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**ON CONTACT PROBLEM OF AN ELASTIC LAMINATED HALF-PLANE WITH A BOUNDARY NORMAL TO LAYERING****Dariusz M. Perkowski, Stanisław J. Matysiak, Roman Kulchytsky-Zhyhailo**Faculty of Mechanical Engineering, Białystok University of Technology, 15-351 Białystok,  
Wiejska St., 45C, Poland**Abstract**

The paper deals with the plane contact problem of an infinite long cylinder pressured into an elastic laminated semi-space. The nonhomogeneous body is composed of periodically repeated two constituent laminae and its boundary is assumed to be normal to the layering. The homogenized model with microlocal parameters given by Woźniak (1987), Matysiak and Woźniak (1987) is applied to find an approximate solution to the problem. The problem is reduced to a well-known dual integral equations and the stresses are obtained in the form of Fourier integrals. Numerical results, which show the influence of geometrical and mechanical properties of composite constituents on stress distributions are presented in figures.

**Keywords:** laminate theory, elastic properties, stress concentration, deformation**1. Introduction**

The knowledge of stress distributions in contact to two bodies play an important role in many technological applications. For this reason, considerable progress has been made with the analysis of contact problems in the theory of elasticity (see, for example monographs Shtaerman (1949), Galin (1953, 1980), Gladwell (1980), Goryacheva, Dobykhin (1988), Johnson (1985), Mossakowskii et al. (1985), Raous et al. (1995), Rvachev, Protsenko (1977), Fabrikant (1989, 1991). Some contact problems of anisotropic bodies were investigated by Wills (1966), Hwu and Fan (1998), Rogowski (1982, 2001), W.C. Liao (2001).

Composite materials with periodic structures are widely utilized in building engineering, machine elements, and aviation structures. Some composite structural members are constructed to work in contact with others. The two-dimensional contact problems of periodically layered half-space with the layering parallel to the boundary were considered by Kaczyński, Matysiak (1988, 1993, 2001). The above-mentioned papers are based on the homogenized model with microlocal parameters given by Woźniak (1987), and in the case of

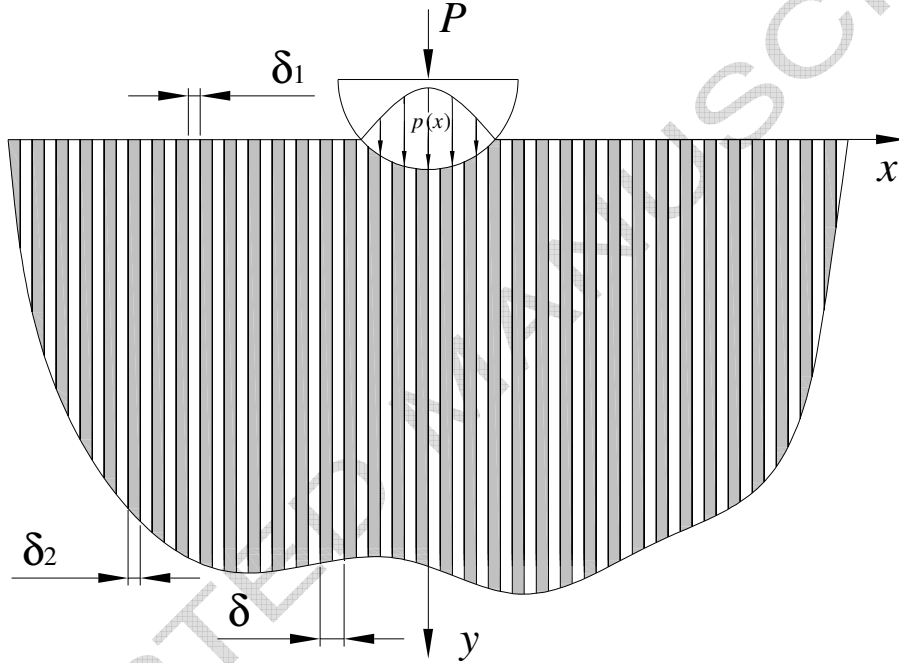
elastic laminates by Matysiak, Woźniak (1987). This model was derived by using the concepts of the nonstandard analysis combined with some postulated a priori physical assumptions. The governing equations of the homogenized model are expressed in terms of unknown macrodisplacements and certain extra unknowns called microlocal parameters and they replace the partial differential equations with strongly oscillating coefficients of the theory of elasticity. It should be emphasized that the continuity conditions on interfaces are fulfilled within the framework of the homogenized model.

This paper deals with the two-dimensional contact problem of a periodically two-layered elastic half-space with an infinitely long, rigid cylinder. The boundary of the composite is assumed to be normal to the layering, and the radius of the pressured cylinder is much greater than the width of contact region. The considered problem will be solved within the framework of the homogenized model with microlocal parameters (see Woźniak (1987), Matysiak and Woźniak (1987)). Moreover, the boundary condition describing the stress component normal to the boundary plane, discontinuous on interfaces is replaced by some averaging condition. This approach allows to reduce the considered mixed boundary problem to dual integral equations and to determine the contact pressures, the width of contact region, and displacements, stresses in the composite half-space. It is not possible to get an exact, closed form of the solution of the contact problem, so this paper presents some approximate solution (the exact closed form solution within the framework of the homogenized model with microlocal parameters). Other approximation models are known. They were derived by using some averaging procedures: the classical asymptotic homogenization (cf. Bensoussan et al. (1978), Sanchez-Palencia (1980), Bakhvalov and Panasenko (1984), Jikov et al. (1994)); the procedure based of the concept applied in the theory of thick plates (cf. Achenbach (1975)); the procedure based on the mixture theory (Bedford and Stern (1971)); the procedure based on some physical assumptions Guz et al. (1980) or on the matrix method (Bufler (1971), (2000)) or on the tolerance averaged procedures proposed by Woźniak and Wierzbicki (2000). However, the homogenized model with microlocal parameters seems to be useful to describe the contact problem, because: (1°) the model leads to two partial differential equations with constant coefficients in the two-dimensional case, (2°) the continuity conditions on interfaces are satisfied, (3°) many boundary value problems for periodically layered composites have been solved within the framework of the model. Moreover the substitution of the averaging boundary condition and the application of homogenized model with microlocal parameters leads to a good approximation of stresses for some problems of a semi-infinite laminated

layer (see R. Kulchytsky-Zhyhailo et al. (2006), where the results obtained within the framework of the homogenized model and the classical elasticity are compared).

## 2. Formulation and solution of the problem

Let us consider a microperiodic laminated half-space, the middle cross section of which is shown in Fig. 1. A repeated fundamental layer (lamina) of thickness  $\delta$  is composed of two homogeneous isotropic elastic sublayers, with thicknesses  $\delta_1$  and  $\delta_2$ . They are characterized by Lamé constants  $\lambda_j, \mu_j, j=1, 2$ .



**Fig. 1** The scheme of considered problem.

The boundary plane of half-space is assumed to be normal to the layering. Perfect bonding between the layers is assumed. A Cartesian coordinate system  $(x, y, z)$  such that the  $y$  axis is parallel to the layering and the  $x$  axis coincides with the straight line normal to the layering is introduced. Restricting the considerations to the plane-strain state (independent on the variable  $z$ ), denote at the point  $(x, y, 0)$  the displacement vector by  $[u(x, y), v(x, y), 0]$  and the stresses by  $\sigma_{xx}^{(j)}, \sigma_{xy}^{(j)}, \sigma_{yy}^{(j)}, \sigma_{zz}^{(j)}, j=1, 2$ , (in the following, all quantities pertaining to the sublayers of the first kind will be denoted with the index  $j=1$  and of the second kind with  $j=2$ ).

Consider an infinitely long rigid cylinder pressured into the laminated half-space, Fig. 1. The cross-section of punch is assumed to be in the shape of a parabola (it can be used to the approximation of a circular punch with large radius  $R$ ):

$$g(x) = \frac{x^2}{2R}, \quad R = \text{const.} > 0. \quad (2.1)$$

The problem of indentation of the periodically two-layered half-space is frictionless and has an unknown width of contact zone.

The basis of considerations is the homogenized model with microlocal parameters (see Woźniak (1987), Matysiak and Woźniak (1987)). Such an approach has proved to be useful and effective for solving a variety of boundary problems (see Matysiak (1995)). Below we recall below some relevant results from this theory. The homogenized model with microlocal parameters in the case of plane-strain state for the stratified body can be determined by following relations (see for example Kaczyński, Matysiak (1988, 1993, 2001)):

$$\begin{aligned} u(x, y) &= U(x, y) + h(x)q_x(x, y) \cong U(x, y), \\ v(x, y) &= V(x, y) + h(x)q_y(x, y) \cong V(x, y), \\ \sigma_{yy}^{(j)}(x, y) &\cong (\lambda_j + 2\mu_j)V_{,y} + \lambda_j(U_{,x} + h_j q_x), \\ \sigma_{xx}^{(j)}(x, y) &\cong (\lambda_j + 2\mu_j)(U_{,x} + h_j q_x) + \lambda_j V_{,y}, \\ \sigma_{xy}^{(j)}(x, y) &\cong \mu_j(U_{,y} + V_{,x} + h_j q_y), \quad j=1, 2, \end{aligned} \quad (2.2)$$

where  $U$ ,  $V$  and  $q_x$ ,  $q_y$  are unknown functions interpreted as the macro-displacement and microlocal parameters, respectively, and  $h(x)$  is a priori given  $\delta$ -periodic function (called the shape function) defined as follows

$$h(x) = \begin{cases} x - 0.5\delta_1, & 0 \leq x \leq \delta_1, \\ \frac{-\eta x}{1-\eta} - 0.5\delta_1 + \frac{\delta_1}{1-\eta}, & \delta_1 \leq x \leq \delta, \end{cases} \quad (2.3)$$

$$h(x + \delta) = h(x),$$

and

$$\eta = \delta_1 / \delta. \quad (2.4)$$

The shape function given by Eq. (2.3) is chosen in such way, that the continuity conditions for the stress vector on the interfaces are satisfied. The symbol  $h_j$ ,  $j=1, 2$ , denotes the derivative of the function  $h(x)$  in the  $j$ -th kind of composite component and

$$h_1 = 1, \quad h_2 = -\frac{\eta}{1-\eta}. \quad (2.5)$$

The four equations of the homogenized model with microlocal parameters are expressed in terms of four unknown macro-displacements  $U$ ,  $V$  and the microlocal parameters  $q_x$ ,  $q_y$ . The microlocal parameters are determined by derivatives of the functions  $U$ ,  $V$  (see, for example Kaczyński, Matysiak, (1988)). So, the governing relations of the homogenized model given in terms of macro-displacements (after eliminating the microlocal parameters) can be written in the form

$$\begin{aligned} A_1 \frac{\partial^2 U}{\partial x^2} + C \frac{\partial^2 U}{\partial y^2} + (B+C) \frac{\partial^2 V}{\partial x \partial y} &= 0, \\ C \frac{\partial^2 V}{\partial x^2} + A_2 \frac{\partial^2 V}{\partial y^2} + (B+C) \frac{\partial^2 U}{\partial x \partial y} &= 0, \end{aligned} \quad (2.6)$$

and

$$\begin{aligned} \sigma_{xy}^{(j)} &= C \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right), \quad \sigma_{xx}^{(j)} = A_1 \frac{\partial U}{\partial x} + B \frac{\partial V}{\partial y} \\ \sigma_{yy}^{(j)} &= D_j \frac{\partial U}{\partial x} + E_j \frac{\partial V}{\partial y}, \quad \sigma_{zz}^{(j)} = \frac{\lambda_j}{\lambda_j + 2\mu_j} (\sigma_{xx}^{(j)} + \sigma_{yy}^{(j)}), \quad j=1, 2, \end{aligned} \quad (2.7)$$

where

$$\begin{aligned} \tilde{\lambda} &= \eta\lambda_1 + (1-\eta)\lambda_2, \quad [\lambda] = \eta(\lambda_1 - \lambda_2), \quad \hat{\lambda} = \eta\lambda_1 + \frac{\eta^2}{1-\eta}\lambda_2, \\ \tilde{\mu} &= \eta\mu_1 + (1-\eta)\mu_2, \quad [\mu] = \eta(\mu_1 - \mu_2), \quad \hat{\mu} = \eta\mu_1 + \frac{\eta^2}{1-\eta}\mu_2, \\ A_1 &= \tilde{\lambda} + 2\tilde{\mu} - \frac{([\lambda] + 2[\mu])^2}{\hat{\lambda} + 2\hat{\mu}} > 0, \quad A_2 = \tilde{\lambda} + 2\tilde{\mu} - \frac{[\lambda]^2}{\hat{\lambda} + 2\hat{\mu}} > 0, \\ B &= \tilde{\lambda} - \frac{[\lambda]([\lambda] + 2[\mu])}{\hat{\lambda} + 2\hat{\mu}} > 0, \quad C = \tilde{\mu} - \frac{[\mu]^2}{\hat{\mu}} > 0, \\ D_j &= \frac{\lambda_j}{\lambda_j + 2\mu_j} A_1, \quad E_j = \frac{4\mu_j(\lambda_j + \mu_j)}{\lambda_j + 2\mu_j} + \frac{\lambda_j}{\lambda_j + 2\mu_j} B, \quad j=1, 2. \end{aligned} \quad (2.8)$$

From equations (2.7) it follows that the continuity conditions on interfaces are satisfied. However the components of stress tensor  $\sigma_{yy}^{(j)}$ ,  $\sigma_{zz}^{(j)}$  are discontinuous on interfaces.

The equations (2.6) can be separated by introducing the potentials  $\Psi_1$ ,  $\Psi_2$  (see Kulchitsky-Zhyhailo et al., 2006) as follows

$$U = \kappa_1 \frac{\partial \Psi_1}{\partial x} + \kappa_2 \frac{\partial \Psi_2}{\partial x}, \quad V = \frac{\partial \Psi_1}{\partial y} + \frac{\partial \Psi_2}{\partial y}, \quad (2.9)$$

where

$$\kappa_j = \frac{A_2 \gamma_j^2 - C}{B + C}, \quad (2.10)$$

and  $\gamma_i^2$ ,  $i = 1, 2$ , are the solutions of characteristic equations

$$A_2 C \gamma_j^4 + (B^2 + 2BC - A_1 A_2) \gamma_j^2 + A_1 C = 0. \quad (2.11)$$

Thus, we have

$$\gamma_j^2 \frac{\partial^2 \Psi_j}{\partial x^2} + \frac{\partial^2 \Psi_j}{\partial y^2} = 0, \quad j = 1, 2. \quad (2.12)$$

The characteristic equation (2.11) have four real roots  $\pm \gamma_1$ ,  $\pm \gamma_2$  in the form

$$\begin{aligned} \gamma_1 &= \left( \frac{A_1 A_2 - 2BC - B^2 - \sqrt{\Delta}}{2A_2 C} \right)^{\frac{1}{2}}, \\ \gamma_2 &= \left( \frac{A_1 A_2 - 2BC - B^2 + \sqrt{\Delta}}{2A_2 C} \right)^{\frac{1}{2}}, \end{aligned} \quad (2.13)$$

$$\Delta = (B^2 + 2BC - A_1 A_2)^2 - 4A_1 A_2 C^2 > 0.$$

The considered contact problem is described by the following boundary conditions

$$\begin{aligned} \frac{\partial V(x, 0)}{\partial x} &= -\frac{x}{R}, \quad \text{for } |x| \leq a, \\ \sigma_{yy}^{(j)}(x, 0) &= 0, \quad \text{for } |x| > a, \\ \sigma_{xy}^{(j)}(x, 0) &= 0, \quad \text{for } x \in \mathbf{R}, \quad j = 1, 2. \end{aligned} \quad (2.14)$$

Keeping in mind equations (2.7) one can observe that boundary condition (2.14)<sub>2</sub> may be rewritten in the form

$$D_i \frac{\partial U}{\partial x} + E_i \frac{\partial V}{\partial y} = 0, \quad i = 1, 2, \quad |x| > a, \quad y = 0. \quad (2.15)$$

The condition (2.15) is periodic and it leads to a rather complicated boundary value problem.

For this reason the periodic boundary condition (2.15) will be averaged. Thus, we obtain<sup>1</sup>:

$$B \frac{\partial U}{\partial x} + A_2 \frac{\partial V}{\partial y} = 0, \quad |x| > a. \quad (2.16)$$

The condition (2.16) together with (2.14)<sub>1,3</sub>, and the regularity conditions of infinity

<sup>1</sup> It can be shown that

$$\tilde{D} = D_1 \eta + (1 - \eta) D_2 = \frac{\lambda_1 \eta}{\lambda_1 + 2\mu_1} A_1 + \frac{\lambda_2 (1 - \eta)}{\lambda_2 + 2\mu_2} A_1 = B,$$

$$\begin{aligned} \tilde{E} &= E_1 \eta + E_2 (1 - \eta) = \eta \left( \frac{4\mu_1 (\lambda_1 + \mu_1)}{\lambda_1 + 2\mu_1} + \frac{\lambda_1}{\lambda_1 + 2\mu_1} \left( \tilde{\lambda} - \frac{[\lambda]([\lambda] + 2[\mu])}{\hat{\lambda} + 2\hat{\mu}} \right) \right) + \\ &+ (1 - \eta) \left( \frac{4\mu_2 (\lambda_2 + \mu_2)}{\lambda_2 + 2\mu_2} + \frac{\lambda_2}{\lambda_2 + 2\mu_2} \left( \tilde{\lambda} - \frac{[\lambda]([\lambda] + 2[\mu])}{\hat{\lambda} + 2\hat{\mu}} \right) \right) = A_2. \end{aligned}$$



$$\sigma_{xx}^{(j)}, \sigma_{xy}^{(j)}, \sigma_{yy}^{(j)} \rightarrow 0 \text{ for } x^2 + y^2 \rightarrow \infty, \quad (2.17)$$

and the equations (2.6) stand for the mixed boundary value problem for a half-plane described approximately the considered contact problem of periodically half-plane. Denoting the Fourier transform of function  $f$  with respect to the variable  $x$  by

$$\tilde{f}(s, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x, y) \exp(-ixs) dx, \quad i^2 = -1, \quad (2.18)$$

using equation (2.12), (2.9) and regularity conditions (2.17) we obtain

$$\begin{aligned} \tilde{U} &= is(\kappa_1 a_1(s) \exp(-|s|\gamma_1 y) + \kappa_2 a_2(s) \exp(-|s|\gamma_2 y)), \\ \tilde{V} &= -|s|(\gamma_1 a_1(s) \exp(-|s|\gamma_1 y) + \gamma_2 a_2(s) \exp(-|s|\gamma_2 y)), \end{aligned} \quad (2.19)$$

where functions  $a_1(s)$ ,  $a_2(s)$  will be determined from boundary conditions (2.14)<sub>1,3</sub> and (2.16).

To this aim, the Fourier transforms of stress components given in (2.7) are calculated

$$\begin{aligned} \tilde{\sigma}_{xx}^{(j)}(s, y) &= isA_1 \tilde{U} + B \frac{\partial \tilde{V}}{\partial y}, \\ \tilde{\sigma}_{xy}^{(j)}(s, y) &= C \left( \frac{\partial \tilde{U}}{\partial y} + is\tilde{V} \right), \\ \tilde{\sigma}_{yy}^{(j)}(s, y) &= isD_j \tilde{U} + E_j \frac{\partial \tilde{V}}{\partial y}. \end{aligned} \quad (2.20)$$

Using equations (2.20), (2.19) and assuming that

$$B \frac{\partial U}{\partial x} + A_2 \frac{\partial V}{\partial y} = -p(x) \text{ for } |x| < a, \quad (2.21)$$

where  $p(x)$  is an unknown function<sup>2</sup>, we obtain the following system of linear algebraic equations for  $a_1(s)$ ,  $a_2(s)$ :

$$\begin{cases} a_1(s)(A_2 \gamma_1^2 - \kappa_1 B) + a_2(s)(A_2 \gamma_2^2 - \kappa_2 B) = -\tilde{p}(s)s^{-2} \\ \gamma_1(1 + \kappa_1)a_1(s) + \gamma_2(1 + \kappa_2)a_2(s) = 0 \end{cases} \quad (2.22)$$

where

$$\tilde{p}(s) = \frac{1}{\sqrt{2\pi}} \int_{-a}^a p(x) e^{-ixs} ds. \quad (2.23)$$

The solution of equations (2.22) takes the form

<sup>2</sup> The function  $p(x)$  stands for the averaged contact pressure.

$$\begin{aligned}
a_1(s) &= \frac{\tilde{p}(s)\gamma_2}{s^2(\gamma_1 - \gamma_2)} \frac{B+C}{(A_2\gamma_1^2 + B)C}, \\
a_2(s) &= \frac{-\tilde{p}(s)\gamma_1}{s^2(\gamma_1 - \gamma_2)} \frac{B+C}{(A_2\gamma_2^2 + B)C}.
\end{aligned} \tag{2.24}$$

Now, from equations (2.24), (2.19) we can determine the displacement normal to the boundary for  $y=0$  in terms of the unknown function  $\tilde{p}(s)$ . Assuming that  $p(-x) = p(x)$ ,  $|x| < a$  we have

$$V(x, 0) = \sqrt{\frac{2}{\pi}} C_v \int_0^{\infty} s^{-1} \tilde{p}(s) \cos(xs) ds, \tag{2.25}$$

where

$$C_v = \frac{\sqrt{A_1 A_2} (\gamma_1 + \gamma_2)}{A_1 A_2 - B^2}. \tag{2.26}$$

To determine the function  $\tilde{p}(s)$  we return to the boundary conditions (2.14)<sub>1</sub>, (2.16) (conditions (2.14)<sub>2</sub> were satisfied in (2.22)). We obtain the following dual integral equations

$$\begin{aligned}
\sqrt{\frac{2}{\pi}} \int_0^{\infty} \tilde{p}(s) \sin(xs) ds &= \frac{x}{C_v R} \quad \text{for } 0 \leq x \leq a, \\
\sqrt{\frac{2}{\pi}} \int_0^{\infty} \tilde{p}(s) \cos(xs) ds &= 0 \quad \text{for } x > a.
\end{aligned} \tag{2.27}$$

The dual integral equations (2.27) are well known, see Sneddon (1951, 1966). The solution of equations (2.27) takes the form

$$\tilde{p}(s) = \sqrt{\frac{\pi}{2}} \frac{a}{C_v R} \frac{J_1(as)}{s}, \tag{2.28}$$

where  $J_1(as)$  is the Bessel function of the first kind.

The knowledge of function  $\tilde{p}(s)$  permits to find the distribution of averaged contact pressure

$$p(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \tilde{p}(s) \cos(xs) ds = \frac{1}{C_v R} \sqrt{a^2 - x^2}, \quad |x| \leq a, \tag{2.29}$$

The unknown half of the width of contact zone  $a$  can be determined by using the equilibrium condition

$$\int_{-a}^a p(x)dx = P, \quad (2.30)$$

where  $P$  is given the total normal force acting on the punch cross-section.

Substituting equation (2.28) into (2.30) we obtain

$$a^2 = \frac{2C_v RP}{\pi}. \quad (2.31)$$

Denoting the mean value of contact pressure by

$$p_0 = \frac{1}{2a} \int_{-a}^a p(x)dx = \frac{P}{2a}, \quad (2.32)$$

from (2.29) it follows that

$$\frac{p(x)}{p_0} = \frac{4}{\pi} \sqrt{1 - (x/a)^2}. \quad (2.33)$$

Using (2.19), (2.20) and (2.24) we obtain that the stresses in  $j$ -th layer, being the composite component take the form

$$\begin{aligned} \sigma_{xx}^{(j)}(x, y) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \tilde{\sigma}_{xx}^{p(j)}(s, y) \tilde{p}(s) \cos(xs) ds \\ \sigma_{yy}^{(j)}(x, y) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \tilde{\sigma}_{yy}^{p(j)}(s, y) \tilde{p}(s) \cos(xs) ds \\ \sigma_{xy}^{(j)}(x, y) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \tilde{\sigma}_{xy}^{p(j)}(s, y) \tilde{p}(s) \sin(xs) ds, \quad j=1, 2. \end{aligned} \quad (2.34)$$

where

$$\begin{aligned} \tilde{\sigma}_{xx}^{p(j)}(s, y) &= \sum_{k=1}^2 (-1)^{k+1} (\gamma_k^2 B - \kappa_k A_k) \exp(-|s| \gamma_k y) G_k^p, \\ \tilde{\sigma}_{yy}^{p(j)}(s, y) &= \sum_{k=1}^2 (-1)^{k+1} (\gamma_k^2 E_j - \kappa_k D_j) \exp(-|s| \gamma_k y) G_k^p, \end{aligned} \quad (2.35)$$

$$\tilde{\sigma}_{xy}^{p(j)}(s, y) = C \sum_{k=1}^2 (-1)^k \gamma_k (1 + \kappa_k) G_k^p \exp(-|s| \gamma_k y),$$

and

$$G_1^p = \frac{\gamma_2}{\gamma_1 - \gamma_2} \frac{B+C}{(A_2\gamma_1^2 + B)C}, \quad G_2^p = \frac{\gamma_1}{\gamma_1 - \gamma_2} \frac{B+C}{(A_2\gamma_2^2 + B)C}. \quad (2.36)$$

**Remark**

Assuming that

$$\lambda_1 = \lambda_2 = \lambda, \quad \mu_1 = \mu_2 = \mu, \quad (2.37)$$

then from (2.8) we have

$$\begin{aligned} \tilde{\lambda} &= \lambda, \quad \tilde{\mu} = \mu, \quad [\lambda] = [\mu] = 0, \\ A_1 &= A_2 = \lambda + 2\mu, \quad B = \lambda, \quad C = \mu. \end{aligned} \quad (2.38)$$

From equations (2.13), (2.38) and (2.26) it follows that

$$C_v = \frac{\lambda + 2\mu}{2\mu(\lambda + \mu)} = \frac{1-\nu}{\mu}, \quad \nu = \frac{\lambda}{2(\lambda + \mu)}, \quad (2.39)$$

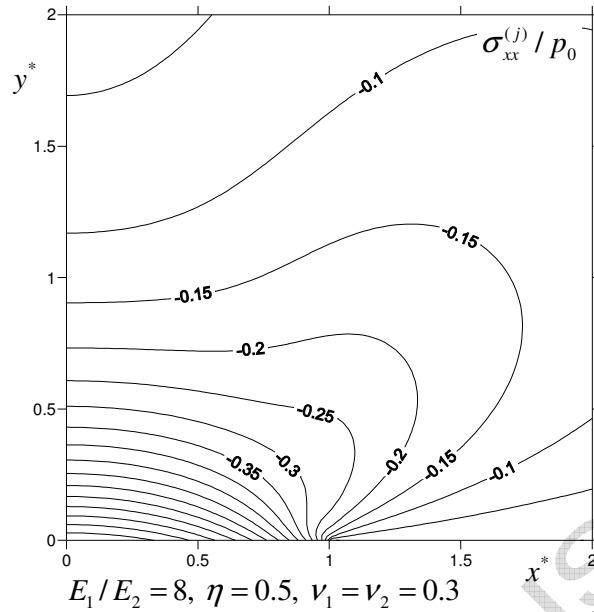
and the contact pressure  $p(x)$  given in (2.29) agrees with the adequate contact pressure for the homogeneous, isotropic semi-space, see Timoshenko, Goodier (1934).

**3. Numerical results**

The stresses at the point  $(x, y)$ , which is located in the  $j$ -th component of the composite, are presented in the form of integrals by equations (2.34) – (2.36). The Fourier integrals can be calculated exactly, but the expressions of their values are rather complicated. For this reason some distributions of stresses will be presented in the form of graphs. Let us introduce the following dimensionless variables

$$x^* = x/a, \quad y^* = y/a. \quad (3.1)$$

The stress components  $\sigma_{xx}^{(j)}$ ,  $\sigma_{xy}^{(j)}$ ,  $j=1, 2$ , are continuous (see equations (2.7)), but the stress components  $\sigma_{yy}^{(j)}$ ,  $j=1, 2$ , are discontinuous on the interfaces. The lines of constant values for dimensionless stresses  $\sigma_{xx}^{(j)}/p_0$  are presented in Fig. 2,

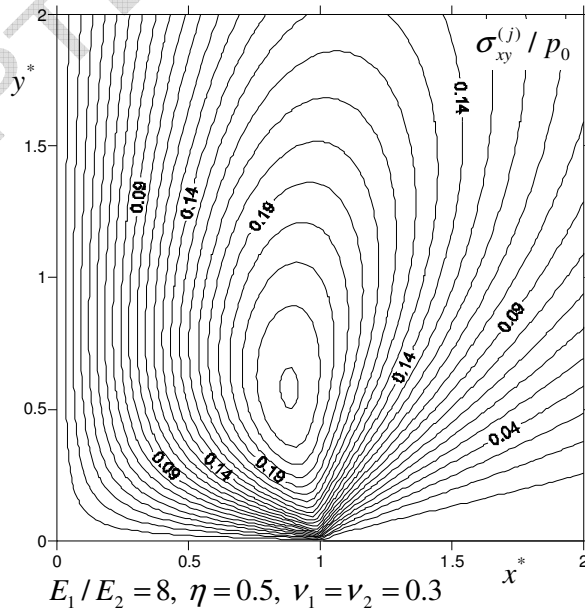


**Fig. 2** The distributions of dimensionless stresses  $\sigma_{xx}^{(j)} / p_0$ .

for  $\eta = 0.5$ , Poisson's ratios  $\nu_1 = \nu_2 = 0.3$  and Young modulus ratios  $E_1 / E_2 = 8$ . The half of contact zone stands for the region  $0 \leq x^* \leq 1$ ,  $y^* = 0$ .

Observe that the dimensionless stresses  $\sigma_{xx}^{(j)} / p_0$  take negative values and the zero stresses is far from the contact zone.

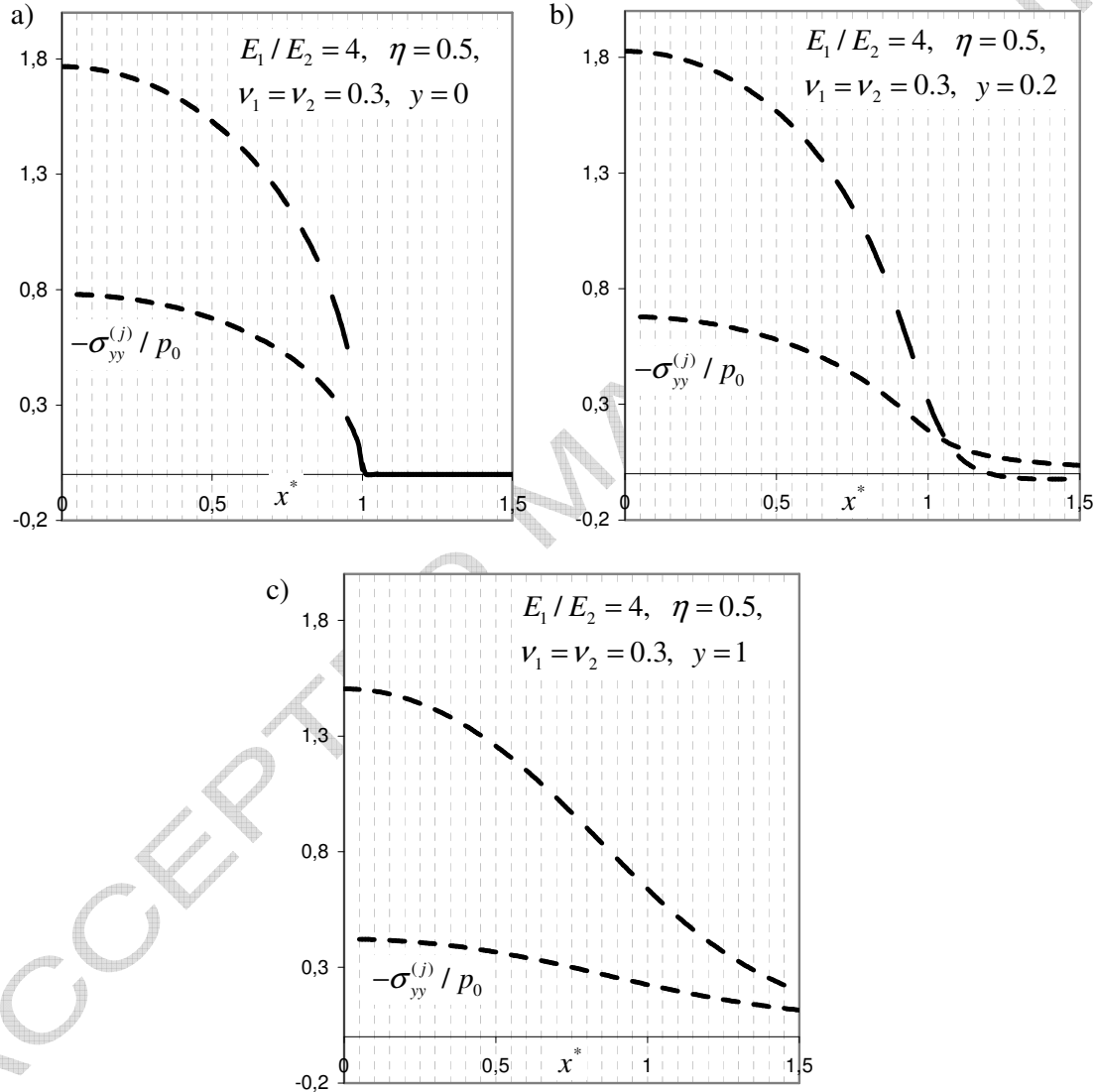
The lines of constant values for dimensionless stresses  $\sigma_{xy}^{(j)} / p_0$  are shown Fig. 3 for  $\eta = 0.5$ ,  $\nu_1 = \nu_2 = 0.3$  and the ratio of Young's  $E_1 / E_2 = 8$ .



**Fig. 3** The distributions of dimensionless stresses  $\sigma_{xy}^{(j)} / p_0$ .

It can be observed that the maximal values of  $\sigma_{xy}^{(j)} / p_0$  are achieved in regions located under the ends of contact zone in some distances for the boundary of the composite half-space.

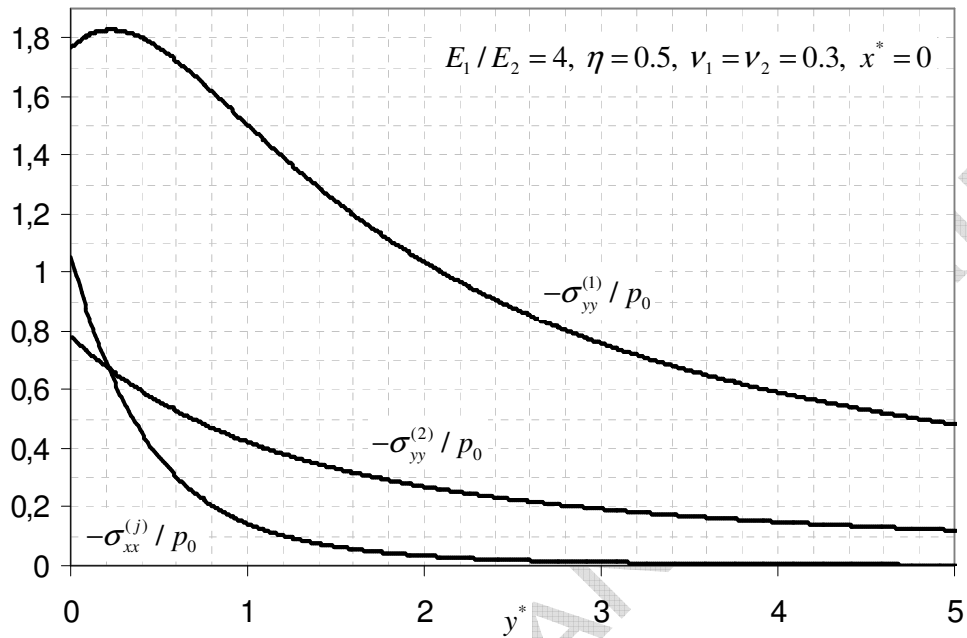
The stress component  $\sigma_{yy}^{(j)} / p_0$  is discontinuous on the interfaces. The distributions of  $\sigma_{yy}^{(j)} / p_0$  for the depths (the distances from the boundary plane)  $y^* = 0.0; 0.2; 1.0$  are presented in Fig. 4.



**Fig. 4** The distributions of dimensionless stresses  $\sigma_{yy}^{(j)} / p_0$ .

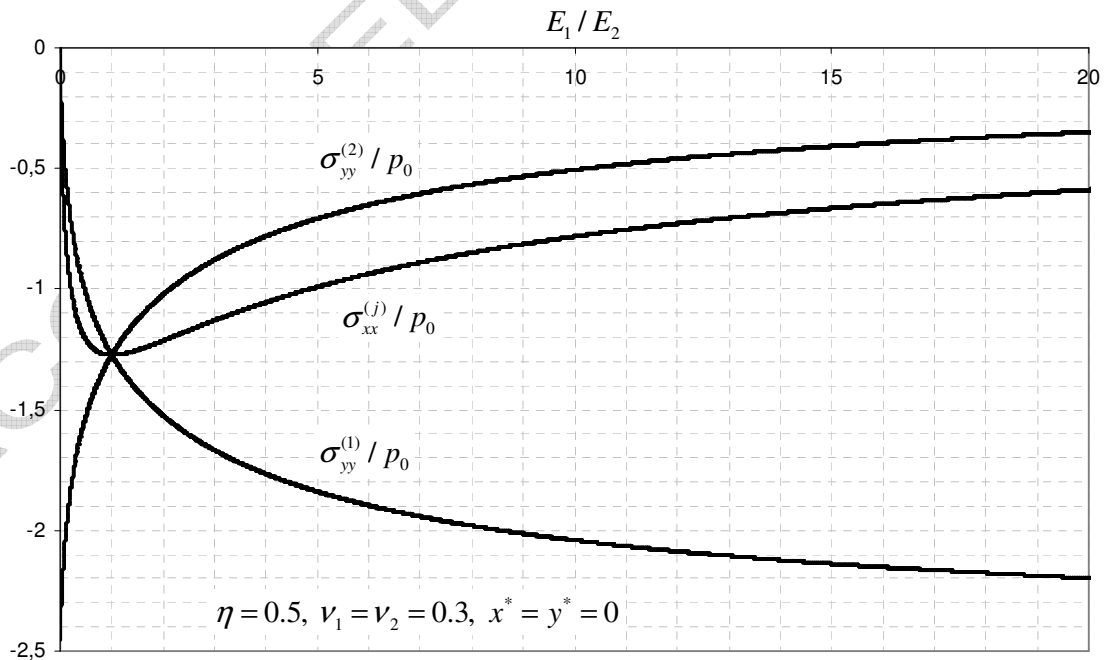
From Fig.4a one can see that the component  $\sigma_{yy}^{(j)}$  on the boundary for  $x^* > 1$ , ( $x > a$ ) is almost equal to zero, which confirms that the approximation of the boundary condition given

in (2.16) is well-founded. Figure 5 shows the dependence of stresses  $\sigma_{yy}^{(j)} / p_0$  and  $\sigma_{xx}^{(j)} / p_0$  under the centre of contact zone with respect to the distance from the boundary.



**Fig. 5** The distributions of dimensionless stresses  $\sigma_{xx}^{(j)} / p_0$  and  $\sigma_{yy}^{(j)} / p_0$ .

The influence of the dimensionless normal stresses on Young's modulus ratio  $E_1 / E_2$  at the contact centre for  $\nu_1 = \nu_2 = 0.3$  and  $\eta = 0.5$  is presented in Fig. 6.



**Fig. 6** The dimensionless normal stresses at the contact centre as a function of ratio  $E_1 / E_2$ .

One can see that for  $E_1/E_2=1$  (the homogeneous half-space) the normal stresses  $\sigma_{xx}^{(j)}$  and  $\sigma_{yy}^{(j)}$  take the same value.

From the point of view of material failure (Tresca yield criterion) it is important to consider the following shear intensity function in  $j$ -th kind of composite constants

$$\tau^{(j)} = \sqrt{\left(\frac{\sigma_{xx}^{(j)} - \sigma_{yy}^{(j)}}{2}\right)^2 + \left(\sigma_{xy}^{(j)}\right)^2}, \quad j=1, 2. \quad (3.2)$$

The shear stress intensity function  $\tau^{(j)}$ ,  $j=1, 2$  is discontinuous on the interfaces, because the stress components  $\sigma_{yy}^{(j)}$  have jumps on these surfaces. The distributions of  $\tau^{(j)}/p_0$  for the depths  $y^* = 0.0; 0.1; 1.0; 2.0$ ; are presented in Fig. 7. The greatest differences between the functions  $\tau^{(1)}/p_0$  and  $\tau^{(2)}/p_0$  are observed under contact zone ( $x^* < 1$ ).

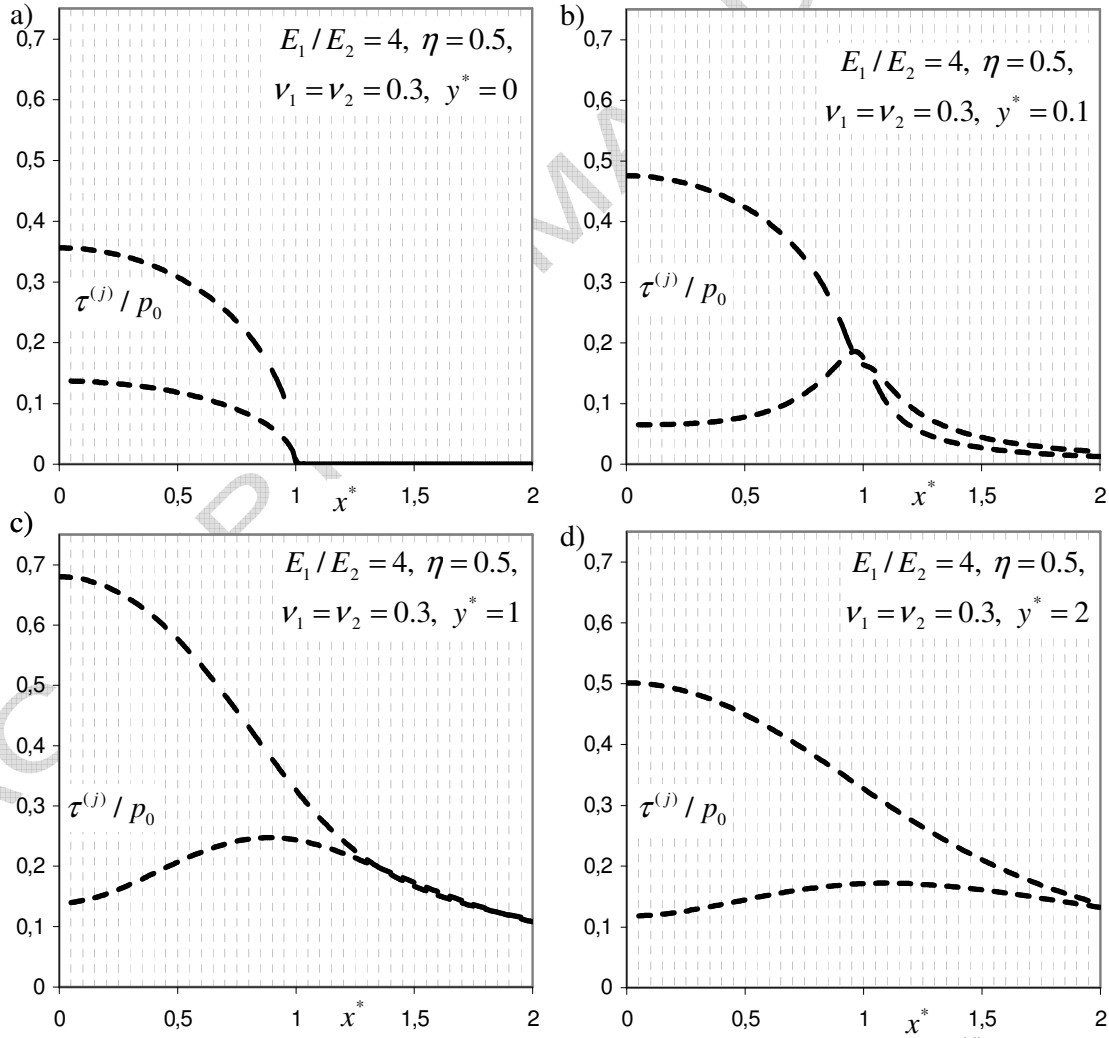


Fig. 7 The distributions of dimensionless shear stress intensity function  $\tau^{(j)}/p_0$ .



#### 4. Final remarks and conclusions

This paper contains the solution of smooth contact problem between the rigid cylindrical indenter and composite half-space with ply-orientation normal to the boundary. The homogenized model with microlocal parameters are applied to an approximate formulation of the problem. It gives the possibilities to find the exact solution within the framework of the homogenized model. The problem has reduced to the well-known dual integral equations. The stress distributions in the composite half space are presented in the form of Fourier integrals, what permits to analyze the influence of mechanical and geometrical properties of composite constituents on the stress behavior. Moreover, the averaged contact pressure and the width of contact zone are determined. It should be emphasized that the obtained results under assumption  $\lambda_1 = \lambda_2$  and  $\mu_1 = \mu_2$  agree with the adequate solution for homogeneous half-space. Moreover the stress  $\sigma_{yy}^{(j)}$  has jumps on the interfaces, what it cannot be obtained within the description by the anisotropic body but it is in a good agreement with the formulation of contact problem within the framework of the classical elasticity.

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