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► **To cite this version:**

Vilmos Prokaj. Unfolding the Skorohod reflection of a semimartingale. *Statistics and Probability Letters*, 2009, 79 (4), pp.534. 10.1016/j.spl.2008.09.029 . hal-00505541

HAL Id: hal-00505541

<https://hal.science/hal-00505541>

Submitted on 24 Jul 2010

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Accepted Manuscript

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PII: S0167-7152(08)00462-8
DOI: 10.1016/j.spl.2008.09.029
Reference: STAPRO 5228

To appear in: *Statistics and Probability Letters*

Received date: 31 March 2008
Revised date: 25 September 2008
Accepted date: 25 September 2008

Please cite this article as: Prokaj, V., Unfolding the Skorohod reflection of a semimartingale. *Statistics and Probability Letters* (2008), doi:10.1016/j.spl.2008.09.029

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Unfolding the Skorohod reflection of a semimartingale

Vilmos Prokaj^{a,b},

^a*Department of Probability Theory and Statistics, Eötvös Loránd University, 1117 Budapest, Pázmány P. sétány 1/C, Hungary*

^b*Computer and Automation Institute of the Hungarian Academy of Sciences, 1111 Budapest, Kende utca 13-17, Hungary*

Abstract

It is well-known that the Skorohod reflection of a Wiener-process is the absolute value of another Wiener process with finer filtration. In other words it can be unfolded to obtain a Wiener process. In this short note a similar statement is proved for continuous semimartingales.

Key words: reflected process, semimartingale

Let W_t be a Wiener process. It is well-known, see e.g. in Revuz and Yor (1994) pp. 230, that

$$|W_t| = \beta_t + \sup_{s \leq t} (-\beta_s) \quad (1)$$

where $\beta = \int \text{sign}(W_s) dW_s$ is another Wiener process. In other words starting with β , the Skorohod reflection of β defined as the right hand side of (1) can be unfolded to a Wiener process W which has finer filtration than β . In this note we give a generalized version of this observation as stated in Theorem 1.

Our motivation comes from the analysis of the continuous time version of the stochastic approximation algorithm with re-initialization. In the course of this the following equation occurred

$$dV_t = v_t dt + dM_t + (V_0 - V_{t-}) dN_t$$

where $V_0 < K$ with some finite $K \in \mathbf{R}$, M is a martingale starting from zero and N is a finite counting process. Without the jump term the remaining equation is easy to handle. Now if we put $U_t = \int_0^t v_s ds + M_t$ then the Skorohod

* Corresponding author.

Email address: prokaj@cs.elte.hu (Vilmos Prokaj).

reflection $Y_t = U_t + \sup_{s \leq t} (-U_s)$ dominates V_t , in the form $V_t \leq K + Y_t$. However the estimation of Y is not so easy. The advantage of Theorem 1 below, is that using it we can reduce the problem to the case when N identically zero, for the details see Gerencsér and Prokaj (2008).

It is clear from the case of the Wiener process, that in general the unfolding can not be carried out on the original probability field of U . So in general we need an enlargement of the probability space. The enlargement of $(\Omega, \mathcal{F}, \mathbf{P})$ is $(\Omega', \mathcal{F}', \mathbf{P}')$ with a measure preserving mapping $\pi : \Omega' \rightarrow \Omega$.

Theorem 1. *Let U_t be a continuous semimartingale starting from zero. Denote $Y_t = U_t + \sup_{s \leq t} (-U_s)$ the Skorohod reflection of U . Then on a suitable enlargement of the probability space there is a continuous semimartingale Y'_t such that $|Y'_t| = Y_t$ and $dY'_t = \text{sign}(Y'_t)dU_t$.*

Proof. First we define the sign of the prospective semimartingale Y' . To this end let $\{\xi_k : k \geq 1\}$ be a sequence of independent variables such that $\mathbf{P}(\xi_k = \pm 1) = 1/2$ and $\sigma(\{\xi_k : k \geq 1\})$ is independent from \mathcal{F}_∞ . This can be achieved by a suitable enlargement of the probability field. On the enlarged probability field we have all the objects we have on the original field and we will use the same notation for them.

Next put $\mathfrak{z} = \{t \geq 0 : Y_t = 0\}$ and denote $\{C_k : k \geq 1\}$ an enumeration of the components of $[0, \infty) \setminus \mathfrak{z}$. This can be done in a measurable way, i.e the event $(t \in C_k)$ belongs to \mathcal{F}_∞ for all $t \geq 0$ and $k \geq 1$. To simplify the notation later let $C_0 = \mathfrak{z}$ and $\xi_0 = 0$. Now put $Z_t = \sum_{k=0}^{\infty} \xi_k \chi_{(t \in C_k)}$ and $\tilde{\mathcal{F}}_t = \mathcal{F}_t \vee \mathcal{F}_t^Z$, where $\mathcal{F}_t^Z = \sigma(Z_s, s \leq t)$.

Since U is a continuous semimartingale it can be written as $U_t = A_t + M_t$ where A is a continuous process with locally bounded total variation and M is a \mathcal{F} local martingale. We claim

Proposition 2. *M is an $\tilde{\mathcal{F}}$ local martingale.*

Proposition 2 gives that Y is a semimartingale with respect to the filtration $\tilde{\mathcal{F}}$, therefore Z can be integrated by Y . For this integral the following expression is true

Proposition 3.

$$Z_t Y_t = \int_0^t Z_s dY_s.$$

Observe that by Proposition 3 the process defined as $Y'_t = Z_t Y_t = \int_0^t Z_s dY_s$ has the property that $|Y'_t| = Y_t$ and $\text{sign}(Y'_t) = Z_t$.

Now put $\varphi_t = Y_t - U_t = \sup_{s \leq t} (-U_s)$. The points of increase of φ is a subset

of $\mathfrak{z} = \{t \geq 0 : Y_t = 0\}$ by the Skorohod reflection lemma, see Revuz and Yor (1994) Chap VI. Lemma 2.1. This finishes the proof as $\int_0^t \chi_{(Y_s \neq 0)} d\varphi_s = 0$, implies that $dY'_t = Z_t dY_t = Z_t dU_t$. \square

Proof of Proposition 2. By stopping if necessary it is enough to prove that if M is an \mathcal{F} martingale then it is also an $\tilde{\mathcal{F}}$ martingale, i.e.

$$\mathbf{E}((M_t - M_s)\chi_A) = 0, \quad \text{for } s \leq t \text{ and } A \in \tilde{\mathcal{F}}_s$$

Since $\tilde{\mathcal{F}}_s = \mathcal{F}_s \vee \mathcal{F}_s^Z$ actually it is enough to prove for all A of the form $B \cap D$ where $B \in \mathcal{F}_s$ and $D = \cap_{i=1}^n (Z_{t_i} = \varepsilon_i)$ with $0 \leq t_1 < \dots < t_n \leq s$, $\varepsilon_i \in \{-1, 0, 1\}$.

Since

$$\mathbf{E}((M_t - M_s)\chi_B\chi_D) = \mathbf{E}((M_t - M_s)\chi_B \mathbf{E}(\chi_D | \mathcal{F}_\infty))$$

it is enough to show that $\mathbf{E}(\chi_D | \mathcal{F}_\infty)$ is \mathcal{F}_s measurable.

D can be written in the following form $D = \cap_{i=1}^n (\xi_{\kappa(t_i)} = \varepsilon_i)$, where $\kappa(t_i)$ is the (random) index of the C_k to which t_i belongs. It is clear that $\kappa(t_i)$ is measurable with respect to the condition \mathcal{F}_∞ while $\{\xi_k : k \geq 0\}$ is independent from it. So

$$\mathbf{E}(\chi_D | \mathcal{F}_\infty) = \mathbf{P}(\xi_{k_i} = \varepsilon_i, i = 1, \dots, n) |_{k_1=\kappa(t_1), \dots, k_n=\kappa(t_n)}$$

Now if m denotes the number of distinct non-zero indices in $\{k_1, \dots, k_n\}$, then

$$\mathbf{P}(\xi_{k_i} = \varepsilon_i, i = 1, \dots, n) = \begin{cases} 0 & \text{if } \varepsilon \text{ contradicts } k \\ 2^{-m} & \text{otherwise.} \end{cases}$$

Here ε contradicts k means, that either there are i, j such that $k_i = k_j$ but $\varepsilon_i \neq \varepsilon_j$ or exists i such that $k_i = 0$ but $\varepsilon_i \neq 0$ or $k_i \neq 0$ but $\varepsilon_i = 0$.

Observe that both the value of m , i.e. the number of the components of $[0, s] \setminus \mathfrak{z}$ that contain some t_i and the condition, i.e. ε contradicts k can be calculated if we know the trajectory of Y up to time s . In other words $\mathbf{E}(\chi_D | \mathcal{F}_\infty)$ is \mathcal{F}_s measurable and the proposition is proved. \square

Proof of Proposition 3. To calculate the integral $\int_0^t Z_s dY_s$ we use a rather standard approximation of the process Z . For $\varepsilon > 0$ let us define the following sequence of stopping times

$$\begin{aligned} \tau_0^\varepsilon &= 0 \\ \tau_{2k+1}^\varepsilon &= \inf \{t > \tau_{2k}^\varepsilon : |Y_t| > \varepsilon\} & k = 0, 1, 2, \dots \\ \tau_{2k+2}^\varepsilon &= \inf \{t > \tau_{2k+1}^\varepsilon : Y_t = 0\} & k = 0, 1, 2, \dots \end{aligned}$$

Put

$$\tilde{Z}_t^\varepsilon = \sum_{k=0}^{\infty} Z_t \chi_{(\tau_{2k+1}^\varepsilon < t \leq \tau_{2k+2}^\varepsilon)}$$

\tilde{Z}^ε is constant on every random interval of the form $(\tau_k^\varepsilon, \tau_{k+1}^\varepsilon]$. Since Y is continuous the sequence (τ_k^ε) can not accumulate in a bounded interval so \tilde{Z}^ε is of bounded variation on every compact interval. This means that Y is Riemann–Stieltjes integrable with respect to \tilde{Z}^ε almost surely and

$$\tilde{Z}_t^\varepsilon Y_t - \tilde{Z}_0^\varepsilon Y_0 = \int_0^t \tilde{Z}_s^\varepsilon dY_s + \int_{(0,t)} Y_s d\tilde{Z}_s^\varepsilon \quad (2)$$

As $\varepsilon \rightarrow 0$ we have that $\tilde{Z}_t^\varepsilon \rightarrow Z_t$ for all t almost surely. From this, it is clear that the left hand side is convergent as $\varepsilon \rightarrow 0$ and its almost sure limit is $Z_t Y_t - Z_0 Y_0 = Z_t Y_t$ since $Y_0 = 0$. Since $|\tilde{Z}^\varepsilon| \leq 1$ the convergence of \tilde{Z}^ε implies as well that

$$\int_0^t \tilde{Z}_s^\varepsilon dY_s \rightarrow \int_0^t Z_s dY_s$$

in probability for all t . So to prove the proposition it is enough to show that the last term in (2) goes to zero as $\varepsilon \rightarrow 0+$.

The last term in (2) is a sum. It is

$$\int_{(0,t)} Y_s d\tilde{Z}_s^\varepsilon = \sum_{\substack{k \\ \tau_{2k+1}^\varepsilon < t}} Y_{\tau_{2k+1}^\varepsilon} Z_{\tau_{2k+1}^\varepsilon} = \varepsilon \sum_{\substack{k \\ \tau_{2k+1}^\varepsilon < t}} Z_{\tau_{2k+1}^\varepsilon}$$

Put $N(t, \varepsilon) = \max\{k : \tau_{2k+1}^\varepsilon < t\}$, which is usually called the number of upcrossing of the interval $[0, \varepsilon]$. So

$$\int_{(0,t)} Y_s d\tilde{Z}_s^\varepsilon = \varepsilon \sum_{l=1}^{N(t,\varepsilon)} \xi_{k_l}$$

where $\{\xi_{k_l} : l = 0, 1, \dots, N(t, \varepsilon)\}$ is an enumeration of the $\tilde{Z}_{\tau_{2k+1}^\varepsilon}^\varepsilon$ values.

Now there are two cases. On the event, where $N(t, \varepsilon)$ remains bounded the limit is clearly zero by trivial estimates.

On the event, where $N(t, \varepsilon)$ goes to infinity as $\varepsilon \rightarrow 0$ we can apply e.g. the Bernoulli law of large numbers to obtain that

$$\frac{\sum_{l=1}^{N(t,\varepsilon)} \xi_{k_l}}{N(t, \varepsilon)} \rightarrow \mathbf{E}(\xi_1) = 0.$$

It is also well known, see e.g. in Revuz and Yor (1994) Chap. VI. Theorem (1.10) that $\lim_{\varepsilon \rightarrow 0+} \varepsilon N(t, \varepsilon) = \frac{1}{2} L_t^0(Y)$, where L_t^0 is the local time of Y at level

zero. So we have in both cases that

$$\int_{(0,t)} Y_s d\tilde{Z}_s^\varepsilon \rightarrow \frac{1}{2} L_t^0 \mathbf{E}(\xi_1).$$

Now as $\mathbf{E}(\xi_1) = 0$ we get the result. \square

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