_{np} = \pm 1$ or $s_{np} = 0$ for non-interacting particles. Using this notation, the different source amplitude are written in Tab. (I). Different possible configurations of the two-mass component system and the corresponding statistical factors are listed in Tab. (II) Eq. (103) can be simplified into:

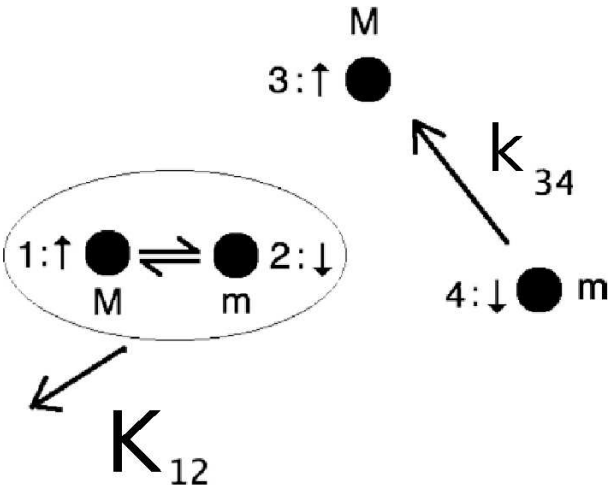

FIG. 6: Schematic representation of the momentum coordinates used for the source amplitude associated with the contact condition between particles 1 and 2 for a system of fermions with two-mass components.

Pair (ij)	Permutations	$\langle\{\boldsymbol{\xi}^{ij}\}\|\mathrm{S}_\Psi^{i\leftrightharpoons j}\rangle$
(12)		$(2\pi)^D\delta(\sum_{n=1}^{4}\mathbf{k}_n)\,\mathrm{F}(\boldsymbol{K}_{12},\boldsymbol{k}_{34})$
(13)	$2\leftrightarrow 3$	$s_{13}(2\pi)^D\delta(\sum_{n=1}^{4}\mathbf{k}_n)\,\mathrm{F}(\boldsymbol{K}_{13},\boldsymbol{k}_{24})$
(14)	$2\leftrightarrow 4$	$s_{14}(2\pi)^D\delta(\sum_{n=1}^{4}\mathbf{k}_n)\,\mathrm{F}(\boldsymbol{K}_{14},\boldsymbol{k}_{32})$
(23)	$1\leftrightarrow 3$	$s_{23}(2\pi)^D\delta(\sum_{n=1}^{4}\mathbf{k}_n)\,\mathrm{F}(\boldsymbol{K}_{23},\boldsymbol{k}_{14})$
(24)	$1\leftrightarrow 4$	$s_{24}(2\pi)^D\delta(\sum_{n=1}^{4}\mathbf{k}_n)\,\mathrm{F}(\boldsymbol{K}_{24},\boldsymbol{k}_{31})$
(34)	$1\leftrightarrow 3,\ 2\leftrightarrow 4$	$s_{34}(2\pi)^D\delta(\sum_{n=1}^{4}\mathbf{k}_n)\,\mathrm{F}(\boldsymbol{K}_{34},\boldsymbol{k}_{12})$

TABLE I: Source terms for each pair (ij) deduced by using the permutation symmetry of the 4-body wave function from the source term of the interacting pair (12) in Eq. (155). The statistical factors s_{ij} are given in Tab. II.

Configuration	(m_1,m_2,m_3,m_4)	$(s_{13},s_{14},s_{23},s_{24},s_{34})$
$BBBB$	(m,m,m,m)	$(1,1,1,1,1)$
$(f\,\text{or}\,b)BBB$	(m,M,M,M)	$(1,1,0,0,0)$
$BbBb$	(M,m,M,m)	$(0,1,1,0,1)$
$F_\uparrow f_\downarrow F_\uparrow f_\downarrow$	(M,m,M,m)	$(0,-1,-1,0,1)$
$Bf_\uparrow Bf_\uparrow$	(M,m,M,m)	$(0,-1,1,0,-1)$
$(f\,\text{or}\,b)F_\uparrow F_\uparrow F_\uparrow$	(m,M,M,M)	$(-1,-1,0,0,0)$

TABLE II: Different possible 4-body configurations. The notation is as follows: B (resp. F) means that the atom is a boson (resp. fermion) of mass M, b (resp. f) means that the atom is a boson (resp. a fermion) of mass f. The fermions have two possible internal states $\uparrow$ and $\downarrow$. For each configuration, interaction is non vanishing only between one type of heterogeneous pair. The statistical factors s_{ij} appears in the integral equation (159).

Pair (ij)	$\mathbf{v}_{ij}$	$\boldsymbol{\sigma}_{ij}$
(13)	$\frac{m_4\mathbf{u}}{M_{3,4}}+\mathbf{v}-\frac{m_4\mathbf{k}}{M_{2,4}}$	$\left(\frac{\mu_{1,2}}{m_1}-\frac{\mu_{3,4}}{m_3}\right)\mathbf{u}-\mathbf{v}+\mathbf{k}$
(14)	$-\frac{m_3\mathbf{u}}{M_{3,4}}+\mathbf{v}+\frac{m_3\mathbf{k}}{M_{2,3}}$	$\left(\frac{\mu_{1,2}}{m_1}-\frac{\mu_{3,4}}{m_4}\right)\mathbf{u}+\mathbf{v}+\mathbf{k}$
(23)	$\frac{m_4\mathbf{u}}{M_{3,4}}+\mathbf{v}+\frac{m_4\mathbf{k}}{M_{1,4}}$	$\left(\frac{\mu_{1,2}}{m_1}-\frac{\mu_{3,4}}{m_4}\right)\mathbf{u}+\mathbf{v}-\mathbf{k}$
(24)	$\frac{m_3\mathbf{u}}{M_{3,4}}-\mathbf{v}+\frac{m_3\mathbf{k}}{M_{1,3}}$	$\left(\frac{\mu_{1,2}}{m_1}-\frac{\mu_{3,4}}{m_3}\right)\mathbf{u}-\mathbf{v}-\mathbf{k}$

TABLE III: Coordinates appearing in Eq. (159).

$$\int \frac{d^D\boldsymbol{k}_{12}}{(2\pi)^D}\sum_{\substack{\mathrm{n<p}\\ \mathrm{p>2}}} s_{np}\frac{\mathrm{F}\left(\boldsymbol{K}_{np},\boldsymbol{k}_{kl}\right)}{\kappa^2+k_{12}^2} \;=\; \frac{-\hbar^2\mathrm{F}(\mathbf{u},\mathbf{v})}{2\mu_{1,2}T_D(E_{\mathrm{col}}^{(12)})}, \tag{156}$$

where $\mathbf{u}=\boldsymbol{K}_{12}$, $\mathbf{v}=\boldsymbol{k}_{34}$. In Eq. (156) the momentum κ is related to the collisional energy by:

$$E_{\mathrm{col}}^{(12)}=\frac{\hbar^2\kappa^2}{2\mu_{1,2}}, \tag{157}$$

and can be expressed as

$$\kappa^2=q^2+\frac{\mu_{1,2}}{\mu_{(12),(34)}}u^2+\frac{\mu_{1,2}}{\mu_{3,4}}v^2. \tag{158}$$

The integral equation which encapsulate the Bethe-Peierls can finally be written as:

$$\int \frac{d^D\mathbf{k}}{(2\pi)^D}\left[s_{13}\frac{\mathrm{F}(\mathbf{k},\mathbf{v}_{13})}{\kappa^2+\sigma_{13}^2}+s_{14}\frac{\mathrm{F}(\mathbf{k},\mathbf{v}_{14})}{\kappa^2+\sigma_{14}^2}+s_{23}\frac{\mathrm{F}(\mathbf{k},\mathbf{v}_{23})}{\kappa^2+\sigma_{23}^2}\right.$$
$$\left.+s_{24}\frac{\mathrm{F}(\mathbf{k},\mathbf{v}_{24})}{\kappa^2+\sigma_{24}^2}+s_{34}\frac{\mathrm{F}(-\mathbf{u},\mathbf{k})}{\kappa^2+k^2}\right]=\frac{-\hbar^2\mathrm{F}(\mathbf{u},\mathbf{v})}{2\mu_{1,2}T_D(E_{\mathrm{col}}^{(12)})}, \tag{159}$$

In the particular case where the system is composed of two heterogeneous dimers made of fermions ($F_\uparrow f_\downarrow F_\uparrow f_\downarrow$), one recovers the same integral equation as in Eq. (16) of Ref. [12]. In 3D depending on the statistics of the particle and of the mass ratio, Eq. (159) is not in general well defined as a consequence of the Efimov-Thomas effect. For systems composed of four particles in 2D, bound states of particles of same mass has been considered in [72]. In the present work, the binding energies of 2D ground tetramers are computed numerically as a function of the mass ratio of the interacting particles in the $BBBb$ configuration. Calculations are restricted to s-wave tetramers by using the ansatz:

$$\mathrm{F}(\mathbf{u},\mathbf{v})=F(u,v,\theta)\quad\text{with }\theta=\angle(\mathbf{u},\mathbf{v}), \tag{160}$$

In order to check the numerical computation, the particular case of four identical bosons which has been already obtained by several authors [72, 73] has been considered. In this configuration, two bound states have been found with the binding energies which are close to already published results: $E_4/E_2=197$ and 24, to be compared with

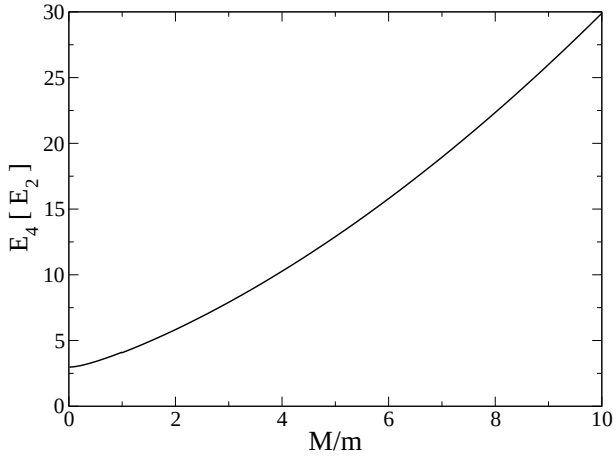

FIG. 7: Ground state branch for the three bosons of mass M interacting with one impurity of mass m in 2D as a function of the mass ratio.

$E_4/E_2=197.3$ and 25.5 in Ref. [73] or with $E_4/E_2=194$ and 24 in Ref. [72]. Results for three identical bosons interacting resonantly with another particle is shown in Fig. (7). In 3D for the same $bBBB$ configuration, similarly to the 3-bosons original STM equation, Eq. (159) does not constitute a well defined problem. The nature and properties of eigenstates of this system in the zero-range limit is yet unsolved. The $bF_\uparrow F_\uparrow F_\uparrow$ or $f_\downarrow F_\uparrow F_\uparrow F_\uparrow$ configurations have been studied recently in Ref. [68] and a pure 4-body Efimov effect (*i.e.* without 3-body Efimov state) involving a four-body parameter has been found for a mass ratio $13.384<M/m<13.607$.

CONCLUSIONS

In this paper, it has been shown how the zero-range approach can be formulated for different problems in a unified framework without technical intricacy for all dimensions $D\leq 3$. In the few-body problem, general integral eigenequations are obtained straightforwardly in the momentum representation. Efimov effect is deduced from the Skorniakov Ter-Martirosian equation by using a nodal condition and a subtracting technique which is close to regularizing schemes used in the Effective Field Theory. Open questions remain to be solved concerning the few-body problem in the limit of zero-range forces. For example, the question whether or not the 3-body parameter is enough for describing 4-bosons properties in 3D is still under debate [69, 74]. The '$N+1$' fermionic problem is another open issue. The question whether or not a pure N-body Efimov effect occurs in 3D and at which critical mass ratio, is yet unsolved for $N\geq 4$. However, the corresponding zero-range integral equation for this last problem can be easily deduced from the present formalism and is given below as a concluding remark. Each of the N polarized fermions of mass M labeled by i ($1\leq i\leq N$) interact with only one impurity (particle

$i = N+1$) of mass m. In the center of mass frame, the source amplitude for the pair $N, (N+1)$ can be written as:

$$\langle\{\boldsymbol{\xi}^{N,(N+1)}\}|\mathrm{S}_\Psi^{N\leftrightharpoons N+1}\rangle = (2\pi)^D\delta\left(\sum_{i=1}^{N+1}\mathbf{k}_i\right) \times F(\mathbf{k}_1,\mathbf{k}_2\dots\mathbf{k}_{N-1}), \quad (161)$$

and other source amplitudes $|\mathrm{S}_\Psi^{i\leftrightharpoons N+1}\rangle$ $(i = 1\cdots N-1)$ are deduced from this ansatz by using the fermionic statistics. The general integral equation for this problem which encapsulates the Bethe-Peierls asymptotic condition is then obtained from Eq. (103):

$$\int\frac{d^D\mathbf{k}_N}{(2\pi)^D}\frac{F(\mathbf{k}_N,\mathbf{k}_2,\mathbf{k}_3,\cdots\mathbf{k}_{N-1})+F(\mathbf{k}_1,\mathbf{k}_N,\mathbf{k}_3,\cdots\mathbf{k}_{N-1})+\cdots+F(\mathbf{k}_1,\mathbf{k}_2,\cdots\mathbf{k}_{N-2},\mathbf{k}_N)}{-E-i0^+ +\frac{\hbar^2}{2M}\sum_{i=1}^N k_i^2+\frac{\hbar^2}{2m}\left(\sum_{i=1}^N\mathbf{k}_i\right)^2} = \frac{F(\mathbf{k}_1,\mathbf{k}_2,\cdots\mathbf{k}_{N-1})}{T_D(E_{\mathrm{col}}+i0^+)} - \int\frac{d^D\boldsymbol{k}_N}{(2\pi)^D}\Psi^{(0)}(\mathbf{k}_1,\cdots\mathbf{k}_N), \quad (162)$$

where $E_{\mathrm{col}} = E - \frac{\hbar^2}{2M}\sum_{i=1}^{N-1}k_i^2 - \frac{\hbar^2}{2(M+m)}\left(\sum_{i=1}^{N-1}\mathbf{k}_i\right)^2$, $\Psi^{(0)}$ is a complementary solution associated with the free Hamiltonian at energy E in the center of mass frame (it is equal to zero for $E<0$) and F is antisymmetric under exchange of two coordinates $(\mathbf{k}_i,\mathbf{k}_j)$.

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