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# SOME GENERALIZED EUCLIDEAN AND 2-STAGE EUCLIDEAN NUMBER FIELDS THAT ARE NOT NORM-EUCLIDEAN

JEAN-PAUL CERRI

ABSTRACT. We give examples of Generalized Euclidean but not norm-Euclidean number fields of degree greater than 2. In the same way we give examples of 2-stage Euclidean but not norm-Euclidean number fields of degree greater than 2. In both cases, no such examples were known.

## 1. INTRODUCTION

In 1985, Johnson, Queen and Sevilla [9] introduced a generalization of the classical notion of Euclidean number field.

**Definition 1.1.** A number field  $K$  is said to be *Generalized Euclidean* or simply *G.E.* if for every  $(\alpha, \beta) \in \mathbb{Z}_K \times \mathbb{Z}_K \setminus \{0\}$  such that the ideal  $(\alpha, \beta)$  is principal, there exists  $\Upsilon \in \mathbb{Z}_K$  such that

$$|N_{K/\mathbb{Q}}(\alpha - \Upsilon\beta)| < |N_{K/\mathbb{Q}}(\beta)|.$$

If  $(\alpha, \beta)$  is principal, we thus have at our disposal the Euclidian algorithm to compute a gcd of  $\alpha$  and  $\beta$  because it is easy to see that  $(\beta, \alpha - \Upsilon\beta)$  is principal again, and so on. Note that if  $K$  is norm-Euclidean then  $K$  is G.E. and that if  $K$  is principal, i.e. has class number 1, then  $K$  is G.E. if and only if  $K$  is norm-Euclidean. If we want to illustrate the difference between “G.E.” and “norm-Euclidean”, the interesting case is when  $K$  is G.E. but not principal (so not norm-Euclidean). The following result was established by Johnson, Queen and Sevilla in [9].

**Theorem 1.1.** *The quadratic number field  $\mathbb{Q}(\sqrt{d})$  is G.E. but not norm-Euclidean for  $d = 10$  and  $d = 65$ . The quadratic number field  $\mathbb{Q}(\sqrt{d})$  is not G.E. for  $d = 15, 26, 30, 35, 39, 51, 78, 87, 102, 115, 195$  and 230.*

Furthermore, Johnson, Queen and Sevilla conjectured that  $K = \mathbb{Q}(\sqrt{d})$  (with  $d > 1$  squarefree) is G.E. if and only if  $K$  is norm-Euclidean or  $d = 10$  or 65.

Another variation on norm-Euclidean number fields has been introduced by Cooke [7].

**Definition 1.2.** Let  $m$  be a rational integer  $\geq 1$ . The number field  $K$  is *m-stage Euclidean* if and only if for every  $\alpha \in \mathbb{Z}_K$  and every  $\beta \in \mathbb{Z}_K \setminus \{0\}$  there exists a positive rational integer  $k \leq m$  and  $k$  pairs  $(q_i, r_i)$  ( $1 \leq i \leq k$ ) of elements of  $\mathbb{Z}_K$

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such that

$$\begin{aligned}\alpha &= \beta q_1 + r_1, \\ \beta &= r_1 q_2 + r_2, \\ &\vdots \\ r_{k-2} &= r_{k-1} q_k + r_k, \\ \text{and } |N_{K/\mathbb{Q}}(r_k)| &< |N_{K/\mathbb{Q}}(\beta)|.\end{aligned}$$

When it is well defined, let us put

$$[q_1, q_2, \dots, q_k] = q_1 + \frac{1}{q_2 + \frac{1}{q_3 + \dots + \frac{1}{q_k}}} = \frac{a_k}{b_k},$$

where  $a_k$  and  $b_k$  are given by

$$\begin{aligned}a_1 &= q_1, & b_1 &= 1, \\ a_2 &= a_1 q_2 + 1, & b_2 &= q_2,\end{aligned}$$

and recursively for  $k \geq 3$  by

$$a_k = a_{k-1} q_k + a_{k-2}, \quad b_k = q_k b_{k-1} + b_{k-2}.$$

Since

$$\frac{\alpha}{\beta} = \frac{a_k}{b_k} + (-1)^{k+1} \frac{r_k}{b_k \beta},$$

this definition is equivalent to the following.

**Definition 1.3.** The number field  $K$  is *m-stage Euclidean* if and only if for every  $\xi \in K$ , there exists a positive rational integer  $k \leq m$ , and  $k$  elements  $q_1, q_2, \dots, q_k \in \mathbb{Z}_K$  such that

$$\left| N_{K/\mathbb{Q}}(\xi - [q_1, q_2, \dots, q_k]) \right| < \frac{1}{|N_{K/\mathbb{Q}}(b_k)|}.$$

As in the previous case, norm-Euclidean implies *m-stage Euclidean*, but contrary to what happens with the G.E. condition, we have the following result [7].

**Theorem 1.2.** *A number field  $K$  with unit rank  $r \geq 1$  (i.e.  $r = \text{rank}(\mathbb{Z}_K^*) \geq 1$ ) is principal if and only if  $K$  is *m-stage Euclidean* for some  $m$ .*

As a consequence, if we want to study the difference between *m-stage Euclidean* and norm-Euclidean, we have to look at number fields with class number 1 and find some example where  $K$  is principal, *m-stage Euclidean* but not norm-Euclidean. The following result was established by Cooke [7].

**Theorem 1.3.** *For  $d = 14, 22, 23, 31, 38, 43, 46, 53, 61, 69, 89, 93, 97$ ,  $\mathbb{Q}(\sqrt{d})$  is 2-stage euclidean but not norm-Euclidean.*

Furthermore, Cooke and Weinberger [8] proved that, under GRH, every principal number field  $K$  with unit rank  $r \geq 1$  is 4-stage Euclidean, and even 2-stage Euclidean if  $K$  has at least one real place.

For both notions (G.E. and  $m$ -stage Euclidean), no examples of number fields of degree greater than 2 were known. Our main results are the following.

**Theorem 1.4.** *None of the totally real number fields enumerated in Table 1 are principal. They all are G.E. except for the second cubic number field of discriminant 3969, defined by  $x^3 - 21x - 35$ , which is neither principal nor G.E.*

$n$	$D_K$	$P(x)$	$h$	$M(K)$
3	1957	$x^3 - x^2 - 9x + 10$	2	2
3	2597	$x^3 - x^2 - 9x + 8$	3	5/2
3	2777	$x^3 - x^2 - 14x + 23$	2	5/3
3	3969 <sup>1</sup>	$x^3 - 21x - 28$	3	4/3
3	3969	$x^3 - 21x - 35$	3	7/3
3	3981	$x^3 - x^2 - 11x + 12$	2	3/2
3	4212	$x^3 - 12x - 10$	3	7/2
3	4312	$x^3 - x^2 - 16x + 8$	3	11/4
3	5684	$x^3 - 14x - 14$	3	9/2
4	21025	$x^4 - 17x^2 + 36$	2	1
4	32625	$x^4 - x^3 - 19x^2 + 4x + 76$	2	1
4	46400	$x^4 - 22x^2 + 116$	2	5/4
4	51200	$x^4 - 20x^2 + 50$	2	7/2

TABLE 1. Here,  $n$  is the degree of the field  $K$ ,  $D_K$  its discriminant,  $P(x)$  its equation,  $h$  its class number and  $M(K)$  its Euclidean minimum.

**Theorem 1.5.** *The totally real number fields of degree 3 and of discriminants  $< 15000$  which are principal but not norm-Euclidean (82 cases) are 2-stage norm-Euclidean. The same is true for degree 4 and discriminants 18432, 34816, 35152 and for degree 5 and discriminant 390625. In all these cases, the number field is principal, not norm-Euclidean, but 2-stage norm-Euclidean.*

Details on the number fields appearing in Theorem 1.5 are available from [6] and are given in the online version of the paper. In Section 2, we recall other definitions and general results. In Section 3 and 4, we study the case of Generalized Euclidean number fields and the case of 2-stage Euclidean number fields, respectively.

## 2. THE ALGORITHM, GENERALITIES

Let  $K$  be a number field of degree  $n$ . We have designed an algorithm which allows us to compute the Euclidean minimum of  $K$ , in particular when  $K$  is totally real [5], but also in the general case [3]. According to theoretical results [4], this algorithm can also give the upper part of the Euclidean spectrum of  $K$  and this yields new examples of number fields with interesting properties.

<sup>1</sup>In [2] and [10] the Euclidean minimum of the number field with discriminant 3969, defined by  $x^3 - 21x - 28$  was erroneously announced to be 1.

From now on, we suppose that  $K$  is totally real and that  $n > 2$ . We denote by  $\mathbb{Z}_K$  the ring of its integers and by  $N_{K/\mathbb{Q}}$  its absolute norm. The *Euclidean minimum* of an element  $\xi \in K$  is

$$m_K(\xi) = \inf_{\Upsilon \in \mathbb{Z}_K} |N_{K/\mathbb{Q}}(\xi - \Upsilon)|$$

and the *Euclidean minimum* of  $K$  is

$$M(K) = \sup_{\xi \in K} m_K(\xi).$$

The set of values taken by  $m_K$  is called the *Euclidean spectrum* of  $K$ . We know the following important result [4].

**Theorem 2.1.** *The Euclidean spectrum of  $K$  is the union of  $\{0\}$  and of a strictly decreasing sequence of rationals  $(r_i)_{i \geq 0}$  with limit 0. For each  $i$ , the set of  $\xi \in K$  such that  $m_K(\xi) = r_i$  is finite modulo  $\mathbb{Z}_K$ .*

In fact, we have a stronger result, which can be formulated in terms of the inhomogeneous spectrum. However, we shall not need this in what follows.

**Corollary 2.2.** *The set of  $\xi \in K$  such that  $m_K(\xi) \geq 1$  is finite modulo  $\mathbb{Z}_K$ .*

Recall now that we have at our disposal an algorithm which can give us all the  $\xi \in K$  such that  $m_K(\xi) \geq 1$ . Without going into details – these can be found in [5] – let us give nevertheless the theorem which justifies the algorithm and the main ideas that are behind it. Let us choose a constant  $k > 0$  and let us embed  $K$  into  $K \otimes_{\mathbb{Q}} \mathbb{R} =: \overline{K}$ , which we can identify with  $\mathbb{R}^n$ , in which  $\mathbb{Z}_K$  is a lattice. Under this identification an element  $\xi$  of  $K$  is viewed as  $(\sigma_i(\xi))_{1 \leq i \leq n}$ , where the  $\sigma_i$  are the embeddings of  $K$  into  $\mathbb{R}$ . The map  $m_K$  extends to a map  $m_{\overline{K}}$  from  $\mathbb{R}^n$  to  $\mathbb{R}^+$  in a natural way:

$$m_{\overline{K}}(x) = \inf_{\Upsilon \in \mathbb{Z}_K} \left| \prod_{i=1}^n (x_i - \sigma_i(\Upsilon)) \right|.$$

Moreover, the product of two elements of  $K$  is extended to the product coordinate by coordinate in  $\mathbb{R}^n$ . This new product of two elements  $x, y \in \mathbb{R}^n$  will be denoted by  $x \cdot y$ . Let finally  $\varepsilon$  be a non-torsion unit of  $\mathbb{Z}_K^*$ .

The main idea is to find in a fundamental domain  $\mathcal{F}$  associated to  $\mathbb{Z}_K$  in  $\mathbb{R}^n$ ,  $s$  distinct bounded sets  $\mathcal{T}_i$  ( $1 \leq i \leq s$ ) with the property that for each such  $\mathcal{T}_i$  there exists an  $X_i \in \mathbb{Z}_K$  and  $s_i$  integers  $n_{i,1}, \dots, n_{i,s_i}$  ( $s_i > 0$ ) such that

$$(1) \quad (\varepsilon \cdot \mathcal{T}_i - X_i) \setminus \mathcal{H} \subset \bigcup_{1 \leq l \leq s_i} \mathcal{T}_{n_{i,l}} \quad (i = 1, \dots, s),$$

where

$$\mathcal{H} = \{x \in \mathbb{R}^n \text{ such that } m_{\overline{K}}(x) \leq k\}.$$

We consider the  $\mathcal{T}_i$  as the vertices of a directed graph  $G$  and represent (1) by  $s_i$  directed edges whose tail is  $\mathcal{T}_i$  and whose respective heads are the  $\mathcal{T}_{n_{i,l}}$  ( $1 \leq l \leq s_i$ ). To describe such an edge of  $G$  we shall use the notation  $\mathcal{T}_i \rightarrow \mathcal{T}_{n_{i,l}}(X_i)$ . The set  $\mathcal{C}$  of simple cycles of  $G$  is nonempty and finite. Each element  $c$  of  $\mathcal{C}$  of length  $j$  is in the form of the circular path,  $\mathcal{T}'_0 \rightarrow \mathcal{T}'_1(X'_0) \dots \rightarrow \mathcal{T}'_{j-1}(X'_{j-2}) \rightarrow \mathcal{T}'_0(X'_{j-1})$ , for some subset  $\{\mathcal{T}'_1, \dots, \mathcal{T}'_{j-1}\} \subseteq \{\mathcal{T}_1, \dots, \mathcal{T}_s\}$ , where  $X'_i$  denotes the element  $X \in \mathbb{Z}_K$

associated to  $\mathcal{T}'_i$ . This defines, in a unique way,  $j$  elements of  $K$ ,  $\xi_0, \dots, \xi_{j-1}$  by the formula:

$$\xi_r = \frac{\varepsilon^{j-1} X'_r + \varepsilon^{j-2} X'_{r+1} + \dots + X'_{j-1+r}}{\varepsilon^j - 1} \quad (r = 0, \dots, j-1),$$

the indices being read modulo  $j$ . In this context, we say that  $\xi_0, \dots, \xi_{j-1}$  are *associated* to the cycle  $c$ .

We denote by  $\mathcal{E}$  the *finite* set of all elements of  $K$  associated to the elements of  $\mathcal{C}$ . The  $\xi_i$  associated to a cycle  $c$  are in the same orbit modulo  $\mathbb{Z}_K$  under the action of  $\mathbb{Z}_K^*$  (in fact  $\xi_{r+1} = \varepsilon \cdot \xi_r - X'_r$ ) and satisfy

$$m_{\overline{K}}(\xi_0) = \dots = m_{\overline{K}}(\xi_{j-1}) =: m(c),$$

which is a rational number. Finally, define

$$m(G) = \max_{c \in \mathcal{C}} m(c) = \max_{\xi \in \mathcal{E}} m_{\overline{K}}(\xi).$$

Let us say that  $G$  is *convenient* if every infinite path of  $G$  is ultimately periodic. The essential result is the following.

**Theorem 2.3.** *Assume that  $G$  is convenient and that there exists  $\mathcal{T} \in \{\mathcal{T}_1, \dots, \mathcal{T}_s\}$  and  $x \in \mathbb{R}^n$  such that  $m_{\overline{K}}(x) > k$ . Then*

- i)  $m_{\overline{K}}(x) \leq m(G)$ .
- ii) *If  $x \in K$ , there exists  $\xi \in \mathcal{E}$  such that  $x \equiv \xi \pmod{\mathbb{Z}_K}$ .*

In this situation we know all the potential  $\xi \in K$  such that  $m_K(\xi) > k$ , and since computing  $m_K(\xi)$  is possible (again see [5] for more details), we know in fact all the  $\xi \in K$  such that  $m_K(\xi) > k$ . To identify the elements  $\xi \in K$  such that  $m_K(\xi) \geq 1$ , it is sufficient to run the algorithm with  $k = 0.999$ , for instance.

### 3. GENERALIZED EUCLIDEAN NUMBER FIELDS

**3.1. Generalities.** From the definition of a G.E. number field and the definition of the map  $m_K$ , we have the following result.

**Proposition 3.1.** *The field  $K$  is G.E. if and only if for every  $(\alpha, \beta) \in \mathbb{Z}_K \times \mathbb{Z}_K \setminus \{0\}$  such that  $m_K(\alpha/\beta) \geq 1$ , the ideal  $(\alpha, \beta)$  is not principal.*

*Remark 1.* Suppose that we have at our disposal the finite set  $S$  of all  $\xi \in K$  (modulo  $\mathbb{Z}_K$ ) such that  $m_K(\xi) \geq 1$ , and that for each such  $\xi$  we have a representative  $u/v$  where  $(u, v) \in \mathbb{Z}_K \times \mathbb{Z}_K \setminus \{0\}$ . Let  $(\alpha, \beta) \in \mathbb{Z}_K \times \mathbb{Z}_K \setminus \{0\}$  such that  $m_K(\alpha/\beta) \geq 1$ . Then there exists  $\xi \equiv u/v$  in  $S$  such that  $\alpha/\beta = u/v + \gamma$  with  $\gamma \in \mathbb{Z}_K$ . Since

$$(\alpha, \beta) = (\beta u/v + \gamma \beta, \beta) = (\beta u/v, \beta) = \beta/v(u, v),$$

it is sufficient, for proving that  $K$  is G.E., to check that for every  $\xi \equiv u/v \in S$ ,  $(u, v)$  is not principal.

**3.2. A first example.** The purpose of this subsection is to study in detail a particular case. Other results, obtained in another way, will be given in the next subsection. Let  $K$  be the normal quartic field generated by any one of the roots of

$$P(X) = X^4 - 20X^2 + 50.$$

The field  $K$  is totally real, its discriminant is 51200, its class number is 2, and a  $\mathbb{Z}$ -basis of  $\mathbb{Z}_K$  is  $(e_1, e_2, e_3, e_4)$  with

$$e_1 = 1, \quad e_2 = \sqrt{2}, \quad e_3 = \sqrt{10 + 5\sqrt{2}}, \quad e_4 = \sqrt{10 - 5\sqrt{2}}.$$

Our algorithm shows that

$$M(K) = \frac{7}{2},$$

and that there is a unique  $\xi \in K$  (modulo  $\mathbb{Z}_K$ ) such that  $m_K(\xi) \geq 1$ . More precisely

$$\xi \equiv \frac{1}{2}(e_3 + e_4).$$

According to Remark 1, if we want to establish that  $K$  is G.E., we have just to prove that the ideal  $(2, e_3 + e_4)$  is not principal.

**Theorem 3.2.** *The field  $K$  is not norm-Euclidean but it is G.E.*

*Proof.* First of all, we note that  $e_3 + e_4 = e_2 \cdot e_3$ , so that we are reduced to proving that the ideal  $(e_2, e_3)$  is not principal. Suppose on the contrary that it is principal so that we have

$$e_2 \mathbb{Z}_K + e_3 \mathbb{Z}_K = \nu \mathbb{Z}_K,$$

with  $\nu \in \mathbb{Z}_K$ . Since  $N_{K/\mathbb{Q}}(e_2) = 4$  and  $N_{K/\mathbb{Q}}(e_3) = 50$ , we have

$$N_{K/\mathbb{Q}}(\nu) \mid 2 = \gcd(4, 50),$$

so that we have two possibilities : either  $\nu \in \mathbb{Z}_K^*$  or  $N_{K/\mathbb{Q}}(\nu) = \pm 2$ .

*First case :*  $\nu$  is a unit and we have in fact  $e_2 \mathbb{Z}_K + e_3 \mathbb{Z}_K = \mathbb{Z}_K$ .

In this case, there exist  $u, v \in \mathbb{Z}_K$  such that

$$(2) \quad 1 = e_2 \cdot u + e_3 \cdot v.$$

Let us write

$$(3) \quad \begin{cases} u &= a + be_2 + ce_3 + de_4 \\ v &= a' + b'e_2 + c'e_3 + d'e_4, \end{cases}$$

where  $a, b, c, d, a', b', c', d' \in \mathbb{Z}$ .

Since  $e_2 \cdot e_3 = e_3 + e_4$ ,  $e_2 \cdot e_4 = e_3 - e_4$  and  $e_3 \cdot e_4 = 5e_2$ , if we substitute (3) into (2) we obtain, by identification of the coefficients in our  $\mathbb{Z}$ -basis, that  $2b + 10c' = 1$ , which is clearly impossible.

*Second case :*  $\nu$  has norm  $\pm 2$ .

If  $\nu = a + be_2 + ce_3 + de_4$  where  $a, b, c, d \in \mathbb{Z}$ , an easy computation leads to

$$\begin{aligned} \pm 2 &= N_{K/\mathbb{Q}}(\nu) \\ &= a^4 + 4b^4 + 50c^4 + 50d^4 - 4a^2b^2 - 20a^2c^2 - 20a^2d^2 - 40b^2c^2 \\ &\quad - 40b^2d^2 + 100c^2d^2 + 40abc^2 - 40abd^2 + 200cd^3 - 200dc^3 + 80abcd. \end{aligned}$$

This implies that

$$\pm 2 \equiv (a^2 - 2b^2)^2 \pmod{5},$$

which is impossible as neither of  $\pm 2$  are quadratic residues (mod 5).  $\square$

**3.3. The Dedekind-Hasse criterion.** In this subsection, we study the link between G.E. and a Euclidean-type map that we shall deduce from the Dedekind-Hasse criterion. This will lead us to define an easy test which allows to find new examples, without requiring detailed calculations as above. First of all, recall the Dedekind-Hasse criterion (see for instance [11]).

**Theorem 3.3.** *A number field  $K$  has class number 1 if and only if for every  $\alpha, \beta \in \mathbb{Z}_K \setminus \{0\}$  such that  $\beta \nmid \alpha$ , there exist  $\gamma, \delta \in \mathbb{Z}_K$  such that*

$$(4) \quad 0 < |N_{K/\mathbb{Q}}(\alpha\gamma - \beta\delta)| < |N_{K/\mathbb{Q}}(\beta)|.$$

This leads to the following natural definition.

**Definition 3.1.** For every  $\xi \in K \setminus \mathbb{Z}_K$  we shall denote by  $h_K(\xi)$  the real number defined by

$$h_K(\xi) = \inf\{m_K(\Upsilon\xi); \Upsilon \in \mathbb{Z}_K \text{ and } \Upsilon\xi \notin \mathbb{Z}_K\}.$$

This map has the following elementary properties, which we give here without proof.

**Proposition 3.4.** *For every  $\xi \in K \setminus \mathbb{Z}_K$  we have*

- (1)  $0 < h_K(\xi) \leq m_K(\xi)$ ;
- (2) For every  $\alpha \in \mathbb{Z}_K$ ,  $h_K(\xi + \alpha) = h_K(\xi)$ ;
- (3) For every  $\varepsilon \in \mathbb{Z}_K^*$ ,  $h_K(\varepsilon\xi) = h_K(\xi)$ .

We can now reformulate Dedekind-Hasse criterion as follows.

**Theorem 3.5.** *A number field  $K$  has class number 1 if and only if for every  $\xi \in K \setminus \mathbb{Z}_K$  we have  $h_K(\xi) < 1$ .*

*Proof.* The norm being multiplicative, (4) can be reformulated as follows: for every  $\xi \in K \setminus \mathbb{Z}_K$  there exist  $\gamma, \delta \in \mathbb{Z}_K$  such that

$$(5) \quad 0 < |N_{K/\mathbb{Q}}(\gamma\xi - \delta)| < 1,$$

which leads to  $m_K(\xi) < 1$ . Since (5) cannot be true if  $\gamma\xi \in \mathbb{Z}_K$ , we have  $h_K(\xi) < 1$ . Conversely, since  $|N_{K/\mathbb{Q}}(\gamma\xi - \delta)| = 0$  implies  $\gamma\xi \in \mathbb{Z}_K$ , which is excluded in the definition of  $h_K$ , we see that if  $h_K(\xi) < 1$  then (5) is true.  $\square$

Now consider a number field  $K$  and put

$$S = \{\xi \in K; m_K(\xi) \geq 1\}.$$

Suppose that  $K$  is not norm-euclidean so that  $S \neq \emptyset$ . We have the following result.

**Theorem 3.6.** *One of the following three possibilities holds:*

- (1) For every  $\xi \in S$ ,  $h_K(\xi) < 1$ . Then  $K$  has class number 1 and is not G.E.
- (2) For every  $\xi \in S$ ,  $h_K(\xi) \geq 1$ . Then  $K$  is G.E. (and not principal).
- (3) There exist  $\xi, \mu \in S$  such that  $h_K(\xi) < 1$  and  $h_K(\mu) \geq 1$ . Then  $K$  is not principal. If in addition, there exists  $\xi = \alpha/\beta \in S$  (with  $\alpha, \beta \in \mathbb{Z}_K$ ) with  $h_K(\xi) < 1$  and such that  $(\alpha, \beta)$  is principal, then  $K$  is not G.E. Otherwise it is G.E.

*Proof.* Clearly we have the three cases.

*Case 1.* The result is a consequence of Theorem 3.5 and of the fact that when the field is principal norm-Euclidean and G.E. are synonymous.



*Case 2.* Theorem 3.5 indicates that  $K$  is not principal. By Proposition 3.1 it is sufficient to prove that for every  $\xi = \alpha/\beta \in S$  where  $\alpha, \beta \in \mathbb{Z}_K$ , the ideal  $(\alpha, \beta)$  is not principal. Otherwise, we have  $(\alpha, \beta) = \nu \mathbb{Z}_K$  with  $\nu \in \mathbb{Z}_K$ . By hypothesis  $h_K(\xi) \geq 1$  so that for every  $X, Y \in \mathbb{Z}_K$  with  $X\xi \notin \mathbb{Z}_K$  we have

$$|N_{K/\mathbb{Q}}(X\alpha - Y\beta)| \geq |N_{K/\mathbb{Q}}(\beta)|.$$

Now  $\nu$  can be written  $\nu = X\alpha - Y\beta$  with  $X, Y \in \mathbb{Z}_K$  and  $X\xi \notin \mathbb{Z}_K$ . Otherwise  $\nu \in \beta \mathbb{Z}_K$  so that  $\beta \mid \nu$ . But this implies that  $\nu$  and  $\beta$  are associates and we have  $(\alpha, \beta) = \beta \mathbb{Z}_K$  which implies  $\beta \mid \alpha$  and  $\xi \in \mathbb{Z}_K$ , which is impossible. We deduce from this that  $|N_{K/\mathbb{Q}}(\nu)| \geq |N_{K/\mathbb{Q}}(\beta)|$ . Since  $N_{K/\mathbb{Q}}(\nu) \mid N_{K/\mathbb{Q}}(\beta)$  we have  $|N_{K/\mathbb{Q}}(\nu)| = |N_{K/\mathbb{Q}}(\beta)|$ , and since  $\nu \mid \beta$ ,  $\nu$  and  $\beta$  are associates, which is impossible by the previous argument.

*Case 3.* Theorem 3.5 indicates that  $K$  is not principal. The second assertion is a consequence of Proposition 3.1. Indeed, as previously, if  $h_K(\xi) \geq 1$  and  $\xi = \alpha/\beta$  then  $(\alpha, \beta)$  is not principal and this case is not an obstruction for  $K$  to be G.E. Finally, the only possibilities for contradicting G.E. come from the  $\xi = \alpha/\beta \in S$  such that  $h_K(\xi) < 1$  and  $(\alpha, \beta)$  is principal.  $\square$

**Corollary 3.7.** *Suppose that  $K$  is not norm-Euclidean and that, with the above notation,  $S$  modulo  $\mathbb{Z}_K$  is composed of a single orbit under the (multiplicative) action of  $\mathbb{Z}_K^*$  modulo  $\mathbb{Z}_K$ , i.e. that if  $\xi, \mu \in S$  there exists an  $\varepsilon \in \mathbb{Z}_K^*$  and an  $\alpha \in \mathbb{Z}_K$  such that  $\mu = \varepsilon\xi + \alpha$ . Then either  $K$  is principal and not G.E. or  $K$  is not principal but is G.E.*

*Proof.* If  $K$  is principal, we are in case 1. Otherwise, since all the elements of  $S$ , which are in the same orbit, have the same image by  $h_K$  (Proposition 3.4), we cannot be in case 3 of Theorem 3.6. Finally, we are in case 2 and  $K$  is G.E.  $\square$

*Remark 2.* To simplify notation and vocabulary, we shall often, by abuse of language, speak of  $\xi \in K$  to mean  $\xi \in K \bmod \mathbb{Z}_K$ . For instance we shall speak of orbits in  $S$  under the action of  $\mathbb{Z}_K^*$ ; in this context  $S$  and these orbits should be understood modulo  $\mathbb{Z}_K$ .

**Corollary 3.8.** *The totally real number fields of degree 3 and discriminants 1957, 2777, 3981 (see Table 1) are G.E. The totally real number fields of degree 4 and discriminants 46400 and 51200 (see Table 1) are G.E.*

*Proof.* In fact, in all these cases, our algorithm establish that we are under the previous hypotheses. For discriminant 1957, we have  $M(K) = 2$  and one orbit with one element in  $S$ . For discriminant 2777, we have  $M(K) = 5/3$  and one orbit with 2 elements in  $S$ . For discriminant 3981, we have  $M(K) = 3/2$  and one orbit with one element in  $S$ . For discriminant 46400, we have  $M(K) = 5/4$  and one orbit with 3 elements in  $S$ . For discriminant 51200, we have  $M(K) = 7/2$  and one orbit with one element in  $S$ .  $\square$

Now, if there are several orbits in  $S$ , and we want to use Theorem 3.6, we have to see whether, for one element  $\xi$  by orbit, and for every orbit, we have  $h_K(\xi) \geq 1$ , in which case necessarily  $K$  is G.E. The problem is now: how can we compute  $h_K(\xi)$ ? Our algorithm gives us every such  $\xi$  by its coordinates in a  $\mathbb{Z}$ -basis of  $\mathbb{Z}_K$ . These coordinates are of the form  $(a_1/d, a_2/d, \dots, a_n/d)$  where  $a_i \in \mathbb{Z}$  for every  $i$  and  $d \in \mathbb{Z}_{>0}$ . Furthermore we can compute  $m_K(\mu)$  for every  $\mu \in K$ . Hence, it is

easy to see that, to compute  $h_K(\xi)$ , it is sufficient to compute  $m_K(\Upsilon\xi)$  for every  $\Upsilon$  with coordinates in  $\{0, 1, \dots, d-1\}$  for our basis, such that  $\Upsilon\xi \notin \mathbb{Z}_K$ . This is easy to check. By definition, the value of  $h_K(\xi)$  will be the minimum of these  $m_K(\Upsilon\xi)$ . Of course if for every  $\xi$  and every such  $\Upsilon$  we have  $\Upsilon\xi \in S \bmod \mathbb{Z}_K$ , then  $K$  is G.E. Using this last approach we have established the following result.

**Theorem 3.9.** *The following totally real number fields of degree  $n$  are G.E. but not norm-Euclidean :*

- when  $n = 3$ , the fields in Table 1 with discriminants 2597, 4212, 4312, 5684;
- when  $n = 4$ , the fields in Table 1 with discriminants 21025, 32625.

*Proof.* We just give a typical example. For  $n = 3$  and discriminant 2597, we have two orbits in  $S$ , the first one  $O_1$  with two elements  $(\pm(e_1 + 2e_2 + 2e_3))/3$  modulo  $\mathbb{Z}_K$  where  $(e_i)$  is the  $\mathbb{Z}$ -basis of  $\mathbb{Z}_K$  returned by PARI [1]) and the second one  $O_2$  with one element  $((e_1 + e_2 + e_3)/2$  modulo  $\mathbb{Z}_K$ ). Then we can easily check that  $\mathbb{Z}_K \cdot O_1 = O_1 \cup \{0\}$  and that  $\mathbb{Z}_K \cdot O_2 = O_2 \cup \{0\}$ . The same thing happens in other cases with sometimes more complicated equalities but always with  $\mathbb{Z}_K \cdot O \subseteq S \cup \{0\}$ .  $\square$

*Remark 3.* If we want to treat all the non-principal number fields of degree 3 and discriminant  $< 6000$ , it remains to study the two number fields with discriminant 3969. In these cases, our previous method does not work, because we have some  $\xi = \alpha/\beta \in S$  such that  $h_K(\xi) < 1$ . The first one,  $K_1$ , is defined by  $x^3 - 21x - 28$ . For this field,  $S$  is composed of five orbits  $O_i$ ,  $1 \leq i \leq 5$ . For four of them, say for  $1 \leq i \leq 4$ , we have  $\mathbb{Z}_K \cdot O_i \subseteq S \cup \{0\}$ ; but for the last one,  $O_5$ , this is not true. Take an element  $\alpha/\beta$  of  $O_5$ : here we can take  $\alpha = 3e_1 + 2e_2 + 2e_3$  and  $\beta = 6$  where  $(e_1, e_2, e_3)$  is the  $\mathbb{Z}$ -basis returned by PARI [1]. We can then prove directly as in Section 3.2 that the ideal  $(\alpha, \beta)$  is not principal. We conclude that  $K_1$  is G.E.

For the second field,  $K_2$ , defined by  $x^3 - 21x - 35$  the situation is different. Here  $S$  is composed of seven orbits  $O_i$ ,  $1 \leq i \leq 7$  and four of them, say  $O_i$  with  $1 \leq i \leq 4$ , are such that  $\mathbb{Z}_K \cdot O_i \subseteq S \cup \{0\}$ . Now if we look at the three others, we find that two of them contain an  $\alpha/\beta$  for which  $(\alpha, \beta)$  is principal. For completeness these  $(\alpha, \beta)$  are  $(7e_1 + 12e_2 + 4e_3, 21)$  and  $(7e_1 + 5e_2 + 11e_3, 21)$  with the usual notation. Consequently  $K_2$  is not G.E. All the computations, which are long and complicated – in particular for  $K_2$  – have been done by hand and checked using PARI [1]. We do not give them here for lack of space; anyway they are not especially enlightening.

Finally, we put all these results together to give us Theorem 1.4.

#### 4. THE 2-STAGE EUCLIDEAN NUMBER FIELDS

Let us begin with an example. Let  $K$  be the totally real cubic number field with discriminant 3988, defined by  $x^3 - 16x - 4$ . Using our algorithm we see that the upper part of the Euclidean spectrum of  $K$  has five elements:

$$\text{sp}(K) \cap [1, \infty) = \{19/8, 11/8, 5/4, 19/16, 133/128\}.$$

The set  $S$  is composed of five orbits, respectively the orbits of  $ae_1 + be_2 + ce_3$  with  $(a, b, c) = (0, 1/2, 1/2)$ ,  $(1/2, 1/2, 0)$ ,  $(1/2, 1/2, 1/2)$ ,  $(0, 3/4, 1/2)$  and  $(0, 3/8, 1/2)$ , where  $(e_1, e_2, e_3)$  is the  $\mathbb{Z}$ -basis of  $\mathbb{Z}_K$  returned by PARI [1]. These orbits have respectively 1, 1, 1, 2 and 4 elements. For one element  $\xi$  by orbit, we try to find

$q_1, q_2 \in \mathbb{Z}_K$  such that

$$(6) \quad \left| N_{K/\mathbb{Q}}\left(\xi - q_1 - \frac{1}{q_2}\right) \right| < \frac{1}{|N_{K/\mathbb{Q}}(q_2)|},$$

by testing “small”  $q_1 \in \mathbb{Z}_K$  and “small”  $q_2 \in \mathbb{Z}_K \setminus \{0\}$ . In each case this is possible, so that for every  $\xi \in S$ , (6) is true. Finally this implies that  $K$  is 2-stage norm-Euclidean. Using exactly the same approach we have established the results of Theorem 1.5.

*Remark 4.* Obviously these fields, which are principal and not norm-Euclidean, are not G.E.

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