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► **To cite this version:**

Sören Christensen, Albrecht Irle. A note on pasting conditions for the american perpetual optimal stopping problem. *Statistics and Probability Letters*, 2009, 79 (3), pp.349. 10.1016/j.spl.2008.09.002 . hal-00504136

HAL Id: hal-00504136

<https://hal.science/hal-00504136>

Submitted on 20 Jul 2010

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Accepted Manuscript

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PII: S0167-7152(08)00418-5
DOI: [10.1016/j.spl.2008.09.002](https://doi.org/10.1016/j.spl.2008.09.002)
Reference: STAPRO 5200

To appear in: *Statistics and Probability Letters*

Received date: 17 March 2008
Revised date: 29 August 2008
Accepted date: 3 September 2008

Please cite this article as: Christensen, S., Irle, A., A note on pasting conditions for the american perpetual optimal stopping problem. *Statistics and Probability Letters* (2008), doi:10.1016/j.spl.2008.09.002

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* Manuscript

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A Note on Pasting Conditions for the American Perpetual Optimal Stopping Problem

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Abstract

The principles of smooth and continuous pasting play an important role in the study of optimal stopping problems with jump processes. This principles state that the optimal stopping boundary is selected so that the value function is smooth and continuous respectively (depending on the behavior of the underlying process at the boundary). Extending the results of Alili & Kyprianou (2005, Ann. Appl. Probab., 1, 157) we show that in the case of an American perpetual put under a Lévy process the optimal stopping point is in fact characterized as the only point that fulfills this smooth/continuous pasting condition.

Key words: Optimal stopping, Lévy processes, American Put, smooth pasting, continuous pasting

1 Introduction

We consider the optimal stopping problem

$$v(x) = \sup_{\tau \in \mathcal{T}} E_x(e^{-q\tau}(K - e^{X_\tau})^+), \quad (\text{AOOSP})$$

where X is a real-valued Lévy process defined on a probability space (Ω, \mathcal{F}, P) with filtration $(\mathcal{F}_t)_{t \geq 0}$, $K, q > 0$ and \mathcal{T} is the set of all stopping times with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$. As usual E_x denotes the expected value with respect to the distribution of X when $X_0 = x$. (AOOSP) is the stopping problem connected with the pricing of an American put option in a financial

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market driven by a Lévy process (see e.g. Cont & Tankov (2004), Chapter 11.4).

The problem is well studied and solutions were published for many special classes of Lévy processes (see e.g. Asmussen et al. (2004), Boyarchenko & Levendorskii (2002)). It is not surprising that a first passage time of the form

$$\tau_y^- = \inf\{t > 0 : X_t < y\}$$

solves the problem. This was proved by E. Mordecki in 2002 for a general Lévy process by random-walk approximation (see Mordecki (2002)). L. Alili and A. E. Kyprianou gave an alternative proof using fluctational theory (see Alili & Kyprianou (2005)). Using a stochastic version of the Pecherskii-Rogozin identity they showed that for $y < \log K$ and $x \geq y$

$$\begin{aligned} v_y(x) &:= E_x(e^{-q\tau_y^-} (K - e^{X_{\tau_y^-}})^+) \\ &= \frac{E((KE(e^{\underline{X}_{e_q}}) - e^{x+\underline{X}_{e_q}})\mathbb{1}_{\{-\underline{X}_{e_q} > x-y\}})}{E(e^{\underline{X}_{e_q}})} \\ &= (K - e^x) + \frac{E((e^{x+\underline{X}_{e_q}} - KE(e^{\underline{X}_{e_q}}))\mathbb{1}_{\{-\underline{X}_{e_q} \leq x-y\}})}{E(e^{\underline{X}_{e_q}})} \end{aligned} \quad (1)$$

where \underline{X} denotes the running infimum of X , i.e. $\underline{X}_t = \inf_{s \leq t} X_s$, and e_q is an exponentially distributed random variable with parameter q , that is independent of X . They concluded that $\tau_{x^*}^-$ is an optimal stopping time, where the optimal stopping point x^* is given by $x^* = \log KE(e^{\underline{X}_{e_q}})$. This implies that

$$v(x) = v_{x^*}(x) = \frac{E((KE(e^{\underline{X}_{e_q}}) - e^{x+\underline{X}_{e_q}})^+)}{E(e^{\underline{X}_{e_q}})}.$$

Using this approach they dealt with the principles of smooth and continuous pasting. The principle of smooth pasting was already introduced in Mikhalevich (1957) and has been applied in a variety of problems, ranging from sequential analysis to mathematical finance. The principle of continuous pasting is more recent and was introduced in Peskir & Shiryaev (2000) and Peskir & Shiryaev (2002) as a variational principle to solve sequential testing and disorder problems for the Poisson process. An excellent overview is given in Peskir & Shiryaev (2006), Chapter IV.9, and one may summarize, see the above reference, p. 49:

"If X enters the interior of the stopping region S immediately after starting on ∂S , then the optimal stopping point x^* is selected so that the value function v is smooth in x^* . If X does not enter the interior of the stopping region immediately, then x^* is selected so that v is continuous in x^* ."

In our example the stopping region S is a half-plane and using the Markov-property of X one sees that entering S immediately means, that $\tau_0^- = 0$ P-a.s. In this case the point 0 is said to be regular for $(-\infty, 0)$. Alili and Kyprianou proved that if 0 is regular for $(-\infty, 0)$, then v is differentiable at x^* and if 0 is not regular for $(-\infty, 0)$, then v is continuous at x^* .

In the next section we will extend this result by proving that x^* is in fact the only "eligible" point with this property. This result underlines the strength of the principle and shows that it is a good idea to use the smooth/continuous pasting condition to find a solution to an optimal stopping problem. We refer to Peskir & Shiryaev (2006) where a wealth of applications is to be found.

2 Smooth and continuous fit

We consider the optimal stopping problem (AOOSP) as described in section 1 and use all notions given there. To make the statement in the last paragraph precise we have to say what an "eligible" stopping point is: We call a point y eligible if $y < \log(K)$. This is suggestive because if $y \geq \log K$, we have $v_y(x) = 0$ for all $x \in [\log K, y]$ and hence y cannot be an optimal stopping point.

Now we cover the case of continuous pasting, i.e. 0 is not regular for $(-\infty, 0)$. First we note that 0 is regular for $(-\infty, 0)$ if and only if $P(\underline{X}_{e_q} = 0) = 0$ because – using the monotonicity of $t \mapsto \underline{X}_t$ – we have

$$\begin{aligned} P(\underline{X}_{e_q} = 0) = 0 &\Leftrightarrow \int_0^\infty P(\underline{X}_t = 0) q e^{-qt} dt = 0 \\ &\Leftrightarrow \forall t > 0 : P(\underline{X}_t < 0) = 1 \Leftrightarrow 0 \text{ is regular for } (-\infty, 0). \end{aligned}$$

The case of continuous pasting was implicitly treated in the proof of Theorem 3 in Alili & Kyprianou (2005). For the sake of completeness and because it is easy and instructive we repeat the proof of this case here.

Theorem 1 (*Continuous fit*)

Suppose that $y < \log(K)$. Then v_y is continuous at y if and only if 0 is regular for $(-\infty, 0)$ or $y = x^$.*

PROOF. For $x < y$ we have $v_y(x) = K - e^x$ and for $x \geq y$ the identity (1)

states

$$v_y(x) = (K - e^x) + \frac{E((e^{x+\underline{X}_{e_q}} - KE(e^{\underline{X}_{e_q}}))\mathbb{1}_{\{-\underline{X}_{e_q} \leq x-y\}})}{E(e^{\underline{X}_{e_q}})}.$$

So v_y is continuous at y if and only if

$$\lim_{x \searrow y} E((e^{x+\underline{X}_{e_q}} - KE(e^{\underline{X}_{e_q}}))\mathbb{1}_{\{-\underline{X}_{e_q} \leq x-y\}}) = 0$$

and by dominated convergence this is the case if and only if

$$P(\underline{X}_{e_q} = 0) = 0 \text{ or } e^y - KE(e^{\underline{X}_{e_q}}) = 0.$$

Recalling that $x^* = \log KE(e^{\underline{X}_{e_q}})$ this proves the result.

The case of smooth fit is not so easy to handle. The key step is the following result on small ball probabilities of \underline{X}_{e_q} , that is proved by using the theory of subadditive functions.

Lemma 2 *Suppose that 0 is regular for $(-\infty, 0)$. Then*

$$\lim_{x \searrow 0} \frac{P(-\underline{X}_{e_q} \leq x)}{x} > 0.$$

PROOF. Write $\phi(x) = P(-\underline{X}_{e_q} > x) = 1 - P(-\underline{X}_{e_q} \leq x)$ for $x \geq 0$. Then $\lim_{x \searrow 0} \frac{P(-\underline{X}_{e_q} \leq x)}{x} = -\phi'(0+)$ and for all $x, y > 0$ we have by the strong Markov property and the lack of memory property of e_q

$$\begin{aligned} \phi(x+y) &= P(-\underline{X}_{e_q} > x+y) \geq P(\tau_x^- < e_q, -(\underline{X}_{e_q} - X_{\tau_x^-}) > y) \\ &= E(\mathbb{1}_{\{\tau_x^- < e_q\}} E(\mathbb{1}_{\{-(\underline{X}_{e_q} - X_{\tau_x^-}) > y\}} | \mathcal{F}_{\tau_x^-})) \\ &= P(\tau_x^- < e_q) P(\tau_y^- < e_q) = \phi(x)\phi(y) \end{aligned}$$

where we used that conditionally on $\mathcal{F}_{\tau_x^-}$ the random variables $-(\underline{X}_{e_q} - X_{\tau_x^-})$ and $-\underline{X}_{e_q}$ have the same distribution on $\{\tau_x^- < e_q\}$.

We obtain that $f := -\log \circ \phi$ is subadditive and because 0 is regular for $(-\infty, 0)$ we have that $\lim_{t \rightarrow 0} f(t) = 0$. It is well known, that every function f with these properties fulfills

$$f'(0) = \lim_{t \rightarrow 0} \frac{f(t)}{t} = \sup_{t > 0} \frac{f(t)}{t}$$

(see Hille & Phillips (1957), Chapter VII). Because $f \neq 0$ we have $f'(0) > 0$ and hence

$$\lim_{x \searrow 0} \frac{P(-\underline{X}_{e_q} \leq x)}{x} = -\phi'(0) = -(e^{-f})'(0) = f'(0)e^{-f(0)} > 0.$$

Now we can prove the smooth fit condition.

Theorem 3 (*Smooth fit*)

Suppose that $y < \log(K)$.

- (i) If 0 is not regular for $(-\infty, 0)$, then v_y is not differentiable at y .
- (ii) If 0 is regular for $(-\infty, 0)$, then v_y is differentiable at y if and only if $y = x^*$.

PROOF. If 0 is not regular for $(-\infty, 0)$ and $y \neq x^*$, then v_y is not continuous in y (see Theorem 1), so it is in particular not differentiable.

We can assume that 0 is regular for $(-\infty, 0)$ or $y = x^*$ and hence v_y is continuous in particular. Write $r = \frac{e^y}{KE(e^{\underline{X}_{e_q}})}$. Then $r = 1$ if and only if $y = x^*$.

For $x < y$ we have $v_y(x) = K - e^x$ and hence $v'_y(y-) = -e^y$. The identity (1) gives

$$\begin{aligned} v_y(x) &= (K - e^x) + \frac{E((e^{x+\underline{X}_{e_q}} - KE(e^{\underline{X}_{e_q}}))\mathbb{1}_{\{-\underline{X}_{e_q} \leq x-y\}})}{E(e^{\underline{X}_{e_q}})} \\ &= -Kr(e^{x-y} - 1)E(e^{\underline{X}_{e_q}}\mathbb{1}_{\{-\underline{X}_{e_q} > x-y\}}) \\ &= -KE((re^{\underline{X}_{e_q}} - 1)\mathbb{1}_{\{-\underline{X}_{e_q} > x-y\}}). \end{aligned}$$

Hence

$$\begin{aligned} \frac{v_y(x) - (K - e^y)}{x - y} &= \frac{v_y(x) - (K - rKE(e^{\underline{X}_{e_q}}))}{x - y} \\ &= -Kr \frac{e^{x-y} - 1}{x - y} E(e^{\underline{X}_{e_q}}\mathbb{1}_{\{-\underline{X}_{e_q} > x-y\}}) \\ &\quad + \frac{KE((re^{\underline{X}_{e_q}} - 1)\mathbb{1}_{\{-\underline{X}_{e_q} \leq x-y\}})}{x - y}. \end{aligned}$$

Let $s_1(x)$ and $s_2(x)$ denote the first and second term respectively. Then

$$\begin{aligned} s_1(0+) &= -KrE(e^{\underline{X}_{e_q}}\mathbb{1}_{\{-\underline{X}_{e_q} > 0\}}) \\ &= -KrE(e^{\underline{X}_{e_q}}) + rKP(-\underline{X}_{e_q} = 0) = v'_y(y-) + rKP(-\underline{X}_{e_q} = 0). \end{aligned}$$

In the case that $r = 1$ we have $\lim_{x \searrow y} s_2(x - y) = 0$, because

$$\begin{aligned} 0 \geq s_2(x - y) &= KE\left(\frac{e^{\underline{X}_{e_q}} - 1}{x - y} \mathbb{1}_{\{-\underline{X}_{e_q} \leq x - y\}}\right) \\ &= KE\left(\frac{e^{\underline{X}_{e_q}} - 1}{x - y} \mathbb{1}_{\{0 < -\underline{X}_{e_q} \leq x - y\}}\right) \\ &\geq K \frac{e^{y-x} - 1}{x - y} P(\underline{X}_{e_q} \in]0, x - y]) \xrightarrow{x \searrow y} 0. \end{aligned}$$

If $r \neq 1$ we can assume that 0 is regular for $(-\infty, 0)$. Hence by Lemma 2 $c := \lim_{x \searrow y} \frac{P(-\underline{X}_{e_q} \leq x - y)}{x - y} > 0$ exists and we have

$$\begin{aligned} s_2(x - y) &= KE\left(\left(re^{\underline{X}_{e_q}} - 1\right) \frac{\mathbb{1}_{\{-\underline{X}_{e_q} \leq x - y\}}}{x - y}\right) \\ &\geq K(re^{y-x} - 1) \frac{P(-\underline{X}_{e_q} \leq x - y)}{x - y} \xrightarrow{x \searrow y} K(r - 1)c \end{aligned}$$

and

$$\begin{aligned} s_2(x - y) &= KE\left(\left(re^{\underline{X}_{e_q}} - 1\right) \frac{\mathbb{1}_{\{-\underline{X}_{e_q} \leq x - y\}}}{x - y}\right) \\ &\leq K(r - 1) \frac{P(-\underline{X}_{e_q} \leq x - y)}{x - y} \xrightarrow{x \searrow y} K(r - 1)c. \end{aligned}$$

Hence $\lim_{x \searrow y} s_2(x - y) = c(r - 1)$, where $\infty \cdot 0 := 0$.
Putting pieces together we have

$$v'_y(y+) = v'_y(y-) + rKP(\underline{X}_{e_q} = 0) + Kc(r - 1).$$

If 0 is irregular for $(-\infty, 0)$ we have $r = 1$ and hence

$$v'_y(y+) = v'_y(y-) + KP(\underline{X}_{e_q} = 0) \neq v'_y(y-)$$

and if 0 is regular for $(-\infty, 0)$

$$v'_y(y+) = v'_y(y-) + Kc(r - 1).$$

Because $c > 0$ we have $v'_y(y+) = v'_y(y-)$ if and only if $r = 1$, i.e. $y = x^*$.

Remark 4 All statements above remain true for $q = 0$ when we assume that $\lim_{t \rightarrow \infty} X_t = \infty$.

3 An Application

The following application of the above results demonstrates two things: On the one hand it shows that in some special cases it is very easy to check the uniqueness of the smooth fit condition directly and on the other hand Theorem 3 can be used to determine the only candidate solution by a little algebra.

In the case when X is a Lévy process with mixed exponential jumps and a Brownian part, E. Mordecki gave explicit formulas for v and x^* in Mordecki (2003). Using the general identities above this is easy, because it is known that \underline{X}_{e_q} has a density of the form

$$g(y) = \left(\sum_{k=1}^{n+1} c_k \gamma_k e^{\gamma_k y} \right) \mathbf{1}_{\{y \leq 0\}}$$

where $\gamma_1, \dots, \gamma_{n+1}$ and c_1, \dots, c_{n+1} are positive constants (see Mordecki (2002)). The identity (1) yields that for all $x > y$ one has

$$v_y(x) = \frac{1}{E(e^{\underline{X}_{e_q}})} \sum_{i=1}^{n+1} c_i \gamma_i \left(\frac{KE(e^{\underline{X}_{e_q}})}{\gamma_i} e^{\gamma_i(y-x)} - \frac{1}{\gamma_i + 1} e^{(\gamma_i+1)y - \gamma_i x} \right)$$

and hence

$$v'_y(y+) = \frac{1}{E(e^{\underline{X}_{e_q}})} \sum_{i=1}^{n+1} c_i \gamma_i (-KE(e^{\underline{X}_{e_q}}) + \frac{\gamma_i}{\gamma_i + 1} e^y).$$

Calculating the optimal stopping point using the smooth pasting condition is very easy now:

$$\begin{aligned} v'_y(y+) = v'_y(y-) &\Leftrightarrow -e^y = \frac{1}{E(e^{\underline{X}_{e_q}})} \sum_{i=1}^{n+1} c_i \gamma_i (-KE(e^{\underline{X}_{e_q}}) + \frac{\gamma_i}{\gamma_i + 1} e^y) \\ &\Leftrightarrow E(e^{\underline{X}_{e_q}}) e^y = KE(e^{\underline{X}_{e_q}}) \sum_{i=1}^{n+1} c_i \gamma_i - \left(\sum_{i=1}^{n+1} \frac{c_i \gamma_i}{\gamma_i + 1} \right) e^y \\ &\Leftrightarrow e^y = KE(e^{\underline{X}_{e_q}}). \end{aligned}$$

Hence $y = x^* = \log(KE(e^{\underline{X}_{e_q}}))$ is the uniquely determined candidate for an optimal stopping point. This reaffirms the well known fact that the principle of smooth pasting is an effective tool to handle optimal stopping problems.

Acknowledgement

We thank the referees for their thoughtful comments.

References

- Alili, L., & Kyprianou, A. E., 2005. Some remarks on first passage of Lévy processes, the American put and pasting principles. *Ann. Appl. Probab.* 15, 2062–2080.
- Asmussen, S., Avram, F. and Pistorius, M., 2004. Russian and American put options under exponential phase-type Lévy models. *Stochastic Process. Appl.* 109(1), 79–111.
- Boyarchenko, S. I. & Levendroskii, S. Z., 2002. Perpetual American options under Lévy processes. *SIAM J. Control Optim.* 40, 1663–1696.
- Cont, R. & Tankov, P., 2004. *Financial modelling with Jump Processes*. Chapman and Hall, Boca Raton, FL.
- Hille, E. & Phillips, R. S., 1957. *Functional analysis and semi-groups*. AMS Colloquium Publications 31, Providence, Rhode Island.
- Mikhalevich, V. S., 1957. Bayesian choice between two hypotheses for the mean value of a normal process. *Visnik Kiiv. Univ.* 1, 101–104.
- Mordecki, E., 2003. Ruin probabilities for Lévy processes with mixed exponential negative jumps. *Theory Probab. Appl.* 48, 170–176.
- Mordecki, E., 2002. Optimal stopping and perpetual options for Lévy processes. *Finance Stoch.* 6, 473–493.
- Peskir, G. & Shiryaev, A. N., 2000. Sequential testing problems for Poisson processes. *Ann. Statist.* 28, 837–859.
- Peskir, G. & Shiryaev, A. N., 2002. Solving the Poisson disorder problem. *Advances in finance and stochastics*, 295–312.
- Peskir, G. & Shiryaev, A. N., 2006. *Optimal stopping and free-boundary problems*. Lectures in Mathematics, ETH Zürich, Birkhäuser, Basel.