

Housing and commuting in an extended monocentric model

Vincent Breteau and Fabien Leurent
Université Paris-Est LVMT Ecole des Ponts ParisTech

Abstract

We model a city in which jobs are exogenous and distributed across an extended business area in which transport has a nonzero cost. Households are homogeneous in terms of utility and gross income, but each household chooses its residential location on the basis of its place of employment, which is deemed to be fixed.

Equilibrium conditions for this residential location market are established. It is shown that there is an equilibrium that is unique (for a closed city with absentee landlords). Households' utility and dwelling size increase the farther the workplace is from the centre, whereas land rent decreases.

Within a simplified framework, the model is resolved analytically and we establish the sensitivity of the endogenous variables to the city's characteristic parameters. Two extreme cases are highlighted: the "quasi-monocentric" city where net income decreases with distance from the centre, versus the "eccentric" city, where net income increases with distance from the centre.

JEL Classification: R21

Keywords: Residential Location, Land Markets, Commuting

1. Introduction

Commuting represents a substantial proportion of total urban traffic, and one that is relatively regular in terms of its origin-destination and timing structure: it provides a framework for transport planning and is relatively easy to study (Kwan, 1999). That is why the distance between home and the workplace, and its variations in space, have been extensively studied, whether in geography and urban planning (e.g. Massot & Korsu, 2006; Shearmur, 2006) or in economics and regional sciences (e.g. Hamilton, 1982; Crane, 1996; Wheaton, 2004).

Commuting costs represent a fundamental variable in the monocentric city model, where all jobs are concentrated in a single place, the Central Business District (CBD). This model plays a central role in urban economics (Alonso, 1964; Muth, 1969; Mills, 1972). However, the hypothesis of a single, point-wise employment place gives a highly reductive picture of the spread of urban employment (McMillen & McDonald, 1998; Glaeser & Kahn, 2001): some authors have therefore proposed polycentric models, first as an extension to the monocentric model (Ogawa & Fujita, 1980; Fujita, 1985; Grimaud, 1989) then in a more general context (Anas & Kim, 1996; Lucas & Rossi-Hansberg, 2002), whilst Anas, Arnott & Small (1998) provide a general discussion of modern urban structure. Halfway between the two types of extension, Wheaton (2004) has modelled a spatial mix of residential and productive locations within a monocentric framework: from this he deduced that there is a significant concentration of jobs in the agglomeration centre at urban equilibrium, and an even greater concentration at the optimal state.

In this article, we consider a city where jobs are distributed exogenously and in an extended area where the cost of transport is nonzero. In addition to these hypotheses, already present in Sullivan (1983a,b), we also assume households to be homogeneous in terms of utility and gross income and each household to have a fixed place of employment that influences its residential location. Our model is therefore one of medium-term equilibrium, where households participate in the housing market whereas employers remain static. We show that residential location, household utility and living space increase the farther the workplace is from the centre, whereas land rent decreases.

For simple forms of utility function, transport costs, employment density and land use capacity, we resolve residential location, land rent, lot size and residential density analytically on the basis of workplace location. We show that for each workplace location, each of these functions varies monotonically in relation to the characteristic parameters of the city: the radius limiting the location of firms, their local density, the land capacity, the alternative rent, the gross income, and the preferences between living space and the other goods.

The article is structured as follows: section 2 focuses on presenting the model and determining the equilibrium location of households. Section 3 defines the urban equilibrium and describes the necessary and sufficient conditions for it. In section 4, we study the structural properties of the model in its general form. Section 5 looks at the equilibrium in a simplified framework and establishes analytical properties, with closed formulas for the endogenous variables, which enables their sensitivity to urban parameters to be analysed. Section 6 concludes the article.

2. The model

2.1. About firms and workplaces

Consider a city in which firms, and therefore jobs, are distributed around a centre in a disk of radius ρ_f : we call it the CBD or the employment area. Jobs are assumed to be distributed following a radial density $f(\rho)$ that is non-zero on an interval of $[0, \rho_f]$, yielding a cumulative distribution function $F(\rho) = \int_0^\rho f(r)dr$.

The function F is therefore strictly increasing. The number of jobs is fixed at N , therefore $F(\rho_f) = N$. Using a fixed number of jobs places us in the theoretical framework of a closed city. We also make the assumption that the landowners are absent.

2.2. Households

As regards households, let us assume: (H1) that the households are located in a ring around the employment area. (H2) That the households are homogeneous in their preferences. (H3) That each household has only one member that is employed by a firm in the CBD. (H4) That each household receives an income Y from this job, independent of residential and professional location (i.e. employers are indifferent to the location of their personnel): this is a very reductive hypothesis for the statistical distribution of incomes, since empirical research has shown the existence of an income gradient within cities (Eberts, 1981; McMillen & Singell, 1992; Timothy & Wheaton, 2001). Nevertheless, Glaeser & Kahn (2001) have shown that employment deconcentration tends to generate equalisation of incomes.

Finally, (H5) households are differentiated by their place of employment. The choice of a residential location in r by a household whose workplace is ρ , leaves them a net income of $I = Y - T(\rho, r)$, where $T(\cdot, \cdot)$ is the cost of transport.

A household's utility U depends on the size of its living space, s , and on the quantity z of a composite consumer product treated as numeraire. The function U is assumed to be increasing and continually differentiable into each of its variables. In general terms, each household is deemed to be a rational decision maker, which seeks to maximise its utility within its budget constraint.

2.3. Spatial structure of the residential area

We are interested here in the land use market within the residential zone only: the unit price of land in r , or land rent, is denoted $R(r)$. The opportunity cost of the land, corresponding to an alternative use (e.g. agricultural), is denoted R_A . The economic programme of the household employed in ρ is expressed as follows:

$$\begin{aligned} \max_{r,z,s} U(z,s) \\ \text{s.t. } z + R(r).s \leq Y - T(\rho, r) \end{aligned} \quad (2.1)$$

To analyse land occupancy, the density of the households in r is denoted $h(r)$. This is the main endogenous variable in our model. By way of hypothesis, $h(r) = 0$ where $r \in [0, r_0[$. Following Fujita (1989), it can be shown that if local land capacity $L(r)$ is strictly positive, then $h(r) > 0$ between r_0 and the radius of the city r_A , from which land rent is equal to agricultural rent. The cumulative household distribution function, $H(r) = \int_{r_0}^r h(r)dr$, is thus strictly increasing.

Regarding the transport cost, it is assumed that $T(\rho, r)$ is a decreasing function in ρ and increasing in r .

2.4. Equilibrium location of households

To analyse the choice of household location, we follow Von Thünen (1826) and Alonso (1964) in defining the bid-rent function of a household employed in ρ , $\Psi_\rho(r, u)$, as the maximum unit rent that that household is willing to pay to live in r whilst maintaining a utility level u . In other words:

$$\Psi_\rho(r, u) = \max_{z, s} \left\{ \frac{Y - T(\rho, r) - z}{s} \mid U(z, s) = u \right\} \quad (2.2)$$

This fixed ρ formulation possesses the same properties of continuity and ingress in r and in u as in the basic model. The dwelling size $S_\rho(r, u)$ at maximum bid in the previous optimisation problem, is increasing in r and in u .

On the basis of land rent R and available income $I_\rho = Y - T(\rho, r)$, we can also set the indirect utility function for a household, which plays a fundamental role later on, since it does not depend on ρ :

$$V(R, I_\rho) = \max_{z, s} \{ U(z, s) \mid z + Rs \leq I_\rho \} \quad (2.3)$$

The function $V(R, I_\rho)$ is continuous in R and I_ρ , increasing in I_ρ and decreasing in R . This latter property implies that:

$$V(R(r), I_\rho) \underset{<}{>} V(\Psi_\rho(r, u), I_\rho) \text{ as } R(r) \underset{>}{<} \Psi_\rho(r, u) \quad (2.4)$$

As $V(\Psi_\rho(r, u), I_\rho) = u$, property (2.4) implies a rule a la Fujita (1989), characterising the optimum location of a household employed in ρ and encountering supply conditions $R(r)$:

Rule 1 (Optimisation of residential location) *Given land rent function $R(r)$, a household employed in ρ optimises its utility \tilde{u}_ρ in a place of residence \tilde{r}_ρ if and only if*

$$R(\tilde{r}_\rho) = \Psi_\rho(\tilde{r}_\rho, \tilde{u}_\rho) \text{ and } R(r) \geq \Psi_\rho(r, \tilde{u}_\rho), \forall r.$$

If R and Ψ_ρ are differentiable in \tilde{r}_ρ , then the rule requires that the two curves are tangent at this point. By applying the envelope theorem to the bid-rent function (2.2), we deduce Muth's relation:

$$\dot{R}(\tilde{r}_\rho) = - \frac{\partial T(\rho, \tilde{r}_\rho)}{\partial r} / S_\rho(\tilde{r}_\rho, \tilde{u}_\rho) \quad (2.5)$$

We use $\dot{g}(r)$ to denote the derivative of a function $g(r)$ relative to r .

Carrying out this analysis for a particular household makes it possible to compare households $i \in \{1, 2\}$ with different places of employment. We show in the paragraph 8.1 of the appendix that if $\rho_1 < \rho_2$, then the bid curve $\Psi_1(r, u_1)$ of household 1 is steeper than the curve $\Psi_2(r, u_2)$ of household 2, so the Fujita (1989, p. 28) rule 2.3 applies and produces the following result.

Proposition 1 (on the order of households in the residential zone) *If two household indexed 1 and 2 are such that $\rho_1 < \rho_2$ then $\tilde{r}_{\rho_1} < \tilde{r}_{\rho_2}$: the order of households in the residential zone is the same as the order of their respective jobs in the employment zone.*

The converse implication directly arises from this.

Corollary 1. *All households living in r work in the same place.*

So at equilibrium, a household's number in the employment order, $F(\rho)$, coincides with its number in the residence order, $H(r)$. We define the function r_ω which associates the place of residence of the corresponding household r with a workplace ρ , i.e. $\rho \mapsto r_\omega(\rho)$ such that $H(r_\omega(\rho)) = F(\rho)$. Formally, the functions are linked by the following relation:

$$H = F \circ r_\omega^{-1} \quad (2.6)$$

Let us call r_ω the function for the commute from work to home, and its inverse $\rho_\omega = r_\omega^{-1}$ the function for the commute from home to work. By composition of increasing functions, these two functions are increasing.

3. The urban equilibrium

Having stated the main characteristics of the model and established certain properties of household location, let us look at urban equilibrium, restricted here to the equilibrium of the household location system. The aim is to determine two things at the same time: on the demand side, household location expressed by the density function $h(r)$ or its primitive $H(r)$, and on the supply side, land rent $R(r)$.

Our objective is to define urban equilibrium in an appropriate mathematical form to demonstrate the existence and uniqueness of the equilibrium. The appropriate form is a differential system that links residential distribution and land rent to residential position.

3.1. The household location market

On the demand side, for a fixed land rent function $R(r)$, a household employed in ρ chooses a residential location r that maximises its utility. In any potential position r the household would require a certain living space and a certain quantity of consumption goods, which would optimise its utility under price condition $R(r)$ within budget constraint $I_{\rho,r} = Y - T(\rho, r)$. The level of utility associated with r is therefore $V(R(r), I_{\rho,r})$. We denote this as follows:

$$W(r, \rho) = V(R(r), I_{\rho,r}) \quad (3.1)$$

A household requiring a residential location seeks a position of maximum utility for itself: so its microeconomic programme is:

$$\max_r W(r, \rho) \quad (3.2)$$

Let \hat{r}_ρ denote the optimised position, \tilde{u}_ρ the optimal utility and \tilde{s}_ρ the living space obtained by the household under these conditions. If the distribution of households is optimized, necessarily $H(\hat{r}_\rho) = F(\rho)$, therefore:

$$\tilde{r}_\rho = H^{-1} \circ F(\rho) = r_\omega(\rho) \quad (3.3)$$

On the supply side, if the distribution of households is set at $H(r)$ with radial density $h(r) \geq 0$, without providing details on the landowners, each local unit price reaches the maximum value that a household is willing to pay, or the agricultural price R_A . The associated microeconomic condition is:

$$R(r) = \max\{R_A, \max_\rho \Psi_\rho(r, \tilde{u}_\rho)\} \quad (3.4)$$

In addition, beyond the boundary radius r_A such that $H(r_A) = N$ the total number of households, land rent is equal to agricultural rent:

$$R(r) = R_A = \Psi_{\rho_f}(r_A, \tilde{u}_{\rho_f}), \quad \forall r \geq r_A \quad (3.5)$$

Finally, in any position $r \geq r_0$ such that $R(r) > R_A$, radial land capacity $L(r)$ is saturated, therefore:

$$h(r)\tilde{s}_\rho = L(r) \quad \text{if } r = H^{-1} \circ F(\rho) = r_\omega(\rho) \quad (3.6)$$

Let us summarise these considerations with a formal definition:

Definition 1 (Urban Equilibrium) This is a pair of functions (H, R) on $Z_H = [r_0, +\infty[$ such that H increases from 0 to N until $r_A \geq r_0$ then remains constant, which verifies conditions (3.2-6).

3.2. Alternative definition

By denoting $Z_F = [0, \rho_f[$, condition (3.2) becomes:

$$W(r_\omega(\rho), \rho) \geq W(r', \rho), \quad \forall (r', \rho) \in Z_H \times Z_F \quad (3.7)$$

In equilibrium, $r_\omega = H^{-1} \circ F$ is an increasing function as the composition of two increasing functions. The inverse mapping ρ_ω is also increasing. Let us set $\rho \in Z_F$ and $r' = r_\omega(\rho)$, therefore $\rho = \rho_\omega(r')$. Condition (3.7) implies:

$$W(r', \rho_\omega(r')) \geq W(r, \rho_\omega(r')), \quad \forall (r, r') \in Z_H \times r_\omega(Z_F) \quad (3.8)$$

Definition 2 (Supply-demand equilibrium of the residential location system) This is a pair of functions (H, R) on Z_H such that H increases from 0 to N until $r_A \geq r_0$ then remains constant, which verifies conditions (3.8), (3.3-6).

Proposition 2 (equivalence of the equilibrium definitions) *An urban equilibrium is a supply-demand equilibrium of the residential location system, and reciprocally.*

Proof. We deduced (3.8) from (3.2), therefore the system defining a supply-demand equilibrium is verified by an urban equilibrium. Reciprocally, for $\rho \in Z_f$, condition (3.8) for $r' = r_\omega(\rho)$ produces (3.7) therefore (3.2): therefore a supply-demand equilibrium is an urban equilibrium.

3.3. Characteristic differential system

Let there be a supply-demand equilibrium such that functions R and H are “sufficiently regular”. Then the first-order necessary optimality condition associated with (3.8) is $\partial W(r, \rho) / \partial r = 0$ at point $\rho = \rho_\omega(r)$, therefore:

$$\dot{R}(r) \frac{\partial V}{\partial R} = \frac{\partial T(\rho_\omega(r), r)}{\partial r} \frac{\partial V}{\partial I} \quad (3.9)$$

By Roy’s identity, the (Marshallian) surface demand function verifies $\hat{s}(R, I) = -\frac{\partial V}{\partial R} / \frac{\partial V}{\partial I}$, therefore the previous relation is reformulated:

$$\dot{R}(r) = -\frac{\partial T(\rho_\omega(r), r)}{\partial r} / \hat{s}(R, I_{\rho, r}) \quad (3.10)$$

We are back to Muth’s relation.

On the supply side, (3.6) results in $\hat{s}(R, I) = L(r) / h(r)$ for $h = \dot{H}$, therefore:

$$\dot{H}(r) = L(r) / \hat{s}(R, I_{\rho, r}) \quad (3.11)$$

Since $\rho_\omega = F^{-1} \circ H$, the relations (3.10) and (3.11) constitute a differential system in (H, R) :

$$\begin{cases} \dot{R}(r) = -\frac{\partial T(\rho_\omega(r), r)}{\partial r} / \hat{s}(R, I_{\rho, r}) \\ \dot{H}(r) = L(r) / \hat{s}(R, I_{\rho, r}) \end{cases} \text{ where } \begin{cases} \rho_\omega(r) = F^{-1} \circ H(r) \\ I_{\rho, r} = Y - T(\rho_\omega(r), r) \end{cases} \quad (3.12)$$

Proposition 3 (Necessary condition on urban equilibrium) *At urban equilibrium the pair (H, R) verifies the differential system (3.12). In addition, the boundary conditions have an initial value $H(r_0) = 0$ for H and a terminal value $R(r_A) = R_A$ in r_A such that $H(r_A) = N = F(\rho_f)$.*

Let us now reverse the perspective, considering (H, R) obtained by integrating the differential system (3.12). We interpret $R(r)$ at the unit prices for dwelling space at location r . Then:

Proposition 4 (Sufficient conditions for urban equilibrium) *Let there be a pair of functions (H, R) which verifies (3.12).*

(i) Under price conditions $R(r)$, if the dwelling space is a normal good, then a household employed in ρ and seeking accommodation has an optimal location in $r_\omega(\rho)$ where $r_\omega = \rho_\omega^{-1}$.

In other words, (3.12a) leads to (3.8).

(ii) Assuming furthermore (3.5), then condition $R(r)$ coincides with the economic behaviour that optimises the supply of living space in r : each supplier of living space selects the maximum bid by households for its product. In other words, conditions (3.12) and (3.10) imply condition (3.4).

(iii) Under these microeconomic conditions, in any position r such that $R(r) > R_A$ local land capacity $L(r)$ is equal to local demand by all households.

(iv) In all, conditions (3.12) and (3.10) describe an urban equilibrium.

The proof is provided in paragraph 8.2 of the appendix.

3.4. Properties of the differential system

Proposition 5 (Formal properties of the differential system) Where $R_0 = R(r_0)$ is fixed, the differential system determines:

(i) a function H which increases with r . Therefore the function $r \mapsto \rho_\omega(r)$ is increasing.

(ii) a function R which decreases with r .

(iii) a utility function \tilde{u}_ρ which increases with ρ : equivalently, the function $r \mapsto V(R(r), I_{\rho_\omega(r), r})$ where $I_{\rho r} = Y - T(\rho, r)$, is increasing.

(iv) a lot size function (dwelling size) $r \mapsto S(r) = \hat{s}(R(r), I_{\rho_\omega(r), r})$ which increases with r if the dwelling space is a normal good.

For proof, see paragraph 8.3 of the appendix, whereas the next proposition is demonstrated in paragraph 8.4.

Proposition 6 (Algorithmic property of the differential system) For two initial values R_{01} and $R_{02} > R_{01}$, the respective solutions (H_1, R_1) and (H_2, R_2) of the differential system satisfy, in each position r : $R_1 \leq R_2$; $H_1 \leq H_2$; $S_1 \geq S_2$; $\rho_{\omega 1} \leq \rho_{\omega 2}$; $I_{\rho_{\omega 1}(r), r} \leq I_{\rho_{\omega 2}(r), r}$.

Proposition 7 (Ending integration) System integration is ended when R attains R_A or H attains N . If $\hat{s} \geq \varepsilon > 0$ such that $\int_{r_0}^{\hat{r}} L(r) dr \geq \varepsilon N$ for acceptable \hat{r} , the integration necessarily produces such a state.

Proof. Starting from $R_0 \geq R_A$, since R is decreasing, we either obtain R_A or R remains $> R_A$. If R_A is obtained in r_A , as the value of H increases from 0 in r_0 , the value it reaches in r_A is less than N , otherwise one would have stopped earlier. If R_A is not reached, the additional condition ensures that H attains N , and as soon as this value is reached, the integration can be ended.

3.5. Existence and uniqueness of the equilibrium

Proposition 8 (Existence and uniqueness of urban equilibrium) (i) *There exists a value \hat{R}_0 such that the integration of the differential system (3.12) produces an urban equilibrium.* (ii) *This value is unique, as is the urban equilibrium.*

The proof is provided in paragraph 8.5 of the appendix, with a dichotomous algorithm. Equivalently with this dichotomous algorithm, we can define a function which at any initial integration value R_0 of the differential system, associates the position $\hat{r}(R_0)$ in which $H(\hat{r}(R_0)) = N$. The condition in Property 7 ensures the existence of such a position. Proposition 6 ensures that R_0 a $\hat{r}(R_0)$ is a decreasing function, because for $R_{01} < R_{02}$, it is verified in any r that $H_1(r) \leq H_2(r)$, therefore H attains the value N for R_{02} in a position $\hat{r}(R_{02})$ that is lower than the position for R_{01} .

We can also associate the value $\gamma(R_0) = R(\hat{r}(R_0)|R_0)$ of the integrated rent function with R_0 , starting from the initial value of R_0 . The function γ is increasing in R_0 because (r, x) a $R(r|x)$ is a decreasing function in r and increasing in x , and as $\hat{r}(R_0)$ decreases with R_0 , through the composition of two decreasing functions, R_0 a $R(\hat{r}(R_0)|R_0)$ is an increasing function.

By the same arguments as used in the proof of proposition 8, the function \hat{r} takes low values where R_0 is high and high values where R_0 is low, therefore γ takes high values where R_0 is high and low where R_0 is low. Urban equilibrium is characterised by the condition that $\gamma(\hat{R}_0) = R_A$. Therefore, our function γ plays a similar role in our model as the rent boundary function used by Fujita (1989) to tackle the classical model with a point-wise CBD and a homogeneous population of households. We call it the *terminal rent function*.

4. Structural properties of urban equilibrium

We have already established, for any initial rent value R_0 , that maximum utility and dwelling space are functions that increase with residential position. In this part, we examine the variation in the net income function and analyse the sensitivity of the urban equilibrium to alternative rent (agricultural).

4.1. Variations in net income

Net income $\hat{I}(r) = I_{\rho_\omega(r), r}$ where $I_{\rho, r} = Y - T(\rho, r)$ varies with location r firstly directly and decreasingly, and secondly indirectly via $\rho_\omega(r)$ and increasingly. The two opposing influences become superimposed with a variable result, which we can illustrate in characteristic situations.

The total influence is determined by the total derivative:

$$\begin{aligned} \frac{d}{dr} \hat{I}(r) &= \frac{\partial}{\partial r} \hat{I} + \dot{\rho}_\omega \frac{\partial}{\partial \rho} \hat{I} \\ &= \left(-\frac{\partial T}{\partial \rho}\right) [\dot{\rho}_\omega + \frac{\partial T / \partial r}{\partial T / \partial \rho}] \end{aligned}$$

By assumption, $\partial T/\partial \rho \leq 0$ and $\partial T/\partial r \geq 0$, therefore $d\hat{I}/dr$ is positively proportional to the sum of a positive term, $\dot{\rho}_\omega$ and of a negative term, $\frac{\partial T}{\partial r}/\frac{\partial T}{\partial \rho}$.

For the first term, let us define the *concentration* of housing relative to jobs as the condition that $\dot{r}_\omega \leq 1$, which is equivalent to $\dot{\rho}_\omega \geq 1$. Symmetrically, the *deconcentration* of housing relative to jobs is the condition that $\dot{r}_\omega \geq 1$, equivalent to $\dot{\rho}_\omega \leq 1$.

As regards the second term, we know the *congested* situation of many urban road networks, where traffic time and costs, per journey and per unit of distance, are higher in the centre than on the outskirts: then $|\frac{\partial T}{\partial \rho}| \geq \frac{\partial T}{\partial r}$ therefore $\frac{\partial T}{\partial r}/\frac{\partial T}{\partial \rho} \geq -1$.

Conversely, in the absolute it should be possible to invest in the transport system by prioritising very massive and efficient transport modes near the city centre, where volumes are highest, in order to ensure $|\frac{\partial T}{\partial \rho}| \leq \frac{\partial T}{\partial r}$ therefore $\frac{\partial T}{\partial r}/\frac{\partial T}{\partial \rho} \leq -1$, in the situation of *sufficient massification*.

Our first characteristic situation is a city where housing is deconcentrated relative to jobs and transport massification is sufficient: in this case, net income falls the further residential location is from the centre. Indeed, $\frac{d}{dr}\hat{I}(r) \leq 0$ since $\dot{\rho}_\omega \leq 1 \leq -\frac{\partial T}{\partial r}/\frac{\partial T}{\partial \rho}$.

The second characteristic situation is a city where housing is concentrated relative to jobs and the transport network is congested: here, net income increases with distance from the centre, since $\dot{\rho}_\omega \geq 1 \geq -\frac{\partial T}{\partial r}/\frac{\partial T}{\partial \rho}$ therefore $\frac{d}{dr}\hat{I}(r) \geq 0$.

4.2. Influence of alternative rent on urban equilibrium

Alternative (agricultural) rent R_A is the main boundary condition restricting the agglomeration in a closed city model. As the terminal rent is an increasing function, we know that for two given values $R_A^1 < R_A^2$ the respective urban equilibriums correspond to $\hat{R}_0^1 < \hat{R}_0^2$, therefore to terminal radii $\hat{r}(\hat{R}_0^1) > \hat{r}(\hat{R}_0^2)$: a higher alternative rent results in a smaller agglomeration.

Since the integration conditions of both differential systems are identical apart from the initial values $\hat{R}_0^1 < \hat{R}_0^2$, proposition 6 applies and notably leads to the result that $R_1 \leq R_2$, $S_1 \geq S_2$ and $\rho_{\omega 1} \leq \rho_{\omega 2}$: the second city is denser than the first in every location, households are more concentrated relative to jobs, and land rent is higher in every location.

5. A particular set of specifications

We did not demonstrate any other generic properties for our model in its general assumptions. However, we have been able to establish analytical properties by restricting ourselves to a particular case, by means of simplifying assumptions.

5.1. Specific assumptions

Regarding the geographical aspects of the model, we suppose that the N jobs are distributed uniformly over an interval $[0, \rho_f]$, with a density b :

$$\begin{aligned} f(\rho) &= b \cdot 1_{\{0 \leq \rho \leq \rho_f\}}, \\ F(\rho) &= b \min\{\rho, \rho_f\} \text{ for } \rho \geq 0, \text{ and} \\ F^{-1}(n) &= \min\{\frac{n}{b}, \rho_f\} \text{ for } n \geq 0 \end{aligned}$$

We also suppose that land capacity is uniform from a radius of r_0 :

$$L(r) = \lambda \text{ for } r \geq r_0 \text{ or otherwise } 0$$

In addition, we postulate that transport cost is an affine function of r and ρ :

$$T(\rho, r) = a_0 + ar - a'\rho$$

If $r_0 \leq \rho_f$, one would expect a' and a to be very close, but this specification allows us to distinguish between a central zone primarily devoted to employment, and a ring zone primarily devoted to housing. In principle, we restrict ourselves to positive transport costs, although we put no absolute value in the T function. When $\rho_f = r_0$ we expect that $T(\rho, r) \geq a(r - r_0) + a'(r_0 - \rho)$ and therefore that $a_0 \geq (a' - a)r_0$.

Concerning the economic aspects, we assume that the N households each have one working member, a gross income Y and the same utility function, a Cobb-Douglas function with two parameters $\alpha, \beta > 0$:

$$U(z, s) = v_0 z^\alpha s^\beta$$

Let $\alpha' = \alpha/(\alpha + \beta)$ and $\beta' = \beta/(\alpha + \beta)$ be the reduced parameters. It is easy to show that the indirect utility function, the demand for dwelling space and the Solow type bid-rent function are, respectively:

$$\begin{aligned} V(R, I) &= v_0 R^{-\beta} I^{\alpha+\beta} \text{ for } v_0 = v_0 \alpha'^\alpha \beta'^\beta \\ \hat{s}(R, I) &= \beta' I / R \\ \psi(I, u) &= \left(\frac{u}{v_0}\right)^{-1/\beta} I^{1/\beta'} \end{aligned}$$

5.2. Analytical solution of the differential system

Let us note:

- $R_0 = R_A + aN / \lambda$ rent in r_0 ,
- $A = \frac{Y_0}{a} + \frac{a'}{ba^2} \lambda R_0$ for $Y_0 = Y - a_0$
- $B = \frac{\beta}{\alpha} \frac{a'}{ba^2} \lambda R_0$
- $W = A + B - r_0 = \frac{Y_0}{a} + \frac{a'}{\alpha ba^2} \lambda R_0 - r_0$
-

In the appendix, we resolve the differential system and obtain the inverse function of the rent function:

$$r = A + B \frac{R(r)}{R_0} - W \left(\frac{R(r)}{R_0} \right)^{\beta'} \quad (5.1)$$

In addition:

$$H(r) = \frac{\lambda}{a} (R_0 - R) \quad (5.2)$$

Since $H(r) = F(\rho)$ and $F(\rho) = b\rho$, we link residential rent with place of employment:

$$\frac{R(r)}{R_0} = 1 - \frac{\rho}{\tilde{\rho}} \text{ for } \tilde{\rho} = \frac{\lambda R_0}{ab} \quad (5.3)$$

By combining (5.1) and (5.3), we obtain residential location with respect to the place of employment:

$$r = A + B \left(1 - \frac{\rho}{\tilde{\rho}} \right) - W \left(1 - \frac{\rho}{\tilde{\rho}} \right)^{\beta'} \quad (5.4)$$

All the variables of interest in our urban system can be deduced from the place of employment: we will successively look at residential location and the urban fringe distance, lot size and residential density, land rent, and household expenses and utilities.

5.3. Residential location and the urban fringe distance

In the appendix, we demonstrate the following property.

Proposition 9. *For a given workplace ρ , the residential position $r_\omega(\rho)$:*

- (i) *Increases with minimum residential radius r_0 .*
- (ii) *Increases with gross income Y and falls with a_0 .*
- (iii) *Increases with a' and decreases with a if a' is fixed. If $a' = a$ the total effect of a remains decreasing.*
- (iv) *Increases with job density b ; but decreases with job radius ρ_f and with land density λ .*
- (v) *Decreases with alternative rent R_A .*
- (vi) *Increases with β and decreases with α .*

This proposition applies for any residential location, therefore, because of the monotony of the relations, for any interval of average residential location $[\rho_1, \rho_2]$.

The proposition is immediately transposed for the distance between home and work, $D(\rho) = r_\omega(\rho) - \rho$. The influence of the parameters other than a_0 , a and a' is also transposed onto the cost of transport, $\tilde{T}(\rho) = T(\rho, r_\omega(\rho)) = a_0 + ar_\omega(\rho) - a'\rho$.

The average residential location is:

$$\begin{aligned} \frac{1}{N} \int_0^{\rho_f} r_\omega(\rho) dF(\rho) &= \frac{1}{\rho_f} \int_0^{\rho_f} r_\omega(\rho) d\rho \\ &= A + B \left(1 - \frac{\rho_f}{2\tilde{\rho}}\right) - \frac{w}{\rho_f} \left[-\left(1 - \frac{\rho}{\tilde{\rho}}\right)^{1+\beta'} \frac{\tilde{\rho}}{1+\beta'} \right]_0^{\rho_f} \\ \bar{r} &= \frac{Y_0}{a} + \frac{a'}{a} \left(\frac{\tilde{\rho}}{\alpha'} - \frac{\beta \rho_f}{\alpha 2} \right) - \left(\frac{Y_0}{a} + \frac{a' \tilde{\rho}}{a \alpha'} - r_0 \right) \frac{\tilde{\rho}}{\rho_f} \frac{1 - \left(1 - \frac{\rho_f}{\tilde{\rho}}\right)^{1+\beta'}}{1+\beta'} \end{aligned}$$

The average commuting distance can be deduced from this by subtracting $\frac{1}{2}\rho_f$. The average transport cost is, with $t = 1 - \frac{\rho_f}{\tilde{\rho}} = R_A/R_0$:

$$\bar{T} = Y + \frac{a'}{\alpha'} \left(\tilde{\rho} - \frac{\rho_f}{2} \right) - \left(Y_0 + \frac{a' \tilde{\rho}}{\alpha'} - ar_0 \right) \frac{\tilde{\rho}}{\rho_f} \frac{1 - t^{1+\beta'}}{1+\beta'}$$

In our model, the centre of the agglomeration is primarily a reference used to identify the spatial position of each location. Since the central point does not represent the workplace, for a household established in r , the commuting distance is not the same as r , by contrast with the basic monocentric model.

Figure 1 shows residential location in relation to the workplace for several values of a and a' . The application parameters are inspired by the Paris region: 1.7 million jobs in a central area with a radius of 5 km; an urbanisation boundary radius of 20 km, land capacity of around 6 km² per radial kilometre (restricted to households employed in the centre), alternative rent of €8/m² per month, gross income of 2,500 euros per month, utility parameters of $\alpha' = 0.72$ and $\beta' = 0.28$. We estimated a unit transport cost of some €0.25/km for each outward or return journey.

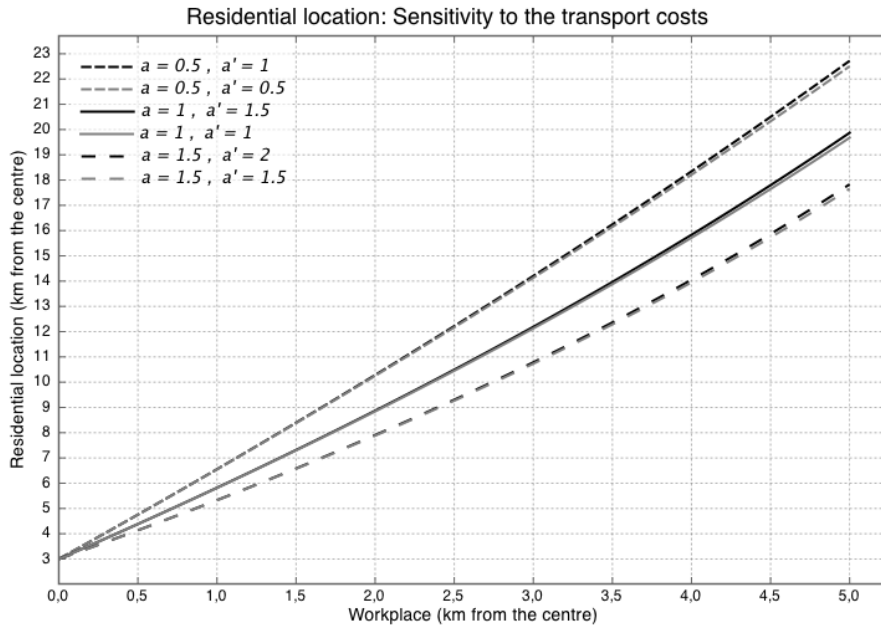


Figure 1 - Effect of the transport costs on the residential location with respect to the workplace

Figure 2 shows the influence of the land capacity on the urban fringe distance and the average lot size of households. It emphasizes the choice of a land capacity of 6 km² per radial kilometre, as it gives appropriate urban fringe distance (20 km) and average lot size (70 m²).

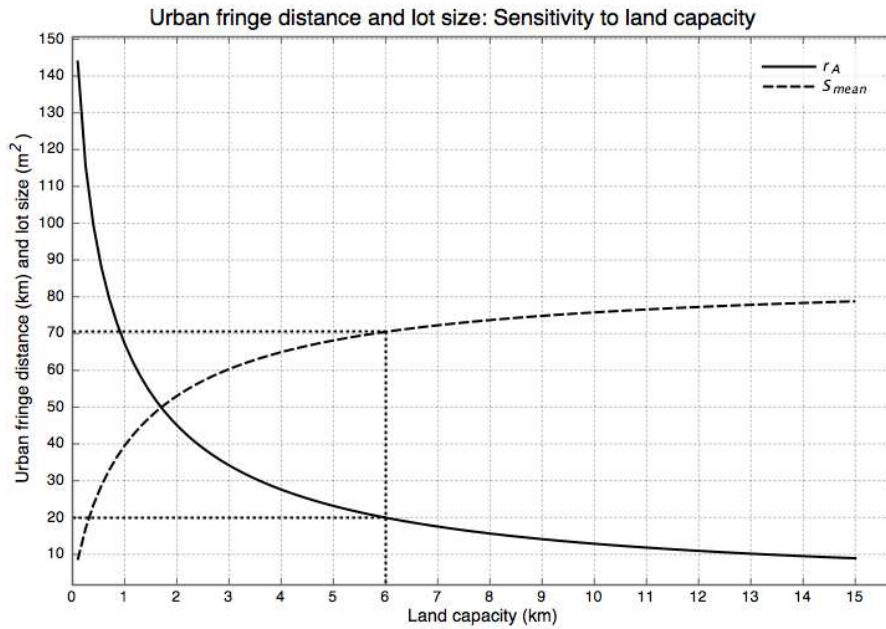


Figure 2 - Effect of the land capacity on the urban fringe distance and lot size

With a fixed population N , the increasing influence of job density b on residential location is reinforced by the decreasing influence of the employment fringe distance, which is itself decreasing, since the product $b \cdot \rho_f = N$ is fixed. Figure 3 shows the influence of the employment fringe distance on the urban fringe distance and the average home-to-work distance.

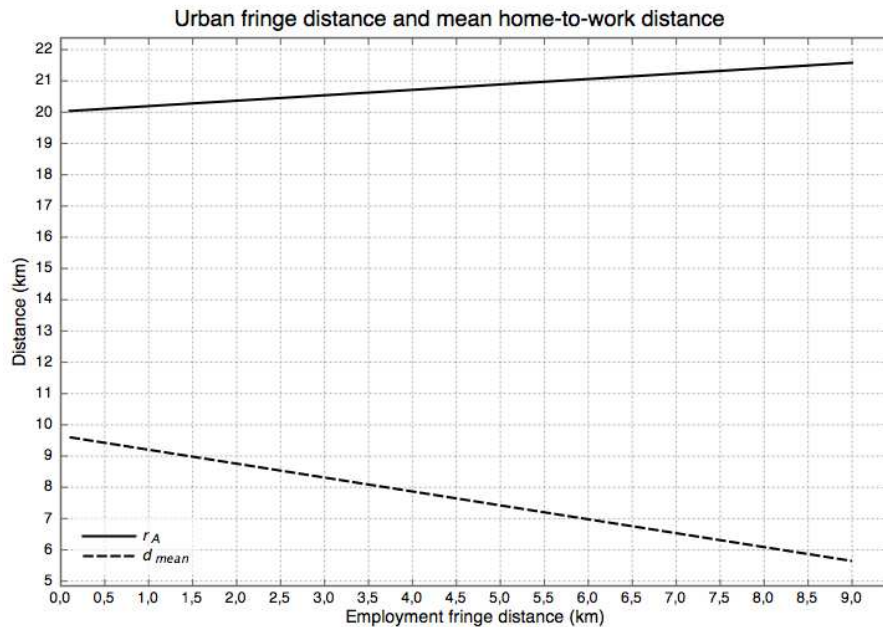


Figure 3 - Effect of the employment fringe distance on the average home-to-work distance and the urban fringe distance

5.4. Residential density and lot size

From the correspondence between places of employment and of residence in $r = r_\omega(\rho)$ we know the *residential density*:

$$\dot{H}(r) = \frac{f(\rho)}{\dot{r}_\omega} \quad \text{with} \quad \dot{r}_\omega = -\frac{B}{\tilde{\rho}} + \frac{\beta'}{\tilde{\rho}} W(1 - \frac{\rho}{\tilde{\rho}})^{-\alpha}$$

$$\dot{H}(r) = \frac{b\tilde{\rho}}{\beta'W(1 - \rho/\tilde{\rho})^{-\alpha} - B}$$

Density has a hyperbolic shape based on ρ . Function \dot{r}_ω is positive and increasing, therefore $\dot{H} \circ r_\omega$ decreases with ρ and likewise \dot{H} decreases with r : residential density decreases with distance from the city centre.

Lot size (dwelling space) is simply deduced from residential density, since land capacity per unit of radial distance is constant:

$$s(r) = \frac{L(r)}{\dot{H}(r)} = \frac{\lambda}{\dot{H}(r)} = \frac{\lambda}{b} \dot{r}_\omega \quad \text{at} \quad r = r_\omega(\rho).$$

Individual living space increases as the workplace becomes further from the centre, hence also as residential location becomes further from the centre.

The following proposition is demonstrated in the appendix.

Proposition 10 (Sensitivity of lot size and residential density). *For a given workplace ρ , function $\dot{r}_\omega(\rho)$ varies depending on a parameter as does residential location $r_\omega(\rho)$. Therefore lot size, which is positively proportional to $\dot{r}_\omega(\rho)$, is affected by a parameter in the same direction as residential location.*

Local residential density, which is the reciprocal of lot size, varies in the opposite direction to residential position, therefore:

- (i) *It diminishes with minimum residential radius r_0 .*
- (ii) *It diminishes with gross income Y and increases with a_0 .*
- (iii) *It diminishes with a' and increases with a if a' is fixed. If $a' = a$ the total effect of a remains increasing.*
- (iv) *It increases with land density λ .*
- (v) *It increases with alternative rent R_A .*
- (vi) *It decreases with β and increases with α .*

Since these properties are valid for any workplace, they also apply for the average lot size on any interval $[\rho_1, \rho_2]$, in particular for the average lot size in the city. Since total residential density is inversely proportional to this average size, it varies with respect to a parameter as local residential density does, except for the influence of λ , for $s = \lambda/h$.

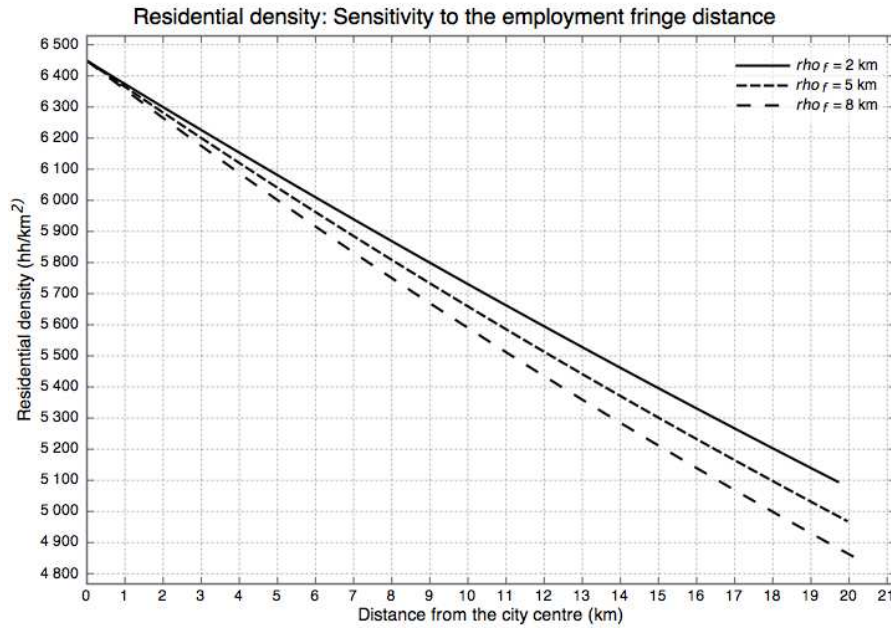


Figure 4 - Effect of employment fringe distance on residential density

5.5. Land rent

Condition (5.3) which links the land rent to the place of employment becomes:

$$R(r) = \tilde{R}(\rho) = R_0 \left(1 - \frac{\rho}{\tilde{\rho}}\right) = R_A + \frac{a}{\lambda} (N - b\rho) \quad (5.5)$$

Rent increases linearly with the place of employment. Therefore it also decreases with residential position, although in a less regular way.

Where $\alpha = \beta$, therefore $\alpha' = \beta' = \frac{1}{2}$, condition (5.1) is reduced to $r = A + B \frac{R}{R_0} - W \sqrt{\frac{R}{R_0}}$, therefore $\sqrt{R/R_0}$ solves the second-degree equation $x^2 - \frac{W}{B}x + \frac{A-r}{B} = 0$, in which we take the decreasing solution with r : $x^* = \frac{W}{2B} \left(1 - \sqrt{1 - \frac{4B}{W^2}(A-r)}\right)$. Finally:

$$R(r) = R_0 \left(\frac{W}{2B}\right)^2 \left(1 - \sqrt{1 - \frac{4B}{W^2}(A-r)}\right)^2.$$

The other variables in the model are deduced from this, beginning with $\rho_\omega(r)$ then density \dot{H} , lot size $s(r)$ etc.

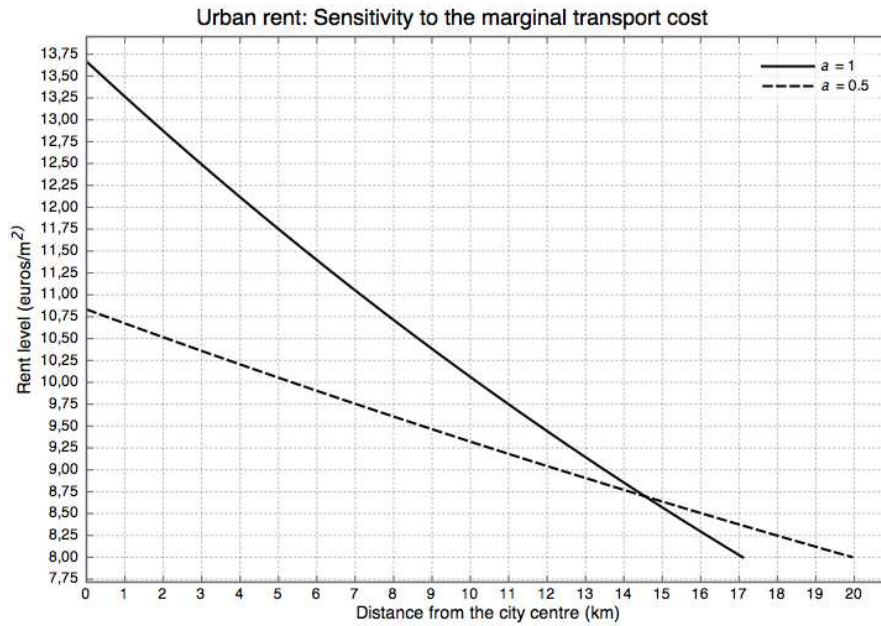


Figure 5 - Influence of transport cost on land rent

The Figure 6 illustrates a result that the canonical model does not allow, the variation in land rent in response to variation in job density (with N constant):

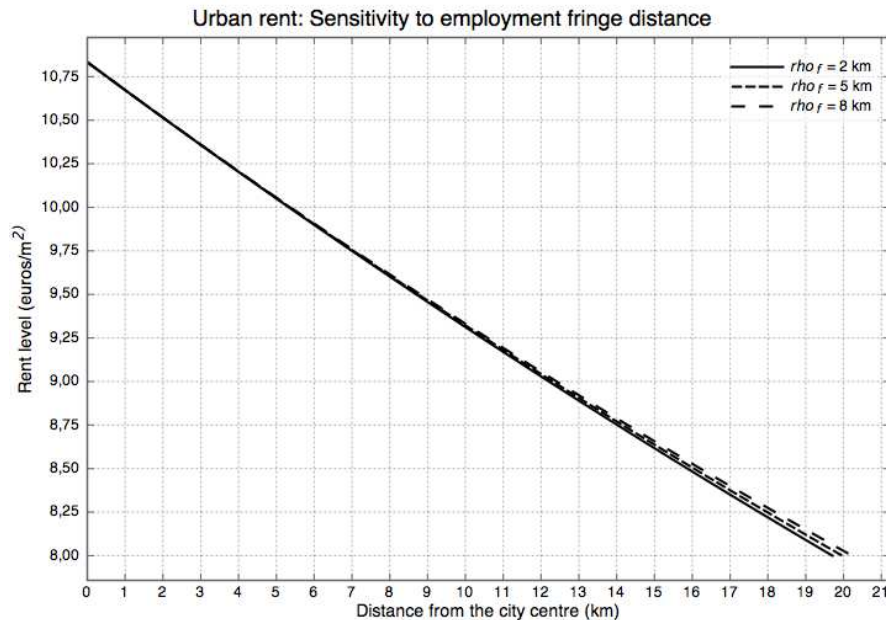


Figure 6 - Influence of employment fringe distance on land rent

5.6. Net household income and expenditure

We have already specified the cost of transport for a household employed in ρ . This household has a net income:

$$\begin{aligned}\tilde{I}(\rho) &= Y - \tilde{T}(\rho) = Y - [a_0 - a'\rho + Y_0 + a'\tilde{\rho} + \frac{\beta}{\alpha}a'\tilde{\rho}(1 - \frac{\rho}{\tilde{\rho}}) - aW(1 - \frac{\rho}{\tilde{\rho}})^\beta] \\ &= (Y_0 + \frac{a'}{\alpha}\tilde{\rho} - ar_0)(1 - \frac{\rho}{\tilde{\rho}})^\beta - \frac{a'}{\alpha}\tilde{\rho}(1 - \frac{\rho}{\tilde{\rho}})\end{aligned}$$

In addition, $\frac{d\tilde{I}(\rho)}{d\rho} = a' - ar'_\omega(\rho) = \frac{a'}{\alpha} - \frac{\beta}{\tilde{\rho}}aWx^{-\alpha}$.

Since the household utility function is a Cobb-Douglas function, housing expenditure is $\tilde{s}(\rho) \cdot \tilde{R}(\rho) = \beta\tilde{I}(\rho)$, and other goods consumption is $\tilde{z}(\rho) = \alpha\tilde{I}(\rho)$.

The formula $d\tilde{I}(\rho)/d\rho$ enables us to find the two typical cases where net income decreases (respectively increases) with residential location.

In the first extreme case, $d\tilde{I}(\rho)/d\rho \leq 0$ is equivalent to $a'\tilde{\rho}x^\alpha \leq \alpha\beta'aW$ at any point $x = 1 - \rho/\tilde{\rho}$. The constraint is greatest in $x=1$ i.e. $\rho=0$. In this case it becomes $a'\tilde{\rho} \leq \beta'(Y_0 - ar_0)$, or else:

$$a'\rho_f [1 + \frac{\lambda R_A}{aN}] \leq \beta'(Y - a_0 - ar_0)$$

It is sufficient to consider a gross income large enough to meet this condition, or a low enough unit cost a' , i.e. sufficient massification of transport, or a sufficiently small job radius resulting in a deconcentration of housing relative to jobs. This kind of city is *quasi-monocentric*.

At the other extreme, the condition is that $d\tilde{I}(\rho)/d\rho \geq 0$ in any ρ , therefore $a'\tilde{\rho}x^\alpha \geq \alpha\beta'aW$ for any $x \in [t, 1]$ where $t = \frac{R_A}{R_0} = 1 - \frac{\rho_f}{\tilde{\rho}}$. The condition imposes the strongest constraint when $x = t$, at which point it implies that $a'\tilde{\rho}t^\alpha \geq \alpha\beta'aW$ i.e. $a'\tilde{\rho}t^\alpha \geq \beta'[\alpha'(Y_0 - ar_0) + a'\tilde{\rho}]$, which is equivalent to:

$$\frac{a'}{a} \frac{\lambda R_0}{b} [t^{\alpha'} - \beta'] \geq \alpha'\beta'(Y_0 - ar_0)$$

This condition requires that $t^{\alpha'} \geq \beta'$. Assuming this last condition to be true (which means that $R_A(\beta'^{-1/\alpha'} - 1) \geq aN/\lambda$), it is sufficient to consider a'/a large enough or b small enough or λ large enough for the condition to be met. A high λ/b corresponds to a concentration of housing relative to jobs, whereas a relatively high a'/a corresponds to city centre congestion. This kind of city is *eccentric*, which means that a household's location becomes increasingly favourable the further it is from the centre.

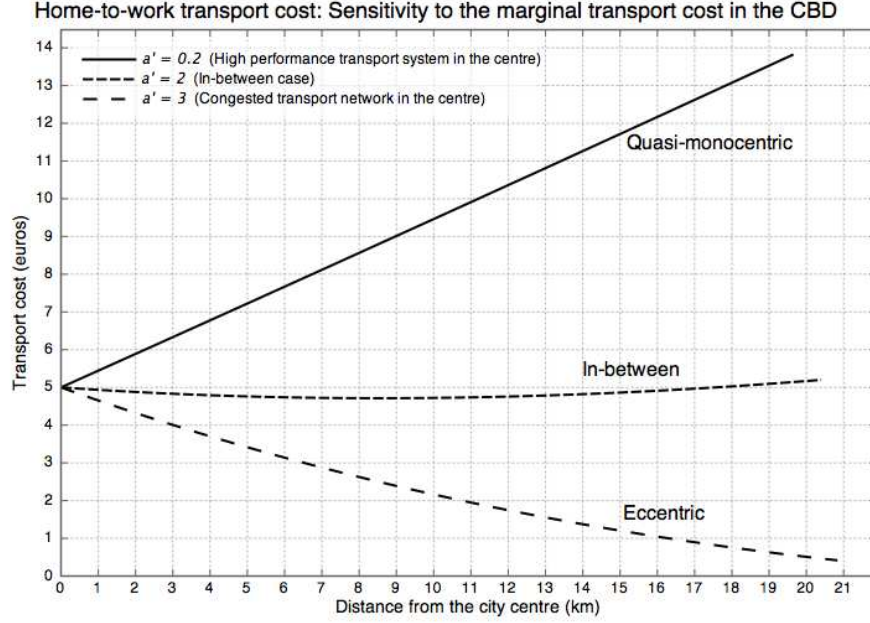


Figure 7 - The stylized cases: Transport cost changes with distance from the city centre

5.7. On utility distribution

A household employed in ρ obtains through its choice of residential location a utility of $\tilde{u}_\rho = V(\tilde{R}(\rho), \tilde{I}(\rho))$, therefore in the simplified model:

$$\begin{aligned} \tilde{u}_\rho &= v_0 \tilde{R}(\rho)^{-\beta} \tilde{I}(\rho)^{\alpha+\beta} \\ &= v_0 \frac{[aW(1-\frac{\rho}{\bar{\rho}})^{\beta'} - \frac{a'}{\alpha'} \tilde{\rho}(1-\frac{\rho}{\bar{\rho}})]^{\alpha+\beta}}{[R_A + \frac{a}{\lambda} N - \frac{ab}{\lambda} \rho]^\beta} \end{aligned}$$

Because of the properties of the general model, we know that this utility function increases with ρ , therefore as the workplace moves away from the city centre. Put another way, in addition to gross income, employment location constitutes a factor of utility, a factor of endowment that leads to a location reference on the residential market.

Net income is not the appropriate indicator by which to evaluate the utility of the place of employment in monetary terms, since it is only an indirect factor. It is better to evaluate the equivalent gross income that would need to be allocated to a reference household for it to achieve utility \tilde{u}_ρ . We can decide the reference household arbitrarily, for example the initial household for which $\rho=0$, or the “median” household employed in $\rho_f/2$. The equivalent income is $\tilde{Y}(\rho) = \tilde{T}(\rho_{\text{ref}}) + E(R_{\text{ref}}, \tilde{u}_\rho)$ where $E(R, u)$ denotes the expenditure function which indicates the net income that the household needs in order to obtain utility level u where the price of living space is R . For a Cobb-Douglas type utility function,

$$E(R, u) = R^{\beta'} \left(\frac{u}{v_0} \right)^{1/(\alpha+\beta)}.$$

Relative to the initial household employed in 0 and living in r_0 , the equivalent income of the household employed in ρ and living in $r = r_\omega(\rho)$ is (see appendix for calculation):

$$\begin{aligned}\tilde{Y}(\rho) &= \tilde{T}(\rho_0) + E(R_0, \tilde{u}_\rho) \\ &= Y + \frac{d'}{\alpha} \tilde{\rho} \left[1 - \left(\frac{R}{R_0} \right)^\alpha \right]\end{aligned}$$

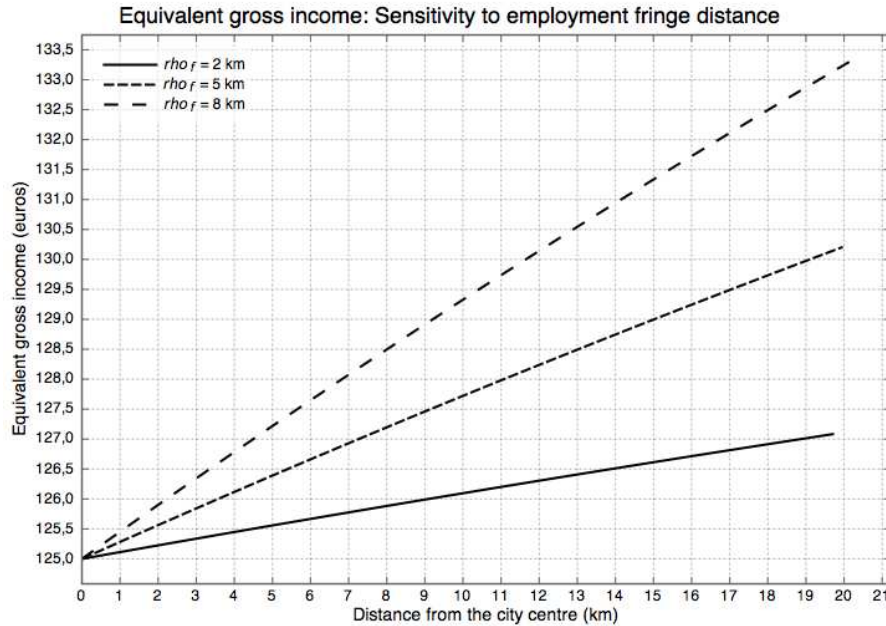


Figure 8 - Influence of job density on equivalent gross income

6. Conclusions

We examined the equilibrium of residential locations for households in a closed city with absentee landlords, where jobs are spatially distributed. We assumed that jobs are established exogenously and bring identical gross income to households, which are homogeneous in their preferences but all have a fixed individual place of employment. Using these assumptions, we studied the influence of place of employment on residential location, dwelling space, household utility, as well as land rent, residential density and commuting distance. We showed that utility and dwelling space increase with distance from the centre, but that net income does not always behave monotonically. By simplifying the specifications, we described two extreme cases: on the one hand a quasi-monocentric city where housing is deconcentrated relative to jobs and transport becomes increasingly efficient as it nears the centre, with the result that net income decreases with distance from the centre. And on the other hand, an eccentric city where housing is constricted relative to jobs or transport is congested in the centre: in this kind of town, net income increases with the distance of residential location from the centre.

We applied our model to the Paris region, in a very stylised way. In the future, we plan to explore the effects of urban policies (for location or transport), in order to characterise its effects, possibly differentiating between quasi-monocentric or eccentric cases. Potential future areas of research would be to model the link between transport congestion and the volume of local commuter journeys, and also employers' wage policies.

7. References

- ALONSO, W. (1964). *Location and Land Use; Toward a General Theory of Land Rent*. Cambridge, Massachusetts, USA: Harvard University Press.
- ANAS, A., ARNOTT, R., & SMALL, K. A. (1998). Urban Spatial Structure. *Journal of Economic Literature*, 36 (3), 1426-1464.
- ANAS, A., & KIM, I. (1996). General Equilibrium Models of Polycentric Urban Land Use with Endogenous Congestion and Job Agglomeration. *Journal of Urban Economics*, 40 (2), 232-256.
- CRANE, R. (1996). The Influence of Uncertain Job Location on Urban Form and Journey to Work. *Journal of Urban Economics*, 39, 342-356.
- EBERTS, R. W. (1981). An Empirical Investigation of Intraurban Wage Gradients. *Journal of Urban Economics*, 10 (1), 50-60.
- FUJITA, M. (1985). Towards General Equilibrium Models of Urban Land Use. *Revue Économique*, 36 (1), 135-168.
- FUJITA, M. (1989). *Urban Economic Theory: Land Use and City Size*. New York, New York, USA: Cambridge University Press.
- GLAESER, E. L., & KAHN, M. E. (2001). Decentralized Employment and the Transformation of the American City. *Brookings-Wharton Papers on Urban Affairs*, 1-63.
- GRIMAUD, A. (1989). Agglomeration Economies and Building Height. *Journal of Urban Economics*, 25 (1), 17-31.
- HAMILTON, B. W., & ROËLL, A. (1982). Wasteful Commuting. *Journal of Political Economy*, 90 (5), 1035-1053.
- KWAN, P. (1999). Gender, the Home-Work Link and Space-Time Patterns of Non-Employment Activities. *Economic Geography*, 75 (4), 370-394.
- KORSU, E., & MASSOT, M.-H. (2006). Rapprocher les ménages de leurs lieux de travail: les enjeux pour la régulation de l'usage de la voiture en Île-de-France. *Les Cahiers Scientifiques du Transport*, 50, 61-90.
- LUCAS, R. E., & ROSSI-HANSBERG, E. (2002). On the Internal Structure of Cities. *Econometrica*, 70 (4), 1445-1476.
- MCMILLEN, D. P., & McDONALD, J. F. (1998). Suburban Subcenters and Employment Density in Metropolitan Chicago. *Journal of Urban Economics*, 43 (2), 157-180.
- MCMILLEN, D. P., & SINGELL, L. (1992). Work Location, Residence Location, and the Intraurban Wage Gradient. *Journal of Urban Economics*, 32, 195-213.
- MILLS, E. S. (1967). An Aggregative Model of Resource Allocation in a Metropolitan Area. *The American Economic Review*, 57 (2), 197-210.

- MUTH, R. F. (1969). *Cities and Housing*. Chicago: University of Chicago Press.
- OGAWA, H., & FUJITA, M. (1980). Equilibrium Land use Patterns in a Non-Monocentric City. *Journal of Regional Science* , 20 (4), 445-475.
- RICHARDSON, H. W. (1988) Monocentric vs. Policentric Models: The Future of Urban Economics in Regional Science. *The Annals of Regional Science*, 22 (2), 1-12.
- SHEARMUR, R. (2006). Travel from Home: An Economic Geography of Commuting Distances in Montreal. *Urban Geography* , 27 (4), 330-359.
- SULLIVAN, A. M. (1983). A General Equilibrium Model with External Scale Economies in Production. *Journal of Urban Economics*, 13 (2), 235-255.
- SULLIVAN, A. M. (1983a). The General Equilibrium Effects of Congestion Externalities. *Journal of Urban Economics*, 14 (1), 80-104.
- TIMOTHY, D., & WHEATON, W. C. (2001). Intra-Urban Wage Variations, Employment Location, and Commuting Times. *Journal of Urban Economics*, 50 (2), 338-366.
- WHEATON, W. C. (2004). Commuting, Congestion and Employment Dispersal in Cities with Mixed Land-Use. *Journal of Urban Economics* , 58, 417-438.

8. Appendix 1: Properties of the general model

8.1. Proof of Proposition 1

To be able to apply Fujita's rule 2.3 (1989), it is sufficient to show that the bid-rent curves of a household working in ρ_1 are steeper than those of a household working in ρ_2 , if $\rho_1 < \rho_2$. Let $\Psi_1(x, u_1)$ and $\Psi_2(x, u_2)$ be two bid-rent curves for households working in ρ_1 and ρ_2 respectively, with $\rho_1 < \rho_2$, which intersect in x , i.e.:

$$\exists \bar{R}, \Psi_1(x, u_1) = \Psi_2(x, u_2) \text{ (not necessarily an } R(r))$$

Since $T(\rho, r)$ is decreasing with ρ , we have:

$$T(\rho_1, x) < T(\rho_2, x) \text{ therefore } I_1 = Y - T(\rho_1, x) > Y - T(\rho_2, x) = I_2$$

Let us define the uncompensated demand for space as $\hat{s}(R, I_i)$, corresponding to the solution of (2.1) for household i . Assuming that dwelling space is a normal good, the effect of income on uncompensated demand is positive, therefore:

$$S_1(x, u_1) = \hat{s}(\bar{R}, I_1) < \hat{s}(\bar{R}, I_2) = S_2(x, u_2)$$

Finally, applying the envelope theorem to the bid-rent function (2.2) gives us the equality:

$$\frac{\partial \Psi_i(r, u_i)}{\partial r} = - \frac{\partial T / \partial r}{S_i(r, u_i)}$$

By combining the latter two relations, we get, assuming that $\partial T / \partial r$ is constant or decreasing with ρ for a given r :

$$- \frac{\partial \Psi_1(r, u_1)}{\partial r} = \frac{\partial T / \partial r}{S_1(r, u_1)} > \frac{\partial T / \partial r}{S_2(r, u_2)} = - \frac{\partial \Psi_2(r, u_2)}{\partial r}$$

This demonstrates that the bid-rent curve of the household working in ρ_1 is steeper than that of the household working in $\rho_2 > \rho_1$. Therefore the point at which the land rent curve $R(r)$, unique to the urban equilibrium, meets a bid curve $\Psi_i(x, u_i)$ is closer to the city centre for household 1 than for household 2, and if we consider optimal bids, the equilibrium location of household 1 is closer to the city centre than that of household 2.

8.2. Proof of proposition 4

(i) In (3.10), since $\partial T / \partial r \geq 0$ and $\hat{s} \geq 0$, necessarily $\dot{R} \leq 0$. Equation (3.10) is equivalent to (3.9), $\frac{\partial W(r, \rho)}{\partial r} = 0$ at point $\rho = \rho_\omega(r)$. With fixed ρ , we need to show that the function $r \mapsto W(r, \rho)$ is maximal in $r = r_\omega(\rho)$. By differentiating $W(r, \rho) = V(R(r), Y - T(\rho, r))$ we get:

$$\frac{\partial W(r, \rho)}{\partial r} = \dot{R} \frac{\partial V}{\partial R} - \frac{\partial V}{\partial I} \frac{\partial T}{\partial r} \quad (8.1)$$

The function $g^{(\rho)} : r \mapsto \frac{\partial W(r, \rho)}{\partial r} / \frac{\partial V}{\partial I}$ has the same sign as $\frac{\partial W(r, \rho)}{\partial r}$ since $\frac{\partial V}{\partial I} \geq 0$. Now $g^{(\rho)}(r) = -\dot{R} \hat{s}(R, I) - \frac{\partial T}{\partial r}$ since $\hat{s}(R, I) = -\frac{\partial V}{\partial R} / \frac{\partial V}{\partial I}$. As $\rho \mapsto T(\rho, r)$ is a decreasing function, $\rho \mapsto Y - T(\rho, r) = I_{\rho, r}$ is increasing, and therefore $\rho \mapsto \hat{s}(R, I_{\rho, r})$ is also increasing, since \hat{s} increases with income if dwelling space is a normal good. Moreover $\dot{R} \leq 0$ does not

vary with ρ . Thus the function $\rho \mapsto \varphi^{(r)}(\rho) = g^{(\rho)}(r)$ is an increasing function of ρ . This function cancels out at $\rho = \rho_\omega(r)$ if (3.10) is verified. From this we successively deduce:

$$\varphi^{(r)}(\rho) \underset{>}{\leq} \varphi^{(r)}(\rho_\omega(r)) \text{ if } \rho \underset{>}{\leq} \rho_\omega(r)$$

$$\varphi^{(r)}(\rho) \underset{>}{\leq} 0 \text{ if } \rho \underset{>}{\leq} \rho_\omega(r)$$

$$g^{(\rho)}(r) \underset{>}{\leq} 0 \text{ if } \rho \underset{>}{\leq} \rho_\omega(r) \text{ i.e. } r_\omega(\rho) \underset{>}{\leq} r$$

$$\frac{\partial W(r, \rho)}{\partial r} \underset{>}{\leq} 0 = \frac{\partial W(r, \rho_\omega(r))}{\partial r} \text{ if } r_\omega(\rho) \underset{>}{\leq} r$$

Therefore, with fixed ρ , the function $r \mapsto \frac{\partial W(r, \rho)}{\partial r}$ is positive if $r \leq r_\omega(\rho)$ then negative if $r \geq r_\omega(\rho)$: $r \mapsto W(r, \rho)$ increases until $r_\omega(\rho)$ then decreases, therefore $r_\omega(\rho)$ is the sole maximal argument. We can therefore unambiguously define the maximum utility function for a household employed in ρ and subject to price conditions $R(r)$:

$$\rho \mapsto \tilde{u}_\rho = W(r_\omega(\rho), \rho) \text{ on } Z_F$$

(ii) We need to demonstrate that the function $r \mapsto R(r)$ obtained by the differential system, and which is assumed to constitute the price conditions, does indeed coincide with optimised supply conditions: a supplier of living space in r seeks to optimise their unit rent $R(r)$ by selecting the highest bids by households. For this, we simply need to show (a) that conditionally to the utilities $(\tilde{u}_\rho)_{\rho \in Z_F}$, each $R(r)$ is equal to a bid by a household who wants to bid in r because it is here that they obtain maximum utility, and (b) that this bid is higher than that of the other households in the same position.

(a) Let us show first that $r \mapsto R(r)$ coincides with bids by households working at ρ in their optimised residential location $r_\omega(\rho)$. Condition (3.5) ensures the equation for ρ_f , in $r_\omega(\rho_f) = r_A$:

$$R_A = R(r_A) = \Psi_{\rho_f}(r_A, \hat{u}_{\rho_f})$$

Let us show that the derivatives $\dot{R}(r)$ and $d\Psi/dr$ for $r \mapsto \Psi_{\rho_\omega(r)}(r, \hat{u}_{\rho_\omega(r)})$ coincide in any location, which will ensure that the functions are equal at any point. Let $\varphi(r, \rho) = \Psi_\rho(r, \tilde{u}_\rho)$, $\psi(I, u)$ denote Solow's bid function and $s(I, u)$ Solow's surface demand function for a net income I and a utility u . Since households have homogeneous preferences, functions ψ and s do not depend on ρ . At the maximum bid point:

$$\Psi_\rho(r, \tilde{u}_\rho) = \frac{Y - T(\rho, r) - Z(s, u)}{s} \text{ for } s = s(I, u) \text{ and } u = \tilde{u}_\rho$$

From this, we deduce $\partial\varphi/\partial\rho$ by the envelope theorem, with fixed s :

$$\frac{\partial\varphi}{\partial\rho} = \frac{1}{s} \left(-\frac{\partial T}{\partial\rho} - \frac{\partial Z}{\partial u} \frac{\partial\tilde{u}_\rho}{\partial\rho} \right)$$

Since $V(R, I) = \max U(z, s)$ for $sR + z = I$, $V(R, I) = U(I - sR, s)$ in $s = \hat{s}(R, I)$ and by the envelope theorem, $\partial V(R, I)/\partial I = \partial U/\partial z$. Moreover, since Z is the inverse function of U with fixed s , $\frac{\partial Z}{\partial u} = 1/\frac{\partial U}{\partial z}$. Therefore $\frac{\partial Z}{\partial u} = 1/\frac{\partial V}{\partial z}$. In addition, when condition (3.10) is met,

therefore also (3.9), then by the definition of \tilde{u}_ρ the result is that $\frac{\partial \tilde{u}_\rho}{\partial \rho} = \frac{\partial V}{\partial I} \Big|_{r_\omega(\rho)} \cdot \frac{\partial I}{\partial \rho} \Big|_{r_\omega(\rho)}$.

Bringing it all together, we find that :

$$\frac{\partial \varphi}{\partial \rho} = -\frac{V_I|_r}{V_R|_r} \left[-\frac{\partial T(\rho, r)}{\partial \rho} - \frac{V_I|_{r_\omega(\rho)}}{V_I|_r} \frac{\partial I}{\partial \rho} \Big|_{r_\omega(\rho)} \right]$$

But $\partial I / \partial \rho = -\partial T(\rho, r) / \partial \rho$, therefore in $r = r_\omega(\rho)$, $\partial \varphi / \partial \rho = 0$.

Let us now look at the function $r \mapsto \hat{\varphi}(r) = \varphi(r, r_\omega(\rho))$:

$$\begin{aligned} \frac{d\hat{\varphi}}{dr} &= \frac{\partial \varphi}{\partial r} + \frac{\partial \varphi}{\partial \rho} \frac{d\rho_\omega}{dr} \\ &= \frac{\partial \varphi}{\partial r} \\ &= -\frac{1}{s} \frac{\partial T}{\partial r} \end{aligned}$$

This expression coincides with $\dot{R}(r)$ as defined by (3.10), which ensures the equality of the two primitive functions. Thus $R(r)$ gives the prices bid by households: $\forall r \in r_\omega(Z_F)$, $R(r) = \Psi_\rho(r, \tilde{u}_\rho)$ at $r = r_\omega(\rho)$.

(b) Let us next show that these prices exceed bids by the other households. Since the indirect utility function V is decreasing in the rent variable R , a household employed in ρ and facing conditions $R(r)$ fulfils the following property (Fujita property 2.4, 1989):

$$\forall \rho, \forall r, V(R(r), I_{\rho, r}) \underset{>}{<} V(\Psi_\rho(r, u), I_{\rho, r}) \quad \text{if} \quad R(r) \underset{>}{<} \Psi_\rho(r, u)$$

Since \tilde{u}_ρ is the maximum utility for ρ under conditions R , we deduce that:

$$\forall \rho, \forall r, R(r) \geq \Psi_\rho(r, \tilde{u}_\rho)$$

By combining with the condition $R(r) = \Psi_\rho(r, \tilde{u}_\rho)$ at $\rho = \rho_\omega(r)$, we get the result that

$$\forall r \in r_\omega(Z_F), R(r) = \max_{\rho \in Z_F} \Psi_\rho(r, \tilde{u}_\rho)$$

To summarise, the curve R obtained by integrating the differential system (3.12) under condition (3.5) does indeed define the optimised supply conditions.

(iii) It remains for us to show that the microeconomic conditions are compatible with the physical supply of living space, $L(r)$, both at local and global level. By relating the two terms of (3.12) to each other at point r , we identify that $h(r)s = L(r)$, which ensures local compatibility (3.6) between the demand for living space by all households and local land capacity, under supply conditions R and demand conditions \tilde{u}_ρ . As regards global compatibility, condition (3.5) ensures that households residing between r_0 and r_A , therefore under conditions R , are equal in number to N .

(iv) In all, conditions (3.5) and (3.12), which include condition (3.3), ensure conditions (3.2), (3.4) and (3.6), and are therefore sufficient to characterise (H, R) as a supply-demand equilibrium.

8.3. Proof of proposition 5

(i) We know the signs of $\partial T/\partial r \geq 0$, $\hat{s} \geq 0$, $L \geq 0$ and $f \geq 0$. By (3.11) $\dot{H} \geq 0$ therefore H is increasing. And $\rho_\omega = F^{-1} \circ H$ is increasing, through composition of two increasing functions.

(ii) By (3.10), $\dot{R} \leq 0$ therefore R decreases with r .

(iii) The total derivative of the function $r \mapsto W(r, \rho_\omega(r))$ is

$$\frac{dW}{dr} = \frac{\partial W}{\partial r} + \frac{\partial W}{\partial \rho} \dot{\rho}_\omega(r)$$

With $\partial W/\partial r = 0$ according to (8.1) and (3.10), whereas $\dot{\rho}_\omega(r) \geq 0$ since ρ_ω is increasing, and $\frac{\partial W}{\partial \rho} = \frac{\partial V}{\partial I} \left(-\frac{\partial T}{\partial \rho}\right)$ is non-negative as a product of two non-negative factors.

(iv) We note $\dot{I} = \frac{dI}{dr} = \frac{\partial I}{\partial r} + \frac{\partial I}{\partial \rho} \dot{\rho}_\omega(r)$. We also have: $\frac{dI}{dr} = \dot{R} \hat{s} + \frac{\partial I}{\partial \rho} \dot{\rho}_\omega(r)$.

The total derivative of the function $r \mapsto S(r)$ is $\frac{dS}{dr} = \frac{\partial \hat{S}}{\partial R} \dot{R} + \frac{\partial \hat{S}}{\partial I} \dot{I}$, therefore

$$\frac{dS}{dr} = \left(\frac{\partial \hat{S}}{\partial R} + \hat{s} \frac{\partial \hat{S}}{\partial I}\right) \dot{R} + \frac{\partial \hat{S}}{\partial I} \frac{\partial I}{\partial \rho} \dot{\rho}_\omega(r)$$

By the Slutsky equation, $\frac{\partial \hat{S}}{\partial R} + \hat{s} \frac{\partial \hat{S}}{\partial I} = \frac{\partial \tilde{S}}{\partial R}$ which is negative. Therefore the first term in dS/dr is positive as a product of two negative terms. The second term is a product of three non-negative terms: $\dot{\rho}_\omega \geq 0$, $\frac{\partial I}{\partial \rho} = -\frac{\partial T}{\partial \rho} \geq 0$ and $\frac{\partial \hat{S}}{\partial I}$ which is non-negative if dwelling space is a normal good. In total, $\frac{dS}{dr} \geq 0$.

8.4. Proof of proposition 6

At the start of residential locations r_0 , $R_{02} > R_{01}$ whereas $I_{01} = I_{02} = Y - T(0, r_0)$: therefore $S_1(R, I) > S_2(R, I) \geq 0$, which implies that $h_2(r_0) > h_1(r_0)$ and $\dot{R}_1(r_0) \leq \dot{R}_2(r_0)$. Let us assume that the statement's hypothesis on the current state is verified up to r and consider $r' = r + \delta r$ for a marginal increment δr . If $S_1|_r \geq S_2|_r$, then $h_2(r) \geq h_1(r)$, the difference $H_2 - H_1$ increases: therefore $\rho_{\omega 1} \leq \rho_{\omega 2}$ in r' since $\rho_{\omega i} = F^{-1} \circ H_i$ where F^{-1} is increasing. Since $T(\rho, r)$ decreases with ρ , we deduce that $T(\rho'_1, r') \geq T(\rho'_2, r')$ and therefore $I_{\rho'_1, r'} \leq I_{\rho'_2, r'}$.

For land rent, if $R_1 < R_2$ then $R'_1 = R_1(r') < R_2(r') = R'_2$ even if $\dot{R}_1 \geq \dot{R}_2$: otherwise, if we take $\hat{r} \in]r, r']$ such that $R_1(\hat{r}) = R_2(\hat{r})$, since $I_1(\hat{r}) \leq I_2(\hat{r})$ then $S_1(\hat{r}) \leq S_2(\hat{r})$ therefore between r and \hat{r} the S_1 and S_2 curves would have intersected at a position \tilde{r} such that $\hat{s}(\tilde{R}_1, \tilde{I}_1) = \hat{s}(\tilde{R}_2, \tilde{I}_2)$ given that $\tilde{R}_1 < \tilde{R}_2$, therefore from this point $\dot{R}_1 \leq \dot{R}_2$ which makes the inequality $R'_1 \geq R'_2$ impossible. Thus the inequality for R is maintained from r to $r' = r + \delta r$, and therefore also the inequality for S . In all, the inequalities are true throughout the integration path.

8.5. Proof of proposition 8

(i) Algorithmic proof by a dichotomy method. Let us take an interval $[R_{01}, R_{02}]$ where $R_{01} < R_{02}$. Initially we set $[R_{01}, R_{02}] = [R_A, R_\infty]$ for a number R_∞ arbitrarily large which makes

lot size very small and fills up condition $H = N$ before R has significantly decreased. Let us apply integration to $R' = (R_{01} + R_{02})/2$: if termination produces R_A in r_A such that $H(r_A) < N$, then update the interval to $[R', R_{02}]$, otherwise to $[R_{01}, R']$. According to proposition 6, any initial value $R_0 \in [R_A, R_{01}]$ is too low, whereas any initial value $R_0 \in [R_{02}, R_\infty]$ is too high. Since the interval gradually shrinks, the algorithm converges towards a satisfactory solution \hat{R}_0 .

(ii) The uniqueness comes from the conservation of the inequalities between the two state variables: if $R_{01} < R_{02}$ then $H_1(r_{A1}) = N > H_2(r_{A1})$ whereas $R_1(r_{A1}) = R_A < R_2(r_{A1})$, which prevents R_{02} from producing an equilibrium.

9. Appendix 2: Resolution of the simplified differential system

9.1. Link between land rent and residential distribution

In the model specified in section 5, condition (3.12a) is expressed $\dot{R} = -\frac{a}{\lambda}\dot{H}$, therefore $\Delta R = -\frac{a}{\lambda}\Delta H$, and, where R_0 denotes the value of R at r_0 :

$$R(r) = R_0 - \frac{a}{\lambda}H(r) \quad \text{for } r \geq r_0 \quad (9.1)$$

The boundary condition gives $R_A = R_0 - \frac{a}{\lambda}N$, therefore $R_0 = R_A + \frac{a}{\lambda}N$.

The rent at the initial location arises simply from the parameters of the model.

9.2. Reduction to a simple differential equation

By replacing R by its expression depending on H in condition (3.12b), we obtain a simple differential equation in H :

$$H = \frac{1}{\beta'} \frac{\lambda R_0 - aH}{Y - T} \quad \text{where } Y - T = Y_0 - ar + a'H/b \quad (9.2)$$

9.3. Change of variable, transformed differential equation

Let us change the variable by taking $r = H^{-1}(n)$ for $n \in [0, N]$, therefore $\dot{r} = 1/\dot{H}$. Condition (9.2) becomes

$$\dot{r} = \beta' \frac{Y_0 - ar + \frac{a'}{b}n}{\lambda R_0 - an}$$

Or:

$$\dot{r}(\lambda R_0 - an) + \beta' ar = \beta'(Y_0 + \frac{a'}{b}n) \quad (9.3)$$

9.4. Homogeneous equation

The homogeneous equation associated with (9.3) is:

$$\frac{\dot{r}}{r} = \frac{-\beta'a}{\lambda R_0 - an}$$

It allows a solution in the following form, where \tilde{r}_0 is a constant:

$$\tilde{r} = \tilde{r}_0 \left[\frac{\lambda R_0 - an}{\lambda R_0} \right]^{\beta'} \quad (9.4)$$

9.5. Variation of the “constant”

We now look for a solution to the transformed differential equation (9.3), in the form $r = \nu \cdot \tilde{r}$

by varying the function ν . Since $\dot{\tilde{r}} = \frac{-\beta'a}{\lambda R_0 - an} \tilde{r}$,

$$\nu \dot{\tilde{r}} = \frac{-\beta'a}{\lambda R_0 - an} r = \dot{r} - \beta' \frac{Y_0 + \frac{a'}{b} n}{\lambda R_0 - an}$$

If we compare it with the expression $\dot{r} = \dot{\nu} \cdot \tilde{r} + \nu \cdot \dot{\tilde{r}}$ obtained by deriving the definition of r , we get:

$$\begin{aligned} \dot{\nu} &= \beta' \frac{Y_0 + \frac{a'}{b} n}{\tilde{r} \lambda R} \\ &= \frac{\beta'}{\tilde{r}_0} (\lambda R_0)^{\beta'} (\lambda R)^{-\beta'-1} [Y_0 + \frac{a'}{ba} (\lambda R_0 - \lambda R)] \\ &= \frac{\beta'}{\tilde{r}_0} (\lambda R_0)^{\beta'} \left((Y_0 + a' \tilde{\rho}) (\lambda R)^{-\beta'-1} - \frac{a'}{ba} (\lambda R)^{-\beta'} \right) \end{aligned}$$

in which we noted $\lambda R = \lambda R_0 - an$ and $\tilde{\rho} = \lambda R_0 / (ab)$. By integrating with respect to n , we obtain:

$$\nu = \nu_1 + \frac{\beta'}{\tilde{r}_0} (\lambda R_0)^{\beta'} \left[(Y_0 + a' \tilde{\rho}) \frac{(\lambda R)^{-\beta'}}{a \beta'} + \frac{a'}{ba^2} \frac{1}{\alpha'} (\lambda R)^{\alpha'} \right]$$

9.6. Solution of the transformed equation

By noting $W = -\nu_1 \tilde{r}_0$, we finally obtain:

$$\begin{aligned} r = \nu \tilde{r} &= \nu_1 \tilde{r}_0 \left(\frac{\lambda R}{\lambda R_0} \right)^{\beta'} + \frac{1}{a} (Y_0 + a' \tilde{\rho}) + \frac{a'}{ba^2} \frac{\beta'}{\alpha'} \lambda R \\ &= -W \left(\frac{R}{R_0} \right)^{\beta'} + \frac{Y_0}{a} + \frac{a'}{a} \tilde{\rho} \left(1 + \frac{\beta'}{\alpha} \frac{R}{R_0} \right) \end{aligned}$$

At r_0 , $R = R_0$ and $r_0 = -W + \frac{1}{a} Y_0 + \frac{a'}{a} \tilde{\rho} \left(1 + \frac{\beta'}{\alpha} \right)$, therefore $W = \frac{1}{a} Y_0 + \frac{a'}{a} \tilde{\rho} \left(1 + \frac{\beta'}{\alpha} \right) - r_0$.

9.7. Residential location depending on the place of employment

By setting $A = \frac{1}{a} Y_0 + \frac{a'}{a} \tilde{\rho}$ and $B = \frac{\beta'}{\alpha} \frac{a'}{a} \tilde{\rho}$, then $W = A + B - r_0$, the solution of the transformed equation takes the following form:

$$r = A + B \frac{R}{R_0} - W \left(\frac{R}{R_0} \right)^{\beta'} \quad (9.5)$$

This form is the most appropriate for linking residential position r with the place of employment ρ . This is because R and H are linked by (9.1) whereas H and ρ are linked by $H(r) = F(\rho) = b\rho$, therefore $R(r) = R_0 - \frac{ab}{\lambda} \rho$. As a result:

$$r = A + B \left(1 - \frac{\rho}{\tilde{\rho}} \right) - W \left(1 - \frac{\rho}{\tilde{\rho}} \right)^{\beta'} \quad (9.6)$$

9.8. Calculation of equivalent income

$$\begin{aligned} \tilde{Y}(\rho) &= \tilde{T}(\rho_0) + E(R_0, \tilde{u}_\rho) \\ &= a_0 + ar_0 + R_0^{\beta'} R(\rho)^{-\beta'} \tilde{T}(\rho) \\ &= a_0 + ar_0 + \left(\frac{R}{R_0} \right)^{-\beta'} \left[aW \left(\frac{R}{R_0} \right)^{\beta'} - \frac{a'}{\alpha'} \tilde{\rho} \left(\frac{R}{R_0} \right) \right] \\ &= a_0 + ar_0 + \left[Y_0 + \frac{a'}{\alpha'} \tilde{\rho} - ar_0 - \frac{a'}{\alpha'} \tilde{\rho} \left(\frac{R}{R_0} \right)^{\alpha'} \right] \\ &= Y + \frac{a'}{\alpha'} \tilde{\rho} \left[1 - \left(\frac{R}{R_0} \right)^{\alpha'} \right] \end{aligned}$$

10. Appendix 3: Sensitivity of residential location

This appendix presents the proofs of propositions 9 and 10.

10.1. Simple influences

To reveal the direction of variation of r_ω in response to parameter changes, we can adapt expression (5.4). We note $x = 1 - \rho/\tilde{\rho}$, which is non-negative because $\rho \leq \rho_f \leq \tilde{\rho}$.

First of all, $r_\omega = K_0 + r_0 x^{\beta'}$ where K_0 is independent of r_0 , therefore residential location increases with r_0 : this proves point (i) of the proposition.

Then $r_\omega = \frac{1}{a} Y_0 (1 - x^{\beta'}) + K_1$ where K_1 is independent of Y_0 , hence point (ii).

Next, $r_\omega = \frac{a'}{a} K_2 + K_3$ where $K_2 \geq 0$ and K_3 are independent of a' .

In addition, for point (vi), we reformulate (5.4) as follows:

$$r_\omega - r_0 = (A - r_0)(1 - x^{\beta'}) + \frac{\beta}{\alpha} \frac{a'}{ba^2} (x - x^{\beta'})$$

Since function $x^{\beta'}$ decreases with β and increases with α , functions $1 - x^{\beta'}$ and $x - x^{\beta'}$ increase with β and decrease with α , as does function β/α . This influence is conserved by the product and sum of positive functions having the same property, therefore $r_\omega - r_0$ is an increasing function of β and a decreasing function of α .

10.2. Framework of analysis for a complex influence

Parameters b , a , λ , R_A have more complex influences, both direct and indirect via $\tilde{\rho}$. We can determine the influence of a parameter X on $\Delta r = r_\omega - r_0$ by considering the derivative function $\partial\Delta r/\partial X$: this function is well-defined because Δr is sufficiently regular, and whether it has a positive or negative sign determines whether X has an increasing or decreasing influence.

Since $\Delta r = B(x-1) + W(1-x^{\beta'})$,

$$\frac{\partial\Delta r}{\partial X} = \frac{\partial W}{\partial X}(1-x^{\beta'}) + \frac{\partial B}{\partial X}(x-1) + (B-\beta'Wx^{-\alpha'})\frac{\partial x}{\partial X}$$

According to the expression $x = 1 - \rho/\tilde{\rho}$,

$$\frac{\partial x}{\partial X} = \frac{\rho}{\tilde{\rho}^2} \frac{\partial \tilde{\rho}}{\partial X} = \frac{1-x}{\tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial X}$$

Therefore $\partial x/\partial X = 0$ at point $x = 1$. Therefore if $\frac{\partial^2\Delta r}{\partial x\partial X} \geq 0$ on $[0,1]$ then $\frac{\partial\Delta r}{\partial X} \leq 0$.

The next calculation is:

$$\frac{\partial^2\Delta r}{\partial x\partial X} = -\beta'x^{-\alpha'}\frac{\partial W}{\partial X} + \frac{\partial B}{\partial X} + \alpha'\beta'Wx^{-\alpha'-1}\frac{\partial x}{\partial X} + \frac{\partial(\partial x/\partial X)}{\partial x}(B-\beta'Wx^{-\alpha'})$$

The notation $\frac{\partial}{\partial x}(\partial x/\partial X)$ corresponds to a two-step operation: first $\frac{\partial x}{\partial X} = \frac{\partial x}{\partial \tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial X}$, second the derivation of x in response to variations of ρ , with $\tilde{\rho}$ fixed: so $\delta x = -\delta\rho/\tilde{\rho}$. A more rigorous expression is :

$$\frac{\delta}{\delta x} \left(\frac{\partial x}{\partial \tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial X} \right) = \frac{\delta}{\delta x} \frac{1-x}{\tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial X} = -\frac{1}{\tilde{\rho}} \frac{\partial \tilde{\rho}}{\partial X}$$

Therefore

$$\frac{\partial^2\Delta r}{\partial x\partial X} = -\beta'x^{-\alpha'}\frac{\partial W}{\partial X} + \frac{\partial B}{\partial X} + \alpha'\beta'Wx^{-\alpha'-1}\frac{\partial x}{\partial X} - (B-\beta'Wx^{-\alpha'})\frac{1}{\tilde{\rho}}\frac{\partial \tilde{\rho}}{\partial X}$$

10.3. Influence of job density

Let us study the total influence of b , including via $N = b\rho_f$:

$$R_0 = R_A + \frac{aN}{b} = R_A + ab\frac{1}{\lambda}\rho_f$$

$$\frac{\partial B}{\partial x} = -\frac{\beta' a' \lambda R_A}{\alpha a b^2} = -\frac{B}{b} \text{ where } t = R_A/R_0 < 1$$

$$\frac{\partial W}{\partial b} = \frac{1}{\beta'} \frac{\partial B}{\partial b}$$

$$\frac{\partial \tilde{\rho}}{\partial b} = -\frac{\lambda}{ab^2} R_A = -\frac{\tilde{\rho}}{b} t \text{ hence } \frac{\partial x}{\partial b} = -\frac{t}{b}(1-x) \text{ and } \frac{\partial}{\partial x} \frac{\partial x}{\partial b} = \frac{t}{b}$$

Therefore:

$$\begin{aligned} \frac{\partial^2\Delta r}{\partial x\partial b} &= -\alpha'\beta'W(1-x)x^{-\alpha'-1}\frac{t}{b} - B\frac{t}{b}(1-x^{-\alpha'}) + \frac{t}{b}[B(1-x^{-\alpha'}) - \beta'x^{-\alpha'}(\frac{Y_0}{a} - r_0)] \\ &= -\alpha'\beta'W(1-x)x^{-\alpha'-1}\frac{t}{b} - \frac{t}{b}\beta'x^{-\alpha'}(\frac{Y_0}{a} - r_0) \end{aligned}$$

The cross derivative is the sum of two negative terms, and therefore negative, which implies that b has an increasing influence on residential location.

10.4. Influence of job radius

As regards the influence of ρ_f , including via $N = b\rho_f$, we first calculate:

$$\frac{\partial \tilde{\rho}}{\partial \rho_f} = 1 \text{ therefore } \frac{\partial x}{\partial \rho_f} = \frac{1-x}{\rho_f} \text{ and } \frac{\partial}{\partial x} \frac{\partial x}{\partial \rho_f} = -\frac{1}{\tilde{\rho}}.$$

Next:

$$\frac{\partial B}{\partial \rho_f} = \frac{\beta a'}{\alpha a} = \frac{B}{\tilde{\rho}} \text{ and } \frac{\partial W}{\partial \rho_f} = \frac{1}{\beta'} \frac{\partial B}{\partial \rho_f} = \frac{1}{\beta'} \frac{B}{\tilde{\rho}}.$$

Therefore:

$$\begin{aligned} \frac{\partial^2 \Delta r}{\partial x \partial \rho_f} &= \frac{B}{\tilde{\rho}} (1-x^{-\alpha'}) + \alpha' \beta' W x^{-\alpha'-1} \frac{1-x}{\tilde{\rho}} - \frac{1}{\tilde{\rho}} [B - \beta' W x^{-\alpha'}] \\ &= \alpha' \beta' W x^{-\alpha'-1} \frac{1-x}{\tilde{\rho}} + \frac{\beta'}{\tilde{\rho}} x^{-\alpha'} \left(\frac{Y_0}{a} - r_0 \right) \end{aligned}$$

The cross derivative is the sum of two positive terms, and therefore positive, which implies that ρ_f has a decreasing influence on residential location. Thus, where job density b is constant, an increase of ρ_f is equivalent to an increase in the number of households, therefore to increasing land pressure, which increases residential density.

10.5. Influence of the unit cost of transport

Let us now look at the unit cost of transport in the residential area, a , initially independently of a' . We start by calculating:

$$\frac{\partial \tilde{\rho}}{\partial a} = -\frac{\tilde{\rho}}{a} t \text{ therefore } \frac{\partial x}{\partial a} = -\frac{1-x}{a} t \text{ and } \frac{\partial}{\partial x} \frac{\partial x}{\partial a} = \frac{t}{a}.$$

Next:

$$\frac{\partial B}{\partial a} = -\frac{\beta a'}{\alpha b a^2} (2\lambda R_A + \rho_f) = -\frac{B}{a} (1+t) \text{ and } \frac{\partial W}{\partial a} = -Y_0 a^{-2} + \frac{1}{\beta'} \frac{\partial B}{\partial a}.$$

This gives us:

$$\frac{\partial^2 \Delta r}{\partial x \partial a} = -\frac{B}{a} (1+t) (1-x^{-\alpha'}) + \beta' Y_0 a^{-2} x^{-\alpha'} - t \frac{1-x}{a} \alpha' \beta' W x^{-\alpha'-1} + \frac{t}{a} [B - \beta' W x^{-\alpha'}]$$

which we analyse as the sum of three terms influenced respectively by B , Y_0 and r_0 .

The term in B is:

$$\frac{B}{a} \left\{ -(1+t)(1-x^{-\alpha'}) - t\alpha'(1-x)x^{-\alpha'-1} + t(1-x^{-\alpha'}) \right\} = -\frac{B}{a} \left\{ (1-x^{-\alpha'}) + \alpha't(1-x)x^{-\alpha'-1} \right\}$$

As $t = \frac{R_A}{R_0} = \min_{\rho} \frac{R}{R_0} = \min_{\rho} x$, on the effective domain $\rho \in [0, \rho_f]$ we know that $x \in [t, 1]$.

Therefore $1-x^{-\alpha'} + \alpha't(1-x)x^{-\alpha'-1} \leq 1-x^{-\alpha'} + \alpha'(x^{-\alpha'} - x^{\beta'})$.

Let us use $\varphi(x) = 1 - \beta'x^{-\alpha'} - \alpha'x^{\beta'}$ to denote this function, which takes the value $\varphi(1) = 0$ and has the derivative $\frac{d\varphi}{dx} = \alpha'\beta'[x^{-\alpha'-1} - x^{-\alpha'}]$ which is positive on $[0, 1]$: therefore $\varphi(x) \leq 0$ on this interval, which ensures that the term in B is positive.

The term in Y_0 is

$$\beta' Y_0 a^{-2} \left\{ x^{-\alpha'} (1-t) - \alpha't(1-x)x^{-\alpha'-1} \right\} = \beta' Y_0 a^{-2} x^{-\alpha'-1} \left\{ x(1-t) - \alpha't(1-x) \right\}$$

The expression in brackets is reformulated:

$$x(1-t) - \alpha't(1-x) = x - tx - \alpha't + \alpha'tx = x(1 - \beta't) - \alpha't = x - t(\alpha' + \beta'x)$$

When $x \leq 1$, so is $\alpha' + \beta'x \leq 1$ and, as $t \leq x$, this term is positive, and therefore the term in Y_0 is also positive.

Finally, the term in r_0 is $\beta'r_0 t \frac{1-x}{a} x^{-\alpha'-1} + r_0 \frac{t}{a} \beta' x^{-\alpha'} = \beta'r_0 \frac{t}{a} x^{-\alpha'-1} \geq 0$.

In all, as the sum of three positive terms, the cross derivative is positive, which ensures that the derivative is negative: residential location is closer to the centre when the unit cost of transport in the residential zone is higher.

In the event that $a' = a$, the parameter a continues to have a decreasing influence. Indeed, the effect on $\tilde{\rho}$ is maintained, whereas:

$$B = \frac{\beta}{\alpha} \tilde{\rho} \text{ therefore } \frac{\partial B}{\partial a} = -\frac{B}{a} t \text{ and } \frac{\partial W}{\partial a} = -Y_0 a^{-2} + \frac{1}{\beta'} \frac{\partial B}{\partial a}.$$

As a result:

$$\begin{aligned} \frac{\partial^2 \Delta r}{\partial x \partial a} &= -\frac{B}{a} t (1 - x^{-\alpha'}) + \beta' Y_0 a^{-2} x^{-\alpha'} + \alpha' \beta' W x^{-\alpha'-1} \left(-t \frac{1-x}{a}\right) + \frac{t}{a} [B - \beta' W x^{-\alpha'}] \\ &= \beta' Y_0 a^{-2} x^{-\alpha'-1} \{x(1-t) - \alpha't(1-x)\} + r_0 \beta' \frac{t}{a} x^{-\alpha'} \left(\alpha' \frac{1-x}{x} + 1\right) \end{aligned}$$

The coefficient of r_0 is positive, as is that of Y_0 since $1 \geq x \geq t$ therefore $1-t \geq 1-x \geq 0$, and also $\alpha' \leq 1$.

Therefore the cross derivative is positive, which ensures that the total influence of a on Δr is decreasing.

10.6. Influence of alternative rent

As regards the parameter R_A :

$$\tilde{\rho} = \frac{\lambda R_A + aN}{b} \text{ therefore } \frac{\partial \tilde{\rho}}{\partial R_A} = \frac{\lambda}{ab} = \frac{\tilde{\rho}}{R_0}, \text{ then } \frac{\partial x}{\partial R_A} = \frac{1-x}{R_0} \text{ and } \frac{\partial}{\partial x} \frac{\partial x}{\partial R_A} = -\frac{1}{R_0}.$$

$$\text{Next: } \frac{\partial B}{\partial R_A} = \frac{\beta}{\alpha} \frac{a'}{a} \frac{\partial \tilde{\rho}}{\partial R_A} = \frac{B}{R_0} \text{ and } \frac{\partial W}{\partial R_A} = \frac{1}{\beta'} \frac{B}{R_0}.$$

This gives us:

$$\begin{aligned} \frac{\partial^2 \Delta r}{\partial x \partial R_A} &= \frac{B}{R_0} (1 - x^{-\alpha'}) + \alpha' \beta' W x^{-\alpha'-1} \frac{1-x}{R_0} - \frac{1}{R_0} [B - \beta' W x^{-\alpha'}] \\ &= \alpha' \beta' W x^{-\alpha'-1} \frac{1-x}{R_0} + \frac{\beta'}{R_0} x^{-\alpha'} \left(\frac{Y_0}{a} - r_0\right) \end{aligned}$$

This is the sum of two positive terms, therefore the cross derivative is positive and the derivative is negative: an increase in R_A brings residential location closer to the centre.

10.7. Influence of land capacity

The parameter λ has exactly the same effect as parameter R_A on the model's results, because both parameters appear in the formulas solely in the grouped form λR_A .

10.8. Sensitivity of lot size (Proposition 10)

We analyse the sensitivity of $\dot{r}_\omega(\rho)$ to a parameter X by considering the function $X \mapsto \frac{\partial}{\partial X} \dot{r}_\omega$. Since $\dot{r}_\omega = \frac{\partial}{\partial \rho} \Delta r$, formally we have:

$$\frac{\partial}{\partial X} \dot{r}_\omega(\rho) = \frac{\partial^2 \Delta r}{\partial X \partial \rho} = \frac{\partial^2 \Delta r}{\partial \rho \partial X}$$

as r_ω is a sufficiently regular function. Moreover, we have already established the variations of $\partial^2 \Delta r / (\partial x \partial X)$ with respect to x that would correspond solely to variations in ρ , according to the relation $\delta x = -\delta \rho / \tilde{\rho}$. Therefore:

$$\frac{\partial^2 \Delta r}{\partial \rho \partial X} = \frac{\delta}{\delta \rho} \frac{\partial \Delta r}{\partial X} = -\frac{1}{\tilde{\rho}} \frac{\delta}{\delta x} \frac{\partial \Delta r}{\partial X} = -\frac{1}{\tilde{\rho}} \frac{\partial^2 \Delta r}{\partial x \partial X}$$

The sign of $\partial^2 \Delta r / (\partial \rho \partial X)$, therefore of $\frac{\partial}{\partial X} \dot{r}_\omega(\rho)$, is the opposite of that of $\partial^2 \Delta r / (\partial x \partial X)$, therefore identical to that of $\partial \Delta r / \partial X = \partial r_\omega / \partial X$. Thus the function $X \mapsto \dot{r}_\omega(\rho)$ varies according to X as $X \mapsto r_\omega(\rho)$, and the results of Proposition 9 are also true for $\dot{r}_\omega(\rho)$.