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Laurent Decreusefond, Jean-Stephane Dhersin, Pascal Moyal, Viet Chi Tran. Large graph limit for an SIR process in random network with heterogeneous connectivity. 2010. hal-00505167v1

**HAL Id: hal-00505167**

**<https://hal.science/hal-00505167v1>**

Preprint submitted on 22 Jul 2010 (v1), last revised 25 Jun 2011 (v3)

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# Large graph limit for an SIR process in random network with heterogeneous connectivity

L. Decreusefond      J.-S. Dhersin      P. Moyal      V. C. Tran

July 22, 2010

## Abstract

We consider an SIR epidemic model propagating on a Configuration Model network, where the degree distribution of the vertices is given and where the edges are randomly matched. The evolution of the epidemic is summed up into three measure-valued equations that describe the degrees of the susceptible individuals and the number of edges from an infectious or removed individual to the set of susceptibles. These three degree distributions are sufficient to describe the course of the disease. The limit in large population is investigated. As a corollary, this provides a rigorous proof of the equations obtained by Volz (2008).

**Keywords:** Configuration Model graph, SIR model, mathematical model for epidemiology, measure-valued process, large network limit.

**AMS codes:** 60J80, 05C80, 92D30, 60F9960J80, 05C80, 92D30, 60F99.

## 1 Introduction and notation

In this work, we investigate an epidemic spreading on a random graph with fixed degree distribution and evolving according to a SIR model as follows. Every individual not yet infected is assumed to be susceptible. Infected individuals stay infected during random exponential times with mean  $1/\beta$  during which they infect each of their susceptible neighbors with rate  $r$ . At the end of the infectious period, the individual becomes removed and is no longer susceptible to the disease. Contrarily to the classical mixing compartmental SIR epidemic models (*e.g.* [17, 5] see also [2] Chapter 2 for a presentation), heterogeneity in the number of contacts makes it difficult to describe the dynamical behavior of the epidemic. Mean field approximations (*e.g.* [23, 4, 10]) or large population approximations (*e.g.* [3], see also Eq. (3) of [1] in discrete time) provide a set of denumerable equations to describe our system. We are here inspired by the paper of Volz [26], who proposes a low-dimensional system of five differential equations for the dynamics of an SIR model on a Configuration Model (CM) graph [19]. We refer to Volz' article for a bibliography about SIR models on graphs (see also Newman [20, 21], Durrett [10] or Barthélemy *et al.* [4]). Starting from a random model in finite population, Volz derives deterministic equations by increasing the size of the network, following in this respect works of Newman for instance ([21]). The convergence of the continuous-time stochastic SIR model to its deterministic limit for large graphs was however not proved. In this paper, we prove the convergence that was left open by Volz. To achieve this, we provide a rigorous individual-based description of the epidemic on a random graph. Three degree distributions are sufficient to describe the epidemic dynamics. We describe these distributions by equations in the space of measures on  $\mathbb{N}$ , of which Volz' equations are a by-product. Starting with a node-centered description, we show that the individual dimension is lost in the large graph limit. Our construction heavily relies on the choice of a CM for the graph underlying the epidemic, which was also made in [26].

The beginning of the epidemic and computation of the reproduction number, when the numbers of infected individuals and of contaminating edges are small and when Volz's deterministic approximation does not hold, makes the object of another study [9].

The size  $N$  of the population is fixed. The individuals are related through a random network and are represented by the *vertices* of an undirected graph. Between two neighbors, we place an *edge*. The graph is non-oriented and an edge between  $x$  and  $y$  can be seen as two directed edges, one from  $x$  to  $y$  and the other from  $y$  to  $x$ . If we consider an edge as emanating from the vertex  $x$  and directed to the vertex  $y$ , we call  $x$  the *ego* of the edge and  $y$  the *alter*. The number of neighbors of a given individual is the *degree* of the associated vertex. The degree of  $x$  is denoted  $d_x$ . It varies from an individual to another one.

The population is partitionned into the classes of susceptible, infectious or removed individuals. At time  $t$ , we denote by  $S_t$ ,  $I_t$  and  $R_t$  the set of susceptible, infectious and removed nodes. With a slight abuse, we will say that a susceptible individual is of type S (accordingly of type I or R) and that an edge linking an infectious ego and susceptible alter is of type IS (accordingly RS, II or IR). For  $x \in I$  (respectively R),  $d_x(S)$  represents the number of half-edges with  $x$  as ego and susceptible alter. We denote by  $S_t$ ,  $I_t$  and  $R_t$  the sizes of these classes at time  $t$ . The numbers of edges with susceptible ego (resp. of edges of types IS and RS) will be denoted by  $N_t^S$  (resp.  $N_t^{IS}$  and  $N_t^{RS}$ ).

The space of real bounded functions on  $\mathbb{N}$  is denoted by  $\mathcal{B}_b(\mathbb{N})$ . For any  $f \in \mathcal{B}_b(\mathbb{N})$ , set  $\|f\|_\infty$  the supremum of  $f$  on  $\mathbb{N}$ . For all such  $f$  and  $y \in \mathbb{N}$ , we denote by  $\tau_y f$  the function  $x \mapsto f(x - y)$ . For all  $n \in \mathbb{N}$ ,  $\chi^n$  is the function  $x \mapsto x^n$ , and in particular,  $\chi \equiv \chi^1$  is the identity function, and  $\mathbf{1} \equiv \chi^0$  is the function constantly equal to 1.

We denote by  $\mathcal{M}_F(\mathbb{N})$  the set of finite measure on the set of non-negative integers, which is embedded with the topology of weak convergence. For all  $\mu \in \mathcal{M}_F(\mathbb{N})$  and  $f \in \mathcal{B}_b(\mathbb{N})$ , we write

$$\langle \mu, f \rangle = \sum_{k \in \mathbb{N}} f(k) \mu(\{k\}).$$

With some abuse of notation, for all  $\mu \in \mathcal{M}_F(\mathbb{N})$  and  $k \in \mathbb{N}$ , we denote  $\mu(k) = \mu(\{k\})$ . For  $x \in \mathbb{N}$ , we write  $\delta_x$  for the Dirac measure at point  $x$ . Note, that some additional notation is provided in Appendix A, together with several topological results, that will be used in the sequel.

The plan of the paper and the main results are described below. In Section 2, we describe the mechanisms underlying the propagation of the epidemic on the CM graph. We explain how the course of the epidemic can be entirely described by the following three elements of  $\mathcal{M}_F(\mathbb{N})$ , for  $t \geq 0$ :

$$\mu_t^S = \sum_{x \in S_t} \delta_{d_x}, \quad \mu_t^{IS} = \sum_{x \in I_t} \delta_{d_x(S)}, \quad \mu_t^{RS} = \sum_{x \in R_t} \delta_{d_x(S)}. \quad (1.1)$$

The degree distribution  $\mu_t^S$  of susceptible individuals is needed to describe the degrees of the new infected individuals. The measure  $\mu_t^{IS}$  provides information on the number of edges from  $I_t$  to  $S_t$ , through which the disease can propagate. Similarly, the measure  $\mu_t^{RS}$  is used to describe the evolution of the set of edges linking  $S_t$  to  $R_t$ .

We can see that  $N_t^S = \langle \mu_t^S, \chi \rangle$  and  $S_t = \langle \mu_t^S, \mathbf{1} \rangle$  (and accordingly for  $N_t^{IS}$ ,  $N_t^{RS}$ ,  $I_t$  and  $R_t$ ).

In Section 3, we study the large graph limit obtained when the number of vertices tends to infinity, the degree distribution being unchanged. The degree distributions mentioned above can then be approximated by the solution  $(\bar{\mu}_t^S, \bar{\mu}_t^{IS}, \bar{\mu}_t^{RS})_{t \geq 0}$  of the following system of deterministic

measure-valued equations, that is addressed in the sequel: for any  $f \in \mathcal{B}_b(\mathbb{N})$ ,

$$\langle \bar{\mu}_t^S, f \rangle = \sum_{k \in \mathbb{N}} \bar{\mu}_0^S(k) \theta_t^k f(k) \quad \text{where } \theta_t = \exp\left(-r \int_0^t \bar{p}_s^I ds\right), \quad (1.2)$$

$$\begin{aligned} \langle \bar{\mu}_t^{IS}, f \rangle &= \langle \bar{\mu}_0^{IS}, f \rangle - \int_0^t \beta \langle \bar{\mu}_s^{IS}, f \rangle ds \\ &+ \int_0^t \sum_{k \in \mathbb{N}} r k \bar{p}_s^I \sum_{\substack{j, \ell, m \in \mathbb{N} \\ j+\ell+m=k-1}} \binom{k-1}{j, \ell, m} (\bar{p}_s^I)^j (\bar{p}_s^R)^\ell (\bar{p}_s^S)^m f(m) \bar{\mu}_s^S(k) ds \\ &+ \int_0^t \sum_{k \in \mathbb{N}} r k \bar{p}_s^I (1 + (k-1) \bar{p}_s^I) \sum_{k' \in \mathbb{N}^*} (f(k') - 1) \frac{k' \bar{\mu}_s^{IS}(k')}{\bar{N}_s^{IS}} \bar{\mu}_s^S(k) ds, \end{aligned} \quad (1.3)$$

$$\begin{aligned} \langle \bar{\mu}_t^{RS}, f \rangle &= \int_0^t \beta \langle \bar{\mu}_s^{IS}, f \rangle ds \\ &+ \int_0^t \sum_{k \in \mathbb{N}} r k \bar{p}_s^I (k-1) \bar{p}_s^R \sum_{k' \in \mathbb{N}^*} (f(k') - 1) \frac{k' \bar{\mu}_s^{RS}(k')}{\bar{N}_s^{RS}} \bar{\mu}_s^S(k) ds, \end{aligned} \quad (1.4)$$

where the initial conditions  $\bar{\mu}_0^S$  and  $\bar{\mu}_0^{IS}$  on degrees are finite measures on  $\mathbb{N}$ , where for all  $t \geq 0$ ,  $\bar{N}_t^S = \langle \bar{\mu}_t^S, \chi \rangle$  (resp.  $\bar{N}_t^{IS} = \langle \bar{\mu}_t^{IS}, \chi \rangle$  and  $\bar{N}_t^{RS} = \langle \bar{\mu}_t^{RS}, \chi \rangle$ ) are the continuous number of edges with ego in s (resp. in IS edges, in RS edges), and  $\bar{p}_t^I = \bar{N}_t^{IS}/\bar{N}_t^S$  (resp.  $\bar{p}_t^R = \bar{N}_t^{RS}/\bar{N}_t^S$  and  $\bar{p}_t^S = (\bar{N}_t^S - \bar{N}_t^{IS} - \bar{N}_t^{RS})/\bar{N}_t^S$ ) are the proportions of edges with infectious (respectively removed, susceptible) alter among those having susceptible ego. Remark that  $\theta_t$  defined in (1.2) is also considered in [26] and is the probability that a degree one node remains susceptible until time  $t$ . Choosing  $f(k) = \mathbb{1}_i(k)$ , we obtain the following countable system of ordinary differential equations (ODEs).

$$\begin{aligned} \bar{\mu}_t^S(i) &= \bar{\mu}_0^S(i) \theta_t^i, \\ \bar{\mu}_t^{IS}(i) &= \bar{\mu}_0^{IS}(i) + \int_0^t \left\{ r \bar{p}_s^I \sum_{j, \ell \geq 0} (i+j+\ell+1) \bar{\mu}_s^S(i+j+\ell+1) \binom{i+j+\ell}{i, j, \ell} (\bar{p}_s^S)^i (\bar{p}_s^I)^j (\bar{p}_s^R)^\ell \right. \\ &\quad \left. + \left( r (\bar{p}_s^I)^2 \langle \bar{\mu}_s^S, \chi^2 - \chi \rangle + r \bar{p}_s^I \langle \bar{\mu}_s^S, \chi \rangle \right) \frac{(i+1) \bar{\mu}_s^{IS}(i+1) - i \bar{\mu}_s^{IS}(i)}{\langle \bar{\mu}_s^{IS}, \chi \rangle} - \beta \bar{\mu}_s^{IS}(i) \right\} ds, \\ \bar{\mu}_t^{RS}(i) &= \int_0^t \left\{ \beta \bar{\mu}_s^{IS}(i) + r \bar{p}_s^I \langle \bar{\mu}_s^S, \chi^2 - \chi \rangle \bar{p}_s^R \frac{(i+1) \bar{\mu}_s^{RS}(i+1) - i \bar{\mu}_s^{RS}(i)}{\langle \bar{\mu}_s^{RS}, \chi \rangle} \right\} ds, \end{aligned} \quad (1.5)$$

It is noteworthy to say that this system is similar but not identical to that in Ball and Neal [3]. Our equations differ since our mechanism is not the same (compare Section 2.2 with Section 5 in [3]). We emphasize that the number of links of an individual to s decreases as the epidemic progresses, which modifies its infectivity.

The system (1.2)-(1.4) allows us to recover the equations proposed by Volz [26]. More precisely, the dynamics of the epidemic is obtained by solving the following four ODEs, referred to as Volz' equations in the sequel:

**Proposition 1.1** (Volz [26]). *Let*

$$g(z) = \sum_{k \in \mathbb{N}} \bar{\mu}_0^S(k) z^k \quad (1.6)$$

*be the generating function for the initial degree distribution of the susceptible individual  $\bar{\mu}_0^S$ , and*

let  $\theta_t = \exp(-r \int_0^t \bar{p}_s^i ds)$ . Then, the epidemic can be described by the four following ODEs:

$$\bar{S}_t := \langle \bar{\mu}_t^S, \mathbf{1} \rangle = g(\theta_t), \quad (1.7)$$

$$\bar{I}_t := \langle \bar{\mu}_t^{IS}, \mathbf{1} \rangle = \bar{I}_0 + \int_0^t \left( r \bar{p}_s^i \theta_s g'(\theta_s) - \beta \bar{I}_s \right) ds, \quad (1.8)$$

$$\bar{p}_t^i = \bar{p}_0^i + \int_0^t \left( \int_{\mathbb{N}} r \bar{p}_s^i \bar{p}_s^S \theta_s \frac{g''(\theta_s)}{g'(\theta_s)} - r \bar{p}_s^i (1 - \bar{p}_s^i) - \beta \bar{p}_s^i \right) ds, \quad (1.9)$$

$$\bar{p}_t^S = \bar{p}_0^S + \int_0^t r \bar{p}_s^i \bar{p}_s^S \left( 1 - \theta_s \frac{g''(\theta_s)}{g'(\theta_s)} \right) ds. \quad (1.10)$$

In (1.8), we see that the classical contamination terms  $r \bar{S}_t \bar{I}_t$  (mass action) or  $r \bar{S}_t \bar{I}_t / (\bar{S}_t + \bar{I}_t)$  (frequency dependence) of mixing SIR models (*e.g.* [2, 8]) are replaced by  $r \bar{p}_t^i \theta_t g'(\theta_t)$  where the graph structure appears through the generating function  $g$ . Notice that  $\theta_t g'(\theta_t) = \bar{N}_t^S$  (see (3.48) below) is the number of edges with susceptible ego.

## 2 SIR model on a CM graph

In this section, we introduce CM graphs and describe the propagation of SIR on such graphs.

### 2.1 Configuration Model graph

Graphs at large can be mathematically represented as matrices with integer entries: to each graph corresponds an adjacency matrix, the  $(x, y)$ -th coefficient of which is the number of edges between the vertices  $x$  and  $y$ . Defining the distribution of a random graph thus amounts to choosing a sigma-field and a probability measure on the space  $\mathbb{N}^{\mathbb{N}^* \times \mathbb{N}^*}$ , where  $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ . Another approach is to construct a random graph by modifying progressively a given graph, as in Erdős-Renyi model. Several other constructions are possible such as the preferential attachment model, the threshold graphs, etc.

Here, we are interested in the Configuration Model (CM) proposed by Molloy and Reed [19] (see also [22, 22, 10]) and which models graphs with specified degree distribution and independence between the degree of neighbors. As shown by statistical tests, these models might be realistic in describing community networks. See for instance Cléménçon *et al.* [7] for dealing with the spread of the HIV-AIDS disease among the homosexual community in Cuba.

We recall its construction (see *e.g.* [10]). Suppose we are given the number of vertices,  $N$ , and i.i.d. random variables  $d_1, \dots, d_N$  representing the degrees of each vertex. To the vertex  $i$  are associated  $d_i$  half-edges. To construct an edge, one chooses two open half-edges at random and pair them together uniformly at random.

Remark that this linkage procedure does not exclude self-loops or multiple edges. In the following, we are interested in a large number of nodes with a fixed degree distribution, hence self-loops and multiple edges become less and less apparent in the global picture (see *e.g.* [10, Theorem 3.1.2]).

Notice that the condition for the existence of a giant component is that the expectation of the size biased distribution is larger than 1:

$$\sum_{k \in \mathbb{N}} (k-1) \frac{k \mathbb{P}(d_1 = k)}{\sum_{k \in \mathbb{N}} k \mathbb{P}(d_1 = k)} > 1.$$

Intuitively, the chosen neighbours will in this case always have further half-edges to keep constructing the connected component (see [10, Section 3.2 p. 75] for details).

## 2.2 SIR epidemic on a CM graph

We now propagate an epidemic on a CM graph of size  $N$ . The disease can be transmitted from infectious nodes to neighboring susceptible nodes and removed nodes cannot be reinfected.

Suppose that at initial time, we are given a set of *susceptible* and *infectious* nodes together with their degrees. The graph of relationships between the individuals is in fact irrelevant for studying the propagation of the disease. The minimal information consists in the sizes of the classes S, I, R and the number of edges to the class S for every infectious or removed node. Thus, each node of class S comes with a given number of half-edges of undertermined types ; each node of class I (resp. R) comes with a number of IS (resp. RS) edges. The number of IR, II and RR edges needs not to be retained.

The evolution of the SIR epidemic on a CM graph can be described as follows. To each IS-type half-edge is associated an independent exponential clock with parameter  $r$  and to each I vertex is associated an independent exponential clock with parameter  $\beta$ . The first of all these clocks that rings determines the next event.

**Case 1** If the clock that rings is associated to an I individual, the latter recovers. Change its status from I to R and the type of its emanating half-edges accordingly: IS half-edges become RS half-edges.

**Case 2** If the clock that rings is associated with a half IS-edge, an infection occurs.

**Step 1'** Match randomly the IS-half-edge that has rung to a half-edge belonging to a susceptible.

**Step 2'** This susceptible is the newly infected. Let  $k$ , its degree. Choose uniformly  $k - 1$  half-edges among all the available half-edges (they either are of type IS, RS, or emanate from s). Let  $m$  (resp.  $j$  and  $\ell$ ) be the number of SS-type (resp. of IS and of RS-type) half-edges drawn among these  $k - 1$  half-edges;

**Step 3'** Match those  $k - 1$  half-edges with the  $k - 1$  free half-edges of the newly infected. This fixes his/her  $k - 1$  other neighbors (additional to the I-vertex which contaminated him/her). Change the status of the  $m$  (resp.  $j$ ,  $\ell$ ) SS-type (resp. IS-type, RS-type) edges created to SI-type (resp. II-type, RI-type);

**Step 4'** Change the status of the newly infected from S to I. □

We then wait for another clock to ring and repeat the procedure.

We only need three descriptors of the system to obtain a Markovian evolution, namely the three degree distributions introduced in (1.1).

For a measure  $\mu \in \mathcal{M}_F(\mathbb{N})$ , we denote by  $F_\mu(m) = \mu(\{0, \dots, m\})$ ,  $m \in \mathbb{N}$ , its cumulative distribution function. We introduce  $F_\mu^{-1}$  its right inverse (see Appendix A). Then, for all  $0 \leq i \leq S_t$  (resp.  $0 \leq i \leq I_t$  and  $0 \leq i \leq R_t$ ),

$$\gamma_i(\mu_t^S) = F_{\mu_t^S}^{-1}(i), \quad \left( \text{resp. } \gamma_i(\mu_t^{IS}) = F_{\mu_t^{IS}}^{-1}(i), \quad \gamma_i(\mu_t^{RS}) = F_{\mu_t^{RS}}^{-1}(i) \right)$$

represents the degree at  $t$  of the  $i^{\text{th}}$  susceptible individual (resp. the number of edges to S of the  $i^{\text{th}}$  infectious individual and of the  $i^{\text{th}}$  removed individual) when individuals are ranked by increasing degrees (resp. by number of edges to S and to R).

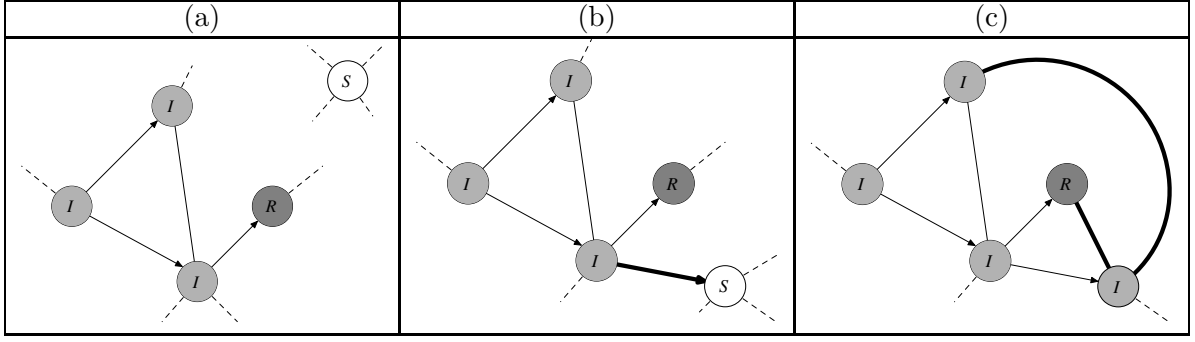


Figure 1: *Infection process.* Arrows provide the infection tree. Susceptible, infectious and removed individuals are colored in white, grey and dark grey respectively. (a) The degree of each individual is known, and for each infectious (resp. removed) individual, we know his/her number of edges of type IS (resp. RS). (b) A contaminating half-edge is chosen, and say that a susceptible of degree  $k$  is infected at time  $t$ . The contaminating edge is drawn in bold line. (c) Once the susceptible individual has been infected, we determine how many of its remaining arrows are linked to the classes I and R. If we denote by  $j$  and  $\ell$  these numbers, then  $N_t^{\text{IS}} = N_{t-}^{\text{IS}} - 1 + (k - 1) - j - \ell$  and  $N_t^{\text{RS}} = N_{t-}^{\text{RS}} - \ell$ .

**Example 1.** Consider for instance the measure

$$\mu = 2\delta_1 + 3\delta_5 + \delta_7.$$

Then, the atoms 1 and 2 are at level 1, the atoms 3, 4 and 5 are at level 5, and the atom 6 is at level 7. We then have that

$$\gamma_1(\mu) = F_\mu^{-1}(1) = 1, \quad \gamma_2(\mu) = 1, \quad \gamma_3(\mu) = \gamma_4(\mu) = \gamma_5(\mu) = 5, \quad \text{and} \quad \gamma_6(\mu) = 7.$$

□

From  $t$ , and because of the properties of exponential distributions the next event will take place in a time exponentially distributed with parameter  $rN_t^{\text{IS}} + \beta I_t$ . Let  $T$  denote the time of this event.

**Case 1** The event corresponds to a removal, i.e., a node goes from status I to status R. Choose uniformly an integer  $i$  in  $N_{T-}^{\text{IS}}$ , then update the measures  $\mu_{T-}^{\text{IS}}$  and  $\mu_{T-}^{\text{RS}}$ :

$$\mu_T^{\text{IS}} = \mu_{T-}^{\text{IS}} - \delta_{\gamma_i(\mu_{T-}^{\text{IS}})} \quad \text{and} \quad \mu_T^{\text{RS}} = \mu_{T-}^{\text{RS}} + \delta_{\gamma_i(\mu_{T-}^{\text{IS}})}.$$

The probability that a given integer  $i$  is drawn is  $1/I_{T-}$ .

**Case 2** The event corresponds to a new infection. We choose uniformly a half-edge with susceptible ego, and this ego becomes infectious. The global rate of infection is  $rN_{T-}^{\text{IS}}$  and the probability of choosing a susceptible individual of degree  $k$  for the new infectious is  $k\mu_{T-}^{\text{S}}/N_{T-}^{\text{S}}$ . The newly infective may have several links with infectious or removed individuals. The probability, given that the degree of the individual is  $k$ , that  $j$  (resp.  $\ell$ ) out of its  $k - 1$  other half-edges (all but the contaminating IS edge) are chosen to be of type II (resp. IR), according to Step 2', is given by the following multivariate hypergeometric distribution:

$$p_{T-}(j, \ell | k - 1) = \frac{\binom{N_{T-}^{\text{IS}} - 1}{j} \binom{N_{T-}^{\text{RS}}}{\ell} \binom{N_{T-}^{\text{S}} - N_{T-}^{\text{IS}} - N_{T-}^{\text{RS}}}{k - 1 - j - \ell}}{\binom{N_{T-}^{\text{S}} - 1}{k - 1}}. \quad (2.1)$$

Finally, to update the values of  $\mu_T^{\text{IS}}$  and  $\mu_T^{\text{RS}}$  given  $k, j$  and  $\ell$ , we have to choose the infectious and removed individuals to which the newly infectious is linked: some of their edges, which were IS or RS, now become II or RI. We draw two sequences  $u = (u_1, \dots, u_{I_{T_-}})$  and  $v = (v_1, \dots, v_{R_{T_-}})$  that will indicate how many links each infectious or removed individual has to the new contaminated individual. The probability of the draw us given by

$$\begin{aligned}\rho(u|j+1, \mu_{T_-}^{\text{IS}}) &= \frac{\prod_{i=1}^{I_{T_-}} \binom{\gamma_i(\mu_{T_-}^{\text{IS}})}{u_i}}{\binom{N_{T_-}^{\text{IS}}}{j+1}} \mathbb{1}_{u \in \mathcal{U}(j+1, \mu_{T_-}^{\text{IS}})} \\ \rho(v|\ell, \mu_{T_-}^{\text{RS}}) &= \frac{\prod_{i=1}^{R_{T_-}} \binom{\gamma_i(\mu_{T_-}^{\text{RS}})}{v_i}}{\binom{N_{T_-}^{\text{RS}}}{\ell}} \mathbb{1}_{v \in \mathcal{U}(\ell, \mu_{T_-}^{\text{RS}})}.\end{aligned}\quad (2.2)$$

Thus,  $u$  and  $v$  provide respectively the number of edges IS which become II per infectious individual and the number of edges RS which become RI per removed individual. We then update the measures as follows.

$$\begin{aligned}\mu_T^{\text{S}} &= \mu_{T_-}^{\text{S}} - \delta_k \\ \mu_T^{\text{IS}} &= \mu_{T_-}^{\text{IS}} + \delta_{k-1-j-\ell} + \sum_{i=1}^{I_{T_-}} \delta_{\gamma_i(\mu_{T_-}^{\text{IS}}) - u_i} - \delta_{\gamma_i(\mu_{T_-}^{\text{IS}})} \\ \mu_T^{\text{RS}} &= \mu_{T_-}^{\text{RS}} + \sum_{i'=1}^{R_{T_-}} \delta_{\gamma_{i'}(\mu_{T_-}^{\text{RS}}) - v_{i'}} - \delta_{\gamma_{i'}(\mu_{T_-}^{\text{RS}})}.\end{aligned}\quad (2.3)$$

Notice that in (2.3), there exists constraints on  $u$  and  $v$ : for all  $i$ , the number of edges that should be removed can not exceed the number of existing edges. Let us define the set

$$\mathcal{U} = \bigcup_{m=1}^{+\infty} \mathbb{N}^m, \quad (2.4)$$

and for all finite integer-valued measure  $\mu$  on  $\mathbb{N}$ , and all integer  $n \in \mathbb{N}$ , we define the subset

$$\begin{aligned}\mathcal{U}(n, \mu) &= \left\{ u = (u_1, \dots, u_{\langle \mu, \mathbf{1} \rangle}) \in \mathcal{U} \quad \text{such that} \right. \\ &\quad \left. \forall i \in \{1, \dots, \langle \mu, \mathbf{1} \rangle\}, u_i \leq F_\mu^{-1}(i) \text{ and } \sum_{i=1}^{\langle \mu, \mathbf{1} \rangle} u_i = n \right\}.\end{aligned}\quad (2.5)$$

A sequence  $u \in \mathcal{U}(n, \mu)$  describes the connections of a degree  $n$  individual in a population where the degree distribution of the other individuals is given by  $\mu$ . The component  $u_i$ , for  $1 \leq i \leq \langle \mu, \mathbf{1} \rangle$ , provides the number of edges that this individual shares with the  $i$ th individual. This number is necessarily smaller than the degree  $\gamma_i(\mu) = F_\mu^{-1}(i)$  of individual  $i$ . Moreover, the  $u_i$ 's sum to  $n$ , which is the degree of the considered individual. Then in (2.3),  $u$  should belong to  $\mathcal{U}(j+1, \mu_{T_-}^{\text{IS}})$  and  $v$  to  $\mathcal{U}(\ell, \mu_{T_-}^{\text{RS}})$ . The probability of a given draw is given in (2.2).

### 2.3 Stochastic differential equations

Here, we propose SDEs driven by PPMs to describe the evolution of the degree distributions (1.1), following the inspiration of [13, 8]. Let us consider the two following PPMs:  $dQ^1(s, k, \theta_1, j, \ell, \theta_2, u, \theta_3, v, \theta_4)$  and  $dQ^2(s, i)$  on  $\mathbb{R}_+ \times E_1$  with  $E_1 := \mathbb{N} \times \mathbb{R}_+ \times \mathbb{N} \times \mathbb{N} \times \mathbb{R}_+ \times \mathcal{U} \times \mathbb{R}_+ \times \mathcal{U} \times \mathbb{R}_+$  and  $\mathbb{R}_+ \times \mathbb{N}$  with intensity measures  $dq^1(s, k, \theta_1, j, \ell, \theta_2, u, \theta_3, v, \theta_4) = ds \otimes$



$dn(k) \otimes d\theta_1 \otimes dn(j) \otimes dn(\ell) \otimes d\theta_2 \otimes dn(u) \otimes d\theta_3 \otimes dn(v) \otimes d\theta_4$  and  $dq^2(s, i) = \beta ds \otimes dn(i)$ , where  $ds$ ,  $d\theta_1$ ,  $d\theta_2$ ,  $d\theta_3$  and  $d\theta_4$  are Lebesgue measures on  $\mathbb{R}_+$ , where  $dn(k)$ ,  $dn(j)$ ,  $dn(\ell)$  are counting measures on  $\mathbb{N}$ , and where  $dn(u)$ ,  $dn(v)$  are counting measures on  $\mathcal{U}$ .

The point measure  $Q^1$  provides the possible times at which an infection may occur. Each atom of this measure is associated with a possible infection time  $s$ , an integer  $k$  which gives the degree of the susceptible being possibly infected, the number  $j+1$  and  $\ell$  of edges that this individual has to the sets  $I_{s-}$  and  $R_{s-}$ . The marks  $u$  and  $v \in \mathcal{U}$  are as in the previous section. The marks  $\theta_1$ ,  $\theta_2$  and  $\theta_3$  are auxiliary variables used for the construction (see (2.7)-(2.8)). The point measure  $Q^2$  gives possible removal times. To each of its atoms is associated a possible removal time  $s$  and the number  $i$  of the individual that may be removed.

The following SDEs describe the evolution of the epidemic: for all  $t \geq 0$ ,

$$\mu_t^S = \mu_0^S - \int_0^t \int_{E_1} \delta_k \mathbb{1}_{\theta_1 \leq \lambda_{s-}(k)} \mu_{s-}^S(k) \mathbb{1}_{\theta_2 \leq p_{s-}(j, \ell | k-1)} \mathbb{1}_{\theta_3 \leq \rho(u|j+1, \mu_{s-}^{IS})} \mathbb{1}_{\theta_4 \leq \rho(v|\ell, \mu_{s-}^{RS})} dQ^1 \quad (2.6)$$

$$\begin{aligned} \mu_t^{IS} = \mu_0^{IS} + \int_0^t \int_{E_1} & \left( \delta_{k-(j+1+\ell)} + \sum_{i=1}^{I_{s-}} (\delta_{\gamma_i(\mu_{s-}^{IS}) - u_i} - \delta_{\gamma_i(\mu_{s-}^{IS})}) \right) \\ & \times \mathbb{1}_{\theta_1 \leq \lambda_{s-}(k)} \mu_{s-}^S(k) \mathbb{1}_{\theta_2 \leq p_{s-}(j, \ell | k-1)} \mathbb{1}_{\theta_3 \leq \rho(u|j+1, \mu_{s-}^{IS})} \mathbb{1}_{\theta_4 \leq \rho(v|\ell, \mu_{s-}^{RS})} dQ^1 \\ & - \int_0^t \int_{\mathbb{N}} \delta_{\gamma_i(\mu_{s-}^{IS})} \mathbb{1}_{i \leq I_{s-}} dQ^2 \end{aligned} \quad (2.7)$$

$$\begin{aligned} \mu_t^{RS} = \int_0^t \int_{E_1} & \left( \sum_{i=1}^{R_{s-}} (\delta_{\gamma_i(\mu_{s-}^{RS}) - v_i} - \delta_{\gamma_i(\mu_{s-}^{RS})}) \right) \\ & \times \mathbb{1}_{\theta_1 \leq \lambda_{s-}(k)} \mu_{s-}^S(k) \mathbb{1}_{\theta_2 \leq p_{s-}(j, \ell | k-1)} \mathbb{1}_{\theta_3 \leq \rho(u|j+1, \mu_{s-}^{IS})} \mathbb{1}_{\theta_4 \leq \rho(v|\ell, \mu_{s-}^{RS})} dQ^1 \\ & + \int_0^t \int_{\mathbb{N}} \delta_{\gamma_i(\mu_{s-}^{RS})} \mathbb{1}_{i \leq I_{s-}} dQ^2, \end{aligned} \quad (2.8)$$

where we have written  $dQ^1$  and  $dQ^2$  instead of  $dQ^1(s, k, \theta_1, j, \ell, \theta_2, u, \theta_3, v, \theta_3)$  and  $dQ^2(s, i)$  to simplify the notation, and where we denote for all  $s \geq 0$

$$\lambda_s(k) = rk \frac{N_s^{IS}}{N_s^S}, \quad \text{and} \quad p_s(j, \ell | k-1) = \frac{\binom{N_s^{IS}-1}{j} \binom{N_s^{RS}}{\ell} \binom{N_s^S - N_s^{IS} - N_s^{RS}}{k-1-j-\ell}}{\binom{N_s^S-1}{k-1}}, \quad (2.9)$$

Notice that  $\lambda_s(k)$  is the rate of infection of a *given* susceptible of degree  $k$  at time  $s$ . This notation was also used in Volz [26].

**Proposition 2.1.** *For any given initial conditions  $\mu_0^S$ ,  $\mu_0^{SI}$  that are integer-valued measures on  $\mathbb{N}$  and for PPMs  $Q^1$  and  $Q^2$ , there exists a unique strong solution to the SDEs (2.6)-(2.8) in the space  $\mathcal{D}(\mathbb{R}_+, (\mathcal{M}_F(\mathbb{N}))^3)$ , the Skohorod space of càdlàg functions with values in  $(\mathcal{M}_F(\mathbb{N}))^3$ .*

*Proof.* For the proof, we notice that for every  $t \in \mathbb{R}_+$ , the measure  $\mu_t^S$  is dominated by  $\mu_0^S$  and the measures  $\mu_t^{IS}$  and  $\mu_t^{RS}$  have a mass bounded by  $\langle \mu_0^S + \mu_0^{IS}, 1 \rangle$  and a support included in  $[0, \max\{\max(\text{supp}(\mu_0^S)), \max(\text{supp}(\mu_0^{IS}))\}]$ . The result then follows the steps of [13] and [25] (Proposition 2.2.6). ■

## 2.4 Description of the epidemic

The course of the epidemic can be described by  $(S_t, I_t, R_t, N_t^S, N_t^{IS}, N_t^{RS})_{t \in \mathbb{R}_+}$ , which can be deduced from (2.7), (2.6) and (2.8). The following semi-martingale decompositions result from

standard stochastic calculus for jump processes and SDE driven by PPMs (e.g. [13, 14, 15]). For the sizes of the class S, I and R, choosing  $f \equiv 1$ , we have for all  $t \geq 0$ ,

$$\begin{aligned} S_t &= \langle \mu_t^S, \mathbf{1} \rangle = S_0 - \int_0^t \sum_{k \in \mathbb{N}} \mu_s^S(k) \lambda_s(k) \, ds + M_t^S, \\ I_t &= \langle \mu_t^{IS}, \mathbf{1} \rangle = I_0 + \int_0^t \left( \sum_{k \in \mathbb{N}} \mu_s^S(k) \lambda_s(k) - \beta I_s \right) \, ds + M_t^I, \\ R_t &= \langle \mu_t^{RS}, \mathbf{1} \rangle = \int_0^t \beta I_s \, ds + M_t^R, \end{aligned} \quad (2.10)$$

where  $M^S$ ,  $M^I$  and  $M^R$  are square-integrable martingales that can be written explicitly with the compensated PPMs of  $Q^1$  and  $Q^2$ , and with respective quadratic variation given for all  $t \geq 0$  by

$$\begin{aligned} \langle M^S \rangle_t &= \int_0^t \sum_{k \in \mathbb{N}} \left( \mu_s^S(k) \lambda_s(k) \right) \, ds, \\ \langle M^I \rangle_t &= \int_0^t \sum_{k \in \mathbb{N}} \left( \mu_s^S(k) \lambda_s(k) + \beta I_s \right) \, ds, \\ \langle M^R \rangle_t &= \int_0^t \beta I_s \, ds. \end{aligned}$$

For any  $t \geq 0$ , the numbers of edges of the different types satisfy

$$\begin{aligned} N_t^S &= \langle \mu_t^S, \chi \rangle = \sum_{i \in S_t} k \mu_t^S(k) = N_0^S - \int_0^t \sum_{k \in \mathbb{N}} \left( k \mu_s^S(k) \lambda_s(k) \right) \, ds + M_t^{N^S}, \\ N_t^{IS} &= N_0^{IS} + \int_0^t \left( \sum_{k \in \mathbb{N}} \lambda_s(k) \mu_s^S(k) \sum_{j, \ell \in \mathbb{N}} (k - 2(j + 1) - \ell) p_s(j, \ell | k - 1) \right) \, ds - \int_0^t \beta N_s^{IS} \, ds + M_t^{N^{IS}}, \\ N_t^{RS} &= - \int_0^t \left( \sum_{k \in \mathbb{N}} \lambda_s(k) \mu_s^S(k) \sum_{j, \ell \in \mathbb{N}} \ell p_s(j, \ell | k - 1) \right) \, ds + \int_0^t \beta N_s^{IS} \, ds + M_t^{N^{RS}}, \end{aligned} \quad (2.11)$$

where  $M^{N^S}$ ,  $M^{N^{IS}}$  and  $M^{N^{RS}}$  are square integrable martingales that can be written explicitly with the PPMs  $Q^1$  and  $Q^2$ , and having quadratic variations given for all  $t$  by

$$\begin{aligned} \langle M^{N^S} \rangle_t &= \int_0^t \sum_{k \in \mathbb{N}} \left( k^2 \mu_s^S(k) \lambda_s(k) \right) \, ds, \\ \langle M^{N^{IS}} \rangle_t &= \int_0^t \left( \sum_{k \in \mathbb{N}} \lambda_s(k) \mu_s^S(k) \sum_{j, \ell \in \mathbb{N}} (k - 2(j + 1) - \ell)^2 p_s(j, \ell | k - 1) + \beta \sum_{k \in \mathbb{N}} k^2 \mu_s^{IS}(k) \right) \, ds, \\ \langle M^{N^{RS}} \rangle_t &= \int_0^t \left( \sum_{k \in \mathbb{N}} \lambda_s(k) \mu_s^S(k) \sum_{j, \ell \in \mathbb{N}} \ell^2 p_s(j, \ell | k - 1) + \beta \sum_{k \in \mathbb{N}} k^2 \mu_s^{IS}(k) \right) \, ds. \end{aligned}$$

We finally address the following proportion, studied in Volz [26] and representing the proportion

of edges with susceptible ego and infectious alter among all edges started at a susceptible ego.

$$\begin{aligned}
p_t^I &= \frac{N_t^{\text{IS}}}{N_t^{\text{S}}} \\
&= \frac{N_0^{\text{IS}}}{N_0^{\text{S}}} - \int_0^t \int_{E_2} \frac{\gamma_i(\mu_{s_-}^{\text{IS}})}{N_{s_-}^{\text{S}}} \mathbb{1}_{i \leq I_{s_-}} \, dQ^2 \\
&\quad + \int_0^t \int_{E_1} \left( \frac{N_{s_-}^{\text{IS}} + k - 2(j+1) - \ell}{N_{s_-}^{\text{S}} - k} - \frac{N_{s_-}^{\text{IS}}}{N_{s_-}^{\text{S}}} \right) \\
&\quad \quad \times \mathbb{1}_{\theta_1 \leq \lambda_{s_-}(k) \mu_{s_-}^{\text{S}}(k)} \mathbb{1}_{\theta_2 \leq p_{s_-}(j, \ell | k-1)} \mathbb{1}_{\theta_3 \leq \rho(u|j+1, \mu_{s_-}^{\text{IS}})} \mathbb{1}_{\theta_4 \leq \rho(v|\ell, \mu_{s_-}^{\text{RS}})} \, dQ^1 \\
&= p_0^I + \int_0^t \left( \sum_{k \in \mathbb{N}} \sum_{j+\ell+1 \leq k} \left[ \frac{N_s^{\text{IS}} + k - 2(j+1) - \ell}{N_s^{\text{S}} - k} - p_s^I \right] \lambda_s(k) \mu_s^{\text{S}}(k) p_{s_-}(j, \ell | k-1) - \beta p_s^I \right) \, ds \\
&\quad + M_t^{p^I},
\end{aligned} \tag{2.12}$$

where  $M^{p^I}$  is a square integrable martingale with quadratic variation given by

$$\begin{aligned}
\langle M^{p^I} \rangle_t &= \int_0^t \left( \beta \frac{\langle \mu_s^{\text{IS}}, \chi^2 \rangle}{(N_s^{\text{S}})^2} \right. \\
&\quad \left. + \sum_{k \in \mathbb{N}} \sum_{j+\ell+1 \leq k} \left( \frac{N_s^{\text{IS}} + k - 2(j+1) - \ell}{N_s^{\text{S}} - k} - p_s^I \right)^2 \lambda_s(k) \mu_s^{\text{S}}(k) p_{s_-}(j, \ell | k-1) \right) \, ds.
\end{aligned}$$

### 3 Large graph limit

Following Volz [26], we aim at developing a parcimonious deterministic approximation to describe the epidemic dynamics when the population is large. Notice that in [26], the stochastic processes are not clearly written and the convergence of the SDEs to the 4 ODEs that Volz proposes is stated but not proved. We provide here mathematical proofs, which underline that  $\mu^{\text{S}}$ ,  $\mu^{\text{IS}}$  and  $\mu^{\text{RS}}$  are the three distributions at the core of the problem.

#### 3.1 Law of Large Numbers scaling

We consider sequences of measures  $(\mu^{n,\text{S}})_{n \in \mathbb{N}}$ ,  $(\mu^{n,\text{IS}})_{n \in \mathbb{N}}$  and  $(\mu^{n,\text{RS}})_{n \in \mathbb{N}}$  such that for any  $n \in \mathbb{N}^*$ ,  $\mu^{n,\text{S}}$ ,  $\mu^{n,\text{IS}}$  and  $\mu^{n,\text{RS}}$  satisfy (2.6)-(2.8) with initial conditions  $\mu_0^{n,\text{S}}$ ,  $\mu_0^{n,\text{IS}}$  and  $\mu_0^{n,\text{RS}}$ . We denote by  $S_t^n$ ,  $I_t^n$  and  $R_t^n$  the subclasses of susceptible, infectious or removed individuals at time  $t$ , and by  $N_t^{n,\text{S}}$ ,  $N_t^{n,\text{IS}}$  and  $N_t^{n,\text{RS}}$ , the number of edges with susceptible ego, infectious ego and susceptible alter, removed ego and susceptible alter. The number of vertices of each class are denoted  $I_t^n$ ,  $S_t^n$  and  $R_t^n$ . Without loss of generality we assume that  $\mu_0^{n,\text{RS}} = 0$  (see Remark 1 below). As a consequence, the total size of the population is finite and equal to  $S_0^n + I_0^n$ . Remark that the size of the population and the number of edges tend to infinity with  $n$ .

We scale the measures the following way. For any  $n \geq 0$ , we set

$$\mu_t^{(n),\text{IS}} = \frac{1}{n} \mu_t^{n,\text{IS}}$$

for all  $t \geq 0$  (and accordingly,  $\mu_t^{(n),\text{S}}$  and  $\mu_t^{(n),\text{RS}}$ ). Then, we denote

$$N_t^{(n),\text{IS}} = \langle \mu^{(n),\text{IS}}, \chi \rangle = \frac{1}{n} N_t^{n,\text{IS}}, \quad \text{and} \quad I_t^{(n)} = \langle \mu_t^{(n),\text{IS}}, \mathbf{1} \rangle = \frac{1}{n} I_t^n$$

and accordingly,  $N_t^{(n),\text{S}}$ ,  $N_t^{(n),\text{RS}}$ ,  $S_t^{(n)}$  and  $R_t^{(n)}$ .

We assume that the initial conditions satisfy

**Assumption 3.1.** *The sequences  $(\mu_0^{(n),s})_{n \in \mathbb{N}}$  and  $(\mu_0^{(n),is})_{n \in \mathbb{N}}$  converge to  $\bar{\mu}_0^s$  and  $\bar{\mu}_0^{is}$  in  $\mathcal{M}_F(\mathbb{N})$  embedded with the weak convergence topology, where  $\bar{\mu}_0^{is}$  is not the null measure  $\mathbf{0}$ .*

*Remark 1.* • Assumption 3.1 entails that the initial (susceptible and infectious) population size is of order  $n$ . In [9], we study the beginning of the epidemic and consider when the Assumption 3.1 occurs, in which case the deterministic approximation of Volz becomes valid.

- We acknowledge that by the time the population of infective individuals reaches a size of order  $\langle \bar{\mu}_0^{is}, \mathbf{1} \rangle n$ , positive removals have taken place with positive probability. We however start from the initial condition  $\bar{\mu}_0^{rs} = \mathbf{0}$ . This means that we do not consider initially removed individuals. Indeed, the latter have no influence in the dynamics for  $t \geq 0$ . One could nonetheless add initially removed individuals in the network, without changing its dynamics, only by changing  $\bar{\mu}_0^s$ .
- In case the distributions underlying the measures  $\mu_0^{n,s}$  and  $\mu_0^{n,is}$  do not depend on the total number of vertices (*e.g.* Poisson, power-laws or discrete exponential distributions), Assumption 3.1 can be viewed as a law of large numbers. When the distributions depend on the total number of vertices  $N$  (as in Erdős-Renyi graphs), there may be scalings under which Assumption 3.1 holds. For Erdős-Renyi graphs for instance, if the probability  $p_N$  of connecting two vertices satisfies  $\lim_{N \rightarrow +\infty} N p_N = \lambda$ , then we obtain in the limit a Poisson distribution with parameter  $\lambda$ .
- We denote by  $\bar{S}_0$  (resp.  $\bar{I}_0$ ) the mass of the measure  $\bar{\mu}_0^s$  (resp.  $\bar{\mu}_0^{is}$ ). Notice that  $\bar{\mu}_0^s / \bar{S}_0$  (resp.  $\bar{\mu}_0^{is} / \bar{I}_0$ ) is the probability degree distribution of the susceptible individuals at time zero (resp. of the degree of the infectious individuals towards the susceptible ones).
- In Equation (1.3), notice the appearance of the degree biased probability measure  $k \bar{\mu}_s^s(k) / N_s^s$ . The latter reflects the fact that, in the CM, individuals having large degrees have higher probability to connect than individuals having small degrees. Thus, there is no reason why the degree distributions of the susceptible individuals  $\bar{\mu}_0^s / \langle \bar{\mu}_0^s, \mathbf{1} \rangle$  and the distribution  $\sum_{k \in \mathbb{N}} p_k \delta_k$  underlying the CM should coincide. Assumption 3.1 tells us indeed that the initial infectious population size is of order  $n$ . Even if  $\langle \bar{\mu}_0^{is}, \mathbf{1} \rangle / \langle \bar{\mu}_0^s, \mathbf{1} \rangle$  is very small, the biased distributions that appear imply that the degree distribution  $\bar{\mu}_0^{is} / \langle \bar{\mu}_0^{is}, \mathbf{1} \rangle$  should have a larger expectation than the degree distribution  $\bar{\mu}_0^s / \langle \bar{\mu}_0^s, \mathbf{1} \rangle$ .  $\square$

The rescaled SDEs corresponding to (2.6)-(2.8) then read for all  $t \geq 0$

$$\begin{aligned} \mu_t^{(n),S} = & \mu_0^{(n),S} - \frac{1}{n} \int_0^t \int_{E_1} \delta_k \mathbb{1}_{\theta_1 \leq \lambda_{s_-}^n(k) n \mu_{s_-}^{(n),S}(k)} \mathbb{1}_{\theta_2 \leq p_{s_-}^n(j, \ell | k-1)} \\ & \times \mathbb{1}_{\theta_3 \leq \rho(u|j+1, n \mu_{s_-}^{(n),IS})} \mathbb{1}_{\theta_4 \leq \rho(v|\ell, n \mu_{s_-}^{(n),RS})} dQ^1, \end{aligned} \quad (3.1)$$

$$\begin{aligned} \mu_t^{(n),IS} = & \mu_0^{(n),IS} + \frac{1}{n} \int_0^t \int_{E_1} \left( \delta_{k-(j+1+\ell)} + \sum_{i=1}^{I_{s_-}^n} (\delta_{\gamma_i(n \mu_{s_-}^{(n),IS)} - u_i} - \delta_{\gamma_i(n \mu_{s_-}^{(n),IS})}) \right) \\ & \times \mathbb{1}_{\theta_1 \leq \lambda_{s_-}^n(k) n \mu_{s_-}^{(n),S}(k)} \mathbb{1}_{\theta_2 \leq p_{s_-}^n(j, \ell | k-1)} \mathbb{1}_{\theta_3 \leq \rho(u|j+1, n \mu_{s_-}^{(n),IS})} \mathbb{1}_{\theta_4 \leq \rho(v|\ell, n \mu_{s_-}^{(n),RS})} dQ^1 \\ & - \frac{1}{n} \int_0^t \int_{\mathbb{N}} \delta_{\gamma_i(n \mu_{s_-}^{(n),IS})} \mathbb{1}_{i \in I_{s_-}^n} dQ^2, \end{aligned} \quad (3.2)$$

$$\begin{aligned} \mu_t^{(n),RS} = & \frac{1}{n} \int_0^t \int_{E_1} \left( \sum_{i=1}^{R_{s_-}^n} (\delta_{\gamma_i(n \mu_{s_-}^{(n),RS)} - v_i} - \delta_{\gamma_i(n \mu_{s_-}^{(n),RS})}) \right) \\ & \times \mathbb{1}_{\theta_1 \leq \lambda_{s_-}^n(k) n \mu_{s_-}^{(n),S}(k)} \mathbb{1}_{\theta_2 \leq p_{s_-}^n(j, \ell | k-1)} \mathbb{1}_{\theta_3 \leq \rho(u|j+1, n \mu_{s_-}^{(n),IS})} \mathbb{1}_{\theta_4 \leq \rho(v|\ell, n \mu_{s_-}^{(n),RS})} dQ^1 \\ & + \frac{1}{n} \int_0^t \int_{\mathbb{N}} \delta_{\gamma_i(n \mu_{s_-}^{(n),IS})} \mathbb{1}_{i \in I_{s_-}^n} dQ^2, \end{aligned} \quad (3.3)$$

where we denote for all  $s \geq 0$

$$\lambda_s^n(k) = rk \frac{N_s^{n,IS}}{N_s^{n,S}}, \quad \text{and} \quad p_s^n(j, \ell | k-1) = \frac{\binom{N_s^{n,IS}-1}{j} \binom{N_s^{n,RS}}{\ell} \binom{N_s^{n,S}-N_s^{n,IS}-N_s^{n,RS}}{k-1-j-\ell}}{\binom{N_s^{n,S}-1}{k-1}}. \quad (3.4)$$

Several semi-martingale decompositions will be useful in the sequel. We focus on  $\mu^{(n),IS}$  but similar decompositions hold for  $\mu^{(n),S}$  and  $\mu^{(n),RS}$ , which we do not detail since they can be deduced by direct adaptation of the following computation.

**Proposition 3.2.** *For all  $f \in \mathcal{B}_b(\mathbb{N})$ , for all  $t \geq 0$ ,*

$$\langle \mu_t^{(n),IS}, f \rangle = \sum_{k \in \mathbb{N}} f(k) \mu_0^{(n),IS}(k) + A_t^{(n),IS,f} + M_t^{(n),IS,f}, \quad (3.5)$$

where the finite variation part  $A_t^{(n),IS,f}$  of  $\langle \mu_t^{(n),IS}, f \rangle$  reads

$$\begin{aligned} A_t^{(n),IS,f} = & \int_0^t \sum_{k \in \mathbb{N}} \lambda_s^n(k) \mu_s^{(n),S}(k) \sum_{j+\ell+1 \leq k} p_s^n(j, \ell | k-1) \\ & \times \left( f(k - (j+1+\ell)) + \sum_{u \in \mathcal{U}} \rho(u|j+1, \mu_s^{n,IS}) \sum_{i=1}^{I_s^n} (f(\gamma_i(\mu_s^{n,IS}) - u_i) - f(\gamma_i(\mu_s^{n,IS}))) \right) ds, \end{aligned} \quad (3.6)$$

and where the martingale part  $M_t^{(n),IS,f}$  of  $\langle \mu_t^{(n),IS}, f \rangle$  is a square integrable martingale starting from 0 with quadratic variation

$$\begin{aligned} \langle M^{(n),IS,f} \rangle_t = & \frac{1}{n} \int_0^t \sum_{k \in \mathbb{N}} \lambda_s^n(k) \mu_s^{(n),S}(k) \sum_{j+\ell+1 \leq k} p_s^n(j, \ell | k-1) \sum_{u \in \mathcal{U}} \rho(u|j+1, \mu_s^{n,IS}) \\ & \times \left( f(k - (j+1+\ell)) + \sum_{i=1}^{I_s^n} (f(\gamma_i(\mu_s^{n,IS}) - u_i) - f(\gamma_i(\mu_s^{n,IS}))) \right)^2 ds. \end{aligned}$$

*Proof.* The proof proceeds from (3.2) and standard stochastic calculus for jump processes (see e.g. [13]).  $\blacksquare$

### 3.2 Convergence of the normalized process

We aim to study the limit of the system when  $n \rightarrow +\infty$ . For any  $\varepsilon > 0$  and any  $A > 0$ , we define the following closed set of  $\mathcal{M}_F(\mathbb{N})$  as

$$\mathcal{M}_{\varepsilon, A} = \{\nu \in \mathcal{M}_F(\mathbb{N}) ; \langle \nu, \mathbf{1} + \chi^5 \rangle \leq A \text{ and } \langle \nu, \chi \rangle \geq \varepsilon\}. \quad (3.7)$$

Topological properties of these spaces are given in Appendix A.

Our main result is the following Theorem.

**Theorem 1.** *Assume that there exists  $0 < \varepsilon < A$  such that*

$$\left( \bar{\mu}_0^{s,(n)}, \bar{\mu}_0^{\text{IS},(n)}, \bar{\mu}_0^{\text{RS},(n)} \right) \text{ belongs to } \mathcal{M}_{0, A} \times \mathcal{M}_{2\varepsilon, A} \times \mathcal{M}_{0, A} \text{ for any } n. \quad (3.8)$$

*Assume also that Assumption 3.1 holds. Then,*

1. *there exists  $T_\varepsilon > 0$  and a solution  $(\bar{\mu}^s, \bar{\mu}^{\text{IS}}, \bar{\mu}^{\text{RS}})$  to the deterministic system of measure-valued equations (1.2)-(1.4) for  $t \in [0, T_\varepsilon]$ .*
2. *this solution is unique,*
3. *the sequence  $(\mu^{(n),s}, \mu^{(n),\text{IS}}, \mu^{(n),\text{RS}})_{n \in \mathbb{N}}$  converges in distribution in  $\mathcal{D}([0, T_\varepsilon], \mathcal{M}_{0, A}^3)$  to the solution  $(\bar{\mu}^s, \bar{\mu}^{\text{IS}}, \bar{\mu}^{\text{RS}})$  of (1.2)-(1.4) when  $n$  tends to infinity.*

Let us define, for all  $\delta > 0$ ,

$$t_\delta := \inf\{t \geq 0, \langle \bar{\mu}_t^{\text{IS}}, \chi \rangle < \delta\} \quad (3.9)$$

and the following stopping times for  $n \in \mathbb{N}^*$ :

$$\tau_\delta^n = \inf\{t \geq 0, \langle \mu_t^{(n),\text{IS}}, \chi \rangle < \delta\}. \quad (3.10)$$

We start by providing a lower bound for the horizon  $T_\varepsilon$  of convergence in Theorem 1.

**Proposition 3.3.** *Under the Assumptions of Theorem 1,*

$$T_\varepsilon \geq \bar{\tau}_\varepsilon := \frac{\log(\langle \bar{\mu}_0^s, \chi^2 \rangle + \bar{N}_0^{\text{IS}}) - \log(\langle \bar{\mu}_0^s, \chi^2 \rangle + 2\varepsilon)}{\max(\beta, r)}. \quad (3.11)$$

*Proof of Proposition 3.3.* In the course of Step 4 of the proof of Theorem 1, we show in fact that the convergence holds at least on  $[0, t_{2\varepsilon}]$ . So it suffices to show that  $t_{2\varepsilon} \geq \bar{\tau}_\varepsilon$ . In that purpose, fix  $\xi > 0$  and  $n \in \mathbb{N}$ . Define the following event.

$$\mathcal{A}_\xi^n := \left\{ \bigcap_{\alpha=1}^{+\infty} \{ |\mu_0^{n,s}(\alpha) - n\bar{\mu}_0^s(\alpha)| \leq \xi \} \cap \{ |\mu_0^{n,\text{IS}}(\alpha) - n\bar{\mu}_0^{\text{IS}}(\alpha)| \leq \xi \} \right\} \\ \cap \{ |N_0^{n,\text{IS}} - n\langle \bar{\mu}_0^{\text{IS}}, \chi \rangle| \leq \xi \} \cap \{ |I_0^n - n\langle \bar{\mu}_0^{\text{IS}}, \mathbf{1} \rangle| \leq \xi \}.$$

When a susceptible individual of degree  $\alpha$  is contaminated, at most  $\alpha$  IS-edges are removed from the graph (the maximal number of infectious neighbors of the newly infected individual). When an infectious individual of degree  $\alpha$  is removed, at most  $\alpha$  IS-edges are removed from the graph (the maximal number of susceptible neighbors of the newly removed individual). Consequently, denoting by  $Z_t^n$  the number of IS-edges removed from the graph up to  $t$ , we have that

$$Z_t^n \leq \sum_{\alpha \in \mathbb{N}} \alpha (X_t^{n,\alpha} + Y_t^{n,\alpha}), \quad (3.12)$$

where  $X_t^{n,\alpha}$  denotes the number of edges with alter of initial degree  $\alpha$  having transmitted the disease up to  $t$  and  $Y_t^{n,\alpha}$  denotes the number of I-vertices initially of degree  $\alpha$  having recovered up to  $t$ . Denote  $N_0^{n,\text{IS},\alpha}$ , the number of edges initially starting from an infectious ego and leading to a susceptible alter having degree  $\alpha$ . Notice, that  $N_0^{n,\text{IS},\alpha}$  is less than the number of edges initially leading to susceptible individuals of degree  $\alpha$ , thus for all  $t \geq 0$ ,

$$X_t^{n,\alpha} \leq N_0^{n,\text{IS},\alpha} \leq \alpha \mu_0^{n,\text{S}}(\alpha). \quad (3.13)$$

On another hand,

$$Y_t^{n,\alpha} \leq \mu_0^{n,\text{IS}}(\alpha), \quad (3.14)$$

so that

$$\sup_{n \in \mathbb{N}} Z_t^n \leq \sup_{n \in \mathbb{N}} \left\{ \sum_{\alpha \in \mathbb{N}} \alpha (\alpha \mu_0^{n,\text{S}}(\alpha) + \mu_0^{n,\text{IS}}(\alpha)) \right\} \leq \sup_{n \in \mathbb{N}} \{ \langle \mu_0^{n,\text{S}}, \chi^2 \rangle + \langle \mu_0^{n,\text{IS}}, \chi \rangle \} < +\infty, \quad (3.15)$$

in view of (3.8).

Now, a given IS-edge transmits the disease before  $t$  provided its exponential clock rings (this happens with probability  $1 - e^{-rt}$ ). Consequently, for all  $\alpha$ ,  $X_t^{n,\alpha}$  is stochastically dominated by the sum of  $N_0^{n,\text{IS},\alpha}$  independent r.v.'s, independent of  $N_0^{n,\text{IS},\alpha}$ , having Bernoulli distribution of probability  $1 - e^{-rt}$ . On  $\mathcal{A}_\xi^n$ , in view of (3.13), the latter is in turn stochastically dominated by a Binomial r.v.  $\tilde{X}_t^{n,\alpha}$  with parameters  $\alpha \lfloor n \bar{\mu}_0^{\text{S}}(\alpha) + \xi \rfloor$  and  $1 - e^{-rt}$ . With an abuse of terminology, we say that a real r.v.  $Y$  stochastically dominates another r.v.  $X$  on an event  $\mathcal{A}$  whenever for all  $x \in \mathbb{R}$ ,

$$\mathbb{P}(\{Y \leq x\} \cap \mathcal{A}) \leq \mathbb{P}(\{X \leq x\} \cap \mathcal{A}).$$

All the same, from (3.14), on  $\mathcal{A}_\xi^n$ ,  $Y_t^n$  is stochastically upper bounded by a Binomial r.v.  $\tilde{Y}_t^n$  with parameters  $\lfloor n \bar{\mu}_0^{\text{IS}}(\alpha) + \xi \rfloor$  and  $1 - e^{-\beta t}$ , and independant of  $\tilde{X}_t^n$ .

Therefore, recalling the definition (3.10), we have for all  $t \geq 0$

$$\begin{aligned} \mathbb{P}(\{\tau_{2\varepsilon}^n \leq t\} \cap \mathcal{A}_\xi^n) &= \mathbb{P}(\{Z_t^n > N_0^{n,\text{IS}} - 2n\varepsilon\} \cap \mathcal{A}_\xi^n) \\ &\leq \mathbb{P}(\{Z_t^n > n\bar{N}_0^{\text{IS}} - \xi - 2n\varepsilon\} \cap \mathcal{A}_\xi^n) \\ &\leq \mathbb{P}\left(\left\{\sum_{\alpha \in \mathbb{N}} \alpha (\tilde{X}_t^{n,\alpha} + \tilde{Y}_t^{n,\alpha}) > n\bar{N}_0^{\text{IS}} - \xi - 2n\varepsilon\right\} \cap \mathcal{A}_\xi^n\right) \\ &\leq \mathbb{P}\left(\sum_{\alpha \in \mathbb{N}} \alpha (\tilde{X}_t^{n,\alpha} + \tilde{Y}_t^{n,\alpha}) \geq n(\bar{N}_0^{\text{IS}} - 2\varepsilon) - \xi\right) \\ &\leq \mathbb{P}\left(\sum_{\alpha \in \mathbb{N}} \alpha \tilde{Z}_t^{n,\alpha} \geq n(\bar{N}_0^{\text{IS}} - 2\varepsilon) - \xi\right), \end{aligned} \quad (3.16)$$

where the fourth inequality is true for a large enough  $n$ , and  $\tilde{Z}_t^{n,\alpha}$  is a Binomial r.v. with parameters  $\alpha \lfloor n \bar{\mu}_0^{\text{S}}(\alpha) + \xi \rfloor + \lfloor n \bar{\mu}_0^{\text{IS}}(\alpha) + \xi \rfloor$  and  $1 - e^{-(\max(\beta, r))t}$ . The bound (3.15), and the Law

of Large Numbers applied to the r.h.s. of (3.16) yield the following.

$$\begin{aligned}
& \mathbb{P}\left(\sum_{\alpha \in \mathbb{N}} \alpha \tilde{Z}_t^{n,\alpha} \geq n(\bar{N}_0^{\text{IS}} - 2\varepsilon) - \xi\right) \\
& \stackrel{\sim}{n \rightarrow \infty} \mathbb{1}\left\{\sum_{\alpha \in \mathbb{N}} \alpha \left((1 - e^{-(\max(\beta, r))t}) (\alpha \lfloor n\bar{\mu}_0^{\text{S}}(\alpha) + \xi \rfloor + \lfloor n\bar{\mu}_0^{\text{IS}}(\alpha) + \xi \rfloor)\right) \geq n(\bar{N}_0^{\text{IS}} - 2\varepsilon) - \xi\right\} \\
& \leq \mathbb{1}\left\{(1 - e^{-(\max(\beta, r))t}) \sum_{\alpha \in \mathbb{N}} \alpha (n\alpha\bar{\mu}_0^{\text{S}}(\alpha) + \alpha\xi + n\bar{\mu}_0^{\text{IS}}(\alpha) + \xi) \geq n(\bar{N}_0^{\text{IS}} - 2\varepsilon) - \xi\right\} \\
& \stackrel{\sim}{n \rightarrow \infty} \mathbb{1}\left\{(1 - e^{-(\max(\beta, r))t}) \sum_{\alpha \in \mathbb{N}} \alpha (\alpha\bar{\mu}_0^{\text{S}}(\alpha) + \bar{\mu}_0^{\text{IS}}(\alpha)) \geq (\bar{N}_0^{\text{IS}} - 2\varepsilon)\right\} \\
& = \mathbb{1}\left\{1 - e^{-(\max(\beta, r))t} \geq \frac{(\bar{N}_0^{\text{IS}} - 2\varepsilon)}{\langle \bar{\mu}_0^{\text{S}}, \chi^2 \rangle + \bar{N}_0^{\text{IS}}}\right\} \\
& = \mathbb{1}_{\{t \geq \bar{\tau}_\varepsilon\}}, \tag{3.17}
\end{aligned}$$

where  $\bar{\tau}_\varepsilon$  is defined by (3.11). For all  $t < \bar{\tau}_\varepsilon$ , it thus follows from Assumption 3.1, together with (3.16) and (3.17) that

$$\lim_{n \rightarrow +\infty} \mathbb{P}(\tau_{2\varepsilon}^n \leq t) \leq \lim_{n \rightarrow +\infty} \left(\mathbb{P}(\{\tau_\varepsilon^n \leq t\} \cap \mathcal{A}_\xi^n) + \mathbb{P}(\overline{\mathcal{A}}_\xi^n)\right) = 0,$$

so that

$$\lim_{n \rightarrow +\infty} \mathbb{P}(\tau_{2\varepsilon}^n < \bar{\tau}_\varepsilon) = 0. \tag{3.18}$$

Finally, assuming that  $t_{2\varepsilon} < \bar{\tau}_\varepsilon$  would imply that for some  $\eta > 0$ ,

$$\inf_{t \in [0, \bar{\tau}_\varepsilon - \eta]} \bar{N}_t^{\text{IS}} < 2\varepsilon.$$

But it is easily seen given (3.18) that Theorem 1 holds on  $[0, \bar{\tau}_\varepsilon]$ . Therefore, from (ii) of Theorem 1, and applying the same argument as that leading to (3.42) below, we would have that

$$\begin{aligned}
1 &= \mathbb{1}_{\{\inf_{t \in [0, \bar{\tau}_\varepsilon - \eta]} \bar{N}_t^{\text{IS}} < 2\varepsilon\}} \leq \lim_{n \rightarrow \infty} \mathbb{P}\left(\inf_{t \in [0, \bar{\tau}_\varepsilon - \eta]} N_t^{(n), \text{IS}} < 2\varepsilon\right) \\
&\leq \lim_{n \rightarrow \infty} \mathbb{P}\left(\left\{\inf_{t \in [0, \bar{\tau}_\varepsilon - \eta]} N_t^{(n), \text{IS}} < 2\varepsilon\right\} \cap \{\tau_{2\varepsilon}^n \geq \bar{\tau}_\varepsilon\}\right) + \mathbb{P}(\tau_{2\varepsilon}^n < \bar{\tau}_\varepsilon) \xrightarrow{n \rightarrow \infty} 0,
\end{aligned}$$

from (3.18). This absurdity shows that  $t_{2\varepsilon} \geq \bar{\tau}_\varepsilon$ , which concludes the proof.  $\blacksquare$

*Proof of Theorem 1.* The scheme of the proof is as follows. We will prove the tightness of the sequence  $(\mu^{(n), \text{S}}, \mu^{(n), \text{IS}}, \mu^{(n), \text{RS}})_{n \in \mathbb{N}^*}$  in  $\mathcal{D}(\mathbb{R}_+, \mathcal{M}_{0,A}^3)$ . This implies the tightness of the sequence of stopped processes  $(\mu_{\cdot \wedge \tau_\varepsilon^n}^{(n), \text{S}}, \mu_{\cdot \wedge \tau_\varepsilon^n}^{(n), \text{IS}}, \mu_{\cdot \wedge \tau_\varepsilon^n}^{(n), \text{RS}})_{n \in \mathbb{N}^*}$ . We prove that the accumulation point of the latter sequence satisfy an evolution equation for which the solution is unique.

**Step 1** Let us prove that  $(\mu^{(n), \text{S}}, \mu^{(n), \text{IS}}, \mu^{(n), \text{RS}})_{n \in \mathbb{N}^*}$  is tight. Let  $t \in \mathbb{R}_+$  and  $n \in \mathbb{N}^*$ . By hypothesis, we have that

$$\langle \mu_t^{(n), \text{S}}, \mathbf{1} + \chi^5 \rangle + \langle \mu_t^{(n), \text{IS}}, \mathbf{1} + \chi^5 \rangle + \langle \mu_t^{(n), \text{RS}}, \mathbf{1} + \chi^5 \rangle \leq \langle \mu_0^{(n), \text{S}}, \mathbf{1} + \chi^5 \rangle + \langle \mu_0^{(n), \text{IS}}, \mathbf{1} + \chi^5 \rangle \leq A. \tag{3.19}$$

Thus the sequences  $(\mu_t^{(n), \text{S}})_{n \in \mathbb{N}^*}$ ,  $(\mu_t^{(n), \text{IS}})_{n \in \mathbb{N}^*}$  and  $(\mu_t^{(n), \text{RS}})_{n \in \mathbb{N}^*}$  are tight for each  $t \in \mathbb{R}_+$ . Now by the criterion of Roelly [24], it remains to prove that for each  $f \in \mathcal{B}_b(\mathbb{N})$ , the sequence  $(\langle \mu^{(n), \text{S}}, f \rangle, \langle \mu^{(n), \text{IS}}, f \rangle, \langle \mu^{(n), \text{RS}}, f \rangle)_{n \in \mathbb{N}^*}$  is tight in  $\mathcal{D}(\mathbb{R}_+, \mathbb{R}^3)$ . Since we have semi-martingale decompositions of these processes, it is sufficient, by using the Rebolledo criterion, to prove



that the finite variation part and the bracket of the martingale part satisfy the Aldous criterion (see *e.g.* [16]). For the sake of simplicity, we only prove that  $\langle \mu^{(n),\text{IS}}, f \rangle$  is tight. For the other components, the computations are similar to what follows.

The Rebolledo-Aldous criterion is satisfied if for all  $\alpha > 0$  and  $\eta > 0$  there exists  $n_0 \in \mathbb{N}$  and  $\delta > 0$  such that for all  $n > n_0$  and for all stopping times  $S_n$  and  $T_n$  such that  $S_n < T_n < S_n + \delta$ ,

$$\mathbb{P}(|A_{T_n}^{(n),\text{IS},f} - A_{S_n}^{(n),\text{IS},f}| > \eta) \leq \alpha, \quad \text{and} \quad \mathbb{P}(|\langle M^{(n),\text{IS},f} \rangle_{T_n} - \langle M^{(n),\text{IS},f} \rangle_{S_n}| > \eta) \leq \alpha. \quad (3.20)$$

For the finite variation part,

$$\mathbb{E} \left[ |A_{T_n}^{(n),\text{IS},f} - A_{S_n}^{(n),\text{IS},f}| \right] \leq \mathbb{E} \left[ \int_{S_n}^{T_n} \sum_{k \in \mathbb{N}} \lambda_s^n(k) \mu_s^{(n),\text{S}}(k) \sum_{j+\ell \leq k-1} p_s^n(j, \ell | k-1) (2j+1) \|f\|_\infty \, ds \right].$$

The term  $\sum_{j+\ell \leq k-1} j p_s^n(j, \ell | k-1)$  is the mean number of links to  $i_{s-}^n$  that the newly infected individual has, given that this individual is of degree  $k$ . It is bounded by  $k$ . Then, with (3.4),

$$\mathbb{E} \left[ |A_{T_n}^{(n),\text{IS},f} - A_{S_n}^{(n),\text{IS},f}| \right] \leq \mathbb{E} \left[ \int_{S_n}^{T_n} \sum_{k \in \mathbb{N}} r(2k^2 + k) \|f\|_\infty \mu_s^{(n),\text{S}}(k) \, ds \right] \leq \delta r \|f\|_\infty \langle \mu_0^{(n),\text{S}}, 2\chi^2 + \chi \rangle,$$

by using the fact that  $\mu_s^{(n),\text{S}}(k) \leq \mu_0^{(n),\text{S}}(k)$  for all  $k$  and  $s \geq 0$ . From (3.8), the r.h.s. is finite. Using Markov's inequality,

$$\mathbb{P}(|A_{T_n}^{(n),\text{IS},f} - A_{S_n}^{(n),\text{IS},f}| > \eta) \leq \frac{\delta r \|f\|_\infty \langle \mu_0^{(n),\text{S}}, 2\chi^2 + \chi \rangle}{\eta} \leq \frac{3A\delta r \|f\|_\infty}{\eta},$$

which is smaller than  $\alpha$  for  $\delta$  small enough.

We use the same arguments for the bracket of the martingale:

$$\begin{aligned} & \mathbb{E} \left[ |\langle M^{(n),\text{IS},f} \rangle_{T_n} - \langle M^{(n),\text{IS},f} \rangle_{S_n}| \right] \\ & \leq \frac{1}{n} \mathbb{E} \left( \int_{S_n}^{T_n} \sum_{k \in \mathbb{N}} \lambda_s^n(k) \sum_{j+\ell \leq k-1} (2j+3)^2 p_s^n(j, \ell | k-1) \|f\|_\infty^2 \mu_s^{(n),\text{S}}(k) \, ds \right) \\ & \leq \frac{1}{n} \mathbb{E} \left( \int_{S_n}^{T_n} \sum_{k \in \mathbb{N}} r k (2k+3)^2 \|f\|_\infty^2 \mu_s^{(n),\text{S}}(k) \, ds \right) \\ & \leq \frac{\delta r \|f\|_\infty^2 \langle \mu_0^{(n),\text{S}}, \chi(2\chi+3)^2 \rangle}{n} \\ & \leq \frac{25 A \delta r \|f\|_\infty^2}{n}, \end{aligned} \quad (3.21)$$

using Assumption (3.8). The r.h.s. can be made smaller than  $\eta\alpha$  for a small enough of  $\delta$ , so the second inequality of (3.20) follows again from Markov's inequality. By [24], this provides the tightness in  $\mathcal{D}(\mathbb{R}_+, \mathcal{M}_{0,A}^3)$ .

By Prohorov theorem (*e.g.* [11], p.104) and Step 1, the distributions of  $(\mu^{(n),\text{S}}, \mu^{(n),\text{IS}}, \mu^{(n),\text{RS}})$ , for  $n \in \mathbb{N}^*$ , form a relatively compact family of bounded measures on  $\mathcal{D}(\mathbb{R}_+, \mathcal{M}_{0,A}^3)$ , and so do the laws of the stopped processes  $(\mu_{\cdot \wedge \tau_\varepsilon^n}^{(n),\text{S}}, \mu_{\cdot \wedge \tau_\varepsilon^n}^{(n),\text{IS}}, \mu_{\cdot \wedge \tau_\varepsilon^n}^{(n),\text{RS}})_{n \in \mathbb{N}^*}$  (recall (3.10)). Let  $\bar{\mu} := (\bar{\mu}^{\text{S}}, \bar{\mu}^{\text{IS}}, \bar{\mu}^{\text{RS}})$  be an accumulation point in  $\mathcal{C}(\mathbb{R}_+, \mathcal{M}_{0,A}^3)$  and let us consider a subsequence again denoted by  $\mu^{(n)} := (\mu^{(n),\text{S}}, \mu^{(n),\text{IS}}, \mu^{(n),\text{RS}})_{n \in \mathbb{N}^*}$ , with an abuse of notation, and that converges to  $\bar{\mu}$ . Because the limiting values are continuous, the convergence of  $(\mu^{(n)})_{n \in \mathbb{N}^*}$  to  $\bar{\mu}$  holds for the uniform

convergence on every compact subset of  $\mathbb{R}_+$ , in particular on  $[0, t_\varepsilon - \eta]$  for every small  $\eta > 0$  (e.g. [6] p.112).

Now, let us define for all  $t \in \mathbb{R}_+$  and for all bounded function  $f$  on  $\mathbb{N}$ , the mappings  $\Psi_t^{S,f}$ ,  $\Psi_t^{IS,f}$  and  $\Psi_t^{RS,f}$  from  $\mathcal{D}(\mathbb{R}_+, \mathcal{M}_{0,A}^3)$  into  $\mathcal{D}(\mathbb{R}_+, \mathbb{R})$  such that (1.2)-(1.4) read

$$(\langle \bar{\mu}_t^S, f \rangle, \langle \bar{\mu}_t^{IS}, f \rangle, \langle \bar{\mu}_t^{RS}, f \rangle) = \left( \Psi_t^{S,f}(\bar{\mu}^S, \bar{\mu}^{IS}, \bar{\mu}^{RS}), \Psi_t^{IS,f}(\bar{\mu}^S, \bar{\mu}^{IS}, \bar{\mu}^{RS}), \Psi_t^{RS,f}(\bar{\mu}^S, \bar{\mu}^{IS}, \bar{\mu}^{RS}) \right). \quad (3.22)$$

Our purpose is to prove that the limiting values are the unique solution of Equations (1.2)-(1.4).

*Remark 2.* Before proceeding to the proof, a remark is in order. A natural way of reasoning would be to prove that  $\Psi^{S,f}$ ,  $\Psi^{IS,f}$  and  $\Psi^{RS,f}$  are Lipschitz continuous in some spaces of measures. It turns that this only can be done by considering the set of measures with moments of any order, which is too small a set for applications. We circumvent this difficulty by first proving that the mass and the first two moments of any solution of the system are the same. Once this is done, we prove that the generating functions of these measures satisfy a partial differential equation known to have a unique solution.

**Step 2** We now prove that for any  $T$ , the differential system (1.2)-(1.4) has at most one solution in  $\mathcal{C}([0, T], \mathcal{M}_{\varepsilon,A}^3)$ . Let  $\bar{\mu}^i = (\bar{\mu}^{S,i}, \bar{\mu}^{IS,i}, \bar{\mu}^{RS,i})$ ,  $i \in \{1, 2\}$  be two solutions of (1.2)-(1.4), started with the same initial conditions and driven by the same PPMs. Set

$$\Upsilon_t = \sum_{j=0}^2 \left( |\langle \bar{\mu}_t^{S,1}, \chi^j \rangle - \langle \bar{\mu}_t^{S,2}, \chi^j \rangle| + |\langle \bar{\mu}_t^{IS,1}, \chi^j \rangle - \langle \bar{\mu}_t^{IS,2}, \chi^j \rangle| + |\langle \bar{\mu}_t^{RS,1}, \chi^j \rangle - \langle \bar{\mu}_t^{RS,2}, \chi^j \rangle| \right).$$

Let us first remark that for all  $0 \leq t < T$ ,

$$\begin{aligned} |\bar{p}_t^{1,1} - \bar{p}_t^{1,2}| &= \left| \frac{\bar{N}_t^{IS,1}}{\bar{N}_t^{S,1}} - \frac{\bar{N}_t^{IS,2}}{\bar{N}_t^{S,2}} \right| \leq \frac{A}{\varepsilon^2} |\bar{N}_t^{S,1} - \bar{N}_t^{S,2}| + \frac{1}{\varepsilon} |\bar{N}_t^{IS,1} - \bar{N}_t^{IS,2}| \\ &= \frac{A}{\varepsilon^2} |\langle \bar{\mu}_t^{S,1}, \chi \rangle - \langle \bar{\mu}_t^{S,2}, \chi \rangle| + \frac{1}{\varepsilon} |\langle \bar{\mu}_t^{IS,1}, \chi \rangle - \langle \bar{\mu}_t^{IS,2}, \chi \rangle| \leq \frac{A}{\varepsilon^2} M_t. \end{aligned} \quad (3.23)$$

The same computations show a similar result for  $|\bar{p}_t^{S,1} - \bar{p}_t^{S,2}|$  and  $|\bar{p}_t^{R,1} - \bar{p}_t^{R,2}|$ .

Using that  $\bar{\mu}^i$  are solutions to (1.2)-(1.3) with the functions  $f \equiv \chi^0$ ,  $f \equiv \chi$  and  $f \equiv \chi^2$  we obtain after some computations that  $M$  satisfies a Gronwall inequality which implies that it is equal 0 for all  $t \leq T$ . We proceed here with only one of the computations, others can be done similarly. According to (1.3), we have

$$\langle \bar{\mu}_t^{IS,1}, \chi^0 \rangle - \langle \bar{\mu}_t^{IS,2}, \chi^0 \rangle = \beta \int_0^t \langle \bar{\mu}_s^{SI,1} - \bar{\mu}_s^{IS,1}, \mathbf{1} \rangle ds + r \int_0^t (\bar{p}_s^{1,1} - \bar{p}_s^{1,2}) \langle \bar{\mu}_s^S, \chi \rangle ds.$$

Hence, with (3.23),

$$\left| \langle \bar{\mu}_t^{IS,1} - \bar{\mu}_t^{IS,2}, \mathbf{1} \rangle \right| \leq \beta \int_0^t |\langle \bar{\mu}_s^{IS,1} - \bar{\mu}_s^{IS,2}, \mathbf{1} \rangle| ds + \frac{rA}{\varepsilon} \int_0^t |\langle \bar{\mu}_s^{IS,1} - \bar{\mu}_s^{IS,2}, \chi \rangle| ds,$$

and thus

$$\left| \langle \bar{\mu}_t^{IS,1} - \bar{\mu}_t^{IS,2}, \mathbf{1} \rangle \right| \leq \beta \int_0^t |\langle \bar{\mu}_s^{IS,1} - \bar{\mu}_s^{IS,2}, \chi^0 \rangle| ds + \frac{rA}{\varepsilon} \int_0^t \Upsilon_s ds.$$

By analogous computations for other quantities, we then show that

$$\Upsilon_t \leq c \int_0^t \Upsilon_s ds,$$

hence  $\Upsilon \equiv 0$ . It follows that for all  $t < T$ , and for all  $j \in \{0, 1, 2\}$ ,

$$\langle \bar{\mu}_t^{s,1}, \chi^j \rangle = \langle \bar{\mu}_t^{s,2}, \chi^j \rangle \quad \text{and} \quad \langle \bar{\mu}_t^{\text{IS},1}, \chi^j \rangle = \langle \bar{\mu}_t^{\text{IS},2}, \chi^j \rangle, \quad (3.24)$$

and in particular,  $\bar{N}_t^{s,1} = \bar{N}_t^{s,2}$  and  $\bar{N}_t^{\text{IS},1} = \bar{N}_t^{\text{IS},2}$ . This implies that  $\bar{p}_t^{s,1} = \bar{p}_t^{s,2}$ ,  $\bar{p}_t^{1,1} = \bar{p}_t^{1,2}$  and  $\bar{p}_t^{\text{R},1} = \bar{p}_t^{\text{R},2}$ . From (1.2) and the continuity of the solutions to (1.2)-(1.4), pathwise uniqueness for  $\bar{\mu}^s$  a.s.

Our purpose is now to prove that  $\bar{\mu}^{\text{IS},1} = \bar{\mu}^{\text{IS},2}$ . Let us introduce the following generating functions: for any  $t \in \mathbb{R}_+$ ,  $i \in \{1, 2\}$  and  $\eta \in [0, 1)$ ,

$$\mathcal{G}_t^i(\eta) = \sum_{k \geq 0} \eta^k \bar{\mu}_t^{\text{IS},i}(k).$$

Since we already know these measures do have the same total mass, it boils down to prove that  $\mathcal{G}^1 \equiv \mathcal{G}^2$ . Let us define

$$\begin{aligned} H(t, \eta) &= \int_0^t \sum_{k \in \mathbb{N}} r k \bar{p}_s^1 \sum_{\substack{j, \ell, m \in \mathbb{N} \\ j + \ell + m = k - 1}} \binom{k-1}{j, \ell, m} (\bar{p}_s^1)^j (\bar{p}_s^{\text{R}})^\ell (\bar{p}_s^{\text{S}})^m \eta^k \bar{\mu}_s^{\text{S}}(k) ds \\ K_t &= \sum_{k \in \mathbb{N}} r k \bar{p}_t^1 (k-1) \bar{p}_t^{\text{R}} \frac{\bar{\mu}_t^{\text{S}}(k)}{\bar{N}_t^{\text{IS}}}. \end{aligned} \quad (3.25)$$

The latter quantities are respectively of class  $\mathcal{C}^1$  and  $\mathcal{C}^0$  with respect to time  $t$  and are well-defined and bounded on  $[0, T]$  since the three processes are supposed to have values in  $\mathcal{M}_{\varepsilon, A}$ . Moreover,  $H$  and  $K$  do not depend on the chosen solution because of (3.24). Apply (1.3) to  $f(k) = \eta^k$  to obtain that

$$\begin{aligned} \mathcal{G}_t^i(\eta) &= \mathcal{G}_0^i(\eta) + H(t, \eta) + \int_0^t \left( K_s \sum_{k' \in \mathbb{N}^*} (\eta^{k'-1} - \eta^{k'}) k' \bar{\mu}_s^{\text{IS}}(k') - \beta \mathcal{G}_s^i(\eta) \right) ds \\ &= \mathcal{G}_0^i(\eta) + H(t, \eta) + \int_0^t \left( K_s (1 - \eta) \partial_\eta \mathcal{G}_s^i(\eta) - \beta \mathcal{G}_s^i(\eta) \right) ds. \end{aligned} \quad (3.26)$$

Then, the functions  $t \mapsto \tilde{\mathcal{G}}_t^i(\eta)$  defined by  $\tilde{\mathcal{G}}_t^i(\eta) = e^{\beta t} \mathcal{G}_t^i(\eta)$ ,  $i \in \{1, 2\}$ , are solutions of the following transport equation:

$$\partial_t g(t, \eta) - (1 - \eta) K_t \partial_\eta g(t, \eta) = \partial_t H(t, \eta) e^{\beta t}. \quad (3.27)$$

In view of the regularity of  $H$  and  $K$ , it is known that this equation admits a unique solution (see *e.g.* [12]). Hence  $\mathcal{G}_t^1(\eta) = \mathcal{G}_t^2(\eta)$  for all  $t \in \mathbb{R}_+$  and  $\eta \in [0, 1)$ . The same method applies to  $\bar{\mu}^{\text{RS}}$ . Thus there is at most one solution to the differential system (1.2)-(1.4).

**Step 3** We now show that  $\mu^{(n)}$  nearly satisfies (1.2)-(1.4) as  $n$  gets large. Recall (3.5) for a bounded function  $f$  on  $\mathbb{N}$ . To identify the limiting values, we establish that for all  $n \in \mathbb{N}^*$  and all  $t \geq 0$ ,

$$\langle \mu_{t \wedge \tau_\varepsilon^n}^{(n), \text{IS}}, f \rangle = \Psi_{t \wedge \tau_\varepsilon^n}^{\text{IS}, f}(\mu^{(n)}) + \Delta_{t \wedge \tau_\varepsilon^n}^{n, f} + M_{t \wedge \tau_\varepsilon^n}^{(n), \text{IS}, f}, \quad (3.28)$$

where  $M^{(n), \text{IS}, f}$  is defined in (3.5) and where  $\Delta_{\cdot \wedge \tau_\varepsilon^n}^{n, f}$  converges to 0 when  $n \rightarrow +\infty$ , in probability and uniformly in  $t$  on compact time intervals.

Let us fix  $t \in \mathbb{R}_+$ . Computation similar to (3.21) give:

$$\mathbb{E}((M_t^{(n),\text{IS},f})^2) = \mathbb{E}(\langle M^{(n),\text{IS},f} \rangle_t) \leq \frac{25 \text{Art} \|f\|_\infty^2}{n}. \quad (3.29)$$

Hence the sequence  $(M_t^{(n),\text{IS},f})_{n \in \mathbb{N}}$  converges in  $L^2$  and in probability to zero (and in  $L^1$  by Cauchy-Schwarz inequality).

We now consider the finite variation part of (3.5), given in (3.6). The sum in (3.6) corresponds to the links to  $\mathfrak{I}$  that the new infected individual has. We separate this sum into cases where the new infected individual only has simple edges to other individuals of  $\mathfrak{I}$ , and cases where multiple edges exist. The latter term is expected to vanish for large populations.

$$A_t^{(n),\text{IS},f} = B_t^{(n),\text{IS},f} + C_t^{(n),\text{IS},f}, \quad (3.30)$$

where

$$\begin{aligned} B_t^{(n),\text{IS},f} &= \int_0^t \sum_{k \in \mathbb{N}} \lambda_s^n(k) \mu_s^{(n),\text{S}}(k) \sum_{j+\ell+1 \leq k} p_s^n(j, \ell | k-1) \\ &\times \left\{ (f(k - (j+1+\ell))) \right. \\ &\quad \left. + \sum_{\substack{u \in \mathcal{U}(j+1, \mu_s^{n,\text{IS}}); \\ \forall i \leq I_{s_-}^n, u_i \leq 1}} \rho(u|j+1, \mu_s^{n,\text{IS}}) \sum_{i=0}^{I_{s_-}^n} \left( f(\gamma_i(\mu_{s_-}^{n,\text{IS}}) - u_i) - f(\gamma_i(\mu_{s_-}^{n,\text{IS}})) \right) \right\} ds \quad (3.31) \end{aligned}$$

and

$$\begin{aligned} C_t^{(n),\text{IS},f} &= \int_0^t \sum_{k \in \mathbb{N}} \lambda_s^n(k) \mu_s^{(n),\text{S}}(k) \sum_{j+\ell+1 \leq k} p_s^n(j, \ell | k-1) \\ &\times \sum_{\substack{u \in \mathcal{U}(j+1, \mu_s^{n,\text{IS}}); \\ \exists i \leq I_{s_-}^n, u_i > 1}} \rho(u|j+1, \mu_s^{n,\text{IS}}) \sum_{i=1}^{I_{s_-}^n} \left( f(\gamma_i(\mu_{s_-}^{n,\text{IS}}) - u_i) - f(\gamma_i(\mu_{s_-}^{n,\text{IS}})) \right) ds. \quad (3.32) \end{aligned}$$

We first show that  $C_t^{(n),\text{SI},f}$  is a negligible term. Let  $q_{j,\ell,s}^n$  denote the probability that the newly infected individual at time  $s$  has a double (or of higher order) edge to some alter in  $I_{s_-}^n$ , given  $j$  and  $\ell$ . The probability to have a multiple edge to a given infectious  $i$  is less than the number of couples of edges linking the newly infected to  $i$ , times the probability that these two particular edges linking  $i$  to a susceptible alter at  $s_-$  actually lead to the newly infected. Hence,

$$\begin{aligned} q_{j,\ell,s}^n &= \sum_{\substack{u \in \mathcal{U}(j+1, \mu_s^{n,\text{IS}}); \\ \exists i \leq I_{s_-}^n, u_i > 1}} \rho(u|j+1, \mu_s^{n,\text{IS}}) \\ &\leq \binom{j}{2} \sum_{x \in I_{s_-}^n} \frac{d_x(S_{s_-}^n)(d_x(S_{s_-}^n) - 1)}{N_{s_-}^{n,\text{IS}}(N_{s_-}^{n,\text{IS}} - 1)} = \binom{j}{2} \frac{1}{n} \frac{\langle \mu_{s_-}^{(n),\text{IS}}, \chi(\chi - 1) \rangle}{N_{s_-}^{(n),\text{IS}}(N_{s_-}^{(n),\text{IS}} - 1/n)} \\ &\leq \binom{j}{2} \frac{1}{n} \frac{A}{\varepsilon(\varepsilon - 1/n)} \quad \text{if } s < \tau_\varepsilon^n \text{ and } n > 1/\varepsilon. \quad (3.33) \end{aligned}$$

Then, since for all  $u \in \mathcal{U}(j+1, \mu_s^{n, \text{IS}})$ ,

$$\left| \sum_{i=1}^{I_{s-}^n} \left( f\left(\gamma_i(\mu_{s-}^{n, \text{IS}}) - u_i\right) - f\left(\gamma_i(\mu_{s-}^{n, \text{IS}})\right) \right) \right| \leq 2(j+1)\|f\|_\infty, \quad (3.34)$$

we have by (3.33) and (3.34), for  $n > 1/\varepsilon$ ,

$$\begin{aligned} |C_{t \wedge \tau_\varepsilon^n}^{(n), \text{IS}, f}| &\leq \int_0^{t \wedge \tau_\varepsilon^n} \sum_{k \in \mathbb{N}} rk \mu_s^{(n), S}(k) \sum_{j+\ell+1 \leq k} p_s^n(j, \ell | k-1) 2(j+1) \|f\|_\infty \frac{j(j-1)A}{2n\varepsilon(\varepsilon-1/n)} ds \\ &\leq \frac{A r t \|f\|_\infty}{n\varepsilon(\varepsilon-1/n)} \langle \mu_0^{(n), S}, \chi^4 \rangle, \end{aligned} \quad (3.35)$$

which tends to zero in view of (3.8) and to the fact that  $\mu_s^{(n), S}$  is dominated by  $\mu_0^{(n), S}$  for all  $s \geq 0$  and  $n \in \mathbb{N}^*$ .

We now aim at proving that  $B_{\cdot \wedge \tau_\varepsilon^n}^{(n), \text{IS}, f}$  is somewhat close to  $\Psi_{\cdot \wedge \tau_\varepsilon^n}^{\text{IS}, f}(\mu^{(n)})$ . First, notice that

$$\begin{aligned} &\sum_{\substack{u \in \mathcal{U}(j+1, \mu_s^{n, \text{IS}}); \\ \forall i \leq I_{s-}^n, u_i \leq 1}} \rho(u | j+1, \mu_s^{n, \text{IS}}) \sum_{i=1}^{I_{s-}^n} \left( f\left(\gamma_i(\mu_{s-}^{n, \text{IS}}) - u_i\right) - f\left(\gamma_i(\mu_{s-}^{n, \text{IS}})\right) \right) \\ &= \sum_{\substack{u \in (I_{s-}^n)^{j+1} \\ u_0 \neq \dots \neq u_j}} \left( \frac{\prod_{k=0}^j d_{u_k}(S_{s-}^n)}{N_{s-}^{n, \text{IS}} \dots (N_{s-}^{n, \text{SI}} - (j+1))} \right) \cdot \sum_{m=0}^j \left( f\left(d_{u_m}(S_{s-}^n) - 1\right) - f\left(d_{u_m}(S_{s-}^n)\right) \right) \\ &= \sum_{m=0}^j \sum_{\substack{u \in (I_{s-}^n)^{j+1} \\ u_0 \neq \dots \neq u_j}} \left( \frac{\prod_{k=0}^j d_{u_k}(S_{s-}^n)}{N_{s-}^{n, \text{IS}} \dots (N_{s-}^{n, \text{SI}} - (j+1))} \right) \cdot \left( f\left(d_{u_m}(S_{s-}^n) - 1\right) - f\left(d_{u_m}(S_{s-}^n)\right) \right) \\ &= \sum_{m=0}^j \left( \sum_{x \in I_{s-}^n} \frac{d_x(S_{s-}^n)}{N_{s-}^{n, \text{IS}}} \left( f\left(d_x(S_{s-}^n) - 1\right) - f\left(d_x(S_{s-}^n)\right) \right) \right) \\ &\quad \times \left( \sum_{\substack{u \in (I_{s-}^n \setminus \{i\})^j \\ u_0 \neq \dots \neq u_{j-1}}} \frac{\prod_{k=0}^{j-1} d_{u_k}(S_{s-}^n)}{(N_{s-}^{n, \text{IS}} - 1) \dots (N_{s-}^{n, \text{IS}} - (j+1))} \right) \\ &= (j+1) \frac{\langle \mu_{s-}^{(n), \text{IS}}, \chi(\tau_1 f - f) \rangle}{N_{s-}^{(n), \text{IS}}} (1 - q_{j-1, \ell, s}^n), \end{aligned} \quad (3.36)$$

where we recall that  $\tau_1 f(k) = f(k-1)$  for every function  $f$  on  $\mathbb{N}$  and  $k \in \mathbb{N}$ . In the third equality, we split the term  $u_m$  from the other terms  $(u_{m'})_{m' \neq m}$ . The last sum in the r.h.s. of this equality is the probability of drawing  $j$  different infectious individuals that are not  $u_m$  and that are all different, hence  $1 - q_{j-1, \ell, s}^n$ .

Denote for  $t > 0$  and  $n \in \mathbb{N}$ ,

$$\begin{aligned} p_t^{n,I} &= \frac{\langle \mu_t^{n,IS}, \chi \rangle - 1}{\langle \mu_t^{n,S}, \chi \rangle - 1}, \\ p_t^{n,R} &= \frac{\langle \mu_t^{n,RS}, \chi \rangle}{\langle \mu_t^{n,S}, \chi \rangle - 1}, \\ p_t^{n,S} &= \frac{\langle \mu_t^{n,S}, \chi \rangle - \langle \mu_t^{n,IS}, \chi \rangle - \langle \mu_t^{n,RS}, \chi \rangle}{\langle \mu_t^{n,S}, \chi \rangle - 1}, \end{aligned}$$

the proportion of edges with infectious (resp. removed and susceptible) alters and susceptible egos among all the edges with susceptible egos but the contaminating edge. For all integers  $j$  and  $\ell$  such that  $j + \ell \leq k - 1$  and  $n \in \mathbb{N}^*$ , denote by

$$\tilde{p}_t^n(j, \ell | k - 1) = \frac{(k - 1)!}{j!(k - 1 - j - \ell)! \ell!} (p_t^{n,I})^j (p_t^{n,R})^\ell (p_t^{n,S})^{k-1-j-\ell},$$

the probability that the multinomial variable counting the number of edges with infectious, removed and susceptible alters, among the edges with susceptible egos but the contaminating edge, equals  $(j, \ell, k - 1 - j - \ell)$ . We have that

$$|\Psi_{t \wedge \tau_\varepsilon^n}^{IS,f}(\mu^{(n)}) - B_{t \wedge \tau_\varepsilon^n}^{(n),IS,f}| \leq D_{t \wedge \tau_\varepsilon^n}^{(n),IS,f} + E_{t \wedge \tau_\varepsilon^n}^{(n),IS,f}, \quad (3.37)$$

where

$$\begin{aligned} D_t^{(n),IS,f} &= \int_0^t \sum_{k \in \mathbb{N}} \lambda_s^n(k) \mu_s^{(n),S}(k) \sum_{j+\ell+1 \leq k} (p_s^n(j, \ell | k - 1) - \tilde{p}_s^n(j, \ell | k - 1)) \\ &\quad \times \left( f(k - (j + \ell + 1)) + (j + 1) \frac{\langle \mu_{s-}^{(n),IS}, \chi(\tau_1 f - f) \rangle}{N_{s-}^{(n),IS}} \right) ds, \\ E_t^{(n),IS,f} &= \int_0^t \sum_{k \in \mathbb{N}} \lambda_s^n(k) \mu_s^{(n),S}(k) \sum_{j+\ell+1 \leq k} p_s^n(j, \ell | k - 1) (j + 1) \frac{\langle \mu_{s-}^{(n),IS}, \chi(\tau_1 f - f) \rangle}{N_{s-}^{(n),IS}} q_{j-1, \ell, s}^n ds. \end{aligned}$$

First,

$$|D_{t \wedge \tau_\varepsilon^n}^{(n),IS,f}| \leq \int_0^{t \wedge \tau_\varepsilon^n} \sum_{k \in \mathbb{N}} rk \alpha_s^n(k) \|f\|_\infty \left( 1 + \frac{2kA}{\varepsilon} \right) \mu_s^{(n),S}(k) ds, \quad (3.38)$$

where for all  $k \in \mathbb{N}$

$$\alpha_t^n(k) = \sum_{j+\ell+1 \leq k} \left| p_t^n(j, \ell | k - 1) - \tilde{p}_t^n(j, \ell | k - 1) \right|.$$

The multinomial probability  $\tilde{p}_s^n(j, \ell | k - 1)$  approximates the hypergeometric one,  $p_s^n(j, \ell | k - 1, s)$ , as  $n$  increases to infinity, in view of the fact that the total population size,  $\langle \mu_0^{n,S}, \mathbf{1} \rangle + \langle \mu_0^{n,IS}, \mathbf{1} \rangle$ , is of order  $n$ . Hence, the r.h.s. of (3.38) vanishes by dominated convergence.

On another hand, using (3.33),

$$\begin{aligned} |E_{t \wedge \tau_\varepsilon^n}^{(n),IS,f}| &\leq \int_0^{t \wedge \tau_\varepsilon^n} \sum_{k \in \mathbb{N}} rk^2 \mu_s^{(n),S}(k) \frac{2\|f\|_\infty A}{\varepsilon} \frac{k^2 A}{2n\varepsilon(\varepsilon - 1/n)} ds \\ &\leq \frac{A^3 rt \|f\|_\infty}{n\varepsilon^2(\varepsilon - 1/n)}, \end{aligned} \quad (3.39)$$

in view of (3.8). Gathering (3.29), (3.30), (3.35), (3.37), (3.38) and (3.39) concludes the proof that the rest of (3.28) vanishes in probability uniformly over compact intervals.

**Step 4** Recall that in this proof,  $\bar{\mu} = (\bar{\mu}^S, \bar{\mu}^{IS}, \bar{\mu}^{RS})$  is the limit of  $\mu_{\cdot \wedge \tau_\varepsilon^n}^{(n)} = \left( \mu_{\cdot \wedge \tau_\varepsilon^n}^{(n),S}, \mu_{\cdot \wedge \tau_\varepsilon^n}^{(n),IS}, \mu_{\cdot \wedge \tau_\varepsilon^n}^{(n),RS} \right)_{n \in \mathbb{N}^*}$ , and recall that these processes take values in the closed set  $\mathcal{M}_{0,A}^3$ . Our purpose is now to prove that  $\bar{\mu}$  satisfy (1.2)-(1.4).

The maps  $\nu := (\nu^1, \nu^2, \nu^3) \mapsto \langle \nu^1, \mathbf{1} \rangle / (\langle \nu_0^1, \mathbf{1} \rangle + \langle \nu_0^2, \mathbf{1} \rangle)$  (resp.  $\langle \nu^2, \mathbf{1} \rangle / (\langle \nu_0^1, \mathbf{1} \rangle + \langle \nu_0^2, \mathbf{1} \rangle)$  and  $\langle \nu^3, \mathbf{1} \rangle / (\langle \nu_0^1, \mathbf{1} \rangle + \langle \nu_0^2, \mathbf{1} \rangle)$ ) are clearly continuous for the Skorokhod topology from  $\mathcal{C}(\mathbb{R}_+, \mathcal{M}_{0,A} \times \mathcal{M}_{\varepsilon,A} \times \mathcal{M}_{0,A})$  into  $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ .

Then, it is a consequence of Lemma A.5 together with the continuity of  $(X^1, X^2) \mapsto X^1/X^2$  from  $\mathcal{C}(\mathbb{R}_+, \mathbb{R}) \times \mathcal{C}(\mathbb{R}_+, \mathbb{R}^*)$  into  $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$  (see e.g. [27]), that the mapping  $\nu \mapsto \langle \nu^1, \chi \rangle / \langle \nu^2, \chi \rangle$  is continuous from  $\mathcal{C}(\mathbb{R}_+, \mathcal{M}_{0,A} \times \mathcal{M}_{\varepsilon,A} \times \mathcal{M}_{0,A})$  into  $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ . The same argument yields the continuity of  $\nu \mapsto \mathbb{1}_{\langle \nu^1, \chi \rangle > \varepsilon} / \langle \nu^2, \chi \rangle$  for the same spaces.

Lemma A.5 also implies the continuity of  $\nu \mapsto \langle \nu^2, \chi (\tau_1 f - f) \rangle$  from  $\mathcal{C}(\mathbb{R}_+, \mathcal{M}_{0,A} \times \mathcal{M}_{\varepsilon,A} \times \mathcal{M}_{0,A})$  into  $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$  for all bounded function  $f$  on  $\mathbb{N}$ .

Since, as well known, the mapping  $y \in \mathcal{D}([0, t], \mathbb{R}) \mapsto \int_0^t y_s ds$  is continuous, we have proven the continuity of the mapping  $\Psi_t^f$  defined in (3.22) on  $\mathcal{D}(\mathbb{R}_+, \mathcal{M}_{0,A} \times \mathcal{M}_{\varepsilon,A} \times \mathcal{M}_{0,A})$ .

By Lemma A.5 applied to  $\varphi \equiv 1$ , the process  $(N_{\cdot \wedge \tau_\varepsilon^n}^{(n),IS})_{n \in \mathbb{N}^*}$  converges in distribution to  $\bar{N}^{IS} = \langle \bar{\mu}^{IS}, \chi \rangle$ . Since the latter process is continuous, the convergence holds in  $(\mathcal{C}([0, T], \mathbb{R}_+), \|\cdot\|_\infty)$  for any  $T > 0$  (see [6] p. 112). As  $y \in \mathcal{D}(\mathbb{R}_+, \mathbb{R}) \mapsto \inf_{t \in [0, T]} y(t) \in \mathbb{R}$  is continuous, the following convergence in distribution holds:

$$\inf_{t \in [0, T]} \bar{N}_t^{IS} = \lim_{n \rightarrow +\infty} \inf_{t \in [0, T]} N_{t \wedge \tau_\varepsilon^n}^{(n),IS} \quad (\geq \varepsilon). \quad (3.40)$$

We consider  $t_{2\varepsilon} = \inf\{t \in \mathbb{R}_+, \bar{N}_t^{IS} < 2\varepsilon\}$ . A difficulty lies in the fact that we do not know yet whether this time is deterministic or integrable. We have the following convergence in distribution.

$$2\varepsilon \leq \inf_{t \in [0, T]} \bar{N}_{t \wedge t_{2\varepsilon}}^{IS} = \lim_{n \rightarrow +\infty} \inf_{t \in [0, T]} N_{t \wedge \tau_\varepsilon^n \wedge t_{2\varepsilon}}^{(n),IS}. \quad (3.41)$$

Thus, using (3.41) and  $\mathbb{P}(\inf_{t \in [0, t_{2\varepsilon}]} \bar{N}_t^{IS} = \varepsilon) = 0$ , since  $\{U \in \mathcal{D}([0, T], \mathbb{R}), \inf_{t \leq t_{2\varepsilon}} U(t) > \varepsilon\}$  is an open set, we have

$$1 = \mathbb{P}\left(\inf_{t \in [0, t_{2\varepsilon}]} \bar{N}_t^{IS} > \varepsilon\right) \leq \lim_{n \rightarrow +\infty} \mathbb{P}\left(\inf_{t \in [0, T \wedge t_{2\varepsilon}]} N_{t \wedge \tau_\varepsilon^n}^{(n),IS} > \varepsilon\right) = \lim_{n \rightarrow +\infty} \mathbb{P}\left(\tau_\varepsilon^n > T \wedge t_{2\varepsilon}\right). \quad (3.42)$$

We have hence

$$\Psi_{\cdot \wedge \tau_\varepsilon^n \wedge t_{2\varepsilon} \wedge T}^{IS, f}(\mu^{(n)}) = \Psi_{\cdot \wedge \tau_\varepsilon^n \wedge T}^{IS, f}(\mu^{(n)}) \mathbb{1}_{\tau_\varepsilon^n \leq t_{2\varepsilon}} + \Psi_{\cdot \wedge t_{2\varepsilon} \wedge T}^{IS, f}(\mu_{\cdot \wedge \tau_\varepsilon^n}^{(n)}) \mathbb{1}_{\tau_\varepsilon^n > t_{2\varepsilon}}. \quad (3.43)$$

Using the estimates of the different terms in (3.28), we know that  $\Psi_{\cdot \wedge \tau_\varepsilon^n \wedge T}^{IS, f}(\mu^{(n)})$  is upper bounded by a moment of  $\mu^{(n)}$  of order 5. In view of (3.8) and (3.42), the first term in the r.h.s. converges in  $L^1$  and hence in probability to zero. Using the continuity of  $\Psi^{IS, f}$  on  $\mathcal{D}(\mathbb{R}_+, \mathcal{M}_{0,A} \times \mathcal{M}_{\varepsilon,A} \times \mathcal{M}_{0,A})$ ,  $\Psi^{IS, f}(\mu_{\cdot \wedge \tau_\varepsilon^n}^{(n)})$  converges to  $\Psi^{IS, f}(\bar{\mu})$  and therefore,  $\Psi_{\cdot \wedge t_{2\varepsilon} \wedge T}^{IS, f}(\mu_{\cdot \wedge \tau_\varepsilon^n}^{(n)})$  converges in distribution to  $\Psi_{\cdot \wedge t_{2\varepsilon} \wedge T}^{IS, f}(\bar{\mu})$ . Thanks to this and (3.42), the second term in the r.h.s. converges in distribution to  $\Psi_{\cdot \wedge t_{2\varepsilon} \wedge T}^{IS, f}(\bar{\mu})$  in  $\mathcal{D}(\mathbb{R}_+, \mathbb{R})$ .

Since the accumulation point  $\bar{\mu}$  is continuous,  $(\langle \mu_{\cdot \wedge \tau_\varepsilon^n \wedge t_{2\varepsilon} \wedge T}^{(n),IS}, f \rangle - \Psi_{\cdot \wedge \tau_\varepsilon^n \wedge t_{2\varepsilon} \wedge T}^{IS, f}(\mu^{(n)}))_{n \in \mathbb{N}^*}$  converges in distribution to  $\langle \bar{\mu}_{\cdot \wedge t_{2\varepsilon} \wedge T}, f \rangle - \Psi_{\cdot \wedge t_{2\varepsilon} \wedge T}^{IS, f}(\bar{\mu})$ . From (3.28), this sequence also converges

in probability to zero. By identification of these limits,  $\bar{\mu}$  solves (1.2)-(1.4) on  $[0, t_{2\varepsilon} \wedge T]$  and thus coincides a.s. with the only continuous deterministic solution of these equations on this time interval. This implies that  $t_{2\varepsilon}$  is deterministic and yields the convergence in probability of  $(\mu_{\cdot \wedge \tau_\varepsilon^n}^{(n)})_{n \in \mathbb{N}^*}$  to  $\bar{\mu}$ , uniformly on  $[0, t_{2\varepsilon}]$ .

We finally prove that the non-localized sequence  $(\mu^{(n)})_{n \in \mathbb{N}^*}$  also converges uniformly and in probability to  $\bar{\mu}$  in  $\mathcal{D}([0, t_{2\varepsilon}], \mathcal{M}_{0,A} \times \mathcal{M}_{\varepsilon,A} \times \mathcal{M}_{0,A})$ . For a small positive  $\eta$ ,

$$\begin{aligned} & \mathbb{P}\left(\sup_{t \in [0, t_{2\varepsilon}]} \left| \langle \mu_t^{(n), \text{IS}}, f \rangle - \Psi_t^{\text{IS}, f}(\bar{\mu}) \right| > \eta\right) \\ & \leq \mathbb{P}\left(\sup_{t \in [0, t_{2\varepsilon}]} \left| \Psi_{t \wedge \tau_\varepsilon^n}^{\text{IS}, f}(\mu^{(n)}) - \Psi_{t \wedge \tau_\varepsilon^n}^{\text{IS}, f}(\bar{\mu}) \right| > \frac{\eta}{2}; \tau_\varepsilon^n \geq t_{2\varepsilon}\right) \\ & \quad + \mathbb{P}\left(\sup_{t \in [0, t_{2\varepsilon}]} \left| \varepsilon_{t \wedge \tau_\varepsilon^n}^{n, f} + M_{t \wedge \tau_\varepsilon^n}^{(n), \text{IS}, f} \right| > \frac{\eta}{2}\right) + \mathbb{P}\left(\tau_\varepsilon^n < t_{2\varepsilon}\right). \end{aligned} \quad (3.44)$$

Using the continuity of  $\Psi^f$  and the uniform convergence in probability proved above, the first term in the r.h.s. of (3.44) converges to zero. We can show that the second term converges to zero by using Doob's inequality together with the estimates of the bracket of  $M^{(n), \text{IS}, f}$  (similar to (3.21)) and of  $\Delta^{n, f}$  (Step 2). Finally, the third term vanishes in view of (3.42).

The convergence of the original sequence  $(\mu^{(n)})_{n \in \mathbb{N}^*}$  is then entailed by the uniqueness of the solution to (1.2)-(1.4), implied by Step 2.

**Step 5** When  $n \rightarrow +\infty$ , by taking the limit in (3.1),  $(\mu^{(n), \text{S}})_{n \in \mathbb{N}^*}$  converges in  $\mathcal{D}(\mathbb{R}_+, \mathcal{M}_{0,A})$  to the solution of the following transport equation, that can be solved in function of  $p^1$ . For every bounded function  $f : (k, t) \mapsto f_t(k) \in \mathcal{C}_b^{0,1}(\mathbb{N} \times \mathbb{R}_+, \mathbb{R})$  of class  $\mathcal{C}^1$  with bounded derivative with respect to  $t$ ,

$$\langle \bar{\mu}_t^{\text{S}}, f_t \rangle = \langle \bar{\mu}_0^{\text{S}}, f_0 \rangle + \int_0^t \langle \bar{\mu}_s^{\text{S}}, rk \bar{p}_s^1 f_s - \partial_s f_s \rangle ds. \quad (3.45)$$

Choosing  $f(k, s) = \varphi(k) \exp(-rk \int_0^{t-s} \bar{p}^1(u) du)$ , we obtain that

$$\langle \bar{\mu}_t^{\text{S}}, \varphi \rangle = \sum_{k \in \mathbb{N}} \varphi(k) \theta_t^k \bar{\mu}_0^{\text{S}}(k). \quad (3.46)$$

where  $\theta_t = \exp(-r \int_0^t \bar{p}^1(u) du)$  is the probability that a given degree 1 node remains susceptible at time  $t$ . This is the announced Equation (1.2).  $\blacksquare$

**Corollary 3.4.** *For all  $0 \leq t \leq t_{2\varepsilon}$*

$$\begin{aligned} \bar{N}_t^{\text{S}} &= \theta_t g'(\theta_t) \\ \bar{N}_t^{\text{IS}} &= \bar{N}_0^{\text{IS}} + \int_0^t r \bar{p}_s^1 \theta_s g'(\theta_s) \left( (\bar{p}_s^{\text{S}} - \bar{p}_s^1) \theta_s \frac{g''(\theta_s)}{g'(\theta_s)} - 1 \right) - \beta \bar{N}_s^{\text{IS}} ds \\ \bar{N}_t^{\text{RS}} &= \int_0^t \left( \beta \bar{N}_s^{\text{IS}} - r \bar{p}_s^{\text{R}} \bar{p}_s^1 \theta_s^2 g''(\theta_s) \right) ds. \end{aligned} \quad (3.47)$$

*Proof.* Because of the moment Assumption (3.8), we can prove that (3.28) also holds for  $f(k) = k$ ,  $k \in \mathbb{N}$ . This is obtained by replacing in (3.29), (3.35), (3.38) and (3.39)  $\|f\|_\infty$  by  $k$  and using the Assumption of boundedness of the moments of order 5 in (3.35) and (3.39). This shows that



$(N_{\wedge \tau_s^z}^{(n), \text{IS}})_{n \in \mathbb{N}}$  converges uniformly in probability to the deterministic and continuous solution  $\bar{N}^{\text{IS}}$  of the second equality in (3.47). Fix  $t \geq 0$ . (1.2) with  $f(k) = k$  reads

$$\bar{N}_t^{\text{S}} = \sum_{k \in \mathbb{N}} \bar{\mu}_0^{\text{S}}(k) k \theta_t^k = \theta_t \sum_{k=1}^{+\infty} \bar{\mu}_0^{\text{S}}(k) k \theta_t^{k-1} = \theta_t g'(\theta_t), \quad (3.48)$$

i.e. the first assertion of (3.47).

Choosing  $f(k) = k$  in (1.3), we obtain

$$\begin{aligned} \bar{N}_t^{\text{IS}} &= \bar{N}_0^{\text{IS}} - \int_0^t \beta \bar{N}_s^{\text{IS}} \, ds \\ &+ \int_0^t \sum_{k \in \mathbb{N}} \lambda_s(k) \sum_{j+\ell \leq k-1} \left[ \frac{(k-1)!}{j!(k-1-j-\ell)! \ell!} (\bar{p}_s^{\text{I}})^j (\bar{p}_s^{\text{R}})^\ell (\bar{p}_s^{\text{S}})^{k-1-j-\ell} \right] (k-2j-2-\ell) \bar{\mu}_s^{\text{S}}(k) \, ds. \end{aligned} \quad (3.49)$$

Notice that the term in the square brackets is the probability to obtain  $(j, \ell, k-1-j-\ell)$  from a draw in the multinomial distribution of parameters  $(k-1, (\bar{p}_s^{\text{I}}, \bar{p}_s^{\text{R}}, \bar{p}_s^{\text{S}}))$ . Hence,

$$\sum_{j+\ell \leq k-1} j \times \left( \frac{(k-1)!}{j!(k-1-j-\ell)! \ell!} (\bar{p}_s^{\text{I}})^j (\bar{p}_s^{\text{R}})^\ell (\bar{p}_s^{\text{S}})^{k-1-j-\ell} \right) = (k-1) \bar{p}_s^{\text{I}}$$

as we recognize the mean number of edges to  $\text{I}_s$  of an individual of degree  $k$ . This is similar for the other terms. Hence, using the definition of  $\lambda_s(k)$  in (2.9),

$$\bar{N}_t^{\text{IS}} = \bar{N}_0^{\text{IS}} + \int_0^t r \bar{p}_s^{\text{I}} \left( \langle \bar{\mu}_s^{\text{S}}, \chi^2 - 2\chi \rangle - (2\bar{p}_s^{\text{I}} + \bar{p}_s^{\text{R}}) \langle \bar{\mu}_s^{\text{S}}, \chi(\chi-1) \rangle \right) \, ds - \int_0^t \beta \bar{N}_s^{\text{IS}} \, ds. \quad (3.50)$$

But since

$$\begin{aligned} \langle \bar{\mu}_t^{\text{S}}, \chi(\chi-1) \rangle &= \sum_{k \in \mathbb{N}} \bar{\mu}_0^{\text{S}}(k) k(k-1) \theta_t^k = \theta_t^2 g''(\theta_t) \\ \langle \bar{\mu}_t^{\text{S}}, \chi^2 - 2\chi \rangle &= \langle \bar{\mu}_t^{\text{S}}, k(k-1) \rangle - \langle \bar{\mu}_t^{\text{S}}, k \rangle = \theta_t^2 g''(\theta_t) - \theta_t g'(\theta_t), \end{aligned}$$

we obtain by noticing that  $1 - 2\bar{p}_s^{\text{I}} - \bar{p}_s^{\text{R}} = \bar{p}_s^{\text{S}} - \bar{p}_s^{\text{I}}$ ,

$$\bar{N}_t^{\text{IS}} = \bar{N}_0^{\text{IS}} + \int_0^t r \bar{p}_s^{\text{I}} \left( (\bar{p}_s^{\text{S}} - \bar{p}_s^{\text{I}}) \theta_s^2 g''(\theta_s) - \theta_s g'(\theta_s) \right) \, ds - \int_0^t \beta \bar{N}_s^{\text{IS}} \, ds \quad (3.51)$$

which is the second assertion of (3.47). The third equation of (3.47) is obtained similarly.  $\blacksquare$

### 3.3 Proof of Proposition 1.1 (Volz' equations)

*Proof of Proposition 1.1.* We begin with the proof of (1.7) and (1.8). Fix again  $t \geq 0$ . For the size of the susceptible population, taking  $\varphi = \mathbf{1}$  in (1.2), we are lead to introduce the same quantity  $\theta_t = \exp(-r \int_0^t \bar{p}_s^{\text{I}} \, ds)$  as Volz and obtain (1.7). For the size of the infective population, setting  $f = \mathbf{1}$  in (1.3) entails

$$\begin{aligned} \bar{I}_t &= \bar{I}_0 + \int_0^t \left( \sum_{k \in \mathbb{N}} r k \bar{p}_s^{\text{I}} \bar{\mu}_s^{\text{S}}(k) - \beta \bar{I}_s \right) \, ds \\ &= \bar{I}_0 + \int_0^t \left( r \bar{p}_s^{\text{I}} \sum_{k \in \mathbb{N}} \bar{\mu}_0^{\text{S}}(k) k \theta_s^k - \beta \bar{I}_s \right) \, ds = \bar{I}_0 + \int_0^t \left( r \bar{p}_s^{\text{I}} \theta_s g'(\theta_s) - \beta \bar{I}_s \right) \, ds \end{aligned} \quad (3.52)$$

by using (1.2) with  $f(k) = k$  for the second equality.

Let us now consider the probability that an edge with a susceptible ego has an infectious alter. Both equations (1.7) and (1.8) depend on  $\bar{p}_t^I$ . It is thus important to obtain an equation for this quantity. In [26], this equation also leads to introduce the quantity  $\bar{p}_t^S$ . From the scaling in Section 3.1, we obtain similarly to (2.12),

$$p_t^{n,I} = p_0^{n,I} + \int_0^t \left( \sum_{k \in \mathbb{N}} \sum_{j+\ell \leq k-1} \left[ \frac{(k-2(j+1)-\ell) + k p_s^{n,I}}{N_s^{(n),S} - k/n} \right] \right. \\ \left. \times \lambda_s^n(k) \mu_s^{(n),S}(k) p_s^n(j, \ell | k-1) - \beta p_s^{n,I} \right) ds + M_t^{n,p^{n,I}},$$

where  $M^{n,p^{n,I}}$  is a square integrable martingale. Notice that

$$\left| \frac{(k-2(j+1)-\ell) + k p_{s-}^{n,I}}{N_{s-}^{n,S} - k} \right| p_s^n(j, \ell | k-1) \\ = \left| \frac{N_{s-}^{n,IS} + k - 2(j+1) - \ell}{N_{s-}^{n,S} - k} - \frac{N_{s-}^{n,IS}}{N_{s-}^{n,S}} \right| p_s^n(j, \ell | k-1) \leq 1 + 1$$

since for admissible  $j$  and  $\ell$ , each of the fraction corresponds to the proportion of edges IS among the edges with susceptible ego (after and before) the infection. Using additionally that  $(k-2(j+1)-\ell) + k p_s^{n,I} \leq 2K$ , we can upper bound the bracket of  $M^{n,p^{n,I}}$  for all  $t$ :

$$\langle M^{n,p^{n,I}} \rangle_t = \int_0^t \left( \sum_{k \in \mathbb{N}} \sum_{j+\ell \leq k-1} \left[ \frac{(k-2(j+1)-\ell) + k p_s^{n,I}}{N_s^{n,S} - k} \right]^2 \right. \\ \left. \times \lambda_s^n(k) \mu_s^{n,S}(k) p_s^n(j, \ell | k-1) + \beta \frac{\langle \mu_s^{n,IS}, \chi^2 \rangle}{(N_s^{n,S})^2} \right) ds \\ \leq \int_0^t \left( \sum_{k \leq N_s^{n,S}/2} \left[ \frac{2k}{N_s^{n,S} - k} \right]^2 r k \mu_s^{n,S}(k) + \sum_{k > N_s^{n,S}/2} r k \mu_s^{n,S}(k) + \beta \frac{\langle \mu_s^{n,IS}, \chi^2 \rangle}{(N_s^{n,S})^2} \right) ds \\ \leq \int_0^t \left( 16r \frac{\langle \mu_s^{n,S}, \chi^3 \rangle}{(N_s^{n,S})^2} + \frac{\langle \mu_s^{n,S}, \chi^3 \rangle}{(N_s^{n,S}/2)^2} + \beta \frac{\langle \mu_s^{n,IS}, \chi^2 \rangle}{(N_s^{n,S})^2} \right) ds \\ \leq \frac{1}{n} \int_0^t \left( 16r \frac{\langle \mu_s^{(n),S}, \chi^3 \rangle}{(N_s^{(n),S})^2} + \frac{\langle \mu_s^{(n),S}, \chi^3 \rangle}{(N_s^{(n),S}/2)^2} + \beta \frac{\langle \mu_s^{(n),IS}, \chi^2 \rangle}{(N_s^{(n),S})^2} \right) ds, \quad (3.53)$$

which vanishes when  $n \rightarrow +\infty$  by (3.8). To the limit,

$$\bar{p}_t^I = \bar{p}_0^I + \int_0^t \left( \sum_{k \in \mathbb{N}} \sum_{j+\ell \leq k-1} \left( \frac{(k-1)!}{j!(k-1-j-\ell)! \ell!} (\bar{p}_s^I)^j (\bar{p}_s^R)^\ell (\bar{p}_s^S)^{k-1-j-\ell} \right) \right. \\ \left. \times \frac{(k-2(j+1)-\ell) + k \bar{p}_s^I}{\bar{N}_s^S} \lambda_s(k) \bar{\mu}_s^S(k) - \beta \bar{p}_s^I \right) ds \\ = \bar{p}_0^I + \int_0^t \left( r \bar{p}_s^I \frac{\langle \bar{\mu}_s^S, \chi^2 \rangle - 2\chi - 2\chi(\chi-1)\bar{p}_s^I - \chi(\chi-1)\bar{p}_s^R + \chi^2 \bar{p}_s^I}{\bar{N}_s^S} - \beta \bar{p}_s^I \right) ds \\ = \bar{p}_0^I + \int_0^t \left( r \bar{p}_s^I \frac{(1-\bar{p}_s^I - \bar{p}_s^R)\theta_s^2 g''(\theta_s) - \theta_s g'(\theta_s) + \bar{p}_s^I \theta_s g'(\theta_s)}{\theta_s g'(\theta_s)} - \beta \bar{p}_s^I \right) ds \\ = \bar{p}_0^I + \int_0^t \left( r \bar{p}_s^I \bar{p}_s^S \theta_s \frac{g''(\theta_s)}{g'(\theta_s)} - r \bar{p}_s^I (1 - \bar{p}_s^I) - \beta \bar{p}_s^I \right) ds. \quad (3.54)$$

This achieves the proof of (1.9).

Let us finally establish (1.10). For this, we notice that

$$\bar{p}_t^S = 1 - \bar{p}_t^I - \bar{p}_t^R. \quad (3.55)$$

Proceeding as above, we can establish the equation satisfied by  $\bar{p}_t^R$ . Since for all  $t \geq 0$  and  $n$ ,

$$p_t^{n,R} = \int_0^t \left( \beta p_s^{n,I} - \sum_{k \in \mathbb{N}} \sum_{j+\ell \leq k-1} p_s^n(j, \ell | k-1) \frac{k p_s^{n,R} - \ell}{N^{(n),S} - k/n} \lambda_s^n(k) \mu_s^{(n),S}(k) \right) ds + M_t^{n,p^R}, \quad (3.56)$$

where the bracket of the martingale  $M^{n,p^R}$  is of order  $1/n$  as in (3.53), we obtain to the limit,

$$\bar{p}_t^R = \int_0^t \left( \beta \bar{p}_s^I - r \bar{p}_s^I \bar{p}_s^R \right) ds \quad (3.57)$$

by using arguments similar as for  $\bar{p}_t^I$ . The proof is then achieved by (3.55).  $\blacksquare$

Notice that the system of Equations (1.7, 1.9, 1.10) is closed. Miller [18] shows that this system can be reduced to only two equations by considering the variables  $\theta_t$  and  $\phi_t := \bar{p}_t^I \theta_t$ , and proves that the equation for  $\theta_t$  in (1.2) together with (1.9, 1.10) are equivalent to the two following ones:

$$\frac{d\theta_t}{dt} = -r\phi_t, \quad \frac{d\phi_t}{dt} = \left( -r - \beta + r \frac{g''(\theta_t)}{g'(1)} \right) \phi_t.$$

## A Finite measures on $\mathbb{N}$

First, some notation is needed in order to clarify the way the atoms of a given element of  $\mathcal{M}_F(\mathbb{N})$  are ranked. For all  $\mu \in \mathcal{M}_F(\mathbb{N})$ , let  $F_\mu$  be its cumulative distribution function and  $F_\mu^{-1}$  be its right inverse defined as

$$\forall x \in \mathbb{R}_+, F_\mu^{-1}(x) = \inf\{i \in \mathbb{N}, F_\mu(i) \geq x\}. \quad (A.1)$$

Let  $\mu = \sum_{n \in \mathbb{N}} a_n \delta_n$  be an integer-valued measure of  $\mathcal{M}_F(\mathbb{N})$ , *i.e.* such that the  $a_n$ 's are integers themselves. Then, for each atom  $n \in \mathbb{N}$  of  $\mu$  such that  $a_n > 0$ , we duplicate the atom  $n$  with multiplicity  $a_n$ , and we rank the atoms of  $\mu$  by increasing values, sorting arbitrarily the atoms having the same value. Then, we denote for any  $i \leq \langle \mu, \mathbf{1} \rangle$ ,

$$\gamma_i(\mu) = F_\mu^{-1}(i), \quad (A.2)$$

the level of the  $i^{\text{th}}$  atom of the measure, when ranked as described above. We refer to Example 1 for a simple illustration.

We now make precise a few topological properties of spaces of measures and measure-valued processes. For  $T > 0$  and a Polish space  $(E, d_E)$ , we denote by  $\mathcal{D}([0, T], E)$  the Skorokhod space of càdlàg (right-continuous left-limited) functions from  $\mathbb{R}$  to  $E$  (*e.g.* [6, 16]) equipped with the Skorokhod topology induced by the metric

$$d_T(f, g) := \inf_{\alpha \in \Delta([0, T])} \left\{ \sup_{\substack{(s,t) \in [0, T]^2 \\ s \neq t}} \left| \log \frac{\alpha(s) - \alpha(t)}{s - t} \right| + \sup_{t \leq T} d_E(f(t), g(\alpha(t))) \right\}, \quad (A.3)$$

where the infimum is taken over the set  $\Delta([0, T])$  of continuous increasing functions  $\alpha : [0, T] \rightarrow [0, T]$  such that  $\alpha(0) = 0$  and  $\alpha(T) = T$ .

Limit theorems are heavily dependent on the topologies considered. We introduce here several technical lemmas on the space of measures related to these questions. For any fixed  $0 \leq \varepsilon < A$ , we define the set

$$\mathcal{M}_{\varepsilon, A} := \{ \nu \in \mathcal{M}_F(\mathbb{N}); \langle \nu, \mathbf{1} + \chi^5 \rangle \leq A \text{ and } \langle \nu, \chi \rangle \geq \varepsilon \}. \quad (\text{A.4})$$

Remark that for any  $\nu \in \mathcal{M}_{\varepsilon, A}$ , and  $i \in \{0, \dots, 5\}$ ,  $\langle \nu, \chi^i \rangle \leq A$  since the support of  $\nu$  is included in  $\mathbb{N}$ .

**Lemma A.1.** *Let  $\mathfrak{J}$  a set and a family  $(\nu_\tau, \tau \in \mathfrak{J})$  of elements of  $\mathcal{M}_{\varepsilon, A}$ . Then, for any real function  $f$  on  $\mathbb{N}$  such that  $f(k) = o(k^5)$ , we have that*

$$\lim_{K \rightarrow \infty} \sup_{\tau \in \mathfrak{J}} |\langle \nu_\tau, f \mathbf{1}_{[K, \infty)} \rangle| = 0.$$

*Proof.* By Markov inequality, for any  $\tau \in \mathfrak{J}$ , for any  $K$ , we have

$$\sum_{k \geq K} |f(k)| \nu_\tau(k) \leq A \sup_{k \geq K} \frac{|f(k)|}{k^5},$$

hence

$$\lim_{K \rightarrow \infty} \sup_{\tau \in \mathfrak{J}} |\langle \nu_\tau, f \rangle| \leq A \limsup_{k \rightarrow \infty} \frac{|f(k)|}{k^5} = 0.$$

The proof is thus complete. ■

**Lemma A.2.** *For any  $A > 0$ , the set  $\mathcal{M}_{\varepsilon, A}$  is a closed subset of  $\mathcal{M}_F(\mathbb{N})$  embedded with the topology of weak convergence.*

*Proof.* Let  $(\mu_n)_{n \in \mathbb{N}}$  be a sequence of  $\mathcal{M}_{\varepsilon, A}$  converging to  $\mu \in \mathcal{M}_F(\mathbb{N})$  for the weak topology, which implies in particular that  $\lim_{n \rightarrow +\infty} \mu_n(k) = \mu(k)$  for any  $k \in \mathbb{N}$ . Denoting for all  $n$  and  $k \in \mathbb{N}$ ,  $f_n(k) = k^5 \mu_n(k)$ , we have that  $\lim_{n \rightarrow +\infty} f_n(k) = f(k) := k^5 \mu(k)$ ,  $\mu$ -a.e., and Fatou's lemma implies

$$\langle \mu, \chi^5 \rangle = \sum_{k \in \mathbb{N}} f(k) \leq \liminf_{n \rightarrow \infty} \sum_{k \in \mathbb{N}} f_n(k) = \liminf_{n \rightarrow \infty} \langle \mu_n, \chi^5 \rangle.$$

Since  $\langle \mu_n, \mathbf{1} \rangle$  tends to  $\langle \mu, \mathbf{1} \rangle$ , we have that  $\langle \mu, \mathbf{1} + \chi^5 \rangle \leq A$ .

Furthermore, by uniform integrability (Lemma A.1), it is also clear that

$$\varepsilon \leq \lim_{n \rightarrow \infty} \langle \mu_n, \chi \rangle = \langle \mu, \chi \rangle,$$

which shows that  $\mu \in \mathcal{M}_{\varepsilon, A}$ . ■

**Lemma A.3.** *The traces on  $\mathcal{M}_{\varepsilon, A}$  of the total variation topology and of the weak topology coincide.*

*Proof.* It is well known that the total variation topology is coarser than the weak topology. In the reverse direction, assume that  $(\mu_n)_{n \in \mathbb{N}}$  is a sequence of weakly converging measures belonging to  $\mathcal{M}_{\varepsilon, A}$ . Since,

$$d_{TV}(\mu_n, \mu) \leq \sum_{k \in \mathbb{N}} |\mu_n(k) - \mu(k)|.$$

according to Lemma A.1, it is then easily deduced that the right-hand-side converges to 0 as  $n$  goes to infinity. ■

**Lemma A.4.** *If the sequence  $(\mu_n)_{n \in \mathbb{N}}$  of  $\mathcal{M}_{\varepsilon, A}^{\mathbb{N}}$  converges weakly to the measure  $\mu \in \mathcal{M}_{\varepsilon, A}$ , then  $(\langle \mu_n, f \rangle)_{n \in \mathbb{N}}$  converges to  $\langle \mu, f \rangle$  for all function  $f$  such that  $f(k) = o(k^5)$  for all large  $k$ .*

*Proof.* Triangular inequality says that:

$$|\langle \mu_n, f \rangle - \langle \mu, f \rangle| \leq |\langle \mu_n, f \mathbb{1}_{[0, K]} \rangle - \langle \mu, f \mathbb{1}_{[0, K]} \rangle| + |\langle \mu, f \mathbb{1}_{(K, +\infty)} \rangle| + |\langle \mu_n, f \mathbb{1}_{(K, +\infty)} \rangle|.$$

We then conclude by uniform integrability and weak convergence. ■

Recall that  $\mathcal{M}_{\varepsilon, A}$  can be embedded with the total variation distance topology, hence the topology on  $\mathcal{D}([0, T], \mathcal{M}_{\varepsilon, A})$  is induced by the distance

$$\rho_T(\mu, \nu) = \inf_{\alpha \in \Delta([0, T])} \left( \sup_{\substack{(s, t) \in [0, T]^2 \\ s \neq t}} \left| \log \frac{\alpha(s) - \alpha(t)}{s - t} \right| + \sup_{t \leq T} d_{TV}(\mu_t, \nu_{\alpha(t)}) \right).$$

**Lemma A.5.** *For any  $p \leq 5$ , the following map is continuous:*

$$\Phi_p : \begin{cases} \mathcal{D}(\mathbb{R}_+, \mathcal{M}_{\varepsilon, A}) & \longrightarrow \mathcal{D}(\mathbb{R}_+, \mathbb{R}) \\ \nu & \longmapsto \langle \nu, \chi^p \rangle. \end{cases}$$

*Proof.* It is sufficient to prove the continuity of the above mappings from  $\mathcal{D}([0, T], \mathcal{M}_{\varepsilon, A})$  to  $\mathcal{D}([0, T], \mathbb{R})$ , for any  $T \geq 0$ , where the latter are equipped with the corresponding Skorokhod topologies. For  $\mu$  and  $\nu$  two elements of  $\mathcal{M}_{\varepsilon, A}$ , for any  $p \leq 5$ , for any positive integer  $K$ , according to Markov inequality,

$$|\langle \mu, \chi^p \rangle - \langle \nu, \chi^p \rangle| \leq 2 \frac{A}{K^p} + |\langle \mu - \nu, \chi^p \mathbb{1}_{[0, K]} \rangle| \leq 2 \frac{A}{K^p} + K^p d_{TV}(\mu, \nu). \quad (\text{A.5})$$

Using (A.3) and (A.5) we have for any  $K > 0$ :

$$d_T(\langle \mu, \chi^p \rangle, \langle \nu, \chi^p \rangle) \leq 2 \frac{A}{K^p} + K^p d_T(\mu, \nu),$$

and hence the continuity of  $\Phi_p$ . ■

**Thanks** This work benefited from the support of the ANR Viroscopy (ANR-08-SYSC-016-03) and MANEGE (ANR-09-BLAN-0215), and from the "Chair Modélisation Mathématique de la Biodiversité of Veolia Environnement-Ecole Polytechnique-Museum National d'Histoire Naturelle-Fondation X". The last author thanks T.L. Parsons for the invitation to the DIMACS *Workshop on Stochasticity in Population and Disease Dynamics* in December 2008 and L.M. Wahl for discussions at this workshop on epidemics on graphs.

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