Stochastic Approximation with Averaging Innovation

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Abstract

The aim of the paper is to establish a convergence theorem for multi-dimensional stochastic approximation when the "innovations" satisfy some "light" averaging properties in the presence of a pathwise Lyapunov function. These averaging assumptions allow us to unify apparently remote frameworks where the innovations are simulated (possibly deterministic like in Quasi-Monte Carlo simulation) or exogenous (like market data) with ergodic properties. We propose several fields of applications and illustrate our results on five examples mainly motivated by Finance.

Keywords *Stochastic Approximation, mixing processes, Gàl-Koksma theorem, stationary processes, quasi-Monte Carlo, sequence with low discrepancy, two-armed bandit algorithm, calibration, optimal asset allocation, VaR-CVaR, ergodic control.*

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1 Introduction

This paper presents convergence results for stochastic approximation of Robbins-Monro type (see [30] for the original paper), namely

$$\theta_{n+1} = \theta_n - \gamma_{n+1} H(\theta_n, Y_n), \tag{1.1}$$

when the "innovation" sequence $(Y_n)_{n\geq 0}$ satisfies some "elementary" averaging assumptions. In fact, we will consider a slightly more general setting which includes an extra noisy term

$$\theta_{n+1} = \theta_n - \gamma_{n+1} \left(H(\theta_n, Y_n) + \Delta M_{n+1} \right), \tag{1.2}$$

when $(\Delta M_n)_{n\geq 1}$ is a sequence of martingale increments. Our aim is to unify several frameworks: the first one is the case of generated (possibly non random) innovations to solve problems coming from Numerical Probability, typically low discrepancy sequences. The second one deal with the case of exogenous data satisfying some properties, light ergodic but sharing a priori no Markov property like in the procedures extensively investigated in [3] and more recently in [7].

To establish the *a.s.* convergence of the sequence $(\theta_n)_{n\geq 0}$ toward its "target" θ^* (to be specified later on), the idea is to make the assumption that the innovation sequence $(Y_n)_{n\geq 0}$ satisfies an *averaging*

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property in a "linear" setting: typically that, for a wide enough class $\mathcal{V}$ of integrable functions (with respect to a probability measure ν),

$$\forall f \in \mathcal{V}, \quad \frac{1}{n}\sum_{k=0}^{n-1} f(Y_k) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^q} f d\nu \tag{1.3}$$

at a common rate of convergence to be specified further on. If $\mathcal{V} \supset \mathcal{C}_b(\mathbb{R}^q, \mathbb{R}^q)$, this implies

$$\frac{1}{n}\sum_{k=0}^{n-1} \delta_{Y_k} \underset{n\to\infty}{\overset{(\mathbb{R}^q)}{\Longrightarrow}} \nu$$

and the sequence $(Y_n)_{n\geq 0}$ is often called "stable" in the literature, at least when it is a Markov chain. If $\mathcal{V} = L^1(\nu)$, the sequence $(Y_n)_{n\geq 0}$ may be called in short "ergodic" although no true ergodic framework comes in the game stricto sensu. The target of our recursive procedure (1.2) is then, as expected, a zero, if any, of the (asymptotic) *mean function* of the algorithm defined as

$$h(\theta) := \int_{\mathbb{R}^q} H(\theta, y)\nu(dy).$$

The key assumption is the existence of *pathwise Lyapunov function* with respect to the innovation *i.e.* a function L satisfying

$$\langle \nabla L(\theta) \,|\, H(\theta, y) - H(\theta^*, y)\rangle \geq 0$$

for every θ and y. This assumption may look very stringent. However such a setting embodies standard framework of Stochastic Approximation with Markov representation of the form (1.1) when the $(Y_n)_{n\geq 0}$ is i.i.d. since its canonical form reads

$$\theta_{n+1} = \theta_n - \gamma_{n+1}\left(h(\theta_n) + \Delta\widetilde{M}_{n+1}\right),$$

where $\Delta\widetilde{M}_{n+1} = \mathbb{E}\left[H(\theta_n, Y_n) \,|\, \mathcal{F}_{n-1}^Y\right] - h(\theta_n) + \Delta M_{n+1}$. Then $H(\theta, \cdot) = h(\theta)$ and the notion of pathwise Lyapunov function reduces to the standard one. The above canonical form has been extensively investigated (and extended) in many textbooks on Stochastic Approximation (see [3], [9], [13], [14]).

Our main theorem (Theorem 2.1) let us retrieve almost entirely the classical results about *a.s.* (and L^p)-boundedness and convergence of this procedure under usual Lyapunov assumption. Many extensions have been developed when $(Y_n)_{n\geq 0}$ or even $(\theta_n, Y_n)_{n\geq 0}$ have a Markovian dynamics (see the seminal textbook [3] and more recent contributions like [7] and several reference therein). The main constraint induced by such an approach is that we need to make assumptions on the solution of the Poisson equation related to this chain.

Our aim is more modest from a theoretical point of view. Nevertheless, our main concern is to take into account innovations with very light "ergodic" or even simply "averaging" properties and sharing no kind of Markov property, in order to include as different types of innovation as real real market data or deterministic Quasi-Monte Carlo sequences (also known as uniformly distributed sequences see [26]). We provide various examples which illustrate that the price to be paid (mainly the pathwise Lyapunov function) is not so high for practical applications. In the treated examples, we use the convergence part of Theorem 2.1. However by slightly relaxing the pathwise Lyapunov equation, we can also derive from Theorem 2.1 a boundedness result which could be the starting point of a pathwise reasoning in the spirit of the ODE method.

As a first field of applications, we are interested in quasi-random numbers. The original idea of replacing by uniformly distributed sequences (with low discrepancy) i.i.d. innovations in recursive

stochastic approximation procedures goes back to the early 1990's in [19], leading to "Quasi-Stochastic Approximation" (referring to QMC for Quasi-Monte Carlo). The framework in [19] was purely one-dimensional and many numerical tests have proved the efficiency of QSA via multi-dimensional setting. The aim is to establish a convergence theorem in this higher dimensional setting under natural regularity assumptions (*i.e.* based on Lipschitz regularity rather than finite variation in the Hardy & Krause sense, often encountered in the QMC world). As concerns the low discrepancy sequences, our framework is probably close to the most general one to get pointwise *a.s.* convergence of stochastic approximation.

As a second setting, we consider the case when $(Y_n)_{n\geq 0}$ is a functional of α-mixing process satisfying a priori no Markov assumption. These processes are stationary and dependent, so more realistic to model inputs made of real data. To describe the class of functions $\mathcal{V}$ we need to prove the convergence of the series of covariance coefficients of the innovations. To this end we use some results in [28] and the covariance inequality for α-mixing process (see[8]). Next with the probabilistic version of the theorem of Gàl-Koksma (see [11] and [1]) we prove that this class is large enough ($L^2(\nu) \subset \mathcal{V}$). Finally we examine the case of homogeneous Markov chain with invariant distribution ν. Some results on Stochastic Approximation in this setting have been proved in [3], but they need to establish the convergence of the algorithm to assume the existence of a solution to Poisson equation with some regularity properties. To describe $\mathcal{V}$ we add an ergodic assumption on the chain which allows us to prove that this class does not depend on the initial value of the chain.

Finally we present several examples of applications illustrated with numerical experiment. First we consider a simple case of calibration: the search of an implicit parameter in a financial model to fit its value to the market. We implement the algorithm with an i.i.d. sequence and a quasi-random one to compare their rate of convergence. The second example is devoted to the recursive computation of risk measures commonly considered in energy portfolio management, namely the Value-at-Risk and the Conditional-Value-at-Risk. We show that the recursive procedure introduced in [2] to compute these quantities can be successfully implemented in a QSA framework. In the third example, we solve numerically a long term investment "toy" problem leading to an ergodic control problem (see [24]). In that problem, the innovation turns out to be the outputs of the Euler scheme with decreasing step of a diffusion as introduced in [15, 22]. These three examples describe stochastic numerical procedures based on *simulated* data. The fourth example is the so-called two-armed bandit introduced in learning automata and mathematical psychology in the 1950's (see [25]). Its *a.s.* behaviour in the i.i.d. setting has been extensively investigated in [18] and [16]. These results have been partially extended in [32] to a more general ergodic framework. We show that the starting point of this extension appears as a consequence of our main result (in the multiplicative case). The last example describes a model of asset allocation fully developed in [21] involving exogenous true market data, *a priori* sharing no Markov property but on which an averaging assumption seems natural (at least within a medium laps of time).

The paper is organized as follows : in Section 2 are stated and proved the two main results: Theorem 2.1 and its counterpart Theorem 2.2, for multiplicative noise. Section 3 is devoted to quasi-Stochastic Approximation, *i.e.* the case where the innovation process is an uniformly distributed deterministic sequence over $[0,1]^q$. Section 4 is devoted to applications to random innovations, namely additive noise, mixing process (functionals of α-mixing process), ergodic homogeneous Markov chain. Section 5 presents five examples of applications including numerical illustrations, mostly in connection with Finance: implicit correlation search, recursive computation of VaR and CVaR, long term investment evaluation, two-armed bandit algorithm and optimal allocation problem (more developed in [21]).

Notations $\langle \cdot \,|\, \cdot \rangle$ denote the Euclidean inner product and $|\cdot|$ its related norm. The almost sure convergence will be denoted $\xrightarrow{a.s.}$ and $\xRightarrow{(\mathbb{R}^q)}$ will denote the weak convergence in $\mathbb{R}^q$.

2 Algorithm design and main theoretical result

In this paper, we consider the following general framework for recursive stochastic algorithms of the following form

$$\theta_{n+1} = \theta_n - \gamma_{n+1} \left(H(\theta_n, Y_n) + \Delta M_{n+1} \right), \quad n \geq 0, \tag{2.4}$$

where θ_0 is an $\mathbb{R}^d$-valued random variable, $(Y_n)_{n\geq 0}$ is a $\mathbb{R}^q$-valued sequence of random variables and ΔM_{n+1} is a martingale increment, all defined on a same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Moreover $\theta_0 \in L^1(\mathbb{P})$ and θ_0 is independent of $(Y_n)_{n\geq 0}$. The step sequence $(\gamma_n)_{n\geq 1}$ is a non-increasing and H is a Borel function from $\mathbb{R}^d \times \mathbb{R}^q$ to $\mathbb{R}^d$. We will denote $\mathcal{F}_n = \sigma(\theta_0, Y_0, Y_1, \ldots, Y_n)$, $n \geq 0$, the natural filtration of the innovation process $(Y_n)_{n\geq 0}$. Note that no Markov assumption is made a priori on the sequence $(Y_n)_{n\geq 0}$.

In the following, we adopt a kind of compromise by assuming that $(Y_n)_{n\geq 0}$ is a process satisfying some averaging properties and where the function $H(\theta^*, \dot{)}$ belongs to a class of functions (which will be specified further on) for which some rate of convergence ($a.s.$ and in L^p) holds in (1.3). Moreover we need to reinforce the Lyapunov condition on the pseudo-mean function H which limits, at least theoretically, the range of application of the method.

However, we will see in the examples that it provides some rather general (and now to our knowledge) results about quasi-Stochastic Approximation which justifies the implementation of stochastic approximation procedures with sequences with low discrepancy. This widely spread practice, especially in Financial Engineering, seemed to rely so far only in some heuristic rules derived from the original stochastic approximation framework.

2.1 Framework and assumptions

Let $(Y_n)_{n\geq 0}$ be an $\mathbb{R}^q$-valued random variables sequence. We will say that the sequence $(Y_n)_{n\geq 0}$ satisfies a ν-stability assumption if

$$\mathbb{P}(d\omega)\text{-}a.s. \quad \frac{1}{n} \sum_{k=0}^{n-1} \delta_{Y_k(\omega)} \xRightarrow[n\to\infty]{(\mathbb{R}^q)} \nu \tag{2.5}$$

where $\xRightarrow{(\mathbb{R}^q)}$ stands for the weak convergence of probability measures on $(\mathbb{R}^q, \mathcal{B}or(\mathbb{R}^q))$.

We will see that the stochastic approximation procedure defined by (2.4) is a recursive zero search of the (asymptotic) mean function

$$h(\theta) := \int_{\mathbb{R}^q} H(\theta, y) \nu(dy). \tag{2.6}$$

Let $p \in [1, \infty)$ and let $(\varepsilon_n)_{n\geq 0}$ be a sequence of non-negative numbers such that $\varepsilon_n \underset{n\to\infty}{\longrightarrow} 0$. We denote by $\mathcal{V}_{\varepsilon_n, p}$ the class of functions which convergence rate in (1.3) in both $a.s.$ and in $L^p(\mathbb{P})$ sense is ε_n^{-1}, namely

$$\mathcal{V}_{\varepsilon_n,p} = \left\{ f :\in L^p(\nu) \,|\, \frac{1}{n} \sum_{k=1}^{n} f(Y_k) - \int f d\nu \overset{\mathbb{P}\text{-}a.s. \ \&\ L^p(\mathbb{P})}{=} O(\varepsilon_n) \right\}. \tag{2.7}$$

2.2 Main result

Now we are in a position to state a first convergence theorem, which is a Robbins-Siegmund like Lemma.

Theorem 2.1. (a) Boundedness*: Let $h : \mathbb{R}^d \to \mathbb{R}^d$ satisfying (2.6), $H : \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}^d$ a Borel function and let $(Y_n)_{n\geq 0}$ be a sequence satisfying (2.5). Assume there exists a continuously differentiable function $L : \mathbb{R}^d \to \mathbb{R}^+$ satisfying*

$$\nabla L \text{ is Lipschitz continuous and } |\nabla L|^2 \leq C\,(1+L)\,. \tag{2.8}$$

and that the pseudo-mean function H satisfies the pathwise Lyapunov *assumption*

$$\forall \theta \in \mathbb{R}^d\backslash\{\theta^*\},\ \forall y \in \mathbb{R}^q, \quad \langle \nabla L(\theta) \,|\, H(\theta, y) - H(\theta^*, y)\rangle \geq 0. \tag{2.9}$$

Let $p \in [1, \infty)$. Assume that

$$H(\theta^*, \cdot) \in \mathcal{V}_{\varepsilon_n, p}. \tag{2.10}$$

Moreover, assume that H satisfies the following (quasi-)linear growth assumption

$$\forall \theta \in \mathbb{R}^d, \forall y \in \mathbb{R}^q, \quad |H(\theta, y)| \leq C_H \phi(y)(1 + L(\theta))^{\frac{1}{2}} \tag{2.11}$$

and that the martingale increments sequence $(\Delta M_{n+1})_{n\geq 0}$ satisfies

$$\begin{aligned} &\forall n \geq 0, \quad \mathbb{E}\left(|\Delta M_{n+1}|^{2\vee\frac{p}{p-1}} \,|\, \mathcal{F}_n\right) \leq C_M \phi(Y_n)^{2\vee\frac{p}{p-1}} (1 + L(\theta_n))^{1\vee\frac{p}{2(p-1)}} \quad \text{if} \quad p > 1 \\ &\forall n \geq 0, \qquad\qquad \operatorname{esssup} \mathbb{E}\left(|\Delta M_{n+1}| \,|\, \mathcal{F}_n\right) \leq C_M \phi(Y_n)(1 + L(\theta_n))^{\frac{1}{2}} \quad \text{if} \quad p = 1 \end{aligned} \tag{2.12}$$

where $\sup_{n\geq 0} \|\phi(Y_n)\|_{2\vee\frac{p}{(p-1)}} < +\infty$.

Let $\gamma = (\gamma_n)_{n\geq 1}$ be a non-negative non-increasing sequence of gain parameters satisfying

$$\sum_{n\geq 1} \gamma_n = +\infty, \quad n\varepsilon_n\gamma_n \underset{n\to\infty}{\longrightarrow} 0, \quad \text{and} \quad \sum_{n\geq 1} n\varepsilon_n \max\left(\gamma_n^2, |\Delta\gamma_{n+1}|\right) < +\infty. \tag{2.13}$$

Then, the recursive procedure defined by (2.4) satisfies $(L(\theta_n))_{n\geq 0}$ is L^1-bounded , $L(\theta_n) \underset{n\to\infty}{\longrightarrow} L_\infty < +\infty$ a.s., $\theta_n - \theta_{n-1} \underset{n\to\infty}{\longrightarrow} 0$ a.s. and

$$\sum_{n\geq 1} \langle \nabla L(\theta_n) \,|\, H(\theta_n, Y_n) - H(\theta^*, Y_n)\rangle < +\infty.$$

(b) *A.s.* convergence*: Furthermore, if $\{\theta^*\}$ is a connected component of $\{L = L(\theta^*)\}$ and the pseudo-mean function H satisfies the* strong pathwise Lyapunov *assumption*

$$\forall \delta > 0,\ \forall \theta \in \mathbb{R}^d\backslash\{\theta^*\},\ \forall y \in \mathbb{R}^q, \quad \langle \nabla L(\theta) \,|\, H(\theta, y) - H(\theta^*, y)\rangle \geq \chi_\delta(y)\Psi_\delta(\theta) \tag{2.14}$$

where $\nu(\chi_\delta) > 0$, Ψ_δ is l.s.c. and positive on $\mathbb{R}^d\backslash\{\theta^*\}$ and $\bigcap_{\delta>0}\{\Psi_\delta = 0\} = \{\theta^*\}$, then

$$\theta_n \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} \theta^*.$$

Remark. The conditions on the sequence of gain parameters $\gamma = (\gamma_n)_{n\geq 1}$ are satisfied for example if we set

$$\gamma_n = \frac{c}{n^a}, \quad a > 1 - \beta, \quad c > 0. \tag{2.15}$$

Proof. We set for notational convenience $\Delta\theta_n = \theta_n - \theta_{n-1}$.
First step: We introduce the function

$$\Lambda(\theta) := \sqrt{1 + L(\theta)}$$

as a Lyapunov function instead of $L(\theta)$ like in the classical case. It follows from the fundamental formula of calculus that there exists $\xi_{n+1} \in (\theta_n, \theta_{n+1})$ such that

$$\begin{aligned}\Lambda(\theta_{n+1}) &= \Lambda(\theta_n) + \langle \nabla\Lambda(\theta_n) \,|\, \Delta\theta_{n+1}\rangle + \langle \nabla\Lambda(\xi_{n+1}) - \nabla\Lambda(\theta_n) \,|\, \Delta\theta_{n+1}\rangle \\ &\le \Lambda(\theta_n) + \langle \nabla\Lambda(\theta_n) \,|\, \Delta\theta_{n+1}\rangle + \langle \nabla\Lambda(\theta_{n+1}) - \nabla\Lambda(\theta_n) \,|\, \Delta\theta_{n+1}\rangle\,.\end{aligned}$$

Lemma 2.1. *The new Lyapunov function Λ satisfies the two following properties*

(i) *$\nabla\Lambda$ is bounded (so that Λ is Lipschitz).*

(ii) $\forall \theta, \theta' \in \mathbb{R}^d, \quad |\nabla\Lambda(\theta') - \nabla\Lambda(\theta)| \le C_L \frac{|\theta' - \theta|}{\Lambda(\theta)}$.

Proof of Lemma 2.1.

(i) $\nabla\Lambda = \frac{\nabla L}{2\sqrt{1+L}}$ is bounded by 2.8, consequently Λ is Lipschitz.

(ii) For $\theta, \theta' \in \mathbb{R}^d$,

$$\begin{aligned}\left|\nabla\Lambda(\theta) - \nabla\Lambda(\theta')\right| &\le \frac{|\nabla\mathcal{L}(\theta) - \nabla\mathcal{L}(\theta')|}{2\sqrt{1+L(\theta)}} + \frac{|\nabla\mathcal{L}(\theta')|}{2}\left|\frac{\sqrt{1+L(\theta')} - \sqrt{1+L(\theta)}}{\sqrt{1+L(\theta)}\sqrt{1+L(\theta')}}\right| \\ &\le \frac{[\nabla L]_{\mathrm{Lip}}}{2\sqrt{1+L(\theta)}}\left|\theta - \theta'\right| + \frac{C}{2\sqrt{1+L(\theta)}}[\Lambda]_{\mathrm{Lip}}\left|\theta - \theta'\right| \\ &\le \frac{1}{2}\left([\nabla L]_{\mathrm{Lip}} + C[\Lambda]_{\mathrm{Lip}}\right)\frac{|\theta - \theta'|}{\Lambda(\theta)} \\ &= C_L\frac{|\theta - \theta'|}{\Lambda(\theta)}. \quad \square\end{aligned}$$

Thus

$$\begin{aligned}\Lambda(\theta_{n+1}) &\le \Lambda(\theta_n) - \gamma_{n+1}\langle \nabla\Lambda(\theta_n) \,|\, H(\theta_n, Y_n)\rangle - \gamma_{n+1}\langle \nabla\Lambda(\theta_n) \,|\, \Delta M_{n+1}\rangle + C_L\frac{|\Delta\theta_{n+1}|^2}{\sqrt{1+L(\theta_n)}} \\ &= \Lambda(\theta_n) - \gamma_{n+1}\langle \nabla\Lambda(\theta_n) \,|\, H(\theta_n, Y_n) - H(\theta^*, Y_n)\rangle - \gamma_{n+1}\langle \nabla\Lambda(\theta_n) \,|\, H(\theta^*, Y_n)\rangle \\ &\quad -\gamma_{n+1}\langle \nabla\Lambda(\theta_n) \,|\, \Delta M_{n+1}\rangle + C_L\gamma_{n+1}^2\frac{|H(\theta_n, Y_n) + \Delta M_{n+1}|^2}{\sqrt{1+L(\theta_n)}}.\end{aligned}$$

We have for every $n \ge 0$,

$$|\gamma_{n+1}\langle \nabla\Lambda(\theta_n) \,|\, H(\theta^*, Y_n)\rangle| \le C_\Lambda\gamma_{n+1}\phi(Y_n) \in L^1(\mathbb{P})$$

since $\nabla\Lambda$ is bounded. Besides $\mathbb{E}\left[\langle \nabla\Lambda(\theta_n) \,|\, \Delta M_{n+1}\rangle \,|\, \mathcal{F}_n\right] = 0$, $n \ge 0$, since ΔM_n is a true martingale increment and $\nabla\Lambda$ is bounded. Furthermore, owing to (2.11) and (2.12)

$$\mathbb{E}\left[\frac{|H(\theta_n, Y_n) + \Delta M_{n+1}|^2}{\sqrt{1+L(\theta_n)}} \,|\, \mathcal{F}_n\right] \le C\phi^2(Y_n)\Lambda(\theta_n)$$

where conditional expectation is defined in the sense of positive random variables. Consequently we get

$$\begin{aligned}\mathbb{E}\left[\Lambda(\theta_{n+1}) \,|\, \mathcal{F}_n\right] \;\leq\; & \Lambda(\theta_n)\left(1 + C_L'\gamma_{n+1}^2\phi(Y_n)^2\right) - \gamma_{n+1}\left\langle \nabla\Lambda(\theta_n) \,|\, H(\theta_n, Y_n) - H(\theta^*, Y_n)\right\rangle \\ & -\gamma_{n+1}\left\langle \nabla\Lambda(\theta_n) \,|\, H(\theta^*, Y_n)\right\rangle.\end{aligned} \tag{2.16}$$

We set $V_n := \dfrac{A_n}{B_n}$, where

$$A_n := \Lambda(\theta_n) + \sum_{k=0}^{n-1}\gamma_{k+1}\left\langle \nabla\Lambda(\theta_k) \,|\, H(\theta_k, Y_k) - H(\theta^*, Y_k)\right\rangle, \quad B_n := \prod_{k=1}^{n}\left(1 + C_L'\gamma_k^2\phi(Y_{k-1})^2\right).$$

Using the mean-reverting assumption (2.14) implies that $(A_n)_{n\geq 0}$ is a non-negative process and $(B_n)_{n\geq 0}$ is $\mathcal{F}_{n-1}$-adapted. Elementary computations show that first

$$\mathbb{E}\left[A_{n+1} \,|\, \mathcal{F}_n\right] \leq A_n\frac{B_{n+1}}{B_n} - \gamma_{n+1}\left\langle \nabla\Lambda(\theta_n) \,|\, H(\theta^*, Y_n)\right\rangle$$

which finally yields

$$\forall n \geq 0, \quad \mathbb{E}\left[V_{n+1} \,|\, \mathcal{F}_n\right] \leq V_n - \Delta W_{n+1}, \tag{2.17}$$

where $W_n := \sum_{k=0}^{n-1}\widetilde{\gamma}_{k+1}\left\langle \nabla\Lambda(\theta_k) \,|\, H(\theta^*, Y_k)\right\rangle$ with $\widetilde{\gamma}_n := \frac{\gamma_n}{B_n}$, $n \geq 0$.

Second step: Now our aim is to prove that the sequence $(W_n)_{n\geq 1}$ is L^1-bounded and $a.s.$ converges. To this end we set $S_n^* := \sum_{k=0}^{n-1} H(\theta^*, Y_k)$, then it follows

$$W_n = \sum_{k=0}^{n-1}\widetilde{\gamma}_{k+1}\left\langle \nabla\Lambda(\theta_k) \,|\, \Delta S_{k+1}^*\right\rangle = \widetilde{\gamma}_n\left\langle \nabla\Lambda(\theta_{n-1}) \,|\, S_n^*\right\rangle - \sum_{k=1}^{n-1}\left\langle S_k^* \,|\, \widetilde{\gamma}_{k+1}\nabla\Lambda(\theta_k) - \widetilde{\gamma}_k\nabla\Lambda(\theta_{k-1})\right\rangle.$$

First, since $\nabla\Lambda$ is bounded, note that

$$\widetilde{\gamma}_n\left|\nabla\Lambda(\theta_{n-1})\right|\left|S_n^*\right| \leq \left\|\nabla\Lambda\right\|_\infty n\varepsilon_n\widetilde{\gamma}_n\frac{|S_n^*|}{n\varepsilon_n} \leq \left\|\nabla\Lambda\right\|_\infty n\varepsilon_n\gamma_n\frac{|S_n^*|}{n\varepsilon_n}$$

which $a.s.$ goes to 0 as n goes to infinity since $n\varepsilon_n\gamma_n \underset{n\to\infty}{\longrightarrow} 0$ by (2.13) and $\left(\dfrac{S_n^*}{n\varepsilon_n}\right)_{n\geq 1}$ remains $a.s.$ bounded. Moreover

$$\mathbb{E}\left[\widetilde{\gamma}_n\left|\nabla\Lambda(\theta_{n-1})\right|\left|S_n^*\right|\right] \leq n\varepsilon_n\gamma_n\left\|\nabla\Lambda\right\|_\infty\left\|\frac{S_n^*}{n\varepsilon_n}\right\|_1$$

which converges also in L^1 because $n\varepsilon_n\gamma_n \underset{n\to\infty}{\longrightarrow} 0$ and $H(\theta^*, \cdot) \in \mathcal{V}_{\varepsilon_n, p}$. On the other hand,

$$\sum_{k=1}^{n-1}\left\langle S_k^* \,|\, \widetilde{\gamma}_{k+1}\nabla\Lambda(\theta_k) - \widetilde{\gamma}_k\nabla\Lambda(\theta_{k-1})\right\rangle = \sum_{k=1}^{n-1}\left\langle S_k^* \,|\, \nabla\Lambda(\theta_k)\right\rangle\Delta\widetilde{\gamma}_{k+1} + \sum_{k=1}^{n-1}\widetilde{\gamma}_k\left\langle S_k^* \,|\, \nabla\Lambda(\theta_k) - \nabla\Lambda(\theta_{k-1})\right\rangle.$$

As $\nabla\Lambda = \dfrac{\nabla L}{\sqrt{1+L}}$ is bounded by construction, we have

$$\sum_{k=1}^{n}\left|\Delta\widetilde{\gamma}_{k+1}\left\langle S_k^* \,|\, \nabla\Lambda(\theta_k)\right\rangle\right| \leq \sum_{k=1}^{n}\left|\Delta\widetilde{\gamma}_{k+1}\right|\left|S_k^*\right|\left\|\nabla\Lambda\right\|_\infty \leq \left\|\nabla\Lambda\right\|_\infty\sum_{k=1}^{n}k\varepsilon_k\left|\Delta\widetilde{\gamma}_{k+1}\right|\left|\frac{S_k^*}{k\varepsilon_k}\right|.$$

Now, using that $\frac{a}{1+a} \leq \sqrt{a},\ a>0$,

$$|\Delta\widetilde{\gamma}_{k+1}| \leq |\Delta\gamma_{k+1}| + \gamma_k \frac{C_L'\gamma_{k+1}^2\phi(Y_k)^2}{B_{k+1}} \leq |\Delta\gamma_{k+1}| + \gamma_k \frac{C_L'\gamma_{k+1}^2\phi(Y_k)^2}{1+C_L'\gamma_{k+1}^2\phi(Y_k)^2} \leq |\Delta\gamma_{k+1}| + C_L'\gamma_k\gamma_{k+1}\phi(Y_k).$$

Hence

$$\sum_{k=1}^n |\Delta\widetilde{\gamma}_{k+1} \langle S_k^* \,|\, \nabla\Lambda(\theta_k)\rangle| \leq \|\nabla\Lambda\|_\infty \left(\sum_{k=1}^n k\varepsilon_k |\Delta\gamma_{k+1}| \left|\frac{S_k^*}{k\varepsilon_k}\right| + C_L' \sum_{k=1}^n k\varepsilon_k\gamma_k\gamma_{k+1}\phi(Y_k) \left|\frac{S_k^*}{k\varepsilon_k}\right| \right).$$

By Hölder's Inequality

$$\mathbb{E}\left(\phi(Y_k)\left|\frac{S_k^*}{k\varepsilon_k}\right|\right) \leq \|\phi(Y_k)\|_{\frac{p}{p-1}} \left\|\frac{S_k^*}{k\varepsilon_k}\right\|_p.$$

As $\left(\frac{S_k^*}{k\varepsilon_k}\right)_{n\geq 0}$ is bounded, $n\varepsilon_n\gamma_n \underset{n\to\infty}{\longrightarrow} 0$, $\sum_{k\geq 1} k\varepsilon_k |\Delta\gamma_{k+1}| < +\infty$, $\sum_{k\geq 1} k\varepsilon_k\gamma_{k+1}^2 < +\infty$ and $\sup_{k\geq 0} \|\phi(Y_k)\|_{\frac{p}{p-1}} < +\infty$, then the series $\sum_{k=1}^n \Delta\widetilde{\gamma}_k \langle S_k^* \,|\, \nabla\Lambda(\theta_k)\rangle$ is absolutely converging in $L^1(\mathbb{P})$.

We study now the series $\sum_{k=1}^n \widetilde{\gamma}_k \langle S_k^* \,|\, \nabla\Lambda(\theta_k) - \nabla\Lambda(\theta_{k-1})\rangle$. We have

$$|\nabla\Lambda(\theta_k) - \nabla\Lambda(\theta_{k-1})| \leq C_L' \frac{|\Delta\theta_k|}{\sqrt{1+L(\theta_{k-1})}} \leq C_L'\gamma_k \frac{|H(\theta_{k-1},Y_{k-1})| + |\Delta M_k|}{\sqrt{1+L(\theta_{k-1})}}.$$

Then we are interested in the L^1-convergence of the series

$$\sum_{k=1}^n \gamma_k^2 |S_k^*| \frac{|H(\theta_{k-1},Y_{k-1})|}{\sqrt{1+L(\theta_{k-1})}} \quad \text{and} \quad \sum_{k=1}^n \gamma_k^2 |S_k^*| \frac{|\Delta M_k|}{\sqrt{1+L(\theta_{k-1})}}.$$

For the first sum, as $\frac{|H(\theta_{k-1},Y_{k-1})|}{\sqrt{1+L(\theta_{k-1})}} \leq C_H\phi(Y_{k-1})$, we then come to $\sum_{k=1}^n C_H\gamma_k^2 \mathbb{E}\left[|S_k^*|\,|\phi(Y_{k-1})|\right]$ and by Hölder's inequality we obtain

$$\mathbb{E}\left[|S_k^*|\,|\phi(Y_{k-1})|\right] \leq \|S_k^*\|_p \|\phi(Y_{k-1})\|_{\frac{p}{p-1}} < +\infty$$

because $\|S_k^*\|_p = O(k\varepsilon_k)$ by (2.7) and $\sup_{n\geq 1} \|\phi(Y_n)\|_{\frac{p}{p-1}} < +\infty$. Furthermore, as $\sum_{k\geq 1} k\varepsilon_k\gamma_k^2 < +\infty$ by (2.13), then the series $\sum_{k=1}^n \gamma_k^2 |S_k^*| \frac{|H(\theta_{k-1},Y_{k-1})|}{\sqrt{1+L(\theta_{k-1})}}$ converges in L^1.
For the second sum, we have by Hölder's inequality

$$\mathbb{E}\left[|S_k^*| \frac{|\Delta M_k)|}{\sqrt{1+L(\theta_{k-1})}}\right] \leq \|S_k^*\|_p \left\|\frac{|\Delta M_k|}{\sqrt{1+L(\theta_{k-1})}}\right\|_{\frac{p}{p-1}} \leq C_M \|S_k^*\|_p \|\phi(Y_{k-1})\|_{\frac{p}{p-1}} < +\infty$$

owing to the same arguments as for the previous series. This yields that $\sum_{k=1}^n \gamma_k^2 |S_k^*| \frac{|\Delta M_k|}{\sqrt{1+L(\theta_{k-1})}}$ converges in L^1 too. Finally we then obtain that $W_n \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} W_\infty$ and $\sup_{n\geq 1} \|W_n\|_1 < +\infty$. Thus we have that

$$(V_n + W_n)^- \leq W_n^- \leq |W_n| \in L^1(\mathbb{P}) \quad \text{since} \quad \sup_{n\geq 1} \|W_n\|_1 < +\infty.$$

As $V_0 = \Lambda(\theta_0) \leq C(1+|\theta_0|) \in L^1$, it follows by induction from (2.17) that, for every $n \geq 0$, $\mathbb{E}\, V_n < \infty$. Hence $S_n := V_n + W_n$, $n \geq 0$, is a true supermartingale with an L^1-bounded negative part. We then deduce that

$$S_n \xrightarrow[n\to\infty]{a.s.} S_\infty \in L^1.$$

Now $W_n \xrightarrow[n\to\infty]{a.s.} W_\infty$ implies $V_n \xrightarrow[n\to\infty]{a.s.} V_\infty < +\infty$ $a.s.$

Third step: Now we show that the product B_n converges $a.s.$ to derive that A_n converges $a.s.$. In fact

$$\sum_{n\geq 1} \gamma_n^2 \phi^2(Y_{n-1}) < +\infty \ a.s.,$$

since $\sup_{n\geq 1} \mathbb{E}\left[\phi^2(Y_n)\right] < +\infty$, which in turn implies that $B_n \xrightarrow[n\to\infty]{a.s.} B_\infty < +\infty$. As a consequence $A_n \xrightarrow[n\to\infty]{a.s.} A_\infty < +\infty$. Therefore using the mean reverting property (2.14) of H with respect to $\nabla\Lambda$, we classically derive that

$$\sum_{n\geq 1} \gamma_n \left\langle \nabla\Lambda(\theta_{n-1}) \,|\, H(\theta_{n-1}, Y_{n-1}) - H(\theta^*, Y_{n-1}) \right\rangle < +\infty \quad a.s. \tag{2.18}$$

Consequently

$$\Lambda(\theta_n) \xrightarrow[n\to\infty]{a.s.} \Lambda_\infty < +\infty \quad a.s.$$

As $\lim_{|\theta|\to+\infty} L(\theta) = +\infty$, $\lim_{|\theta|\to+\infty} \Lambda(\theta) = +\infty$, then the sequence $(\theta_n)_{n\geq 0}$ is $a.s.$-bounded and

$$L(\theta_n) \xrightarrow{a.s.} L_\infty < +\infty \quad a.s.$$

Now let us show that $\Delta\theta_n \underset{n\to\infty}{\longrightarrow} 0$. In fact $|\Delta\theta_{n+1}|^2 \leq C\gamma_{n+1}^2 \left(|H(\theta_n, Y_n)|^2 + |\Delta M_{n+1}|^2\right)$, so $\mathbb{E}\left[|\Delta\theta_{n+1}|^2 \,|\, \mathcal{F}_n\right] \leq C\gamma_{n+1}^2(1 + \mathbb{E}\, L(\theta_n))$. Now $(\Lambda(\theta_n))_{n\geq 0}$ is bounded so that

$$\sum_{n\geq 0} \mathbb{E}\left[|\Delta\theta_{n+1}|^2 \,|\, \mathcal{F}_n\right] < +\infty. \ a.s.$$

which classically implies that $\sum_{n\geq 0} |\Delta\theta_{n+1}|^2 < +\infty$ $a.s.$

Fourth step: To prove the convergence of θ_n toward θ^*, we use Assumptions (2.14) and (4.32) to deduce that

$$\sum_{n\geq 1} \gamma_n \chi_\delta(Y_{n-1}) \Psi_\delta(\theta_{n-1}) < +\infty \quad a.s. \tag{2.19}$$

Now,

$$\sum_{k=0}^{n} \gamma_{k+1} \chi_\delta(Y_k) = \sum_{k=0}^{n} \gamma_{k+1} \Delta S_k^\chi = \gamma_{n+1} S_n^\chi - \sum_{k=1}^{n-1} \Delta\gamma_{k+2} S_k^\chi$$

where $S_n^\chi = \sum_{k=0}^n \chi_\delta(Y_k)$ and we set $S_0^\chi = 0$ and $\Delta S_0^\chi = 0$.

By Assumption (2.5), $\dfrac{S_n^\chi}{n} \to \nu(\chi_\delta) > 0$ as $n \to \infty$. Let n_0 be the smallest integer such that

$$\forall n \geq n_0, \quad \frac{S_n^\chi}{n} \geq \epsilon_0 = \frac{\nu(\chi_\delta)}{2} > 0.$$

Then, the usual discrete integration by part yields

$$\forall n \geq n_0, \quad \sum_{k=n_0}^{n} \gamma_{k+1}\chi_\delta(Y_k) = n\gamma_{n+1}\frac{S_n^\chi}{n} - C_{n_0} + \sum_{k=n_0}^{n-1} k(-\Delta\gamma_{k+2})\frac{S_k^\chi}{k} \quad a.s.,$$

where $C_{n_0} = \gamma_{n_0+1}S_{n_0-1}^\chi$. Therefore, using that the sequence $(-\Delta\gamma_n)_{n\geq 1}$ is non-negative,

$$\begin{aligned}\sum_{k=n_0}^{n} \gamma_{k+1}\chi_\delta(Y_k) &\geq n\gamma_{n+1}\epsilon_0 - C_{n_0} + \sum_{k=n_0}^{n-1} k(-\Delta\gamma_{k+2})\epsilon_0 = \epsilon_0\left(n\gamma_{n+1} + \sum_{k=n_0}^{n-1} k(-\Delta\gamma_{k+2})\right) - C_{n_0} \\ &= \epsilon_0\left(\gamma_{n+1} + n_0\gamma_{n_0+1} + \sum_{k=n_0+1}^{n-1}\gamma_{k+1}\right) - C_{n_0}\end{aligned}$$

by a reverse discrete integration by parts. Finally

$$\sum_{k=n_0}^{n} \gamma_{k+1}\chi_\delta(Y_k) \geq \epsilon_0\left(\gamma_{n+1} + \sum_{k=n_0+1}^{n-1}\gamma_{k+1}\right) - C_{n_0} \to \infty \quad \text{as} \quad n \to \infty$$

since $\sum_{n\geq 1}\gamma_n = +\infty$. We have then shown that

$$\sum_{k\geq 0} \gamma_{k+1}\chi_\delta(Y_k) = +\infty \quad a.s.$$

Combining this fact with (2.19) classically implies that

$$\liminf_n \widetilde{\Psi}_\delta(\theta_n) = 0.$$

Let Θ_∞ be the set of limiting points of the sequence $(\theta_n)_{n\geq 0}$. Θ_∞ is a compact connected set since $(\theta_n)_{n\geq 0}$ is bounded and $\Delta\theta_n \underset{n\to\infty}{\longrightarrow} 0$. So $\Theta_\infty \cap \{\Psi_\delta = 0\}$ is a family of non-empty compact sets which decreases as $\delta \searrow 0$ since it is bounded because Θ_∞ is closed since $\{\Psi_\delta = 0\}$ is because $\Psi_\delta \leq 0$ and l.s.c.. As a consequence, $\bigcap_{\delta>0}(\Theta_\infty \cap \{\Psi_\delta = 0\}) \neq \emptyset$. The other assumption on Ψ_δ implies $\bigcap_{\delta>0}(\Theta_\infty \cap \{\Psi_\delta = 0\}) \subset \bigcap_{\delta>0}\{\Psi_\delta = 0\} = \{\theta^*\}$, so that in fact it is reduced to θ^*. Hence θ^* is a limiting point of $(\theta_n)_{n\geq 0}$ which implies that $L(\theta_n)$ converges towards $L(\theta^*)$. By the assumption on the Lyapunov function L, $\{\theta^*\}$ is a connected component of $\{L = L(\theta^*)\}$ and as Θ_∞ is connected, $\Theta_\infty = \{\theta^*\}$. Therefore

$$\theta_n \xrightarrow{a.s.} \theta^* \text{ as } n \to \infty. \quad \square$$

Remark i.i.d. innovation $(Y_n)_{n\geq 0}$ with distribution ν. We can apply Theorem 2.1 by setting

$$H(\theta, y) = h(\theta) = \int_{\mathbb{R}^q} K(\theta, y)\nu(dy) \quad \text{and} \quad \Delta M_{n+1} = K(\theta_n, Y_{n+1}) - \mathbb{E}\left[K(\theta_n, Y_{n+1}) \,|\, \mathcal{F}_n\right]$$

and by assuming that for all $\theta \in \mathbb{R}^d$, $\mathbb{E}\left[K(\theta, Y_1)\right] < \infty$. If we set $p = 2$, Assumption (2.12) (with $\phi \equiv 1$) implies classically that h satisfies (2.11) owing to Schwarz Inequality. Furthermore it follows from the L^2-law of large numbers (LLN) (at rate $n^{-\frac{1}{2}}$) and the law of the iterated logarithm (LLI) at rate $O\left(\sqrt{\frac{\log\log n}{n}}\right)$ that if one sets $\varepsilon_n = \sqrt{\frac{\log\log n}{n}}$, then $\mathcal{V}_{\varepsilon_n,2} = L^2(\nu)$. As a consequence the condition (2.13) on the step sequence $(\gamma_n)_{n\geq 1}$ is clearly more restrictive than the step assumption in the regular Robbins-Monro Theorem (namely $\sum_{n\geq 1}\gamma_n = +\infty$ and $\sum_{n\geq 1}\gamma_n^2 < +\infty$), however any step of the form $\gamma_n = \frac{c}{n^\alpha}$, $c > 0$, $\frac{3}{4} < \alpha \leq 1$ satisfies (2.13).

2.3 The case of multiplicative noise

If we assume that the function H is of the following form

$$\forall \theta \in \mathbb{R}^d, \forall y \in \mathbb{R}^q, \quad H(\theta, y) = \chi(y)h(\theta) + H(\theta^*, y), \tag{2.20}$$

where χ is a Borel function such that $\nu(\chi) = 1$ and $\chi \in \mathcal{V}_{\varepsilon_n,p}$, h is Lipschitz bounded with $h(\theta^*) = 0$, then we replace the growth assumption (2.11) on H by one on the mean function h, *i.e.*

$$\forall \theta \in \mathbb{R}^d, \forall y \in \mathbb{R}^q, \quad |h(\theta)| \leq C_h \phi(y) \sqrt{1 + L(\theta)} \tag{2.21}$$

and the "local" mean-reverting assumption (2.14) is the classical

$$\forall \theta \in \mathbb{R}^d \setminus \{\theta^*\}, \quad \langle \nabla L \mid h \rangle (\theta) > 0. \tag{2.22}$$

Theorem 2.2. *The recursive procedure (2.4) with the function H defined by (2.20) and the previous assumptions on χ and (2.21)-(2.22) on h satisfies*

$$\theta_n \xrightarrow[n\to\infty]{a.s.} \theta^*.$$

Proof. This setting cannot be reduced to the general setting. We use the same notations as in the proof of Theorem 2.1. With the new form of the function H, we obtain

$$\begin{aligned} \Lambda(\theta_{n+1}) \quad \leq \quad & \Lambda(\theta_n) - \gamma_{n+1} \langle \nabla \Lambda(\theta_n) \mid \chi(Y_n) h(\theta_n) \rangle - \gamma_{n+1} \langle \nabla \Lambda(\theta_n) \mid H(\theta^*, Y_n) \rangle \\ & - \gamma_{n+1} \langle \nabla \Lambda(\theta_n) \mid \Delta M_{n+1} \rangle + C_L \gamma_{n+1}^2 \frac{|H(\theta_n, Y_n) + \Delta M_{n+1}|^2}{\sqrt{1 + L(\theta_n)}}. \end{aligned}$$

By the same arguments as before we get

$$\mathbb{E}\left[\Lambda(\theta_{n+1}) \mid \mathcal{F}_n\right] \leq \Lambda(\theta_n) \left(1 + C'_L \gamma_{n+1}^2 \phi(Y_n)^2\right) - \gamma_{n+1} \langle \nabla \Lambda(\theta_n) \mid \chi(Y_n) h(\theta_n) \rangle - \gamma_{n+1} \langle \nabla \Lambda(\theta_n) \mid H(\theta^*, Y_n) \rangle .$$

We set $V_n := \dfrac{A_n}{B_n}$, where $A_n := \Lambda(\theta_n) + \sum_{k=0}^{n-1} \gamma_{k+1} \langle \nabla \Lambda \mid h \rangle (\theta_k)$ and $B_n := \prod_{k=1}^{n} \left(1 + C'_L \gamma_k^2 \phi(Y_{k-1})^2\right)$. Using the mean-reverting assumption (2.22) implies that $(A_n)_{n\geq 0}$ is a non-negative process whereas $(B_n)_{n\geq 0}$ is still $\mathcal{F}_{n-1}$-adapted. Elementary computations show that

$$\mathbb{E}\left[A_{n+1} \mid \mathcal{F}_n\right] \leq A_n \frac{B_{n+1}}{B_n} - \gamma_{n+1} \langle \nabla \Lambda(\theta_n) \mid H(\theta^*, Y_n) \rangle - \gamma_{n+1} \widetilde{\chi}(Y_n) \langle \nabla \Lambda \mid h \rangle (\theta_n)$$

where $\widetilde{\chi}(Y_n) := \chi(Y_n) - \nu(\chi)$, $n \geq 0$. Finally we have

$$\forall n \geq 0, \quad \mathbb{E}\left[V_{n+1} \mid \mathcal{F}_n\right] \leq V_n - \Delta W_{n+1} - \Delta Z_{n+1}, \tag{2.23}$$

where $W_n := \sum_{k=0}^{n-1} \widetilde{\gamma}_{k+1} \langle \nabla \Lambda(\theta_k) \mid H(\theta^*, Y_k) \rangle$ and $Z_n := \sum_{k=0}^{n-1} \widetilde{\gamma}_{k+1} \widetilde{\chi}(Y_k) \langle \nabla \Lambda \mid h \rangle (\theta_k)$ with $\widetilde{\gamma}_n := \frac{\gamma_n}{B_n}$, $n \geq 0$.

Second step: Following the lines of the proof of Theorem 2.1 we show that the sequence $(W_n)_{n\geq 1}$ is L^1-bounded and *a.s.* converges. Now our aim is to prove the same results for the sequence $(Z_n)_{n\geq 1}$. To this end we set $S_n^{\widetilde{\chi}} := \sum_{k=0}^{n-1} \widetilde{\chi}(Y_k)$, then it follows

$$Z_n = \sum_{k=0}^{n-1} \widetilde{\gamma}_{k+1} \Delta S_{k+1}^{\widetilde{\chi}} \langle \nabla \Lambda \mid h \rangle (\theta_k) = \widetilde{\gamma}_n S_n^{\widetilde{\chi}} \langle \nabla \Lambda \mid h \rangle (\theta_{n-1}) - \sum_{k=1}^{n-1} S_k^{\widetilde{\chi}} \left(\widetilde{\gamma}_{k+1} \langle \nabla \Lambda \mid h \rangle (\theta_k) - \widetilde{\gamma}_k \langle \nabla \Lambda \mid h \rangle (\theta_{k-1}) \right).$$

By the same methods as for the sequence $(W_n)_{n\geq 1}$ (*i.e.* using assumptions on H, Λ and $(\gamma_n)_{n\geq 1}$), we obtain that

$$Z_n \xrightarrow[n\to\infty]{a.s.} Z_\infty \quad \text{and} \quad \sup_{n\geq 1} \|Z_n\|_1 < +\infty.$$

Thus we have that

$$(V_n + W_n + Z_n)^- \leq (W_n + Z_n)^- \leq |W_n + Z_n| \in L^1(\mathbb{P}) \quad \text{since} \quad \sup_{n\geq 1} \|W_n + Z_n\|_1 < +\infty.$$

As $V_0 = \Lambda(\theta_0) \leq C(1+|\theta_0|) \in L^1$, it follows by induction from (2.17) that, for every $n \geq 0$, $\mathbb{E}\, V_n < \infty$. Hence $S_n := V_n + W_n + Z_n$, $n \geq 0$, is a true supermartingale with a L^1-bounded negative part. We then deduce that

$$S_n \xrightarrow[n\to\infty]{a.s.} S_\infty \in L^1.$$

Now $W_n \xrightarrow[n\to\infty]{a.s.} W_\infty$ and $Z_n \xrightarrow{a.s.} Z_\infty$ imply that $V_n \xrightarrow[n\to\infty]{a.s.} V_\infty < +\infty$ *a.s.*

Third step : Like in the proof of Theorem 2.1, we have that $B_n \xrightarrow[n\to\infty]{a.s.} B_\infty < +\infty$ which implies that $A_n \xrightarrow[n\to\infty]{a.s.} A_\infty < +\infty$. Therefore using the pseudo-mean reverting property (2.22) of h with respect to $\nabla\Lambda$, we classically derive that

$$\sum_{n\geq 0} \gamma_{n+1} \nu(\chi) \left\langle \nabla\Lambda \,|\, h \right\rangle (\theta_n) < +\infty \quad a.s. \tag{2.24}$$

The end of the proof follows the lines of the one of Theorem 2.1. □

3 Application to quasi-stochastic approximation

This section is devoted to quasi-random innovations introduced in [19] and we generalize their result to multi-dimensional setting when the function H is not bounded. We first recall some definitions and properties of such sequences and apply Theorem 2.1 when H has bounded variation (in the sense of Hardy & Krause see [26]) thanks to Koksma-Hlawka inequality and when H is Lipschitz with Proinov Theorem (see [29]).

3.1 Definitions and characterizations

Definition 3.1. *A $[0,1]^q$-valued sequence $(\xi_n)_{n\geq 1}$ is uniformly distributed (u.d.) on $[0,1]^q$ if*

$$\frac{1}{n}\sum_{k=1}^{n} \delta_{\xi_k} \overset{(\mathbb{R}^q)}{\Longrightarrow} \mathcal{U}([0,1]^q) \quad \text{as } n \to \infty.$$

The proposition below provides a characterization of uniform distribution.

Proposition 3.1. *Let $(\xi_n)_{n\geq 1}$ be a $[0,1]^q$-valued sequence. Then $(\xi_n)_{n\geq 1}$ is uniformly distributed on $[0,1]^q$ if and only if*

$$D_n^*(\xi) := \sup_{x\in[0,1]^q} \left| \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}_{[\![0,x]\!]}(\xi_k) - \prod_{i=1}^{q} x^i \right| \longrightarrow 0 \quad \text{as } n \to \infty,$$

where $D_n^*(\xi)$ is called the discrepancy at the origin or star discrepancy.

3.2 Application of the convergence theorem

We set here $Y_n = \xi_{n+1}$, $\mathcal{F}_n = \{\emptyset, \Omega\}$, $n \geq 0$, and $\Delta M_{n+1} \equiv 0$. We need the strong Lyapunov condition on H and notice that the function ϕ becomes useless because we place in the case $p = 1$. To apply the convergence theorem, we need to check the assumption of the class of functions $\mathcal{V}_{\varepsilon_n,1}$. We then use two main results which give us two cases of applications which depend on the regularity of the quasi-mean function H.

▷ **The finite variation case** Assume that $u \mapsto H(\theta^*, u)$ has finite variation (in the measure sense or in the Hardy and Krause sense). The Koksma-Hlawka Inequality provides an error bound for the empirical mean $\frac{1}{n}\sum_{k=1}^{n} H(\theta^*, \xi_k)$ as estimator of $\mathbb{E}\left[H(\theta^*, U_1)\right]$.

Proposition 3.2. *(Koksma-Hlawka Inequality) Let $\xi = (\xi_1, \ldots, \xi_n)$ be an n-tuple of $[0,1]^q$-valued vectors and let f be a function with finite variation. Then*

$$\left|\frac{1}{n}\sum_{k=1}^{n} f(\xi_k) - \int_{[0,1]^q} f(u)\lambda_q(du)\right| \leq V(f)D_n^*(\xi).$$

Consequently, the class of functions for the ergodicity assumption becomes

$$\mathcal{V} = \{f : [0,1]^q \to \mathbb{R} \text{ s.t. } V(f) < +\infty\} \subset \mathcal{V}_{\varepsilon_n,1} \quad \text{with} \quad \varepsilon_n = \frac{(\log n)^q}{n}$$

and assume that

$$\forall \theta \in \mathbb{R}^d, \quad H(\theta, \cdot) \in \mathcal{V} \quad \text{and} \quad \ell_n := \max_{1 \leq k \leq n} k\left(D_k^*(\xi)\right) = O((\log n)^q).$$

The assumptions on the sequence of gain parameters then become

$$\sum_{n \geq 1} \gamma_n = +\infty, \ \gamma_n(\log n)^q \to 0, \ \sum_{k \geq 1} \max(|\Delta\gamma_{n+1}|, \gamma_n^2)(\log n)^q < +\infty.$$

Remark. The choice of $\gamma_n := \frac{c}{n^\rho}$, $\frac{1}{2} < \rho \leq 1$ satisfies the assumptions (like in i.i.d. setting).

▷ **The Lipschitz case** If $q \geq 2$ it is difficult to check that $f \in \mathcal{V}$ and these functions become "rare" as q increases. If we assume the more natural regularity assumption $u \mapsto H(\theta^*, u)$ is Lipschitz continuous, the following theorem due to Proinov (see [29]) provides another error bound depending on the continuity modulus of the function.

Theorem 3.1. *(Proinov) Assume $\mathbb{R}^q$ is equipped with the ℓ^∞-norm ($|x|_\infty := \max_{1 \leq i \leq q} |x_i|$, $x = (x^1, \ldots, x^q) \in \mathbb{R}^q$). Let $(\xi_1, \ldots, \xi_n) \in ([0,1]^q)^{\otimes n}$. For every continuous function $f : [0,1]^q \to \mathbb{R}$,*

$$\left|\frac{1}{n}\sum_{k=1}^{n} f\left(\xi_k\right) - \int_{[0,1]^q} f(u)\lambda_q(du)\right| \leq C_q w_f\left(D_n^*\left(\xi_1, \ldots, \xi_n\right)^{\frac{1}{q}}\right)$$

where

$$w_f(\delta) := \sup_{x,y \in [0,1]^q, |x-y|_\infty \leq \delta} |f(x) - f(y)|, \quad \delta \in (0,1),$$

is the uniform continuity modulus of f (with respect to the ℓ^∞-norm) and $C_q \in (0, \infty)$ is a universal constant only depending on q. If $q = 1$, $C_q = 1$ and if $q \geq 2$, $C_q \in [1, 4]$.

Then we have $H(\theta^*,\cdot) \in \mathcal{V}_{\varepsilon_n,1}$ with $\varepsilon_n = \frac{\log n}{n^{\frac{1}{q}}}$. In this case, $(\gamma_n)_{n\geq 1}$ be a non-increasing sequence of gain parameters satisfying

$$\sum_{n\geq 1}\gamma_n = +\infty, \quad \gamma_n(\log n)n^{1-\frac{1}{q}} \to 0, \quad \text{and} \quad \sum_{k\geq 1}\max(|\Delta\gamma_{n+1}|\,,\gamma_n^2)(\log n)n^{1-\frac{1}{q}} < +\infty.$$

Remark. The choice of $\gamma_n := \frac{c}{n}$ is always acceptable ($\gamma_n = cn^{-\rho}$, $1-\frac{1}{q} < \rho \leq 1$).

4 Applications to different types of random innovations

This section is devoted to some first applications of the above theorem. By applications, we mean here printing out some classes of random innovation processes $(Y_n)_{n\geq 0}$ for which the averaging rate assumption (2.7) is naturally satisfied by "large" class $\mathcal{V}_{\varepsilon_n,p}$.

First we present a simple framework of stochastic approximation where the noise is additive which is studied in [5] with some mixing properties, but here we only need (2.5). We showed in [21] how easily our result applies to real life stochastic optimization problem (as far as convergence is concerned).

Afterwards we focus on mixing innovations: we consider that the sequence $(Y_n)_{n\geq 0}$ is a functional of a stationary α-mixing process (satisfying condition on the summability of the mixing coefficients).

The last application is the case of an homogeneous Markov chain which can be seen as a possible more elementary counterpart of some (convergence) result obtained *e.g.* [3]. Some (quasi-optimal) *a.s.* rate of convergence can be obtained if H is smooth enough in θ, but to establish a regular CLT it is most likely that we cannot avoid to deal with the Poisson equation.

4.1 Recursive procedure with additive noise

We consider here the case where the function H is the sum of the mean function h and a noise, namely

$$\forall\theta\in\mathbb{R}^d, \quad \forall y\in\mathbb{R}^q, \quad H(\theta,y) = h(\theta)+y, \quad \text{and} \quad \Delta M_{n+1}\equiv 0.$$

In this framework, the Lyapunov assumption (2.14) becomes classical involving only the mean function h, namely

$$\forall\theta\in\mathbb{R}^d\setminus\{\theta^*\} \quad \langle\nabla L(\theta)\,|\,h(\theta)\rangle > 0.$$

Likewise, the growth control assumption (2.11) amounts to

$$\forall\theta\in\mathbb{R}^d, \quad |h(\theta)| \leq C_h\sqrt{1+L(\theta)},$$

provided the moment assumption $\sup_n \|Y_n\|_{\frac{p}{p-1}} < +\infty$, for some $p\in(1,\infty]$, is satisfied (take $\phi(y) := |y|\vee 1$). The martingale is vanishing in this example. Finally the step assumption (2.13) is ruled by the averaging rate of the sequence $(Y_n)_{n\geq 0}$.

4.2 Functional of a stationary α-mixing process

Here we provide a short background on α-mixing processes and their functionals. Our motivation here is to relax as much as possible our assumption on $(Y_n)_{n\geq 0}$ in order to apply stochastic approximation methods to exogenous data (*e.g.* from financial markets).

We aim now at applying our convergence theorem to input sequences $(Y_n)_{n\geq 0}$ which are (causal) functionals of an α-mixing process. Consider a stationary $\mathbb{R}^q$-valued process $X = (X_k)_{k\in\mathbb{Z}}$ with the

associated filtration $\mathcal{F}_n = \mathcal{F}_n^X := \sigma(X_k; k \leq n)$ and $\mathcal{G}_n = \mathcal{G}_n^X := \sigma(X_k; k \geq n)$. The α-mixing coefficients are defined as follows

$$\alpha_n = \sup\left\{|\mathbb{P}(U \cap V) - \mathbb{P}(U)\mathbb{P}(V)| \,|\, U \in \mathcal{F}_k, V \in \mathcal{G}_{k+n}, k \geq 0\right\}. \tag{4.25}$$

Let f be a measurable mapping from $(\mathbb{R}^q)^{\mathbb{Z}}$ to $\mathbb{R}^q$. Let $(Y_k)_{k\in\mathbb{Z}}$ be a causal functional of the process X, *i.e.*

$$\forall n \in \mathbb{Z}, \quad Y_n := f(\cdots, X_{n-1}, X_n).$$

Then $(Y_n)_{n\geq 0}$ is a stationary process with marginal distribution $\nu = \mathcal{L}(Y_0)$. For notational convenience, we also set $\forall n \in \mathbb{Z}$, $Z_n := H(\theta^*, Y_n)$.

Proposition 4.1. *Assume that $H(\theta^*, \cdot) \in L^{2+\delta}(\nu), \delta > 0$ and one of the following assumption*

1. *for all $n \in \mathbb{Z}$, $Y_n := f(\cdots, X_{n-1}, X_n)$ and X is a stationary α-mixing process satisfying the following condition*

$$\sum_{k=1}^{\infty} \sqrt{\frac{\alpha_k^{\frac{\delta}{2+\delta}}}{k}} < +\infty. \tag{4.26}$$

2. *$Y_n = X_n$ for all $n \geq 0$ and X is a stationary α-mixing process satisfying the condition*

$$\sum_{k\geq 0} \alpha_k^{\frac{\delta}{2+\delta}} < +\infty. \tag{4.27}$$

Then

$$H(\theta^*, \cdot) \in \mathcal{V}_{\varepsilon_n, 2} \quad \text{with } \varepsilon_n = n^{-\beta} \text{ for every } \beta \in \left[0, \frac{1}{2}\right). \tag{4.28}$$

Remark. Condition (4.27) is clearly satisfied as soon as the underlying process X is geometrically α-mixing.

The proof of Proposition 4.1 relies on the Gàl-Koksma Theorem (see [11]) stated and proved in a probabilistic setting in [1].

Theorem 4.1. *(Gàl-Koksma's Theorem) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(Z_n)_{n\geq 1}$ be a sequence of random variables belonging to L^p, $p \geq 1$, satisfying*

$$\mathbb{E}\,|Z_{M+1} + Z_{M+2} + \cdots + Z_{M+N}|^p = O(\Psi(N))$$

uniformly in M where $\frac{\Psi(N)}{N}$, $N \geq 1$, is a non-decreasing sequence. Then for every $\epsilon > 0$,

$$Z_1(\omega) + Z_2(\omega) + \cdots + Z_N(\omega) = o\left((\Psi(N)(\log(N))^{p+1+\epsilon})^{\frac{1}{p}}\right) \quad \mathbb{P}(d\omega)\text{-a.s.}$$

Remark. The conditions on X and Z come from a result established by Dedecker, Merlevède and Volný in [6] : by setting $P_0(Z_k) := \mathbb{E}\,[Z_k \,|\, \mathcal{F}_0] - \mathbb{E}\,[Z_k \,|\, \mathcal{F}_{-1}]$, if

$$\sum_{k\in\mathbb{Z}} \|P_0(Z_k)\|_2 < +\infty \quad \text{then} \quad \sum_{k\in\mathbb{Z}} |\mathrm{Cov}(Z_0, Z_k)| < +\infty. \tag{4.29}$$

Moreover Peligrad and Utev show in [28] that condition (4.29) is satisfied as soon as

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \|\mathbb{E}\,[Z_0 \,|\, \mathcal{G}_k]\|_2 < +\infty. \tag{4.30}$$

Proof of Proposition 4.1. 1. By the ergodic theorem, we have

$$\frac{1}{n}\sum_{k=0}^{n-1} H(\theta^*, Y_k) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^q} H(\theta^*, y)\nu(dy) = 0. \tag{4.31}$$

We simply wish to obtain an estimation of the a.s. convergence rate in (4.31). To this end, we rely on Gal-Koksma's Theorem (see Theorem 4.1). We are going to evaluate $\mathbb{E}_\nu \left|Z_0 + \cdots + Z_{n-1}\right|^2$. Elementary computations lead to

$$\mathbb{E}_\nu \left|Z_0 + \cdots + Z_{n-1}\right|^2 = n\mathbb{E}_\nu Z_0^2 + 2\sum_{k=1}^{n-1}\sum_{j=1}^{k} \mathbb{E}_\nu \left[Z_j Z_0\right] = n\mathbb{E}_\nu Z_0^2 + 2\sum_{k=1}^{n-1} S_k^Z = n\left(\mathbb{E}_\nu Z_0^2 + \frac{2}{n}\sum_{k=1}^{n-1} S_k^Z\right)$$

where $S_k^Z = \sum_{j=1}^{k} \mathbb{E}_\nu \left[Z_j Z_0\right]$.

To establish that S_n^Z converges, we must establish that (4.30) holds. Set $B_2(\mathcal{G}_k) := \{W \in \mathcal{G}_k \,:\, \|W\|_2 \le 1\}$. Then

$$\left\|\mathbb{E}\left(Z_0 \,|\, \mathcal{G}_n\right)\right\|_2 = \sup_{W\in B_2(\mathcal{G}_k)} \mathbb{E}\left(WZ_0\right) \le 8\alpha_k^{\frac{1}{r}} \left\|H(\theta^*, Y_0)\right\|_p$$

owing to the classical covariance inequality for α-mixing process (see [8]) with $\frac{1}{r} + \frac{1}{p} = \frac{1}{2}$, $r, p > 2$. As $H(\theta^*, \cdot) \in L^{2+\delta}(\nu)$, $\delta > 0$, we may set $p = 2 + \delta$, and $r = \frac{2(2+\delta)}{\delta}$. As a consequence

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \left\|\mathbb{E}\left(Z_0 \,|\, \mathcal{G}_k\right)\right\|_2 < +\infty,$$

which implies (through (4.29)) that S_k^Z converges.

Now, by Cesaro's Lemma we have

$$\mathbb{E}_\nu \left|Z_0 + \cdots + Z_{n-1}\right|^2 = O(n).$$

Thus, by Gal-Koksma's Theorem, for all $\epsilon > 0$,

$$Z_0 + \cdots + Z_{n-1} = o\left(\sqrt{n}\left(\ln n\right)^{\frac{3}{2}+\epsilon}\right) \quad \mathbb{P}_\nu\text{-}a.s.,$$

and as a result,

$$n^\beta \left(\frac{1}{n}\sum_{k=0}^{n-1} H(\theta^*, Y_k) - \int H(\theta^*, y)\nu(dy)\right) \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} 0, \quad \text{for every} \quad \beta \in \left[0, \frac{1}{2}\right).$$

So we obtain

$$H(\theta^*, \cdot) \in \mathcal{V}_{\varepsilon_n, 2}, \quad \text{with } \varepsilon_n = n^{-\beta} \text{ for a } \beta \in \left[0, \frac{1}{2}\right). \tag{4.32}$$

2. If we assume that $Y_n = X_n$ for all $n \ge 0$, then we can directly use the covariance inequality for α-mixing process

$$\left|\mathbb{E}_\nu \left[H(\theta^*, Y_j)H(\theta^*, Y_0)\right]\right| = \left|\mathrm{Cov}_\nu \left(H(\theta^*, Y_j), H(\theta^*, Y_0)\right)\right| \le 8\alpha_j^{\frac{1}{r}} \left\|H(\theta^*, Y_0)\right\|_p \left\|H(\theta^*, Y_0)\right\|_q,$$

where $\frac{1}{r} + \frac{1}{p} + \frac{1}{q} = 1$. By symmetry, we take $p = q > 2$ and we get

$$\left|\mathbb{E}_\nu \left[H(\theta^*, Y_j)H(\theta^*, Y_0)\right]\right| \le 8\alpha_j^{1-\frac{2}{p}} \left\|H(\theta^*, Y_0)\right\|_p^2.$$

As $H(\theta^*, \cdot) \in L^{2+\delta}$, $\delta > 0$, we set $p = 2 + \delta$ and we obtain $\alpha_j^{1-\frac{2}{2+\delta}} = \alpha_j^{\frac{\delta}{2+\delta}}$. Then the condition (4.26) can be replace by the less stringent (4.27) called Ibragimov's condition. The rest of the proof is the same as in 1. □

4.3 Homogeneous Markov chain

Assume that the innovation process $(Y_n)_{n\geq 0}$ is an $\mathbb{R}^q$-valued homogeneous Markov chain which transition is $(P(y,dx))_{y\in\mathbb{R}^d}$. Let $\mathcal{F}_n^Y = \sigma(Y_0,\ldots,Y_n)$, $n\geq 0$, the natural filtration of the chain.

4.3.1 Application of the convergence theorem

The associated recursive procedure reads

$$\theta_{n+1} = \theta_n - \gamma_{n+1}K(\theta_n, Y_{n+1}), \quad n\geq 0,$$

where $K(\theta,y) := \mathbb{E}\left[K(\theta,Y_1)\,|\,Y_0=y\right]$ and $\Delta M_{n+1} := K(\theta_n,Y_{n+1}) - \mathbb{E}\left[K(\theta_n,Y_{n+1})\,|\,\mathcal{F}_n\right] = K(\theta_n,Y_{n+1}) - H(\theta_n,Y_n)$.

The growth assumption is required here on the function K, namely

$$\forall\theta\in\mathbb{R}^d, \quad \forall y\in\mathbb{R}^q, \quad |K(\theta,y)| \leq C_K\widetilde{\phi}(y)\sqrt{1+L(\theta)}$$

with $\sup_{n\geq 0}\left\|\widetilde{\phi}(Y_n)\right\|_{\frac{p}{p-1}} < +\infty$ and the Lyapunov assumption (2.14) is on H.

Remark. In this setting, the function ϕ of the convergence theorem is then $\left|\widetilde{\phi}\right| \vee \left|P\widetilde{\phi}\right|$ or $\left|\widetilde{\phi}\right| \vee P\left|\widetilde{\phi}\right|$ or $\left|\widetilde{\phi}\right| + P\left|\widetilde{\phi}\right|$ and we only need the above assumption on $\widetilde{\phi}$.

4.3.2 Ergodic framework description

We will say it is ν-ergodic if for every bounded Borel function $f:\mathbb{R}^q\to\mathbb{R}$,

$$\mathbb{P}_\mu\text{-}a.s. \quad \frac{1}{n}\sum_{k=0}^{n-1} f(Y_k) \underset{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^q} f d\nu. \tag{4.33}$$

Remark. As soon as the transition $(P(y,dx))_{y\in\mathbb{R}^q}$ of $(Y_n)_n$ is Feller, ν is an invariant distribution of the chain, *i.e.* $\nu P = \nu$.

If this property holds, ν is necessarily invariant for P. As a consequence the whole sequence $(Y_n)_{n\geq 0}$ is stationary under $\mathbb{P}_\nu$. Furthermore if (4.33) holds, the chain is ergodic under $\mathbb{P}_\nu$ for the shift operator Θ, *i.e.*, for every functional $F:(\mathbb{R}^{\mathbb{N}},\mathcal{B}or(\mathbb{R}^{\mathbb{N}}))\to\mathbb{R}$, $f\in L^r(\nu)$,

$$\frac{1}{n}\sum_{k=1}^{n} F\circ\Theta^k \underset{n\to\infty}{\longrightarrow} \mathbb{E}_\nu(F) \quad \mathbb{P}_\nu\text{-}a.s. \text{ and in } L^r(\nu).$$

Typically, if ν is an extremal invariant distribution for P then the chain is ergodic under $\mathbb{P}_\nu$ so it is of course the case if ν is unique. Birkhoff's theorem shows that

$$\mathcal{V}_{0^+,p}(\mathbb{P}_\nu) = L^p(\nu).$$

Proposition 4.2. *If $(Y_n)_{n\geq 0}$ is $\mathbb{P}_\nu$-ergodic and $\nu(dy)\sim P(x,dy)$ for every $x\in\mathbb{R}^q$, then for every sequence $(\varepsilon_n)_{n\geq 0}$ such that $\varepsilon_n \underset{n\to+\infty}{\longrightarrow} 0$,*

$$\forall y\in\mathbb{R}^q, \quad \mathcal{V}_{\varepsilon_n,p}(\mathbb{P}_y) = \mathcal{V}_{\varepsilon_n,p}(\mathbb{P}_\nu).$$

See [20] for the proof of this Proposition.

COMMENTS. By contrast with the approach of [3], we do not need to solve the Poisson equation related to the pseudo-transition

$$\Pi_{\theta_n}(x, dy) = \mathbb{P}\left(Y_{n+1} \in dy \,|\, \mathcal{F}_n\right)$$

of the algorithm. Indeed, they assume there exists a function $v_\theta := v(\theta, \cdot)$ solution to

$$Id - \Pi_\theta v_\theta = H(\theta, \cdot) - h(\theta) \tag{4.34}$$

(Assumption (H_4) in [3] page 220). The target θ^* is then a zero of the mean function h (not canonically defined at this stage in [3]). In our setting, $\Pi_\theta(x, dy) = P(x, dy)$ since the dynamics of Y_n does not depend upon θ, so that Condition (4.34) reads

$$v(\theta, x) - \int v(\theta, y) P(x, dy) = H(\theta, x) - h(\theta),$$

where the mean function is naturally defined in our setting by

$$h(\theta) = \int H(\theta, x)\nu(dx),$$

ν is the unique invariant probability measure for P. Then the family of Poisson's equations (indexed by the parameter θ) reads

$$v(\theta, x) - Pv(\theta, x) = H(\theta, x) - h(\theta).$$

A formal solution is given by

$$v(\theta, x) = \sum_{k \geq 0} P^k \left(H(\theta, \cdot) - h(\theta)\right)(x).$$

5 Numerical examples

This section is devoted to several examples (notably in Finance) of application of convergence theorems in the different framework developed Section 3 and 4. They are divided in two parts: the first one concerns the generated innovations and the second one the applications on real data. Primarily we present an implicit parameter search, *i.e.* a procedure to look for the value that reaches some threshold of the mean function with i.i.d. and uniformly distributed sequences. Then we develop some stochastic gradient procedures with a companion procedure to compute risk measures (see [2]) and minimization of potential associated to a diffusion by using some result on the computation of the invariant distribution (see [15] and [22]). Next we consider an example introduced in learning automata, the so-called two-armed bandit, which recursive procedure was studied in [18] and [32]. Finally we examine a problem of optimal allocation (see [21]) which is implemented on real data.

5.1 Application to implicit correlation search

Consider a 2-dimensional Black-Scholes model *i.e.* $X_0^t = e^{rt}$ (riskless asset) and

$$X_t^i = x_0^i e^{(r - \frac{\sigma_i^2}{2})t + \sigma_i W_t^i}, \quad x_0^i > 0, \quad i = 1, 2,$$

for the two risky assets where $\left\langle W^1, W^2 \right\rangle_t = \rho t$, $\rho \in [-1, 1]$. Consider a best-of call option characterized by its payoff

$$\left(\max\left(X_T^1, X_T^2\right) - K\right)_+.$$

We will use a stochastic recursive procedure to solve the inverse problem in ρ

$$P_{BoC}(x_0^1, x_0^2, K, \sigma_1, \sigma_2, r, \rho, T) = P_0^{market}$$

where P_0^{market} is the quoted premium of the option (mark-to-market) with

$$\begin{aligned} P_{BoC}(x_0^1, x_0^2, K, \sigma_1, \sigma_2, r, \rho, T) &:= e^{-rT}\mathbb{E}\left[\left(\max\left(X_T^1, X_T^2\right) - K\right)_+\right] \\ &= e^{-rT}\mathbb{E}\left[\left(\max\left(x_0^1 e^{\mu_1 T + \sigma_1\sqrt{T}Z^1}, x_0^2 e^{\mu_2 T + \sigma_2\sqrt{T}Z^2}\right) - K\right)_+\right] \end{aligned}$$

where $\mu_i = r - \frac{\sigma_i^2}{2}$, $i = 1, 2$, $Z = (Z^1, Z^2) \overset{d}{=} \mathcal{N}(0, I_2)$. We assume from now on that this equation (in ρ) has at least one solution, say ρ^*. The most convenient way to prevent edge effects due to the fact that $\rho \in [-1, 1]$ is to use a trigonometric parametrization of the correlation by setting $\rho = \cos\theta, \theta \in \mathbb{R}$. This introduces an over-parametrization (inside $[0, 2\pi]$) since θ and $\pi - \theta$ yield the same solution, but this is not at all a significant problem for practical implementation (a careful examination shows that in fact one equilibrium is repulsive and one is attractive). From now on, for convenience, we will just mention the dependence of the premium function in the variable θ,, namely

$$\theta \longmapsto P(\theta) := P_{BoC}(x_0^1, x_0^2, K, \sigma_1, \sigma_2, r, \cos(\theta), T).$$

The function P is a 2π-periodic continuous function. Extracting the implicit correlation from the market amounts to solving

$$P(\theta) = P_0^{market} \quad (\text{with } \rho = \cos\theta).$$

We need the following additional assumption

$$P_0^{market} \in (\min_\theta P, \max_\theta P)$$

i.e. that P_0^{market} is not an extremal value of P. It is natural to set for every $\theta \in \mathbb{R}$ and every $z = (z^1, z^2) \in \mathbb{R}^2$

$$H(\theta, z) = e^{-rT}\left(\max\left(x_0^1 e^{\mu_1 T + \sigma_1\sqrt{T}z^1}, x_0^2 e^{\mu_2 T + \sigma_2\sqrt{T}(z^1\cos\theta + z^2\sin\theta)}\right) - K\right)_+ - P_0^{market}$$

and to define the recursive procedure

$$\theta_{n+1} = \theta_n - \gamma_{n+1}H(\theta_n, Z_{n+1}) \quad \text{where} \quad (Z_n)_{n\geq 1} \overset{\mathcal{L}}{\sim} \mathcal{N}(0, I_2)$$

and the gain parameter sequence satisfies (2.13). For every $z \in \mathbb{R}^2$, $\theta \longmapsto H(\theta, z)$ is continuous and 2π-periodic which implies that the mean function $h(\theta) := \mathbb{E}H(\theta, Z^1) = P(\theta) - P_0^{market}$ and $\theta \longmapsto \mathbb{E}\left[H^2(\theta, Z^1)\right]$ are both continuous and 2π-periodic as well (hence bounded).

The main difficulty to apply Theorem 2.1 is to find out the appropriate Lyapunov function. The quoted value P_0^{market} is not an extremum of the function P, hence $\int_0^{2\pi} h^\pm(\theta)d\theta > 0$ where $h^\pm := \max(\pm h, 0)$. We consider θ_0 any (fixed) solution to the equation $h(\theta) = 0$ and two real numbers $\beta_\pm$ such that

$$0 < \beta_+ < \frac{\int_0^{2\pi} h_+(\theta)d\theta}{\int_0^{2\pi} h_-(\theta)d\theta} < \beta_-$$

and we set

$$g(\theta) := \begin{cases} \mathbb{1}_{\{h>0\}}(\theta) + \beta_+\mathbb{1}_{\{h<0\}}(\theta) & if \quad \theta \geq \theta_0 \\ \mathbb{1}_{\{h>0\}}(\theta) + \beta_-\mathbb{1}_{\{h<0\}}(\theta) & if \quad \theta < \theta_0. \end{cases}$$

The function

$$\theta \longmapsto g(\theta)h(\theta) = h_+ - \beta_\pm h_-$$

is continuous and 2π-periodic on (θ_0, ∞) and 2π-periodic (*sic*) on $(-\infty, \theta_0)$. Furthermore $gh(\theta) = 0$ iff $h(\theta) = 0$ so that $gh(\theta_0) = gh(\theta_0-) = 0$ which ensures on the way the continuity of gh on the whole real line. Furthermore

$$\int_0^{2\pi} gh(\theta)d\theta > 0 \quad \text{et} \quad \int_{-2\pi}^{0} gh(\theta)d\theta < 0$$

so that, on the one hand,

$$\lim_{\theta\to\pm\infty} \int_0^\theta gh(u)du = +\infty$$

and, on the other hand, there exists a real constant $C > 0$ such that the function

$$L(\theta) = \int_0^\theta gh(u)du + C$$

is nonnegative. Its derivative is given by $L' = gh$ so that $L'h = gh^2 \geq 0$ and $\{L'h = 0\} = \{h = 0\}$. It remains to prove that L' is Lipschitz continuous. Calling upon the usual arguments, one shows that the function

$$\frac{\partial P}{\partial \theta}(\theta) = \sigma_2\sqrt{T}\mathbb{E}\left(\mathbb{1}_{\{X_T^2 > \max(X_T^1, K)\}} X_T^2(\cos(\theta)Z^2 - \sin(\theta)Z^1)\right)$$

is a continuous 2π-periodic function, hence bounded. Consequently h and $h_\pm$ are Lipschitz continuous which implies in turn that $L' = gh$ is Lipschitz as well.

Moreover, one can show that the equation $P(\theta) = P_0^{market}$ market has finitely many solutions on every interval of length 2π. One may apply Theorem 2.1 to derive that θ_n will converge toward a solution θ^* of the equation $P(\theta) = P_0^{market}$.

Numerical experiment. We set the model parameters to the following values

$$x_0^1 = x_0^2 = 100,\ r = 0.10,\ \sigma_1 = \sigma_2 = 0.30,\ \rho = -0.50$$

and the payoff parameters

$$T = 1,\ K = 100.$$

The implicit correlation search recursive procedure is implemented with a sequence of some quasi-random normal numbers, namely

$$(\zeta_n^1, \zeta_n^2) = \left(\sqrt{-2\log(\xi_n^1)}\sin\left(2\pi\xi_n^2\right), \sqrt{-2\log(\xi_n^1)}\cos\left(2\pi\xi_n^2\right)\right),$$

where $\xi_n = (\xi_n^1, \xi_n^2),\ n \geq 1$, is simply a regular 2-dimensional Halton sequence.

The reference Black-Scholes price 30.75 is used as a market price so that the target of the stochastic algorithm is $\theta^* \in \arccos(-0.5)$. The stochastic approximation procedure parameters are

$$\theta_0 = 0,\ n = 10^5.$$

The choice of θ_0 is blind on purpose. Finally we set $\gamma_n = \frac{8}{n}$. No re-scaling of the procedure has been made in the below example.

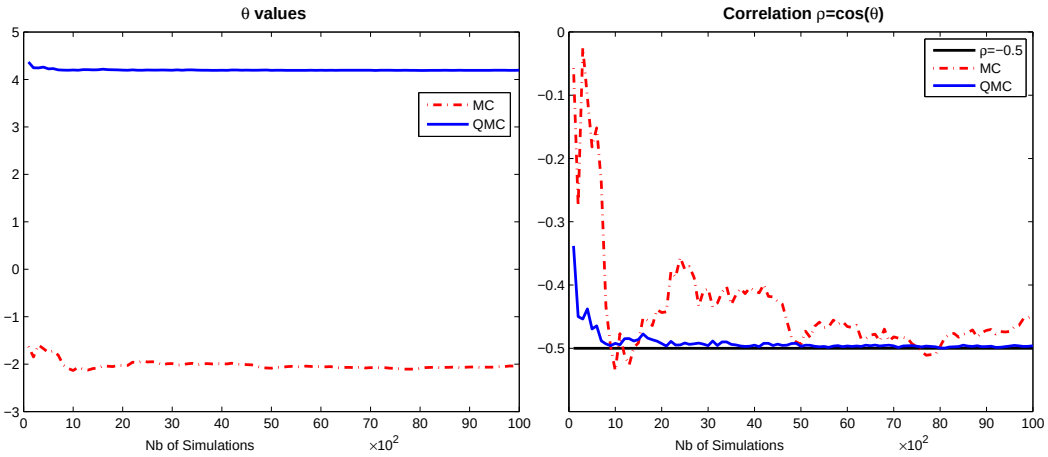

Figure 1: B-S Best-of-Call option. $T = 1$, $r = 0.10$, $\sigma_1 = \sigma_2 = 0.30$, $x_0^1 = x_0^2 = 100$, $K = 100$. Left: convergence of θ_n toward a θ^*. (up to $n = 10000$). Right: convergence of $\rho_n := cos(\theta_n)$ toward -0.5

5.2 Computation of the VaR-CVaR

Another example of application is the recursive computation of financial risk measure which are the best known and the most common: the Value-at-Risk (VaR) and the Conditional Value-at-Risk (CVaR). This risk measures evaluate the extreme losses of a portfolio potentially faced by traders. The recursive computation of the VaR and the CVaR was introduced in [2] based on the formulation as an optimization problem (see [31]) with unconstrained importance sampling developed in [23]. We only present in this section the companion recursive procedure to compute these risk measures without any variance reduction techniques.

5.2.1 Definitions and formulation

Let $Y : (\Omega, A, \mathbb{P}) \to \mathbb{R}$ be a random variable representative of a loss (*i.e.* $Y \geq 0$ stands for a loss equal to Y).

Definition 5.1. *The Value at Risk (at level $\alpha \in (0,1)$) of a given portfolio is the (lowest) α-quantile of the distribution Y i.e.*

$$VaR_\alpha(Y) := \inf \{\theta \,|\, \mathbb{P}(Y \leq \theta) \geq \alpha\}.$$

As soon as the distribution function of Y has no atom, the value at risk satisfies $P(Y \leq VaR_\alpha(Y)) = \alpha$ and if the distribution function F_Y of Y is also increasing (strictly) then, it is the unique solution. As this risk measure is not consistent (see [10]), another consistent risk measure is provided by the Conditional value at Risk when $Y \in L^1(\mathbb{P})$ with a continuous distribution (no atom).

Definition 5.2. *Let $Y \in L^1(\mathbb{P})$ with an atomless distribution. The Conditional value at Risk (at level α) is the conditional expectation of the portfolio losses Y above the $VaR_\alpha(Y)$, i.e.*

$$CVaR_\alpha(Y) := \mathbb{E}\left[Y | Y \geq VaR_\alpha(Y)\right].$$

The following formulation of the $VaR_\alpha(Y)$ and $CVaR_\alpha(Y)$ as solutions to an optimization problem is due to Rockafellar and Uryasev in [31].

Proposition 5.1. *(Rockafellar and Uryasev) Let $Y \in L^1(\mathbb{P})$ with an atomless distribution. The function $L : \theta \mapsto \theta + \frac{1}{1-\alpha}\mathbb{E}\,(Y-\theta)_+$ is convex, and*

$$CVaR_\alpha(Y) = \min_\theta \left(\theta + \frac{1}{1-\alpha}\mathbb{E}\,(Y-\theta)_+\right) \text{ with } VaR_\alpha(Y) = \inf \arg\min_\theta \left(\theta + \frac{1}{1-\alpha}\mathbb{E}(Y-\theta)_+\right).$$

5.2.2 Stochastic gradient for the computation of the $VaR_\alpha(Y)$

This suggests to implement a stochastic gradient descent derived from the above Lyapunov function $L(\theta) = \theta + \frac{1}{1-\alpha}\mathbb{E}(Y-\theta)_+$. Assume that $Y \in L^1(\mathbb{P})$ and that the distribution of Y has a bounded density function f_Y. Then, we set

$$H(\theta, y) := 1 - \frac{1}{1-\alpha}\mathbb{1}_{\{y \geq \theta\}}$$

so that $L'(\theta) = \mathbb{E}[H(\theta, Y)]$ and devise the stochastic gradient descent

$$\theta_{n+1} = \theta_n - \gamma_{n+1} H(\theta_n, Y_{n+1}), \quad n \geq 0, \quad \theta_0 \in L^1(\mathbb{P}).$$

The function L to be minimized satisfies

$$\lim_{\theta \to +\infty} \frac{L(\theta)}{\theta} = 1 \quad \text{and} \quad \lim_{\theta \to +\infty} \frac{L(-\theta)}{\theta} = \frac{\alpha}{1-\alpha} \quad \text{hence} \quad \lim_{\theta \to \pm\infty} L(\theta) = +\infty.$$

Now, the derivative $L'(\theta) = 1 - \frac{1}{1-\alpha}\mathbb{P}(Y > \theta)$ satisfies a Lipschitz property, namely

$$\left|L'(\theta') - L'(\theta)\right| \leq \frac{1}{1-\alpha}\mathbb{P}(\theta \wedge \theta' \leq Y \leq \theta \vee \theta') \leq \frac{\|f_Y\|_\infty}{1-\alpha}\left|\theta' - \theta\right|.$$

Finally it is clear that

$$|H(\theta, y)| \leq 1 \vee \frac{\alpha}{1-\alpha}.$$

So one may apply Theorem 2.1 to conclude that

$$\theta_n \xrightarrow[n\to\infty]{a.s.} \theta^* = VaR_\alpha(Y).$$

5.2.3 Computation of the $CVaR_\alpha(Y)$

The idea to compute the $CVaR_\alpha(Y)$ is to devise a companion procedure of the above stochastic gradient by setting

$$\zeta_{n+1} = \zeta_n - \frac{1}{n+1}\left(\zeta_n - K(\theta_n, Y_{n+1})\right), \quad n \geq 0, \quad \zeta_0 = 0$$

where

$$K(\theta, y) := \theta + \frac{(y-\theta)_+}{1-\alpha}.$$

By rewriting this procedure, one shows that, for every $n \geq 0$, $(n+1)\zeta_{n+1} = n\zeta_n + K(\theta_n, Y_{n+1})$. Martingale argument and Kronecker Lemma imply that

$$\zeta_n \xrightarrow[n\to\infty]{a.s.} CVaR_\alpha(Y).$$

VaR and CVaR are linked to rare events when $\alpha \to 1$, in practice one must add an eventually adaptive importance sampling procedure which is detailed in [2].

5.3 Long term investment evaluation

This example present the case of Markov innovations. To describe the class of functions $\mathcal{V}_{\varepsilon_n,p}$, we use some results on computation of invariant measure of a diffusion developed in [15] and [22] which lead to assumptions on the coefficients of the diffusion and on the regularity of the solution of Poisson equation.

5.3.1 Computation of the invariant distribution of a diffusion

We consider a stochastic recursive algorithm for the computation of the invariant distribution ν introduced in [15] of a Brownian diffusion process

$$dY_t = b(Y_t)dt + \sigma(Y_t)dW_t \tag{5.35}$$

where $b : \mathbb{R}^q \to \mathbb{R}^q$ is a continuous vector field, σ is continuous on $\mathbb{R}^q$ with values in the set $\mathcal{M}_{q,d}$ of matrices with q rows and d columns, and W is a d-dimensional Brownian motion. We denote by $\mathcal{A}$ the infinitesimal generator associated to this diffusion.

First, we compute the Euler discretization of (5.35) with a step γ_n vanishing to 0, *i.e.*

$$\forall n \in \mathbb{N}, \quad \bar{Y}_{n+1} = \bar{Y}_n + \gamma_{n+1} b(\bar{Y}_n) + \sqrt{\gamma_{n+1}}\sigma(\bar{Y}_n)U_{n+1}, \tag{5.36}$$

where $\bar{Y}_0 \in L^0_{\mathbb{R}^q}(\Omega, \mathcal{F}, \mathbb{P})$ and $(U_n)_{n\geq 1}$ is $\mathbb{R}^d$-valued normalized white noise defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, independent of $\bar{Y}_0$. The step sequence $\gamma := (\gamma_n)_{n\geq 1}$ satisfies the conditions

$$\forall n \geq 1, \quad \gamma_n \geq 0, \quad \lim_{n\to\infty} \gamma_n = 0, \quad \text{and} \quad \lim_{n\to\infty} \Gamma_n = \lim_{n\to\infty} \sum_{k=1}^{n} \gamma_k = +\infty. \tag{5.37}$$

Let δ_y denote the Dirac mass at y. For every $n \geq 1$ and every $\omega \in \Omega$, set

$$\nu_n(\omega, dy) := \frac{1}{n} \sum_{k=0}^{n-1} \delta_{\bar{Y}_k(\omega)} \tag{5.38}$$

and use $\nu_n(\omega, f)$ to approximate $\nu(f)$ which can be compute recursively.

Definition 5.3. *(Strong condition of stability) A diffusion with generator $\mathcal{A}$ satisfies a strong condition of stability if it exists a function $V \in \mathcal{C}^2(\mathbb{R}^q, [1, +\infty[)$ such that*

1. $\lim_{|y|\to+\infty} V(y) = +\infty$,
2. $\exists \alpha > 0, \beta > 0$ *s.t.* $\mathcal{A}V \leq -\alpha V + \beta$.

Remark. We will denote that the generator $\mathcal{A}$ satisfies a strong condition of stability of type (V, α).

We assume that b and σ are continuous and the diffusion $(Y_t)_{t\geq 0}$ satisfies a strong condition of stability of type (V, α) with V sub-quadratic et that it exists a unique invariant measure ν. Besides the coefficients b and σ satisfy $|b|^2 + \mathrm{Tr}(\sigma\sigma^t) = O(V)$. Then the Euler scheme $(\bar{Y}_n)_{n\geq 0}$ defined by (5.36) satisfies a strong condition of stability of type (W, n_0) where W is a function depending on V and on the moments of U_1.

Assume that the function f defined on $\mathbb{R}^q$ admits a solution ϕ to the the Poisson equation

$$\mathcal{A}\phi = -(f - \nu(f)), \tag{5.39}$$

sufficiently regular, *i.e.* belonging to the set

$$\mathcal{E}_{p,W} := \left\{ \phi \in \mathcal{C}^p(\mathbb{R}^q, \mathbb{R}),\ \forall j \in \{0, \ldots, p\},\ \forall y \in \mathbb{R}^q,\ \left|D^j\phi(y)\right|^2 = o\left(\frac{W(y)}{V^j(y)}\right) \right\}$$

and satisfying $D^p\phi$ Lipschitz.

Proposition 5.2. *Let $p \geq 2$ such that $U_1 \in L^{2(p+1)}$ and $\phi \in \mathcal{E}_{p,W}$ solution to Poisson equation (5.39) with compact support such that $D^p\phi$ is Lipschitz. If $(v_n)_{n\geq 1}$ is a positive real sequence such that $(\frac{v_n}{n\gamma_n})_{n\geq 1}$ decreases to 0 and that*

$$\sum_{n\geq 1} \frac{v_n}{n} \left|\Delta \frac{1}{\gamma_n}\right| < +\infty, \quad \sum_{n\geq 1} \left(\frac{v_n}{n}\right)^2 \frac{1}{\gamma_n} < +\infty, \quad \textit{and} \quad \lim_n \frac{v_n}{n} \sum_{k=1}^{n} \gamma_k^{\frac{p-1}{2}} = 0,$$

then $f \in \mathcal{V}_{v_n^{-1},2}$.

Remark. This proposition is based on the Proposition V.4 in [22].

5.3.2 Application to the minimization of a potential

We consider the following sequence of gain parameters

$$\forall n \geq 1, \quad \gamma_n := cn^{-\frac{1}{3}}, \quad \text{with} \quad c > 0,$$

then (see [15])

$$a.s. \quad \frac{1}{n}\sum_{k=0}^{n-1} \delta_{\bar{Y}_k} \overset{(\mathbb{R}^q)}{\Longrightarrow} \nu \quad \text{and} \quad n^{\frac{1}{3}} \left(\frac{1}{n}\sum_{k=0}^{n-1} f(\bar{Y}_k) - \nu(f)\right) \xrightarrow{\mathcal{L}} \mathcal{N}(m, \Sigma_c^2),$$

where $\Sigma_c \in \mathcal{G}\ell(d,\mathbb{R}) \cap \mathcal{S}^+(d,\mathbb{R})$. Assume that

1. $\forall y \in \mathbb{R}^q$, $\theta \mapsto K(\theta, y)$ is convex,
2. $\forall \theta \in \mathbb{R}^d$, $K(\theta, \cdot) \in L^1(\nu)$,
3. $\forall \theta \in \mathbb{R}^d$, $\nabla_\theta K(\theta, y)$ exists $\nu(dy)$-*a.s.*,
4. $\forall \theta, \theta' \in \mathbb{R}^d$, $|K(\theta, y) - K(\theta', y)| \leq g_\theta(y)$.

Then

$$\min_{\theta\in\mathbb{R}^d} \int_{\mathbb{R}^q} K(\theta, y)\nu(dy) \Longleftrightarrow \int_{\mathbb{R}^q} \nabla_\theta K(\theta, y)\nu(dy) = 0,$$

consequently we devise the following recursive zero search procedure, also called stochastic gradient,

$$\forall n \geq 0, \quad \theta_{n+1} = \theta_n - \widetilde{\gamma}_{n+1} \nabla_\theta K(\theta_n, \bar{Y}_n),$$

and we can apply Theorem 2.1 to prove its convergence towards the target θ^*.

Numerical example. We consider a long-term investment project (see the example in [24]) which yields payoff at a rate that depends on the installed capacity level an on the value of an underlying state process modeled with an ergodic diffusion. The state process Y can represent an economic indicator such the asset demand or its discounted price. The aim is to determine the capacity expansion strategy that maximizes the long-term average payoff resulting from the project operation. So it is an ergodic control problem applying in a microeconomic framework. In [24] the authors show that this dynamical optimization problem is asymptotically equivalent to a static optimization problem only involving the stationary distribution ν of Y and the running payoff function K. We refer to [24] for further details.

We consider the following running payoff function

$$\forall \theta \in \mathbb{R}_+, \forall y \in \mathbb{R}, \quad K(\theta, y) = y^\alpha \theta^\beta - c\theta,$$

where $\alpha, \beta \in (0,1)$ and $c \in (0,\infty)$. The term $y^\alpha \theta^\beta$ can be identified with the so-called Cobb-Douglas production function, while the term $c\theta$ provides a measure for the cost of capital use. The mean function of the associated recursive procedure then reads

$$\forall \theta \in \mathbb{R}, \quad k(\theta) := \beta \text{sign}(\theta) \, |\theta|^\beta \int_{\mathbb{R}} y^\alpha \nu(dy) - c\theta,$$

where we have extended the function on the whole-real line and we have chosen this form to avoid the explosion at zero. Consequently the associated stochastic gradient procedure to maximize the payoff is given by

$$\forall n \geq 0, \quad \theta_{n+1} = \theta_n + \widetilde{\gamma}_{n+1} \left(\beta \text{sign}(\theta_n) \, |\theta_n|^\beta \, \bar{Y}_n^\alpha - c\theta_n \right).$$

The choice and the extension of the functions k introduces two more equilibrium points: 0 and $-\theta^*$. The literature on traps in Stochastic Approximation does not apply straightforwardly in our setting, but 0 is a repulsive point (because the derivative is infinite at this point) and one checks on simulations that it is never a target for the procedure. On the other hand $-\theta^*$ is a possible target but induces no difficulty.

In [24] the dynamic of the underlying state process is modeled by the one-dimensional CIR diffusion, namely

$$dY_t = \kappa \left(\vartheta - Y_t \right) dt + \sigma \sqrt{Y_t} dW_t \tag{5.40}$$

where $\kappa, \vartheta, \sigma > 0$ are constants satisfying $2\kappa\vartheta > \sigma^2$. The invariant distribution of Y is a Gamma law which density is given by

$$\nu(dy) = \frac{1}{\Gamma\left(\frac{2\kappa\vartheta}{\sigma^2}\right)} y^{\frac{2\kappa\vartheta}{\sigma^2}-1} \exp\left(\frac{2\kappa}{\sigma^2} \left[\vartheta \ln\left(\frac{2\kappa}{\sigma^2} - y \right) \right] \right) \mathbb{1}_{\{y \geq 0\}},$$

where Γ is the gamma function. Thus we can compute the previous integral, namely

$$\int_{\mathbb{R}} y^\alpha \nu(dy) = \frac{\Gamma\left(\frac{2\kappa\vartheta}{\sigma^2} + \alpha\right)}{\Gamma\left(\frac{2\kappa\vartheta}{\sigma^2}\right)} \left(\frac{\sigma^2}{2\kappa} \right)^\alpha,$$

so we have a closed form for θ^* given by

$$\theta^* = \left(\frac{\beta \Gamma\left(\frac{2\kappa\vartheta}{\sigma^2} + \alpha\right)}{c \Gamma\left(\frac{2\kappa\vartheta}{\sigma^2}\right)} \left(\frac{\sigma^2}{2\kappa} \right)^\alpha \right)^{\frac{1}{1-\beta}}.$$

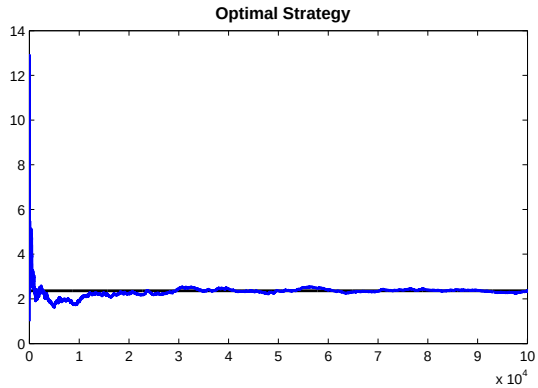

Figure 2: Convergence towards the optimal capacity level of the investment project : $\kappa = 1$, $\vartheta = 1$, $\sigma = 1.5$, $\alpha = 0.8$, $\beta = 0.7$, $c = 0.5$, $n = 10^5$.

If one considers a basket of assets modeled by a Wishart process (see [4] and [12]), a similar long-term ergodic control process can be devised. Closed forms are no longer available for the static optimization problem. However, our numerical approach can be extended straightforwardly (provided one has at hand an efficient method of simulation for Wishart process, see [12]).

5.4 The ergodic two-armed bandit

An application of the case of multiplicative function is the so-called two-armed bandit algorithm introduced in mathematical psychology and learning automata (see [27] and [25]). The criteria on $a.s.$ convergence under i.i.d. assumptions was obtained in [18] and under ergodic assumptions in [32]. A penalized version of this algorithm is also studied in [17]. This algorithm is defined as follows: at each step $n \geq 0$, one plays arm A (resp. arm B) with probability θ_n (resp. $1-\theta_n$), where θ_n is updated according the following rule, for all $n \geq 0$,

$$\begin{cases} \theta_{n+1} = \theta_n + \gamma_{n+1}\left((1-\theta_n)\mathbb{1}_{\{U_{n+1}\leq\theta_n\}\cap A_{n+1}} - \theta_n \mathbb{1}_{\{U_{n+1}>\theta_n\}\cap B_{n+1}}\right) \\ \theta_0 = \theta \in (0,1) \end{cases} \tag{5.41}$$

where $(U_n)_{n\geq 1}$ is a sequence of uniform random variables, $(A_n)_{n\geq 1}$ and $(B_n)_{n\geq 1}$ are two sequences of events evaluating the performances of the arms A and B.

This stochastic procedure can be rewrite as follows

$$\begin{cases} \theta_{n+1} = \theta_n + \gamma_{n+1}\left(\mathbb{1}_{A_{n+1}} - \mathbb{1}_{B_{n+1}}\right)\theta_n(1-\theta_n) + \gamma_{n+1}\Delta M_{n+1} \\ \theta_0 = \theta \in (0,1) \end{cases} \tag{5.42}$$

where $M_n := \sum_{k=1}^{n} \epsilon_k$, $M_0 := 0$, with

$$\epsilon_k := \mathbb{1}_{A_k}(1-\theta_{k-1})\left(\mathbb{1}_{\{U_{n+1}\leq\theta_n\}} - \theta_{k-1}\right) + \mathbb{1}_{B_k}\theta_{k-1}\left((1-\theta_{k-1}) - \mathbb{1}_{\{U_{n+1}>\theta_n\}}\right).$$

Then applying Theorem 2.2 with $h(\theta) = \theta(1-\theta)$, $Y_k := \mathbb{1}_{A_{k+1}} - \mathbb{1}_{B_{k+1}}$, $k \geq 0$, and $\chi(y) = \frac{y}{\nu(A)-\nu(B)}$ where $\nu(A) > \nu(B)$, we show that

$$\theta_n \xrightarrow[n\to\infty]{a.s.} \theta^* \in \{0,1\}.$$

In fact 0 is a trap and 1 is target. Further investigations on θ^* are carried in [32] in the ergodic framework (see also [18] in the purely i.i.d. setting).

5.5 Optimal split of orders across liquidity pools

This example is an application to real data in Finance. It is an optimal allocation problem which leads to a stochastic gradient by using Lagrangian approach developed in [21]. We present numerical results with (pseudo-)real data but simulations with α-mixing innovations are also presented in the reference paper.

5.5.1 Modeling description

The principle of a *Dark pool* is to propose a price with no guarantee of executed quantity at the occasion of an OTC transaction. Usually this price is lower than the one offered on the regular market. So one can model the impact of the existence of N dark pools ($N \geq 2$) on a given transaction as follows: let $V > 0$ be the random volume to be executed, let $\theta_i \in (0,1)$ be the *discount factor* proposed by the dark pool i. Let r_i denote the percentage of V sent to the dark pool i for execution. Let $D_i \geq 0$ be the quantity of securities that can be delivered (or made available) by the dark pool i at price $\theta_i S$.

The rest of the order is to be executed on the regular market, at price S. Then the cost C of the whole executed order is given by

$$C = S\sum_{i=1}^{N}\theta_i \min\left(r_i V, D_i\right) + S\left(V - \sum_{i=1}^{N}\min\left(r_i V, D_i\right)\right) = S\left(V - \sum_{i=1}^{N}\rho_i \min\left(r_i V, D_i\right)\right)$$

where $\rho_i = 1 - \theta_i \in (0,1), i = 1, \ldots, N$. Minimizing the mean execution cost, *given the price* S, amounts to solving the following maximization problem

$$\max\left\{\sum_{i=1}^{N}\rho_i \mathbb{E}\left(S\min\left(r_i V, D_i\right)\right), r \in \mathcal{P}_N\right\} \tag{5.43}$$

where $\mathcal{P}_N := \left\{r = (r_i)_{1\le i\le N} \in \mathbb{R}_+^N \mid \sum_{i=1}^N r_i = 1\right\}$. It is then convenient to *include the price* S *into both random variables* V *and* D_i by considering $\widetilde{V} := V\,S$ and $\widetilde{D}_i := D_i S$ instead of V and D_i.

Let $\mathcal{I}_N = \{1, \ldots, N\}$. We set for all $r = (r_1, \ldots, r_N) \in \mathcal{P}_N$, $\Phi(r_1, \ldots, r_N) := \sum_{i=1}^N \varphi_i(r_i)$, where

$$\forall i \in \mathcal{I}_N, \quad \varphi_i(u) := \rho_i \mathbb{E}\left(\min\left(uV, D_i\right)\right),\ u \in [0,1].$$

We assume that for all $i \in \mathcal{I}_N$,

$$V > 0\ \mathbb{P}-a.s.,\ \mathbb{P}(D_i > 0) > 0 \,\text{and the distribution function of } \tfrac{D_i}{V} \text{ is continuous on } \mathbb{R}_+, \tag{5.44}$$

then φ_i, $i \in \mathcal{I}_N$, are everywhere differentiable on the unit interval $[0,1]$ with

$$\varphi_i'(u) = \rho \mathbb{E}\left(\mathbb{1}_{\{uV \le D_i\}} V\right),\ u \in (0,1], \tag{5.45}$$

and one extends φ_i, $i \in \mathcal{I}_N$, on the whole real line into a concave non-decreasing function with $\lim_{\pm\infty}\varphi_i = \pm\infty$. So we can formally extend Φ on the whole affine hyperplan spanned by $\mathcal{P}_N$ *i.e.* $\mathcal{H}_N := \left\{r = (r_1, \ldots, r_N) \in \mathbb{R}^N \mid \sum_{i=1}^N r_i = 1\right\}$.

5.5.2 Description of the recursive procedure

We aim at solving the following maximization problem $\max_{r\in\mathcal{P}_N}\Phi(r)$. The Lagrangian associated to the sole affine constraint suggests that any $r^* \in \arg\max_{\mathcal{P}_N}\Phi$ iff $\varphi_i'(r_i^*)$ is constant when i runs over $\mathcal{I}_N$ or equivalently if $\varphi_i'(r_i^*) = \frac{1}{N}\sum_{j=1}^N \varphi_j'(r_j^*)$, $i \in \mathcal{I}_N$.

We set $Y^n := (V^n, D_1^n, \ldots, D_N^n)_{n\ge1}$. Then using the representation of the derivatives φ_i' yields

$$r^* \in \arg\max_{\mathcal{P}_N}\Phi \Longleftrightarrow \forall i \in \{1, \ldots, N\},\ \mathbb{E}\left(V\left(\rho_i \mathbb{1}_{\{r_i^* V < D_i\}} - \frac{1}{N}\sum_{j=1}^N \rho_j \mathbb{1}_{\{r_j^* V < D_j\}}\right)\right) = 0.$$

Consequently, this leads to the following recursive zero search procedure

$$r_i^{n+1} = r_i^n + \gamma_{n+1} H_i(r^n, Y^{n+1}),\ r^0 \in \mathcal{P}_N,\ i \in \mathcal{I}_N, \tag{5.46}$$

where for $i \in \mathcal{I}_N$, every $r \in \mathcal{P}_N$, every $V > 0$ and every $D_1, \ldots, D_N \ge 0$,

$$H_i(r, Y) = V\left(\rho_i \mathbb{1}_{\{r_i V < D_i\}} - \frac{1}{N}\sum_{j=1}^N \rho_j \mathbb{1}_{\{r_j V < D_j\}}\right)$$

where $(Y^n)_{n\geq 1}$ is a sequence of random vectors with non negative components such that, for every $n \geq 1$, $(V^n, D_i^n, i = 1, \ldots, N) \stackrel{d}{=} (V, D_i, i = 1, \ldots, N)$.

The underlying idea of the algorithm is to reward the dark pools which outperform the mean of the N dark pools by increasing the allocated volume sent at the next step (and conversely). For sake of simplicity that $\operatorname{argmax}_{\mathcal{P}_N} \Phi = \{r^*\} \subset int(\mathcal{P}_N)$. Our "averaging" assumption is to assume that there exists an exponent $\alpha_i \in (0, 1]$ such that for every $u \in \mathbb{R}_+$,

$$\frac{1}{n}\sum_{k=1}^{n} V^k \mathbf{1}_{\{u<\frac{D_i^k}{V^k}\}} - \mathbb{E}(V\mathbf{1}_{\{u<\frac{D_i}{V}\}}) = O(n^{-\alpha_i}) \quad a.s. \text{ and in } L^2(\mathbb{P}). \tag{5.47}$$

Under additional assumption on the support of $\mathcal{L}(Y^n)$ (see [21]), we can apply Theorem 2.1. Then if the sequence $(\gamma_n)_{n\geq 1}$ satisfies (2.13), we get that the algorithm defined by (5.46) $a.s.$ converges towards $r^* = \operatorname{argmax}_{\mathcal{P}_N} \Phi$.

5.5.3 Numerical Tests

We consider the shortage setting, *i.e.* $\mathbb{E}V > \sum_{i=1}^N \mathbb{E}D_i$ because it is the most interesting case and the most common in the market. Now, we introduce an index to measure the performances of our recursive allocation procedure.

▷ **Relative cost reduction (w.r.t. the regular market):** it is defined as the ratios between the cost reduction of the execution using dark pools and the cost resulting from an execution on the regular market, *i.e.*, for every $n \geq 1$,

$$\frac{CR^{algo}}{V^n} = \frac{\sum_{i=1}^N \rho_i \min\left(r_i^n V^n, D_i^n\right)}{V^n}.$$

We have considered for V the traded volumes of a very liquid security – namely the asset BNP – during an 11 day period. Then we selected the N most correlated assets (in terms of traded volumes) with the original asset. These assets are denoted S_i, $i = 1, \ldots, N$ and we considered their traded volumes during the same 11 day period. Finally, the available volumes of each dark pool i have been modeled as follows using the mixing function

$$\forall\, 1 \leq i \leq N, \quad D_i := \beta_i \left((1-\alpha_i)V + \alpha_i S_i \frac{\mathbb{E}V}{\mathbb{E}S_i}\right)$$

where α_i, $i = 1, \ldots, N$ are the recombining coefficients, β_i, $i = 1, \ldots, N$ some scaling factors and $\mathbb{E}V$ and $\mathbb{E}S_i$ stand for the empirical mean of the data sets of V and S_i. The simulations presented here have been made with four dark pools ($N = 4$). Since the data used here covers 11 days and it is clear that unlike the simulated data, these pseudo-real data are not stationary: in particular they are subject to daily changes of trend and volatility (at least). To highlight this resulting changes in the response of the algorithms, we have specified the days by drawing vertical doted lines. The dark pool pseudo-data parameters are set to $\beta = \begin{pmatrix}0.1 & 0.2 & 0.3 & 0.2\end{pmatrix}^t$, $\alpha = \begin{pmatrix}0.4 & 0.6 & 0.8 & 0.2\end{pmatrix}^t$ and the dark pool trading (rebate) parameters are set to $\rho = \begin{pmatrix}0.01 & 0.02 & 0.04 & 0.06\end{pmatrix}^t$.

We benchmarked the algorithm on the whole data set (11 days) as though it were stationary. In particular, the running means of the performances are computed from the very beginning for the first 1500 data, and by a moving average on a window of 1500 data.

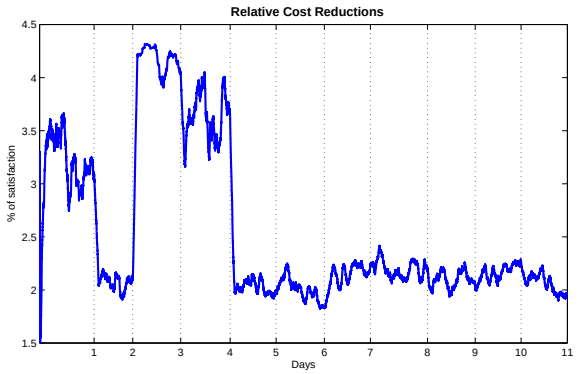

Figure 3: Case $N = 4$, $\sum_{i=1}^{N} \beta_i < 1$, $0.2 < \alpha_i \leq 0.8$ and $r_i^0 = 1/N$, $1 \leq i \leq N$.

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