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# Boundary Conditions at Fluid-Permeable Interfaces in Porous Media: a Variational Approach

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*Nomina sunt consequentia rerum, Flavius Petrus Sabbatius Iustinianus, Institutiones Liber II,7,3*

*In nomine est natura rerum, Anonymous*

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## Abstract

A general set of boundary conditions at fluid-permeable interfaces between dissimilar fluid-filled porous matrices is established starting from an extended Hamilton-Rayleigh principle. These conditions do include friction and inertial effects. Once linearized, they encompass boundary conditions relative to volume Darcy-Brinkman and to surface Saffman-Beavers-Joseph dissipation effects.

*Key words:* Piecewise Diffeomorphisms, Discontinuity Surfaces, Hamilton-Rayleigh Principle, Boundary Conditions at Permeable Interfaces, Galilean Invariance, Porous Media.

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## 1. Introduction

Many interesting mechanical phenomena occur in porous media when the saturating fluid flows under the action of pressure and the solid matrix is deformable. Modeling these phenomena represents an important challenge for engineering sciences.

The aim of this paper is to use the principle of virtual work –when dissipative and inertial effects cannot be neglected– for deducing a set of evolution equations and coherent boundary conditions valid at a fluid-permeable interface between dissimilar fluid-filled porous matrices.

The spirit of the approach adopted here for modeling porous systems is very similar to the one used to develop models for two fluid mixtures by Gavrilyuk *et al.* (1997); Gavrilyuk *et al.* (1998); Gavrilyuk and Perepechko (1998); Gouin and Gavrilyuk (1998); Gavrilyuk and Gouin (1999); Gouin (1990). It also has some similarities with the treatment used to describe fluid saturated porous media by Dormieux and Stolz (1992); Dormieux *et al.* (1991) and Coussy *et al.* (1998).

To frame the results presented here and to compare them with those available in the literature it is necessary to detail some of the features of the model we develop. Slightly modifying the conceptual scheme used by e.g. Dormieux *et al.* (1991) we conceive a kinematical description which seems suitable to model porous systems which are open with respect to the fluid constituent: i.e. systems in which the fluid can freely leave or enter the porous solid matrix through which it flows. Indeed, while we still use as a basic kinematical descriptor the solid-matrix macroscopic placement field  $\chi_s$  we replace the fluid macroscopic placement field  $\chi_f$  with a macroscopic field  $\phi_s$  defined in the homogenized macroscopic solid reference configuration. The field  $\phi_s$  maps any solid material particle  $\mathbf{X}_s$  into a precise particle  $\mathbf{X}_f$  in the fluid reference configuration: it is the fluid material particle which occupies, at the given instant, the same spatial position as  $\mathbf{X}_s$ . The adjective “macroscopic” in the previous sentences is intended to remind to the reader that the model which is used here does not attempt to describe in a detailed way how the complex geometrical structure of the pores varies in the deformation process (see *infra* for a discussion of this point). Obviously the placement field  $\chi_f$  can be easily recovered as it equals  $(\chi_s \circ \phi_s^{-1})$ .

The introduced kinematical description is adapted to describe the evolution of porous systems in which the solid matrix is open to fluid filtration (as it happens in consolidation problems, see e.g. Mandel (1953), Cryer (1963), Madeo *et al.* (2008)). For such open systems, it is necessary to follow the placement of a solid matrix when an unknown amount of fluid, taken from a given reservoir and free to enter or leave the solid matrix, is saturating its pores and flowing through its interconnection canals.

It has to be remarked explicitly that our approach is “purely macroscopic” and “variational”.

The approach is purely “macroscopic” because the kinematical descriptors for the solid matrix and saturating fluid can be regarded as “averaged” placement fields obtained from corresponding “microscopic” ones (see *infra*). Moreover, in the scheme we use, all “microscopic” descriptors which may be relevant are assumed to be given by constitutive equations depending on the two previously introduced basic “macroscopic” placements  $\chi_s$  and  $\phi_s$ . Here, differently to what done for instance in Dormieux and Stolz (1992), we do not attempt to deduce any “macroscopic” constitutive equations from those valid at “microscopic” level.

The approach is “variational” because the evolution equations for the kinematical fields are deduced by paralleling the Hamilton-Rayleigh approach. The variational approach has been successfully adapted to continuous systems in different contexts: see e.g. Seliger and Whitham (1968); Germain (1973); Houlsby and Puzrin (2002); Sonnet *et al.* (2004) for (dissipative or non-dissipative) Cauchy continua, Bedford and Drumheller (1979) for porous media, Bedford and Drumheller (1978) for immiscible mixtures, Bedford and Drumheller (1983) for structured mixtures and in Mobbs (1982) for viscous fluids. In all these papers, the evolution equations of a mechanical system are obtained by assuming the variational principle as a primitive concept. This means that the equations of motions are obtained as a consequence of the variational principle itself. On the other hand, there also exists other approaches to variational principles which does not consider it as a primitive concept. Indeed, if one can obtain the equations of motion for a mechanical system in alternative ways, in order to check consistency of these equations, it is possible to look for a variational principle which is compatible with these equations (see e.g., Altay and Dokmeci (2006)). We decide to adopt the first approach to variational principles and consequently we obtain equations of motions and jump conditions which are intrinsically consistent with the variational principle.

In this paper we construct an action functional accounting for all conservative phenomena occurring in the system and a Rayleigh dissipation function (i.e. a dissipation given in terms of a quadratic pseudopotential). When formulating the principle of virtual work, we assume that the work done by inertial and internal conservative forces can be expressed as the first variation of the action functional and that the work done by dissipative actions can be expressed in terms of the Rayleigh dissipation function (see e.g. Biot (1970) for the discrete formulation of this approach).

The action-based postulation scheme is well posed as the introduced kinematical fields  $\chi_s$  and  $\phi_s$  are both functions defined on the solid matrix reference configuration. To be able to deal with systems in which surface solid material discontinuities are present we allow  $\chi_s$  and  $\phi_s$  to present gradient discontinuities concentrated on surfaces. In our analysis we generalize some results found in Batra *et al.* (1986).

We recover the bulk evolution equations, already available in the literature, which are valid in the regularity points of the kinematical fields. Moreover, we obtain the boundary conditions valid at solid material discontinuity surfaces which are open to fluid flow. These boundary conditions may be interpreted as a “surface balance of force” and a “surface continuity of chemical potential”. An Eulerian form of the first of these conditions (including inertial terms) has been obtained in Dormieux *et al.* (1991) where the principle of virtual work was applied to multiphase systems.

Several authors (see e.g. Ochoa-Tapia and Whitaker (1995a), Jager and Mikelic (2000); Ochoa-Tapia and Whitaker (1995b); Hassanizadeh and Gray (1989)) formulated different boundary conditions to be used at solid material interfaces separating porous media and pure fluid. The main part of their efforts was directed to the justification and discussion of the boundary conditions originally proposed by Beavers and Joseph (1967), and Saffman (1971) for describing dissipation phenomena at the external interface of a porous systems. Some authors also focused on the deduction of such conditions by means of a micro-macro identification method: see e.g. Burrige and Keller (1981); Prat (1988); Chateau and Dormieux (1998); Marle (1982); Chandris and Jamet (2006), Chandris and Jamet (2007), Ochoa-Tapia and Whitaker (1995b) or Valdés-Paradaa *et al.* (2006) and references there cited. On the other hand, in Deresiewicz (1963) a set of boundary conditions valid at interfaces between dissimilar fluid-filled porous media are proposed which assure uniqueness of the solution of field equations proposed by Biot (1956a).

The boundary condition stating the “surface continuity of chemical potential” obtained in the present paper include an inertial term which, to our knowledge, is not deduced in the literature. It generalizes the condition found e.g. in Baek and Srinivasa (2004) (the spirit of which is very similar to the one adopted here). Several versions of “surface balance of force” or “kinematical boundary conditions” can be found in Deresiewicz (1963); Cieszko and Kubik (1998); Cieszko and Kubik (1999); Debergue *et al.* (1999); Goyeau *et al.* (2003); Haber and Mauri (1983); Kuznetsov (1997); Le Bars and Grae Worster (2006); Levy and Sanchez-Palencia (1975); Rajagopal *et al.* (1986); Sharma (2008); Ochoa-Tapia and Whitaker (1998).

Our boundary conditions seem suitable to describe (macroscopically) the behavior of solid material interfaces open to fluid flow. They are deduced without introducing any “small perturbation” assumption, so that they seem suitable also when the assumptions of small deformations of the solid matrix and linearized Stokes fluid flow cannot be accepted. However our results are subject to all the limitations implicit in any Rayleigh-like description of dissipative phenomena based on the introduction of a pseudopotential.

The newly found boundary conditions are effectively Galilean invariant. To check this statement we revisited Reynolds Transport Theorem and Hadamard Conditions to derive some kinematical formulas implicitly used already by Gavriluk *et al.* (1998) and Gavriluk *et al.* (1997). In these papers these formulas were needed to show that some evolution equations and boundary conditions assume the form of conservation laws.

We explicit warn the reader that: i) we decided to introduce an explicit notation for distinguish fields defined on the solid-reference configuration from those defined on the spatial or fluid-reference configuration, ii) we found more convenient to deduce all kinematical formulas using a space-time (Galilean) four-dimensional formalism, iii) we did separate all kinematical deductions and properties from those which are more physical in nature.

Concerning point i) we start remarking that when studying one constituent continua it is possible to proceed in presence of an abuse of notation in which fields defined in different configurations (and therefore corresponding to different mathematical functions) are denoted with the same symbol. This does not seem

careful enough when multicomponent continua are considered. Indeed such an abuse of notation is, in this case, even more risky than usual, as one is dealing with models where it is necessary to introduce many different placement fields and where discontinuity surfaces for at least one of these fields may be present. In fluid-saturated porous media at least three configurations, and therefore domains of definitions for all considered fields, need to be considered. Therefore, we use a notation which is more precise than the usual one, as it allows us to specify clearly for every considered tensor field in which spatial or material domain it is defined. Should the reader be disturbed by the notation which we introduced he is invited to recover the standard one simply ignoring all the circled superscripts.

Concerning point ii) we remark that it is simpler and more convenient to consider (as done for instance by Gavrilyuk and Gouin (1999)) the four-dimensional Galilean space-time as domain for all handled kinematical fields. In this way: a) any moving bi-dimensional surfaces in the physical space becomes a fixed co-dimension one surface in four dimensional space-time, b) piecewise regular spatial fields depending on time when regarded as fields with domain in four-dimensional space-time suffer discontinuities across fixed surfaces, c) space and time differentiation, space gradients and time derivatives, deformation gradients and velocities are dealt with in a more compact and unified manner, d) Hadamard jump conditions and Reynolds transport theorems assume a very simple form, e) as a consequence, some useful –but involved– kinematical relationships are easily seen to stem from elementary differential geometric ones. Indeed, the four-dimensional Galilean space-time is the suitable setting to be used in order to deduce from some well-known results in differential geometry many properties of piecewise differentiable tensor fields. Even if it seems possible to consider weaker regularity conditions (see e.g. Savaré and Tomarelli (1998)) we try to render the presentation the simplest possible still choosing the admissible kinematical fields to be general enough to describe the phenomena we have in mind.

Concerning point iii): in our deduction it was necessary to deal with some important topics in differential geometry, concerning the mathematical properties of tensor fields which can be expressed as gradients of other tensor fields. In the four-dimensional setting we have chosen, this is equivalent to study kinematical properties of multicomponent continua. Sometimes this kinematical study is presented together with topics the nature of which is more specifically mechanical, i.e. related to the postulation scheme -based on phenomenological considerations- which is assumed in a specific modeling situation. We have chosen to keep separate all kinematical considerations. The abuse of notation mentioned at point i) is even more misleading when kinematical assumptions for placement fields are mixed with the phenomenological ones characterizing either the solid or the fluid constituent behavior. The conjunction of all these confusing choices may loose the reader in an indistinct list of properties the origin of which is unclear.

Referring to de Boer (1996); de Boer (2000); de Boer (2005); Rajagopal and Tao (1995); Dormieux *et al.* (2006) for an exhaustive and clear review of the development of porous media theory we limit ourselves to recall the pioneering works of Fillunger (1936) (which were made available to the engineers community by Terzaghi (1943) and Biot (1941)).

The stream of research efforts which were thus originated produced several different families of mathematical models differing in the detail in which they aim to describe the reference and current configurations of solid and fluid constituents.

Indeed, the solid matrix, when displacing from its reference configuration, occupies a different spatial region which delineates a different empty pore region left to the fluid constituent. Such a region can have a very complex time-variable shape: therefore the complete description of its evolution is correspondingly very difficult. Depending on the detail which is required in such a description one can introduce a macroscopic or a microscopic model.

In the context of the theory of porous solids a “*purely microscopic model*” is one in which the kinematical description allows for the complete characterization of the shapes of all matrix internal pores and of fluid density and velocity at any point inside these pores.

In the present paper, instead, we consider a mathematical model for the description of saturating-fluid flow in a porous matrix (having enough interconnected pores so to allow such flow) which is *purely macroscopic* in nature.

In purely macroscopic models the “internal” shape of the porous solid matrix, i.e. the shape of its internal pores, is not described by any kinematical field and therefore the solid matrix kinematical description is

limited to the introduction of a “homogenized” or “macroscopic” placement field  $\chi_s$ . This field is defined on a “homogenized” reference configuration for the solid matrix in which a solid material particle represents a cluster of pores together with that part of solid matrix which is delineating them (for a discussion of the mentioned homogenization procedure see e.g. Marle (1982); de Buhan *et al.* (1998a); de Buhan *et al.* (1998b); Hornung(1997)). The placement of such a macroscopic particle represents the spatial region occupied by the quoted cluster of pores: clearly the Eulerian mass density related to it is related to the solid mass effectively placed in the given Eulerian volume. Thus an “apparent” solid mass density, differing from the mass density of the material constituting the solid matrix, is associated to the introduced macroscopic solid placement field. Similarly the description of the kinematics of the fluid constituent flowing through the pores, delineated by the solid matrix, is obtained in a purely macroscopic model by means of the “homogenized” placement function  $\chi_f$  defined on a “homogenized” fluid reference configuration. The velocity and apparent mass density related to such a macroscopic placement field do not account for the variations of the “microscopic” fluid velocity and mass density fields which occur inside the pores. Recall that in the present paper we prefer to consider the field  $\phi_s$  instead of  $\chi_f$ : this is more convenient as  $\phi_s$  is defined in the same domain as  $\chi_s$ .

One particular aspect of *purely macroscopic models* has been sometimes regarded as their main conceptual weakness. It concerns the physical interpretation which has to be associated to their basic kinematical descriptors, i.e. the “homogenized” placement fields: indeed a “homogenized” solid particle occupies, at a given instant, the same place as a “homogenized” fluid particle. Nevertheless, this circumstance is not surprising if one carefully considers the conceptual modeling assumptions underlying purely macroscopic models. The intuitive interpretation we just came to give to purely macroscopic theories for fluid saturated porous solids gains merit once grounded from a mathematical point of view by the so called “Theory of Homogenization” i.e. the mathematical theory aiming to rigorously deduce macroscopic models from microscopic ones (see e.g. Hornung(1997) with the references there cited, Chateau and Dormieux (1995)).

The importance of the Theory of Homogenization cannot be denied. However, it is always possible and very useful to formulate “directly” a macroscopic theory without being forced to deduce it from a “purely microscopic” one. Moreover, up to now very few results are available about the rigorous deduction of the macroscopic theory of Darcy flow through a deformable porous medium.

Indeed, it is always possible (and often suitable) to develop a macroscopic model independently from any microscopic one. Recall that Cauchy continuum mechanical models for one constituent bodies are formulated in a “direct” way without any reference to “atomistic” or “molecular” models and that very few practical models are rigorously justified by means of homogeneization procedures. In general, a mathematically coherent macroscopic model can be always formulated and supplies a useful guidance to the deduction procedure which starts from microscopic models. These procedures are often used to supply effective macroscopic constitutive equations in terms of the relevant microscopic properties of considered systems (see Allaire (1989a); Allaire (1989b); Allaire (1991a); Allaire (1991b); Allaire (1991c); for deduction of rigorous results concerning Darcy flow and to Pan and Horne (2001); Lee (2004); Kaasschieter and Frijns (2003) and references there cited for those concerning deformable matrices).

One can call “microscopic models” those intermediate models in which the macroscopic kinematical description is refined enough to describe in a more or less detailed way the shapes of the spatial regions separately occupied by solid and fluid constituents and some aspects of the motion of the material occupying these regions. The more detailed is the description of the shape of the solid porous matrix, the more “microscopic” is the formulated model. It is clear that different microscopic features of the pore shapes may be retained in the kinematical description: in some descriptions one could decide to account only for the ratio of volumes of the regions occupied respectively by solid and fluid constituents (thus introducing the solid volume fraction kinematical field) or for the shape of the canals interconnecting the pore (thus introducing a tortuosity tensor field) or for some geometrical features of the pores (thus introducing, for instance, the ratio between two characteristic lengths of the pore). Adding more and more kinematical descriptors one can more and more precisely approximate the purely microscopic theory. The choice of an “approximating” or “intermediate” microscopic theory results from a compromise between the need of a precise description of complex phenomena and the (computational or analytical) difficulties encountered.

We conclude remarking that the aforementioned modeling efforts respond to a strong demand from

applications. Innumerable engineering problems require the design and the control of complex systems in which the flow of a fluid occurs in a region partially occupied by a deformable solid matrix, the pores of which are interconnected.

Soil mechanics, geotechnical engineering and geology must supply the theoretical tools for controlling consolidation and subsidence phenomena, which are often influenced by related fluid filtration or flow phenomena (see e.g. Mandel (1953), Terzaghi (1943)) or the phenomena involved in earthquakes (see e.g. Yang (1999)) or in the bradyseism and in the related micro-earthquakes in the Phlegraean Fields - Campi Flegrei region (South Italy) (see dell'Isola *et al.* (1998), Casertano *et al.* (1976); Orsi *et al.* (1999)).

In biomechanics some phenomena related to the flow of fluids in a deformable porous matrix are also of interest: bone tissues are porous and several different fluids filtrate or flow through those pores which are interconnected. Indeed, it seems now evident that bone tissue growth is regulated by a feed-back control system in which the effect of tissue deformation on fluid flow plays a central role (see e.g. Cowin (2001)).

Underground engineering (e.g. when designing or maintaining underground cavities for stocking nuclear wastes or gas) also has to face relevant problems involving phenomena of fluid filtration and flow in a porous matrix coupled to cracks growth and related increase of pore volume fraction and cracks inteconnection (see e.g. dell'Isola *et al.* (2000); dell'Isola *et al.* (2003) and references there quoted).

Our model is suitable to be applied to all the aforementioned cases both under the hypotheses of small and large deformations of the considered porous medium. In fact, there are practical problems in which the hypothesis of small deformations is acceptable and then a linearized model is refined enough to describe the physical phenomena of interest. Nevertheless, there are physical problems in which the hypothesis of small deformation is no longer acceptable and a more general model is unavoidable. This is the case e.g. when dealing with rubbers, when they can be considered as permeable porous matrices if traversed by fluid solvents as toluene. An interesting application in this sense is given in Baek and Srinivasa (2004) where the filtration of toluene through a gum rubber membrane is investigated. The equations and boundary conditions used can be deduced from ours when inertia terms are negligible. Of great interest would be the study of phenomena in which the inertia terms can not be neglected and we will address further investigations in this sense.

## 2. Notations and Basic Properties of Piecewise Smooth Functions

In this section we recall some basic properties of the gradients of piecewise differentiable vector fields both in their regular and singular points. These properties are necessary to accurately model the macroscopic motion of a porous system with a moving surface discontinuity.

If  $\mathcal{X}$  is an homeomorphism from  $\mathbb{B}_a$  onto  $\mathbb{B}_b$ , given two tensor fields  $\mathbf{t}$  and  $\mathbf{z}$  defined on  $\mathbb{B}_a$  and  $\mathbb{B}_b$  respectively, we denote

$$\mathbf{t}^{\oplus} := \mathbf{t} \circ \mathcal{X}^{-1}, \quad \mathbf{z}^{\oplus} := \mathbf{z} \circ \mathcal{X}, \quad (1)$$

in order to distinguish fields defined in different domains. It is usual practice in mathematical physics, when transporting tensor fields by means of changes of variables, to use the same notation for the different tensor fields. This leads to some difficulties which are overcome when dealing with a one-constituent medium by introducing the adapted notations of material and Eulerian space-time derivatives. Such abuse of notation can be very misleading in the case of multi-constituents media where several diffeomorphisms are present. This is why we introduce this more precise notation which, although burdening, seems to be unavoidable in this context.

We denote the usual gradient and divergence of a  $k$ -th order tensor field  $\mathbf{t}$  defined on  $\mathbb{R}^n$  by  $\nabla \mathbf{t}$  and  $\text{DIV } \mathbf{t}$  which are tensors of order  $k+1$  and  $k-1$  respectively (see appendix A for details).

Let us now assume that  $\mathcal{X}$  is a  $C^2$ -diffeomorphism and denote  $\mathbb{F} := \nabla \mathcal{X}$  and  $\mathbb{J} := \det \mathbb{F}$ . Using these notations the chain rule gives

$$\nabla \mathbf{z}^{\oplus} = (\nabla \mathbf{z})^{\oplus} \cdot \mathbb{F} \quad \text{and} \quad \nabla \mathbf{t}^{\oplus} = (\nabla \mathbf{t})^{\oplus} \cdot (\mathbb{F}^{-1})^{\oplus}, \quad (2)$$

and the well-known change of variables formula reads

$$\int_{\mathbb{B}_b} \mathbf{z} \, d\mathbb{B}_b = \int_{\mathbb{B}_a} \mathbf{z}^{\textcircled{a}} \mathbb{J} \, d\mathbb{B}_a, \quad \text{or} \quad \int_{\mathbb{B}_a} \mathbf{t} \, d\mathbb{B}_a = \int_{\mathbb{B}_b} \mathbf{t}^{\textcircled{b}} (\mathbb{J}^{-1})^{\textcircled{b}} \, d\mathbb{B}_b,$$

where  $d\mathbb{B}_a$  and  $d\mathbb{B}_b$  denote the volume measures on  $\mathbb{B}_a$  and  $\mathbb{B}_b$  respectively and will be omitted in the sequel as no confusion can arise.

It is easy to check (see appendix A) that, given a differentiable tensor field  $\mathbf{z}$  (of order  $\geq 1$ ) defined on  $\mathbb{B}_b$ ,

$$\text{DIV} (\mathbb{J} \mathbf{z}^{\textcircled{a}} \cdot \mathbb{F}^{-T}) = \mathbb{J} (\text{DIV} \mathbf{z})^{\textcircled{a}}, \quad (3)$$

and

$$\text{DIV} (\mathbb{J} \mathbb{F}^{-T}) = 0. \quad (4)$$

Finally, if  $\mathbf{U}$  is a  $C^1$  vector field defined in  $\mathbb{B}_b$  and  $\mathbf{N}_a$  and  $\mathbf{N}_b$  are the outward unit normal vectors to  $\partial\mathbb{B}_a$  and  $\partial\mathbb{B}_b$  respectively, it can be proven that (see appendix A)

$$\int_{\partial\mathbb{B}_b} \mathbf{U} \cdot \mathbf{N}_b = \int_{\partial\mathbb{B}_a} \mathbf{U}^{\textcircled{a}} \cdot (\mathbb{J} \mathbb{F}^{-T} \cdot \mathbf{N}_a).$$

If  $\mathbb{S}_a$  is a smooth hypersurface in  $\mathbb{B}_a \subset \mathbb{R}^n$ , we denote by  $\mathbf{T}_i(\mathbf{x})$ ,  $i = 1, 2, \dots, n-1$ , a family of vectors spanning the tangent space  $\mathcal{T}_{\mathbf{x}}(\mathbb{S}_a)$  of  $\mathbb{S}_a$  at  $\mathbf{x}$  and by  $\mathbf{N}_a(\mathbf{x})$  the unit *normal* vector to  $\mathbb{S}_a$  at  $\mathbf{x}$ .

Let  $\mathbf{t}$  be a tensor field defined on  $\mathbb{B}_a$ . We say that  $\mathbf{t}$  is *piecewise differentiable* (or briefly  $C_{pw}^1$ ) if it is continuous in  $\mathbb{B}_a$  and if its gradient  $\nabla \mathbf{t}$  is continuous except on a smooth hypersurface  $\mathbb{S}_a$ . We denote  $[[\nabla \mathbf{t}]]$  its jump through  $\mathbb{S}_a$  and we say that  $\mathbb{S}_a$  is the *singularity surface* of  $\mathbf{t}$  (see appendix B for more precise definitions). The well-known Hadamard property (see e.g. Kosiński (1986)) states that the jump of the gradient of a  $C_{pw}^1$  tensor field  $\mathbf{t}$  is a rank-one matrix field in the form

$$[[\nabla \mathbf{t}]] = [[\nabla \mathbf{t}]] \cdot \mathbf{N}_a \otimes \mathbf{N}_a = ((\nabla \mathbf{t})^+ \cdot \mathbf{N}_a - (\nabla \mathbf{t})^- \cdot \mathbf{N}_a) \otimes \mathbf{N}_a = \left[ \left[ \frac{\partial \mathbf{t}}{\partial \mathbf{N}_a} \right] \right] \otimes \mathbf{N}_a \quad (5)$$

where  $(\nabla \mathbf{t})^+$  and  $(\nabla \mathbf{t})^-$  are the traces of  $\nabla \mathbf{t}$  on both sides of  $\mathbb{S}_a$ . In other words, for any  $\mathbb{T} \in \mathcal{T}_{\mathbf{x}}(\mathbb{S}_a)$

$$[[\nabla \mathbf{t}(\mathbf{x}) \cdot \mathbb{T}]] = 0. \quad (6)$$

Given a map  $\mathcal{X}$  from  $\mathbb{B}_a$  onto  $\mathbb{B}_b$  which is a piecewise differentiable diffeomorphism with singularity surface  $\mathbb{S}_a \subset \mathbb{B}_a$ , it can be proven (see Appendix B) that the surface  $\mathbb{S}_b := \mathcal{X}(\mathbb{S}_a)$  is a smooth surface in  $\mathbb{B}_b$  with tangent vectors  $(\mathbb{F} \cdot \mathbf{T}_i)^{\textcircled{b}}$ ,  $i \in \{1, 2, \dots, n-1\}$ . Moreover, for any normal vector field  $\mathbf{M}_b$  to  $\mathbb{S}_b$ , the following jump condition holds on  $\mathbb{S}_a$

$$[[\mathbb{J}^{-1} \mathbb{F}^T]] \cdot \mathbf{M}_b^{\textcircled{b}} = 0, \quad (7)$$

where the quantity  $\mathbb{J}^{-1} \mathbb{F}^T \cdot \mathbf{M}_b^{\textcircled{b}}$ , which is continuous through the surface, is orthogonal to  $\mathbb{S}_a$ .

### 3. Kinematics of a Continuum with a Moving Surface Singularity

#### 3.1. Bulk Kinematical Identities and Hadamard Conditions at Moving Boundaries

Let  $\chi : \mathbb{B}_a := B_a \times (0, T) \rightarrow \mathbb{R}^3$  be the placement map of a three dimensional continuum; the fixed domain  $B_a \subset \mathbb{R}^3$  is usually called the reference configuration and the moving volume  $B_b(t) := \chi(B_a, t)$  is called the current configuration. We assume that, at any instant  $t$ ,  $\chi(\cdot, t)$  is a  $C_{pw}^1$  diffeomorphism with singularity surface  $\mathbb{S}_a(t)$ . We denote  $\mathbb{S}_b(t) := \chi(\mathbb{S}_a(t), t)$  the corresponding surface in  $B_b(t)$ .

We also denote  $\mathbf{F} := \nabla \chi$  and  $\mathbf{v} := \partial \chi / \partial t$ , the usual 3D space gradient of the map  $\chi$  in the domain  $B_a$  and the usual velocity field in  $B_a$ . Moreover we denote  $\tau_i$ ,  $i = 1, 2$  a basis of the tangent space to  $\mathbb{S}_a$  and by  $\mathbf{N}_a$  the unit normal vector to  $\mathbb{S}_a$ .

Some kinematical conditions are sometimes derived in an intricate way and are not distinguished from phenomenological assumptions. Let us show that they are simple consequences of the general properties recalled in the previous section. We underline their pure kinematical nature and the fact that they are not based on any physical assumption nor balance principle.

Let us consider the  $C_{pw}^1$ -diffeomorphism  $\mathcal{X}$  defined on  $\mathbb{B}_a := B_a \times (0, T)$  by  $\mathcal{X}(a, t) := (\chi(a, t), t)$  and let us denote by  $\mathbb{B}_b$  its image. The map  $\mathcal{X}$  resumes all needed information about the motion of the considered continuum. In particular,  $\mathbb{B}_b := \cup_{t \in (0, T)} B_b(t) \times \{t\}$ . The singularity surface  $\mathbb{S}_a$  of  $\mathcal{X}$  describes the motion of the singularity surface  $S_a(t)$  by  $\mathbb{S}_a := \cup_{t \in (0, T)} S_a(t) \times \{t\}$ . Analogously the hypersurface  $\mathbb{S}_b := \cup_{t \in (0, T)} S_b(t) \times \{t\}$  resumes the motion of  $S_b(t)$ .

If  $\mathbb{f}$  is a tensor field on  $\mathbb{B}_a$  of order  $k \geq 1$  the components of which are  $\mathbb{f}_{i_1, i_2, \dots, i_k}$ , with  $i_k \in \{1, 2, 3, 4\}$ , we decompose  $\mathbb{f}$  writing  $\mathbb{f} = (\mathbf{f}, f)$ . Here  $\mathbf{f}$  and  $f$  are the tensors of order  $k$  and  $k-1$  defined by

$$\mathbb{f}_{i_1 i_2 \dots i_{k-1} j} = \mathbf{f}_{i_1 i_2 \dots i_{k-1} j}, \quad j = 1, 2, 3, \quad f_{i_1, i_2, \dots, i_{k-1}} = \mathbb{f}_{i_1 i_2 \dots i_{k-1} 4}.$$

Using this decomposition for  $\mathbb{f}$  and  $\mathbb{U}$  we have  $\mathbb{f} \cdot \mathbb{U} = \mathbf{f} \cdot \mathbf{u} + f u$ . Moreover, the 4D space-time gradient and divergence of  $\mathbb{f}$  are related to its 3D gradient and time derivative by

$$\nabla \mathbb{f} = (\nabla \mathbf{f}, \partial \mathbf{f} / \partial t) \quad \text{and} \quad \mathbb{D} \mathbb{f} = \text{div } \mathbf{f} + \partial f / \partial t.$$

In particular, if  $\mathbb{f}$  is a vector field, the  $4 \times 4$  matrix  $\nabla \mathbb{f}$  admits the block decomposition<sup>1</sup>

$$\nabla \mathbb{f} = \begin{pmatrix} \nabla \mathbf{f} & (\partial \mathbf{f} / \partial t)^T \\ \nabla f & \partial f / \partial t \end{pmatrix}.$$

Applying this block decomposition to the gradient of  $\mathcal{X}$  gives

$$\mathbb{F} := \nabla \mathcal{X} = \begin{pmatrix} \mathbf{F} & \mathbf{v}^T \\ \mathbf{0} & 1 \end{pmatrix},$$

and we can remark that  $\mathbb{J} := \det \mathbb{F}$  coincides with  $J := \det \mathbf{F}$ .

Applying equation (3) to  $\mathbb{f} = (\mathbf{f}, 0)$  and  $\mathbb{f} = (\mathbf{0}, f)$  gives

$$\text{div} (J \mathbf{f}^{\textcircled{a}} \cdot \mathbf{F}^{-T}) = J (\text{div } \mathbf{f})^{\textcircled{a}} \quad \text{and} \quad \frac{\partial (J f^{\textcircled{a}})}{\partial t} - \text{div} (J f^{\textcircled{a}} \otimes \mathbf{F}^{-1} \cdot \mathbf{v}) = J \left( \frac{\partial f}{\partial t} \right)^{\textcircled{a}}. \quad (8)$$

Similarly, equation (4) particularizes into

$$\text{div} (J \mathbf{F}^{-T}) = 0, \quad \frac{\partial J}{\partial t} = \text{div} (J \mathbf{F}^{-1} \cdot \mathbf{v}). \quad (9)$$

It is easy to prove that using (9) in (8) gives

$$(\text{div } \mathbf{f})^{\textcircled{a}} = \nabla \mathbf{f}^{\textcircled{a}} : \mathbf{F}^{-1}, \quad \left( \frac{\partial f}{\partial t} \right)^{\textcircled{a}} = \frac{\partial f^{\textcircled{a}}}{\partial t} - \nabla f^{\textcircled{a}} \cdot \mathbf{F}^{-1} \cdot \mathbf{v} \quad (10)$$

which encompass the classical relationships between material and Eulerian derivatives.

Let  $\varphi(\mathbf{s}, t)$  be a time dependent parametrization of the moving singularity surface  $S_a(t)$ , then the singularity hypersurface  $\mathbb{S}_a$  admits the parametrization  $\Phi$  defined by  $\Phi(\mathbf{s}, t) = (\varphi(\mathbf{s}, t), t)$ . A 4D vector  $(\mathbf{m}, m)$  is orthogonal to  $\mathbb{S}_a$  if it is orthogonal to the three tangent vectors  $\partial_1 \Phi = (\tau_1, 0)$ ,  $\partial_2 \Phi = (\tau_2, 0)$  and  $\partial \Phi / \partial t = (\partial \varphi / \partial t, 1)$ ; that is if

$$\tau_1 \cdot \mathbf{m} = 0, \quad \tau_2 \cdot \mathbf{m} = 0, \quad \frac{\partial \varphi}{\partial t} \cdot \mathbf{m} + m = 0.$$

<sup>1</sup>When defining matrices, we identify any vector with the corresponding row matrix.

From the two first equalities we deduce that  $\mathbf{m}$  is proportional to the normal  $\mathbf{N}_a$  to  $S_a$ . Hence the vectors orthogonal to  $\mathbb{S}_a$  are those proportional to  $\mathbb{M}_a = (\mathbf{N}_a, -c_a)$ , where  $c_a := \partial\varphi/\partial t \cdot \mathbf{N}_a$  is usually called the *celerity* of  $S_a(t)$ . As it is well known (see e.g. Kosiński (1986)), this quantity does not depend on the choice of the parametrization  $\varphi$ .

Noticing that the three vectors

$$\mathbb{T}_1 = (\tau_1, 0), \quad \mathbb{T}_2 = (\tau_2, 0), \quad \mathbb{T}_3 = (c_a \mathbf{N}_a, 1)$$

span the tangent hyper-plane to  $\mathbb{S}_a$  and applying equation (6) to the field  $\mathfrak{X}$  gives

$$[[\mathbf{F}]] \cdot \tau_1 = 0, \quad [[\mathbf{F}]] \cdot \tau_2 = 0, \quad [[\mathbf{F}]] \cdot c_a \mathbf{N}_a + [[\mathbf{v}]] = 0. \quad (11)$$

We notice that if  $\mathbf{N}_b$  and  $c_b$  are the unit normal vector and the celerity of  $S_b(t)$  respectively then  $\mathbb{M}_b := (\mathbf{N}_b, -c_b)$  is orthogonal to  $\mathbb{S}_b$ . Hence, equation (7) reads in this context

$$[[\mathbb{J}^{-1}\mathbf{F}^T]] \cdot \mathbb{M}_b^\oplus = \left[ \left( J^{-1}\mathbf{F}^T \cdot \mathbf{N}_b^\oplus, J^{-1}(\mathbf{v} \cdot \mathbf{N}_b^\oplus - c_b^\oplus) \right) \right] = 0,$$

which implies

$$\left[ J^{-1}\mathbf{F}^T \cdot \mathbf{N}_b^\oplus \right] = 0, \quad \left[ J^{-1}(\mathbf{v} \cdot \mathbf{N}_b^\oplus - c_b^\oplus) \right] = 0. \quad (12)$$

As noticed at the end of section 2, the 4D vector  $\mathbb{J}^{-1}\mathbf{F}^T \cdot \mathbb{M}_b^\oplus = \left( J^{-1}\mathbf{F}^T \cdot \mathbf{N}_b^\oplus, J^{-1}(\mathbf{v} \cdot \mathbf{N}_b^\oplus - c_b^\oplus) \right)$  is orthogonal to  $\mathbb{S}_a$  and so colinear to  $(\mathbf{N}_a, -c_a)$ . This implies

$$\mathbf{N}_a = \frac{J^{-1}\mathbf{F}^T \cdot \mathbf{N}_b^\oplus}{\|J^{-1}\mathbf{F}^T \cdot \mathbf{N}_b^\oplus\|} \quad \text{and} \quad c_a = \frac{J^{-1}(c_b^\oplus - \mathbf{v} \cdot \mathbf{N}_b^\oplus)}{\|J^{-1}\mathbf{F}^T \cdot \mathbf{N}_b^\oplus\|}.$$

Considering  $\mathfrak{X}^{-1}$  instead of  $\mathfrak{X}$  gives the symmetrical relationships

$$\mathbf{N}_b = \frac{(J\mathbf{F}^{-T} \cdot \mathbf{N}_a)^\oplus}{\|(J\mathbf{F}^{-T} \cdot \mathbf{N}_a)^\oplus\|} \quad \text{and} \quad c_b = \frac{(Jc_a + J(\mathbf{F}^{-1} \cdot \mathbf{v}) \cdot \mathbf{N}_a)^\oplus}{\|(J\mathbf{F}^{-T} \cdot \mathbf{N}_a)^\oplus\|}. \quad (13)$$

Finally let us apply the rank-one property (5) for the jump of the gradient of the map  $\mathfrak{X}$ . It states the existence of a 4D vector  $\mathbb{U} = (\mathbf{u}, u)$ , such that

$$[[\mathbb{F}]] = \begin{pmatrix} [[\mathbf{F}]] & [[\mathbf{v}^T]] \\ \mathbf{0} & 0 \end{pmatrix} = \mathbb{U} \otimes \mathbb{M}_a,$$

the space-time decomposition of which gives

$$[[\mathbf{F}]] = \mathbf{u} \otimes \mathbf{N}_a \quad \text{and} \quad [[\mathbf{v}]] = -\mathbf{u} c_a.$$

Eliminating  $\mathbf{u}$  implies the following jump condition on  $S_a(t)$

$$c_a [[\mathbf{F}]] = -[[\mathbf{v}]] \otimes \mathbf{N}_a. \quad (14)$$

### 3.2. Balance Equations and Corresponding Jump Conditions in the Space-Time

Any balance equation for a quantity  $f$  on  $B_b(t)$  is of the type  $\text{div } \mathbf{f} + \partial f / \partial t = r$ , where  $\mathbf{f}$  is the corresponding flux and  $r$  is a source term. This equation is written in the time-space, introducing the 4D field  $\mathbb{f} = (\mathbf{f}, f)$  defined on  $\mathbb{B}_b := \cup_{t \in (0, T)} B_b(t) \times \{t\}$  in the simple form

$$\mathbb{D}\text{IV } \mathbb{f} = r, \quad (15)$$

Recalling equation (3), this balance equation is easily transported on the domain  $\mathbb{B}_a$  in the form  $\text{DIV} \left( \mathbb{J} \mathbf{f}^{\textcircled{a}} \cdot \mathbf{F}^{-T} \right) = \mathbb{J} r^{\textcircled{a}}$  which is equivalent to

$$\text{div} \left( J \mathbf{f}^{\textcircled{a}} \cdot \mathbf{F}^{-T} - J f^{\textcircled{a}} \otimes \mathbf{F}^{-1} \cdot \mathbf{v} \right) + \frac{\partial (J f^{\textcircled{a}})}{\partial t} = J r^{\textcircled{a}}. \quad (16)$$

The jump condition on  $\mathbb{S}_b$  associated to this balance equation is easily recovered by considering equation (15) in the sense of distributions. If we do not consider any surface source term, this jump condition reads  $[[\mathbf{f} \cdot \mathbb{M}_b]] = 0$  which, recalling that  $\mathbb{M}_b = (\mathbf{N}_b, -c_b)$ , reduces to the more usual equation

$$[[\mathbf{f} \cdot \mathbf{N}_b - f c_b]] = 0.$$

This jump condition is easily transported on  $S_a$ : it takes the form  $[[\mathbf{f}^{\textcircled{a}} \cdot \mathbf{N}_b^{\textcircled{a}} - f^{\textcircled{a}} c_b^{\textcircled{a}}]] = 0$  which recalling (13) also reads

$$[[J \mathbf{f}^{\textcircled{a}} \cdot (\mathbf{F}^{-T} \cdot \mathbf{N}_a) - J f^{\textcircled{a}} \otimes (\mathbf{F}^{-1} \cdot \mathbf{v}) \cdot \mathbf{N}_a - J c_a f^{\textcircled{a}}]] = 0. \quad (17)$$

Clearly, balance laws symmetrical to (16) and (17) can be obtained starting from the conservation of a quantity on  $\mathbb{B}_a$ .

#### 4. Porous Medium with a Solid-Material Surface Singularity

##### 4.1. Kinematics

As we intend to give a macroscopic description of a porous medium, we consider a continuum made by the superposition of two continuous phases: a fluid one and a solid one.

Let us introduce the domains  $B_s \subset \mathbb{R}^3$  and  $B_f \subset \mathbb{R}^3$  (usually referred to as the Lagrangian configurations of the two constituents) and the maps

$$\chi_s : \mathbb{B}_s := B_s \times (0, T) \rightarrow \mathbb{R}^3, \quad \text{and} \quad \chi_f : \mathbb{B}_f := B_f \times (0, T) \rightarrow \mathbb{R}^3$$

which represent the placement of the solid and fluid constituents. The motion of the fluid inside the solid matrix is described by the function  $\phi_s : \mathbb{B}_s \rightarrow B_f$  which, at any time  $t$ , associates to each solid particle  $\mathbf{X}_s$  that particular fluid material particle  $\mathbf{X}_f = \phi_s(\mathbf{X}_s, t)$  occupying the same physical position as  $\mathbf{X}_s$ . The three introduced maps are related by  $\chi_s = \chi_f \circ \phi_s$ . We can assume, extending  $B_s$  and  $\phi_s$  if necessary, that  $\phi_s$  is an homeomorphism from  $B_s$  to  $B_f$ . This extension and the resulting extension of  $\chi_s$  have no physical sense, but make easier the description of open porous media. It will be mandatory to check that our final equations do not depend on the choice of this extension. Therefore  $\chi_s(B_s, t) = \chi_f(B_f, t)$  and we denote  $B_e(t)$  this time-varying 3D domain referred to as the Eulerian configuration. In the sequel, in order to apply our previous results, we assume that the 4D-counterparts of  $\chi_s$ ,  $\chi_f$  and  $\phi_s$  are piecewise  $C^1$ -diffeomorphisms. We note that this assumption on the regularity of the kinematical fields is a pure mathematical assumption and, while natural in a “continuum mechanics approach”, may lead to ill-posed problems when no physical argument enforces it: for instance when the singularity surface does not remain smooth during some evolution process, when the fluid is non viscous and/or when slip is allowed at the interface.

We still adopt the superscript notation  $\textcircled{\text{S}}$  (respectively  $\textcircled{\text{F}}$  and  $\textcircled{\text{E}}$ ) to denote the transport of a tensor field from the configuration where it is defined to  $B_s$  (resp.  $B_f$ ,  $B_e$ ). For instance, if a tensor  $\mathbf{t}$  is defined on  $B_f$ , then  $\mathbf{t}^{\textcircled{\text{S}}} := \mathbf{t} \circ \phi_s$ , while if it is an Eulerian field defined on  $B_e$ , then  $\mathbf{t}^{\textcircled{\text{S}}} := \mathbf{t} \circ \chi_s$ .

We denote the space gradient of the three placements by

$$\mathbf{F}_s := \nabla \chi_s, \quad \mathbf{F}_f := \nabla \chi_f, \quad \mathbf{G}_s := \nabla \phi_s,$$

and its determinant by

$$J_s := \det \mathbf{F}_s, \quad J_f := \det \mathbf{F}_f, \quad I_s := \det \mathbf{G}_s$$

It is immediate to check, that the chain rule gives  $\mathbf{F}_s = \mathbf{F}_f^{\otimes} \cdot \mathbf{G}_s$  and  $J_s = J_f^{\otimes} I_s$ .

We define now the classical Lagrangian velocity fields  $\mathbf{v}_s$  and  $\mathbf{v}_f$ , associated to the motion of the solid and of the fluid constituent, on  $B_s$  and  $B_f$  and, on  $B_s$ , the time derivative  $\mathbf{u}_s$  of the map  $\phi_s$ , which is not a velocity in the classical sense, but plays a central role in further calculations:

$$\mathbf{v}_s := \frac{\partial \chi_s}{\partial t}, \quad \mathbf{v}_f := \frac{\partial \chi_f}{\partial t}, \quad \mathbf{u}_s := \frac{\partial \phi_s}{\partial t}.$$

By the chain rule we get

$$0 = \frac{\partial}{\partial t} (\phi_s (\phi_s^{-1} (\mathbf{X}_f, t), t)) = \mathbf{G}_s^{\otimes} \cdot \frac{\partial \phi_s^{-1}}{\partial t} + \mathbf{u}_s^{\otimes}.$$

This relationship allows us to link  $\mathbf{v}_s$ ,  $\mathbf{v}_f$  and  $\mathbf{u}_s$ :

$$\mathbf{v}_f = \frac{\partial \chi_f}{\partial t} = \frac{\partial}{\partial t} (\chi_s (\phi_s^{-1} (\mathbf{X}_f, t), t)) = \mathbf{F}_s^{\otimes} \cdot \frac{\partial \phi_s^{-1}}{\partial t} + \mathbf{v}_s^{\otimes} = -\mathbf{F}_s^{\otimes} \cdot (\mathbf{G}_s^{\otimes})^{-1} \cdot \mathbf{u}_s^{\otimes} + \mathbf{v}_s^{\otimes}.$$

Transporting this relationship on  $B_s$  gives

$$\mathbf{v}_f^{\otimes} = \mathbf{v}_s - \mathbf{F}_s \cdot \mathbf{G}_s^{-1} \cdot \mathbf{u}_s. \quad (18)$$

Let us define the acceleration fields  $\gamma_s$ ,  $\gamma_f$  and  $\mathbf{a}_s$  as the time derivatives of  $\mathbf{v}_s$ ,  $\mathbf{v}_f$  and  $\mathbf{u}_s$  respectively. Using Eq. (10) for the diffeomorphism  $\phi_s$ , it is straightforward that

$$\frac{\partial}{\partial t} \mathbf{v}_f^{\otimes} = \gamma_f^{\otimes} + \nabla \mathbf{v}_f^{\otimes} \cdot \mathbf{G}_s^{-1} \cdot \mathbf{u}_s \quad (19)$$

Finally, since it is needed for further calculations, we compute the time derivative of the tensor  $\mathbf{F}_s \cdot \mathbf{G}_s^{-1}$ ; using (10) for the map  $\phi_s$  it is straightforward to recover that

$$\frac{\partial}{\partial t} (\mathbf{F}_s \cdot \mathbf{G}_s^{-1}) = \frac{\partial}{\partial t} \mathbf{F}_f^{\otimes} = \nabla \mathbf{F}_f^{\otimes} \cdot \mathbf{G}_s^{-1} \cdot \mathbf{u}_s + (\nabla \mathbf{v}_f)^{\otimes} = \nabla (\mathbf{F}_s \cdot \mathbf{G}_s^{-1}) \cdot \mathbf{G}_s^{-1} \cdot \mathbf{u}_s + \nabla \mathbf{v}_f^{\otimes} \cdot \mathbf{G}_s^{-1}. \quad (20)$$

In the sequel we focus on a surface  $S_s(t)$  which may be a surface of singularity for  $\chi_s$  and/or  $\phi_s$  and on the image surfaces  $S_f(t) = \phi_s(S_s(t), t)$  and  $S_e(t) = \chi_s(S_s(t), t) = \chi_f(S_f(t), t)$  which are moving surfaces in  $B_f$  and  $B_e(t)$ . We apply to these surfaces the notations and formulas stated in the previous sections. In particular, we introduce the celerities  $c_s$ ,  $c_f$  and  $c_e$  of  $S_s(t)$ ,  $S_f(t)$  and  $S_e(t)$  respectively.

Actually, in this paper we only consider the case in which the surface  $S_s$  is a *solid-material surface discontinuity* which means that it does not depend on time. In other words, from now on, we assume that  $S_s$  is parametrized by a function  $\varphi$  which does not depend on time. Consequently, the celerity  $c_s$  of the surface  $S_s$  is vanishing. This particular case of solid-material surface has many applications. It models all those phenomena in which  $S_s$  divides the solid skeleton in two parts with different mechanical properties (e.g. different porosities, rigidities, etc.) It also models, as a limit case, the boundary of a fluid-filled porous matrix in contact with a pure fluid.

The hypothesis that the surface is solid-material ( $c_s = 0$ ), applying (14) to both  $\chi_s$  and  $\phi_s$ , implies

$$[[\mathbf{v}_s]] = 0 \quad \text{and} \quad [[\mathbf{u}_s]] = 0 \quad \text{on } S_s.$$

We underline that these equations do not imply  $[[\mathbf{v}_f]] = 0$ .

We finally remark that if  $(\mathbf{v}_s^{\otimes} - \mathbf{v}_f^{\otimes}) \cdot \mathbf{N}_e = 0$ , or equivalently by (18) and (13),  $\mathbf{u}_s \cdot \mathbf{N}_f^{\otimes} = 0$ , then from (11)

$$[[\mathbf{F}_f^{\otimes} \cdot \mathbf{u}_s]] = [[\mathbf{v}_s - \mathbf{v}_f^{\otimes}]] = 0.$$

#### 4.2. Balance of Masses

The masses  $\mathcal{M}_s(B)$  and  $\mathcal{M}_f(B)$  of solid skeleton and fluid contained in a part  $B \subset B_e$  of the physical space at time  $t$  are represented by means of two Eulerian densities  $\rho_s$  and  $\rho_f$  respectively in the form

$$\mathcal{M}_s(B) = \int_B \rho_s, \quad \mathcal{M}_f(B) = \int_B \rho_f.$$

These densities are usually called “*apparent densities*” and they do not coincide with the mass densities of the materials which constitute the solid skeleton or the fluid. Simple changes of variables give

$$\mathcal{M}_s(B) = \int_{\chi_s^{-1}(B)} \rho_s^{\oplus} J_s \quad \text{and} \quad \mathcal{M}_f(B) = \int_{\chi_s^{-1}(B)} \rho_f^{\oplus} J_s = \int_{\chi_f^{-1}(B)} \rho_f^{\oplus} J_f,$$

which leads us to introduce the “*solid-Lagrangian apparent densities*”  $\eta_s$ ,  $m_f$  for the solid and the fluid constituent defined on  $B_s$  by  $\eta_s := J_s \rho_s^{\oplus}$ ,  $m_f := J_s \rho_f^{\oplus}$  and the “*fluid-Lagrangian apparent density*”  $\eta_f$  of the fluid constituent on  $B_f$  by  $\eta_f := J_f \rho_f^{\oplus}$ . The densities  $m_f$  and  $\eta_f$  are related by  $m_f = I_s \eta_f^{\oplus}$ .

As we do not intend to model melting, dissolution or erosion phenomena, we assume conservation of mass for each constituent. Mass conservation for the solid skeleton and the fluid take the form of the balance laws

$$\frac{\partial}{\partial t} \eta_s = 0, \quad \text{and} \quad \frac{\partial}{\partial t} \eta_f = 0.$$

which are of the type studied in section 3.2. The results of section 3.2 give the associated jump conditions  $[[\eta_s]] c_s = 0$  and  $[[\eta_f]] c_f = 0$  on  $S_s$  and  $S_f$  respectively. As  $c_s = 0$  the first equation is trivially satisfied. So is the second one if one assumes (which, as it is well known, can be done for a fluid without loss of generality) that  $\eta_f$  is constant in space and time. However, the pull-back on  $B_s$  of the fluid balance and jump equations, using the transport formulas (16) and (17) together with equation (18) gives the non-trivial equations

$$\begin{aligned} \dot{m}_f + \operatorname{div} \mathbf{D} &= 0 \quad \text{on } B_s, \\ [[\mathbf{D}]] \cdot \mathbf{N}_s &= 0 \quad \text{on } S_s, \end{aligned} \tag{21}$$

where  $\mathbf{D} := -m_f \mathbf{G}_s^{-1} \cdot \mathbf{u}_s$  and  $\dot{m}_f := \partial m_f / \partial t$ . The vector  $\mathbf{D}$  is interpreted as the mass fluid flux through the porous medium in the Lagrangian configuration of the skeleton. The quantity  $\mathbf{D} \cdot \mathbf{N}_s$ , which is well defined at the interface  $S_s$ , is the flux (per unit area of  $S_s$ ) of fluid flowing through the interface. We introduce

$$d := \left( \frac{\mathbf{D} \cdot \mathbf{N}_s}{\|J_s \mathbf{F}_s^{-T} \cdot \mathbf{N}_s\|} \right)^{\oplus} = \rho_f (\mathbf{v}_f^{\oplus} - \mathbf{v}_s^{\oplus}) \cdot \mathbf{N}_e = \eta_f^{\oplus} \|(J_f^{-1} \mathbf{F}_f^T)^{\oplus} \cdot \mathbf{N}_e\| (\mathbf{N}_f \cdot \mathbf{u}_s)^{\oplus} \tag{22}$$

which is well defined at the interface  $S_e$  and corresponds to the flux (per unit area of  $S_e$ ) of fluid flowing through the interface.

### 5. Evolution Equations and Associated Jump Conditions in Presence of Dissipation

#### 5.1. Action and Rayleigh Functionals

We start by recalling that the kinematics of the considered porous medium is described by means of the fields  $\chi_s$  and  $\phi_s$  defined on  $B_s$ .

Then, we introduce the kinetic energy

$$\frac{1}{2} \int_{B_e} \left( \rho_s (\mathbf{v}_s^{\oplus})^2 + \rho_f (\mathbf{v}_f^{\oplus})^2 \right) = \int_{B_s} \Lambda \left( \eta_s, m_f, \mathbf{v}_s, \mathbf{v}_f^{\oplus} \right)$$

where  $\Lambda \left( \eta_s, m_f, \mathbf{v}_s, \mathbf{v}_f^{\oplus} \right) = 1/2 \left( \eta_s (\mathbf{v}_s)^2 + m_f (\mathbf{v}_f^{\oplus})^2 \right)$  is the solid-Lagrangian pull-back of the kinetic energy density.

We now assume that the potential energy of the porous medium is characterized by a local density  $\Psi$  on  $B_s$  which depends on the kinematic descriptors  $\chi_s$  and  $\phi_s$  through the placement  $\chi_s$ , the strain tensor  $\varepsilon := 1/2(\mathbf{F}_s^T \cdot \mathbf{F}_s - \mathbf{I})$  and the quantity of fluid contained in the porous medium  $m_f = I_s \eta_f$ . For instance  $\Psi$  can be the sum of a non-homogeneous deformation energy potential  $\Psi_i(\varepsilon, m_f, \mathbf{X}_s)$  and a potential accounting for external body forces  $\Psi_g = (\eta_s + m_f) E_p(\chi_s(\mathbf{X}_s))$ . As we do not intend to model surface tension phenomena, we do not consider any concentration of energy on the singularity surface  $S_s$ . Neither do we consider any dependence of  $\Psi$  on higher gradients of the kinematical fields as done for instance in Sciarra *et al.* (2008). Setting  $\mathbb{B}_s := B_s \times (0, T)$ , we define the action functional  $\mathcal{A}$  for the porous system as

$$\mathcal{A} := \int_{\mathbb{B}_s} (\Lambda - \Psi).$$

It is well known that, in absence of dissipation, imposing the stationarity of the action implies that the kinematic descriptors satisfy the virtual power principle i.e. a weak form of the balance of momentum. As we want to account for dissipation phenomena, we introduce a generalized Rayleigh dissipation pseudo-potential  $\mathcal{R}$  on the Eulerian configuration. In linear thermodynamics the dissipation  $2\mathcal{R}$  is a quadratic form of the velocity fields

$$2\mathcal{R} = \int_{B_e \setminus S_e} \mathcal{D} \left( \mathbf{v}_s^\ominus - \mathbf{v}_f^\ominus, \nabla(\mathbf{v}_s^\ominus), \nabla(\mathbf{v}_f^\ominus) \right) + \int_{S_e} \mathcal{D}_S \left( (\mathbf{v}_s^\ominus)^-, (\mathbf{v}_s^\ominus)^+, (\mathbf{v}_f^\ominus)^-, (\mathbf{v}_f^\ominus)^+ \right) \quad (23)$$

where the volume density  $\mathcal{D}$  is a positive quadratic form, the surface density  $\mathcal{D}_S$  is a Galilean invariant quadratic form (the coefficients of these two forms may also depend on all the static parameters). In the sequel, we limit ourselves to the case

$$\mathcal{D} = (\mathbf{v}_s^\ominus - \mathbf{v}_f^\ominus) \cdot \mathbf{K} \cdot (\mathbf{v}_s^\ominus - \mathbf{v}_f^\ominus) + \nabla(\mathbf{v}_s^\ominus - \mathbf{v}_f^\ominus) : \mathbf{B} : \nabla(\mathbf{v}_s^\ominus - \mathbf{v}_f^\ominus) + \nabla \mathbf{v}_f^\ominus : \mathbf{M} : \nabla \mathbf{v}_f^\ominus \quad (24)$$

$$\mathcal{D}_S = \left[ \left| \mathbf{v}_f^\ominus \right| \right] \cdot \mathbf{S} \cdot \left[ \left| \mathbf{v}_f^\ominus \right| \right] \quad (25)$$

where  $\mathbf{K}$  and  $\mathbf{S}$  are second order symmetric, positive tensors,  $\mathbf{M}$  and  $\mathbf{B}$  are symmetric positive fourth order tensors, the symbol “:” stands for the double contraction product. The tensor  $\mathbf{K}$  accounts for the Darcy dissipation; its inverse (if invertible) is called the Darcy permeability tensor. The tensor  $\mathbf{B}$  accounts for Brinkman dissipation (see e.g. Brinkman (1947)). Classical fluid viscous effects are described by  $\mathbf{M}$ , while  $\mathbf{S}$  describes friction effects at the interface. We already noticed that, as the extension of  $\chi_s$  is arbitrary in a pure fluid region, the model should not depend on  $\mathbf{v}_s$  in this region. Hence,  $\mathbf{K}$  and  $\mathbf{B}$  have to vanish in any pure fluid region.

This Rayleigh dissipation is pulled back on  $B_s$  by simply changing the variables

$$2\mathcal{R} = \int_{B_s \setminus S_s} J_s \mathcal{D}^\ominus + \int_{S_s} \mathcal{D}_S^\ominus \|J_s \mathbf{F}_s^{-T} \cdot \mathbf{N}_s\|.$$

In the application of this formula, the change of variables in  $\mathcal{D}_S$  and in the first term of  $\mathcal{D}$  is straightforward. The terms involving gradients in  $\mathcal{D}$  need to be transported according to formula (2).

We finally introduce respectively, the Darcy friction force  $\kappa$ , the Brinkman stress tensor  $\Pi$ , the fluid viscous stress tensor  $\Pi_f$  and the friction surface force  $\sigma$  as

$$\begin{aligned} \kappa &:= \mathbf{K} \cdot (\mathbf{v}_s^\ominus - \mathbf{v}_f^\ominus), & \Pi &:= \mathbf{B} : \nabla(\mathbf{v}_s^\ominus - \mathbf{v}_f^\ominus), \\ \Pi_f &:= \mathbf{M} : \nabla \mathbf{v}_f^\ominus, & \sigma &:= \mathbf{S} \cdot \left[ \left| \mathbf{v}_f^\ominus \right| \right]. \end{aligned}$$

## 5.2. Equations of Motion

Let us denote by  $\mathbf{q} := (\chi_s, \phi_s)$  the kinematic descriptor of the medium (a field defined on  $\mathbb{B}_s$ ). Hence the action  $\mathcal{A}$  is a functional of  $\mathbf{q}$ . Moreover, let us denote by  $\mathbf{q}_t, \dot{\mathbf{q}}_t$  the fields defined at any instant  $t$  on  $B_s$  by

$\mathbf{q}_t(\mathbf{X}_s) := \mathbf{q}(\mathbf{X}_s, t)$  and  $\dot{\mathbf{q}}_t(\mathbf{X}_s) := \partial \mathbf{q} / \partial t(\mathbf{X}_s, t)$ . The Rayleigh potential  $\mathcal{R}$  is, at each instant  $t$ , a functional of  $(\mathbf{q}_t, \dot{\mathbf{q}}_t)$ .

The physical principle which determines the motion of a system can be alternatively stated in the framework of second Newton's law (balance of momentum), of D'Alembert principle (weak formulation of momentum balance) or of Rayleigh-Hamilton principle. We adopt this last approach which reads

$$\frac{\partial \mathcal{A}}{\partial \mathbf{q}} \mid \delta \mathbf{q} = \int_0^T \left( \frac{\partial \mathcal{R}}{\partial \dot{\mathbf{q}}_t} \mid \delta \dot{\mathbf{q}}_t \right) dt. \quad (26)$$

Here,  $\partial \mathcal{A} / \partial \mathbf{q}$  and  $\partial \mathcal{R} / \partial \dot{\mathbf{q}}_t$  must be understood in the sense of functional differentiation.

Up to now, the functionals we have introduced do not depend on the choice of the reference configuration of the fluid. The choice of admissible test functions is now crucial: if one considers only regular test functions  $\delta \phi_s$ , one gives an undeserved importance to the choice of the reference configuration of the fluid and one can then obtain unphysical equations. Indeed, a  $C_{pw}^1$  change of configuration  $\mathcal{F}$  may transform a continuous test function  $\xi$  in a discontinuous function  $\delta \phi_s = (\nabla \mathcal{F})^{\otimes} \cdot \xi$  which is a test function as legitimate as  $\xi$ . This problem can be overcome by considering all test functions  $\delta \phi_s = (\nabla \mathcal{F})^{\otimes} \cdot \xi$  where  $\xi$  is a  $C^\infty$  function with compact support included in  $\mathbb{B}_s$  and  $\mathcal{F}$  is any time-independent  $C_{pw}^1$  diffeomorphism defined on  $B_f$ , the gradient of which  $\mathbf{\Gamma}$  has constant determinant (which can be set equal to 1 without loss of generality). Indeed, since we decided to work with a constant field  $\eta_f$  in the fluid reference configuration, we need to set  $\det \mathbf{\Gamma} = 1$  to preserve this assumption. We denote  $S_f^*$  the singularity surface of  $\mathcal{F}$  (which is fixed in  $B_f$ ),  $S^* := \phi_s^{-1}(S_f^*)$  (a moving singularity surface in  $B_s$ ) and  $\mathbb{S}^* := \cup_{t \in (0, T)} S^*(t) \times \{t\}$ .

As the reference configuration of the solid has a precise physical meaning, we simply consider test functions  $\delta \chi_s$  of class  $C^\infty$  with compact support included in  $\mathbb{B}_s$ . Note that, as we only consider variations with compact support in  $\mathbb{B}_s$ , we work with fixed initial ( $t = 0$ ) and terminal ( $t = T$ ) conditions. We also work with fixed  $\mathbf{q}$  (Dirichlet type boundary conditions) on the boundary of the considered domain  $B_s$ . This assumption of variations with compact support is the only constraint introduced in our variational principle, but it does not affect the expected result since we are only interested here in the bulk equations and the jump conditions on the singularity surface  $S_s$  inside the domain  $B_s$ . Recalling that  $\eta_s$  and  $\eta_f$  are given fixed fields, that  $\varepsilon$ ,  $m_f$ ,  $\mathbf{v}_s$  and  $\mathbf{v}_f^{\otimes}$  are all defined in terms of  $\chi_s$  and  $\phi_s$ , then  $\mathcal{A}$  and  $\mathcal{R}$  becomes functions of  $\chi_s$  and  $\phi_s$ . We assume that the variational principle (26) holds for all variations  $\delta q = (\delta \chi_s, \delta \phi_s)$  with the regularity described above.

After long computation which we postpone to Appendix C, the extended Hamilton-Rayleigh principle (26) reads

$$\begin{aligned} 0 = & \int_{\mathbb{B}_s \setminus \mathbb{S}_s} \left( \eta_s \gamma_s + m_f \gamma_f^{\otimes} + \frac{\partial \Psi}{\partial \chi_s} - \operatorname{div} \left( \mathbf{F}_s \cdot \frac{\partial \Psi}{\partial \varepsilon} \right) - \operatorname{div} \left( J_s (\Pi_f^{\otimes})^T \cdot \mathbf{F}_s^{-T} \right) \right) \cdot \delta \chi_s \\ & - \int_{\mathbb{B}_s \setminus (\mathbb{S}_s \cup \mathbb{S}^*)} \mathbf{G}_s^{-T} \cdot \left( m_f \mathbf{F}_s \cdot \gamma_f^{\otimes} + m_f \nabla \left( \frac{\partial \Psi}{\partial m_f} \right) - \mathbf{F}_s^T \cdot \left( J_s \kappa^{\otimes} - \operatorname{div} \left( J_s (\Pi^{\otimes} - \Pi_f^{\otimes})^T \cdot \mathbf{F}_s^{-T} \right) \right) \right) \cdot \delta \phi_s \\ & + \int_{\mathbb{S}_s} \left[ \left( \mathbf{F}_s \frac{\partial \Psi}{\partial \varepsilon} - \mathbf{v}_f^{\otimes} \otimes \mathbf{D} + J_s (\Pi_f^{\otimes})^T \cdot \mathbf{F}_s^{-T} \right) \cdot \mathbf{N}_s \right] \cdot \delta \chi_s \\ & + \int_{\mathbb{S}_s \cup \mathbb{S}^*} \left[ \mathbf{G}_s^{-T} \cdot \left( m_f \frac{\partial \Psi}{\partial m_f} \mathbf{N}_s - \frac{1}{2} m_f \left( \mathbf{v}_f^{\otimes} \right)^2 \mathbf{N}_s + \mathbf{F}_s^T \cdot \mathbf{v}_f^{\otimes} \otimes \mathbf{D} \cdot \mathbf{N}_s \right) \cdot \delta \phi_s \right] \\ & + \int_{\mathbb{S}_s \cup \mathbb{S}^*} \left[ \mathbf{G}_s^{-T} \cdot \mathbf{F}_s^T \cdot \left( J_s \left( (\Pi^{\otimes} - \Pi_f^{\otimes})^T \right) \cdot \mathbf{F}_s^{-T} \cdot \mathbf{N}_s - \| J_s \mathbf{F}_s^{-T} \cdot \mathbf{N}_s \| \sigma^{\otimes} \right) \cdot \delta \phi_s \right]. \end{aligned} \quad (27)$$

First let us consider arbitrary test functions  $\delta \chi_s$  and  $\delta \phi_s = \xi$  with compact support included in  $\mathbb{B}_s \setminus \mathbb{S}_s$ . We get the following system of equations valid in  $\mathbb{B}_s \setminus \mathbb{S}_s$

$$- \left( \eta_s \gamma_s + m_f \gamma_f^{\otimes} \right) + \operatorname{div} \left( \mathbf{F}_s \cdot \frac{\partial \Psi}{\partial \varepsilon} \right) - \frac{\partial \Psi}{\partial \chi_s} = - \operatorname{div} \left( J_s (\Pi_f^{\otimes})^T \cdot \mathbf{F}_s^{-T} \right), \quad (28)$$

$$m_f \left( \mathbf{F}_s^T \cdot \gamma_f^{\otimes} + \nabla \left( \frac{\partial \Psi}{\partial m_f} \right) \right) = \mathbf{F}_s^T \cdot \left( J_s \kappa^{\otimes} - \operatorname{div} \left( J_s (\Pi^{\otimes} - \Pi_f^{\otimes})^T \cdot \mathbf{F}_s^{-T} \right) \right). \quad (29)$$

Therefore the first two integrals in (27) vanish. Considering now  $\delta\phi_s = 0$  and arbitrary  $\delta\chi_s$  we get the jump condition valid on  $\mathbb{S}_s$ :

$$\left[ \mathbf{F}_s \cdot \frac{\partial \Psi}{\partial \varepsilon} - \mathbf{v}_f^{\otimes} \otimes \mathbf{D} + J_s(\Pi_f^{\otimes})^T \cdot \mathbf{F}_s^{-T} \right] \cdot \mathbf{N}_s = 0,$$

which we prefer to write, using the continuity of  $\mathbf{v}_s$ ,

$$\left[ \mathbf{F}_s \cdot \frac{\partial \Psi}{\partial \varepsilon} - (\mathbf{v}_f^{\otimes} - \mathbf{v}_s) \otimes \mathbf{D} + J_s(\Pi_f^{\otimes})^T \cdot \mathbf{F}_s^{-T} \right] \cdot \mathbf{N}_s = 0. \quad (30)$$

Dealing with the last two terms in (27) is more tricky. We first take advantage of the continuity of  $\xi = (\mathbf{\Gamma}^{\otimes})^{-1} \cdot \delta\phi_s$  to write

$$\begin{aligned} 0 &= \int_{\mathbb{S}_s \cup \mathbb{S}^*} \left[ \mathbf{G}_s^{-T} \cdot \left( m_f \frac{\partial \Psi}{\partial m_f} \mathbf{N}_s - \frac{1}{2} m_f (\mathbf{v}_f^{\otimes})^2 \mathbf{N}_s + \mathbf{F}_s^T \cdot \mathbf{v}_f^{\otimes} \otimes \mathbf{D} \cdot \mathbf{N}_s \right) \cdot \mathbf{\Gamma}^{\otimes} \right] \cdot \xi \\ &\quad + \int_{\mathbb{S}_s \cup \mathbb{S}^*} \left[ \mathbf{G}_s^{-T} \cdot \mathbf{F}_s^T \cdot \left( J_s \left( (\Pi^{\otimes} - \Pi_f^{\otimes})^T \right) \cdot \mathbf{F}_s^{-T} \cdot \mathbf{N}_s - \|J_s \mathbf{F}_s^{-T} \cdot \mathbf{N}_s\| \sigma^{\otimes} \right) \cdot \mathbf{\Gamma}^{\otimes} \right] \cdot \xi. \end{aligned}$$

Using the arbitrariness of  $\xi$ , we get the local condition on  $\mathbb{S}_s$

$$\begin{aligned} 0 &= \left[ \mathbf{G}_s^{-T} \cdot \left( m_f \frac{\partial \Psi}{\partial m_f} \mathbf{N}_s - \frac{1}{2} m_f (\mathbf{v}_f^{\otimes})^2 \cdot \mathbf{N}_s + \mathbf{F}_s^T \cdot \mathbf{v}_f^{\otimes} \otimes \mathbf{D} \cdot \mathbf{N}_s \right) \cdot \mathbf{\Gamma}^{\otimes} \right] \\ &\quad + \left[ \mathbf{G}_s^{-T} \cdot \mathbf{F}_s^T \cdot \left( J_s \left( (\Pi^{\otimes} - \Pi_f^{\otimes})^T \right) \cdot \mathbf{F}_s^{-T} \cdot \mathbf{N}_s - \|J_s \mathbf{F}_s^{-T} \cdot \mathbf{N}_s\| \sigma^{\otimes} \right) \cdot \mathbf{\Gamma}^{\otimes} \right]. \quad (31) \end{aligned}$$

Then we choose a continuous field  $\mathbf{\Gamma}$  (e.g.  $\mathbf{\Gamma} = \mathbf{I}$ ). We project equation (31) on the tangent plane to  $S_f$  by considering  $\tau_e$  an arbitrary tangent vector to  $S_e$ , and multiplying the equation by the vector  $(\mathbf{\Gamma}^{\otimes})^{-1} \cdot \mathbf{G}_s \cdot \mathbf{F}_s^{-1} \cdot \tau_e^{\otimes}$ , we get

$$\tau_e^{\otimes} \cdot \left[ \left[ \mathbf{v}_f^{\otimes} \right] d^{\otimes} + \tau_e^{\otimes} \cdot \left[ \left( \Pi^{\otimes} - \Pi_f^{\otimes} \right)^T \right] \cdot \mathbf{N}_e^{\otimes} - \|J_s \mathbf{F}_s^{-T} \cdot \mathbf{N}_s\| \tau_e^{\otimes} \cdot \sigma^{\otimes} \right] = 0,$$

which we prefer to write, using the continuity of  $\mathbf{v}_s$  and the definition of  $\sigma$ ,

$$\tau_e^{\otimes} \cdot \left[ \left[ \mathbf{v}_f^{\otimes} - \mathbf{v}_s \right] d^{\otimes} + \tau_e^{\otimes} \cdot \left[ \left( \Pi^{\otimes} - \Pi_f^{\otimes} \right)^T \right] \cdot \mathbf{N}_e^{\otimes} - \|J_s \mathbf{F}_s^{-T} \cdot \mathbf{N}_s\| \tau_e^{\otimes} \cdot \mathbf{S} \cdot \left[ \mathbf{v}_f^{\otimes} - \mathbf{v}_s \right] \right] = 0. \quad (32)$$

Moreover, we also project the jump condition (31) in the direction of the continuous quantity  $-(\mathbf{\Gamma}^{\otimes})^{-1} \cdot \mathbf{u}_s = (\mathbf{\Gamma}^{\otimes})^{-1} \cdot \mathbf{G}_s \cdot \mathbf{F}_s^{-1} \cdot (\mathbf{v}_f^{\otimes} - \mathbf{v}_s)$  (which, owing to (22) is not a tangent vector when  $d \neq 0$ ), we get

$$\left[ \left( \frac{1}{2} (\mathbf{v}_f^{\otimes})^2 - \mathbf{v}_f^{\otimes} \cdot \mathbf{v}_s + \frac{\partial \Psi}{\partial m_f} \right) \right] d^{\otimes} + \left[ (\mathbf{v}_f^{\otimes} - \mathbf{v}_s) \cdot \left( \left( \Pi^{\otimes} - \Pi_f^{\otimes} \right)^T \cdot \mathbf{N}_e^{\otimes} - \sigma^{\otimes} \right) \right] = 0,$$

which we prefer to write, using the continuity of  $\mathbf{v}_s$ ,

$$\left[ \left( \frac{1}{2} (\mathbf{v}_f^{\otimes} - \mathbf{v}_s)^2 + \frac{\partial \Psi}{\partial m_f} \right) \right] d^{\otimes} + \left[ (\mathbf{v}_f^{\otimes} - \mathbf{v}_s) \cdot \left( \left( \Pi^{\otimes} - \Pi_f^{\otimes} \right)^T \cdot \mathbf{N}_e^{\otimes} - \sigma^{\otimes} \right) \right] = 0. \quad (33)$$

Up to now, we used only continuous fields  $\mathbf{\Gamma}$ . We stress the fact that no extra condition can be established by using arbitrary non continuous  $\mathbf{\Gamma}$  when  $d \neq 0$ . Indeed, if  $d \neq 0$  then  $c_f \neq 0$  (owing to (13) and to the fact that  $c_s = 0$ ). If at a time  $t_0$ , the surface  $S^*$  coincides with  $S_s$ , it is moving with the celerity  $-c_f$ , then, since  $S_s$  is fixed, the two surfaces coincide at time  $t_0$  only. Hence, at almost every time,  $\mathbf{\Gamma}$  is continuous across  $S_s$ .

The situation is completely different when  $d = 0$ . Note that in this case equation (33) is useless (it reduces to a particular case of (32)). When  $d = 0$ , we start by simplifying equation (31) using the fact

that  $\mathbf{D} \cdot \mathbf{N}_s = 0$ , that  $\mathbf{v}_f^{\otimes}$  is continuous and consequently that  $\sigma^{\otimes}$  vanishes. We also use the fact that  $[[I_s(\mathbf{\Gamma}^{\otimes})^T \cdot \mathbf{G}_s^{-T} \cdot \mathbf{N}_s]] = 0$  (obtained by applying Eq. (12) to the  $C_{pw}^1$  diffeomorphism  $\phi_s^{-1} \circ \mathcal{F}$ ). We get

$$\left[ \left[ (\mathbf{\Gamma}^{\otimes})^T \cdot \mathbf{G}_s^{-T} \cdot \left( m_f \frac{\partial \Psi}{\partial m_f} \mathbf{N}_s + \mathbf{F}_s^T \cdot J_s(\Pi^{\otimes} - \Pi_f^{\otimes})^T \cdot \mathbf{F}_s^{-T} \cdot \mathbf{N}_s \right) \right] \right] = 0. \quad (34)$$

If  $d = 0$  during a time interval, on a non-negligible part of  $S_s$ , then  $S_f$  is fixed ( $c_f = 0$ ) and one can choose  $\mathcal{F}$  such that  $S^*$  coincides with  $S_s$  on this part: we can choose non continuous fields  $\mathbf{\Gamma}$ . First, let us choose them of the type  $\mathbf{\Gamma} = \beta \tau_f \otimes \mathbf{N}_f$ , with  $\beta$  arbitrary and discontinuous. Using this expression for  $\mathbf{\Gamma}$  and equation (32) with  $d = 0$ , we obtain

$$[[\beta^{\otimes}]] \left( \tau_e^{\otimes} \cdot (\Pi^{\otimes} - \Pi_f^{\otimes})^T \cdot \mathbf{N}_e^{\otimes} \right) \mathbf{N}_f^{\otimes} = 0,$$

which, since  $[[\beta^{\otimes}]]$  is arbitrary gives the following conditions valid on both sides of  $S_s$

$$\left( \tau_e^{\otimes} \cdot (\Pi^{\otimes} - \Pi_f^{\otimes})^T \cdot \mathbf{N}_e^{\otimes} \right)^- = 0, \quad \left( \tau_e^{\otimes} \cdot (\Pi^{\otimes} - \Pi_f^{\otimes})^T \cdot \mathbf{N}_e^{\otimes} \right)^+ = 0. \quad (35)$$

Owing to these conditions, equation (34) becomes

$$\left[ \left[ (\mathbf{\Gamma}^{\otimes})^T \cdot \mathbf{G}_s^{-T} \cdot \mathbf{N}_s \left( m_f \frac{\partial \Psi}{\partial m_f} + \mathbf{N}_s \cdot \mathbf{F}_s^T \cdot J_s(\Pi^{\otimes} - \Pi_f^{\otimes})^T \cdot \mathbf{F}_s^{-T} \cdot \mathbf{N}_s \right) \right] \right] = 0.$$

or, using again the continuity of  $I_s(\mathbf{\Gamma}^{\otimes})^T \cdot \mathbf{G}_s^{-T} \cdot \mathbf{N}_s$ ,

$$\left[ \left[ \left( \frac{\partial \Psi}{\partial m_f} + \frac{1}{m_f} \mathbf{N}_s \cdot \mathbf{F}_s^T \cdot J_s(\Pi^{\otimes} - \Pi_f^{\otimes})^T \cdot \mathbf{F}_s^{-T} \cdot \mathbf{N}_s \right) \right] \right] = 0. \quad (36)$$

In conclusion, the motion of the porous medium is ruled by the equations (28) (29) valid far from  $S_s$  and, if  $d \neq 0$ , by the jump conditions (30), (32) and (33) valid on  $S_s$ . If  $d = 0$ , the last two conditions have to be replaced by the three conditions stated in (35) and (36). In both cases ( $d = 0$ ,  $d \neq 0$ ) it must be recalled that the kinematical condition  $[[\mathbf{v}_s]] = 0$  has been used.

We remark that this system of equations respects Galilean invariance. Indeed, all equations (as well as the criterium  $d = 0$  or  $d \neq 0$ ) involve only Galilean invariant physical quantities.

Equations (28) and (30) encompass the well known Lagrangian balance equation for the total stress and the corresponding jump condition: these equations only involve physical quantities. As for equations (29), (32), (33), (35) and (36), they are not available in the literature.

## 6. The Case of a Deformable Porous Medium Surrounded by a Pure Fluid

In this section we consider the case of a surface discontinuity  $S_s$  separating a porous medium (which occupies the volume  $B_s^+$ ) from a pure fluid (which occupies the volume  $B_s^-$ ). When the fluid is pure (and when external body forces are neglected), its Eulerian energy density, its chemical potential and its pressure are functions of its mass density only. These three real functions are denoted respectively by  $\Psi_f$ ,  $\mu_f$  and  $p_f$ . They are related by

$$\mu_f(y) = \Psi_f'(y) \quad \text{and} \quad p_f(y) = -\Psi_f(y) + y \Psi_f'(y), \quad (37)$$

In the sequel, as no ambiguity can arise, we simply denote  $\mu_f$ ,  $p_f$  and  $\Psi_f$  the fields  $\mu_f(\rho_f)$ ,  $p_f(\rho_f)$  and  $\Psi_f(\rho_f)$  defined on  $B_e$ . The restriction  $\Psi^-$  of the potential  $\Psi$  in  $B_s^-$  is that of a pure fluid: transporting the Eulerian density  $\Psi_f(\rho_f)$  on  $B_s$ , we get

$$\Psi^-(\varepsilon, m_f) = J_s \Psi_f \left( \rho_f^{\otimes} \right) = J_s \Psi_f \left( J_s^{-1} m_f \right), \quad (38)$$

Note that  $J_s$  is a function of  $\varepsilon$  only, as we have  $J_s = \det \mathbf{F}_s = \sqrt{\det(2\varepsilon + \mathbf{I})}$  and that  $\partial J_s / \partial \varepsilon = J_s \mathbf{F}_s^{-1} \cdot \mathbf{F}_s^{-T}$ . Thus we have

$$\frac{\partial \Psi^-}{\partial \varepsilon} = (\Psi_f^\oplus + \mu_f^\oplus \rho_f^\oplus) \frac{\partial J_s}{\partial \varepsilon} = -p_f^\oplus J_s \mathbf{F}_s^{-1} \cdot \mathbf{F}_s^{-T}, \quad \text{and} \quad \frac{\partial \Psi^-}{\partial m_f} = \mu_f^\oplus.$$

As for the porous region, we simply denote  $\Psi^+$  the potential density function in  $B_s^+$ .

In the pure fluid region  $B_s^-$  we clearly have  $\eta_s = 0$  and  $\Pi = 0$ . Moreover, as already noticed,  $\mathbf{K}^\oplus = 0$ ,  $\mathbf{M}^\oplus = 0$ .

Under these assumptions, using equations (9) and (37), we obtain the following expression for the equation of motion (28) in  $B_s^-$  :

$$\begin{aligned} m_f \gamma_f^\oplus &= \operatorname{div} \left( \mathbf{F}_s \cdot \frac{\partial \Psi^-}{\partial \varepsilon} \right) + \operatorname{div} \left( J_s (\Pi_f^\oplus)^T \cdot \mathbf{F}_s^{-T} \right) \\ &= \operatorname{div} \left( J_s \left( -p_f^\oplus \mathbf{I} + (\Pi_f^\oplus)^T \right) \cdot \mathbf{F}_s^{-T} \right) \\ &= J_s \left( \operatorname{div} \left( -p_f \mathbf{I} + \Pi_f^T \right) \right)^\oplus. \end{aligned}$$

As expected, this last equation, when transported on the Eulerian configuration, is the usual Navier-Stokes equation for the motion of a fluid :

$$\rho_f \gamma_f^\oplus = -\nabla p_f + \operatorname{div} \Pi_f^T. \quad (39)$$

The existence of a supplementary equation (29) may seem astonishing. Under the hypotheses we formulated, recalling (38) and (37), Eq. (29) can be rewritten

$$m_f \mathbf{F}_s^T \cdot \gamma_f^\oplus + m_f \nabla \mu_f^\oplus = \mathbf{F}_s^T \cdot \operatorname{div} \left( J_s (\Pi_f^\oplus)^T \cdot \mathbf{F}_s^{-T} \right),$$

which, multiplied on the left by  $\mathbf{F}_s^{-T}$ , and then rewritten on the Eulerian configuration gives

$$\begin{aligned} m_f \gamma_f^\oplus + m_f \mathbf{F}_s^{-T} \cdot \nabla \mu_f^\oplus &= \operatorname{div} \left( J_s (\Pi_f^\oplus)^T \cdot \mathbf{F}_s^{-T} \right), \\ \rho_f \gamma_f^\oplus + \rho_f \nabla \mu_f &= \operatorname{div} \Pi_f^T, \end{aligned}$$

This last equation is clearly equivalent to Eq. (39) owing to the identity  $\nabla p_f = \rho_f \nabla(\mu_f)$ . Hence, as expected, the fluid is governed *only* by the usual Navier-Stokes equation.

Let us consider the Eulerian form of jump condition (30) on the surface  $S_s$  which divides the porous medium region  $B_s^+$  from the pure fluid region  $B_s^-$ :

$$\left( J_s^{-1} \mathbf{F}_s \cdot \frac{\partial \Psi}{\partial \varepsilon} \cdot \mathbf{F}_s^T + (\Pi_f^\oplus)^T \right)^+ \cdot \mathbf{N}_e^\oplus = -(p_f^\oplus)^- \mathbf{N}_e^\oplus + \left[ \left[ \mathbf{v}_f^\oplus - \mathbf{v}_s \right] \right] d^\oplus + ((\Pi_f^\oplus)^T)^- \cdot \mathbf{N}_e^\oplus. \quad (40)$$

The reader can recognize in the left hand side of this equation the total Cauchy stress tensor of the porous medium. It appears clearly that, as predictable, the action of the fluid on the surface do not depend on the extension of  $\chi_s$ .

As for the jump conditions (32) or (35), they remain unchanged, as they do not involve  $\Psi^-$ . Since they are written in Eulerian form, they also do not depend on the extension of  $\chi_s$ .

Considering the jump conditions (33) or (36), we remark that in this special case one just has to replace  $\partial \Psi^- / \partial m_f$  by the chemical potential of the fluid  $\mu_f^\oplus$ . They are both written in Eulerian form if one recognizes that the term  $[(1/m_f \mathbf{N}_s \cdot \mathbf{F}_s^T \cdot J_s (\Pi_f^\oplus - \Pi_f^\oplus)^T \cdot \mathbf{F}_s^{-T} \cdot \mathbf{N}_s)]$  can be rewritten as  $[(1/\rho_f^\oplus \mathbf{N}_e^\oplus \cdot (\Pi_f^\oplus - \Pi_f^\oplus)^T \cdot \mathbf{N}_e^\oplus)]$ .

In conclusion, the motion of the porous medium is driven by two independent equations of the type (28)-(29), while the motion of the pure fluid is driven by a simple equation in the form (39). The differential system is completed by the jump conditions (32) and (33) in the case  $d \neq 0$  and by the jump conditions (35) and (36) if  $d = 0$ .

## 7. Conclusions

In this paper the configuration of a fluid-filled porous medium is characterized by means of the placement fields  $\chi_s$  and  $\phi_s$ . Moreover, suitable action and dissipation functionals are postulated in order to account for the mechanical properties of a deformable *solid* matrix permeable to *fluid* flow. Therefore, the obtained Euler-Lagrange-Rayleigh equations naturally determine the time evolution of the fields  $\chi_s$ ,  $m_f$  and  $\mathbf{u}_s$  which represent the solid placement, the solid-Lagrangian density of the fluid and the solid-fluid relative velocity, respectively.

In the presented model the “solid volume fraction”  $\nu$  does not appear explicitly as a kinematical field: nevertheless, solid volume fraction plays a crucial role, for instance, in the determination of the macroscopic deformation energy of the fluid-filled solid matrix. Indeed, when the macroscopic constitutive equations for such a system need to be postulated, the most natural choice actually is to deduce them starting from the microscopic constitutive equations of the *pure* fluid and solid constituents. This is exactly what is done in the literature stemming from the papers of Fillunger, Terzaghi and Biot (see e.g. Biot (1941); Biot (1956a); Biot (1956b); Biot and Willis (1957); Biot (1962); Biot (1963); Beavers and Joseph (1967); Coussy and Bourbie (1984); Coussy *et al.* (1998); Coussy (2004); de Boer (1996); Deresiewicz (1963); Dormieux and Stolz (1992); Saffman (1971); Wilmanski (2006)).

One suggestive reasoning to explain such an approach and its logical limits can be the following. Let us assume that a macroscopic solid material particle is in a state described by the macroscopic deformation gradient  $\mathbf{F}_s$  and that the macroscopic fluid saturating particle has a (solid referential) density  $m_f$ . Then, the microscopic deformation energy  $\psi_{tot}$  of the fluid-filled porous matrix can be expressed, in terms of the fluid and solid volume fractions  $\nu_f$  and  $\nu$ , as follows:

$$\psi_{tot}(\mathbf{F}_s, m_f, \nu_f, \nu) = \psi_s(\mathbf{F}_s, \nu) + \psi_f\left(\frac{J_s^{-1}m_f}{\nu_f}\right) \quad (41)$$

where  $\psi_s$  is the microscopic deformation energy of a solid matrix when it experiences the macroscopic deformation  $\mathbf{F}_s$  and its solid volume fraction is given by  $\nu$ , while  $\psi_f$  is the microscopic deformation energy of the permeating fluid. In order to obtain from (41) a macroscopic energy density, one needs to assume a kind of “instantaneous local equilibrium hypothesis”. In other words, it must be assumed that, for fixed  $\mathbf{F}_s$  and  $m_f$ , with characteristic times much shorter than those characterizing macro-phenomena, the fluid and solid volume fractions adjust to a local equilibrium value. These equilibrium values are obtained by solving the following local minimization problem: find the functions  $\nu_f(\mathbf{F}_s, m_f)$  and  $\nu(\mathbf{F}_s, m_f)$  such that

$$\psi_{tot}(\mathbf{F}_s, m_f, \nu_f(\mathbf{F}_s, m_f), \nu(\mathbf{F}_s, m_f)) = \min_{\nu_f, \nu} \psi_{tot}(\mathbf{F}_s, m_f, \nu_f, \nu). \quad (42)$$

The minimization problem (42) is crucial and has been solved in very clever ways under physically acceptable assumptions. The resulting macro deformation energy density is thus obtained as follows

$$\Psi(\mathbf{F}_s, m_f) = \psi_{tot}(\mathbf{F}_s, m_f, \nu_f(\mathbf{F}_s, m_f), \nu(\mathbf{F}_s, m_f)). \quad (43)$$

In the present paper we refrain from any attempt of deducing any particular form for  $\Psi$ . Our aim is to find a logically consistent set of evolution equations and boundary conditions for models in which the independent kinematical descriptors are  $\chi_s$  and  $\phi_s$ .

To our knowledge, the inertia terms appearing in the jump conditions (32) and (33) are not found in the literature. Moreover, all presented boundary conditions are valid also when the solid matrix is suffering large deformations and when the Stokes fluid-flow condition is not applicable.

A deduction of a jump condition similar to (33) and valid in the particular case of absence of inertia, of Darcy-Brinkman and Beavers-Joseph dissipation is presented in Baek and Srinivasa (2004).

Other authors (see e.g. Neale G., Nader (1974); Vafai and Thiyagaraja (1987); Vafai and Kim (1990); Poulikakos and Kazmierczak (1987)) based themselves on the pioneering works of Beavers and Joseph (1967) and Saffman (1971), to justify the so-called slip boundary conditions at the interface between a porous matrix and an external viscous fluid. Beavers-Joseph-Saffman conditions include the continuity

of the tangent components of the relative velocity at the interface between the porous medium and the external fluid. Moreover, they express the jump of the gradient of the tangent relative velocity in terms of the common value of tangent velocity at the interface. Nevertheless, they only describe phenomena related to the viscosity of the outflowing fluid with no consideration of inertial effects and Darcy-Brinkman dissipation. Beavers-Joseph-Saffman conditions can be deduced from our jump condition (40), once assuming that the solid matrix is suffering small deformation and when Stokes fluid-flow condition is verified.

The jump conditions deduced in Deresiewicz (1960); Deresiewicz (1962a); Deresiewicz (1962b); Deresiewicz (1962c); Deresiewicz (1963); Deresiewicz (1964a); Deresiewicz (1964b) are suitable to assure that the differential problem of Darcy-Fillunger-Terzaghi-Biot is well-posed (see Fillunger (1936); Terzaghi (1943); Biot (1941); Biot (1956a); Biot (1956b); Biot and Willis (1957); Biot (1962); Biot (1963)). These jump conditions can be obtained as a particular case from Beavers-Joseph-Saffman conditions, once it is possible to neglect dissipative phenomena at the considered interface. In Albers (2006); Wilmanski (1999); Wilmanski (2006); de la Cruz *et al.* (1992); Quiroga-Goode and Carcione (1997) the jump conditions proposed by Deresiewicz (1963) are used to study wave propagation phenomena at discontinuity surfaces in porous media.

In Coussy and Bourbie (1984); Coussy *et al.* (1998); Rasolofosaon and Coussy (1985a); Rasolofosaon and Coussy (1985b); Rasolofosaon and Coussy (1986); Coussy (2004), a variational approach is used to study some wave propagation phenomena of interest in oil industry: the boundary conditions proposed by Deresiewicz (1963) are examined there with a view towards the applications.

In Kubik and Cieszko (2005) a dissipative Rayleigh surface potential is proposed which is suitable to produce a particular form of Beavers-Joseph-Saffman boundary conditions: many versions of them are compared in Alazmi and Vafai (2001).

As for our jump conditions (30), (32) and (33), they also allow for describing phenomena in which inertial effects are relevant. The inertia terms, which are here newly introduced, are at least quadratic in the relative velocity fields at the interface: when Stokes fluid-flow conditions hold (and when the solid matrix is subjected to “small deformations”) they may be negligible. Indeed, when the equations are linearized in the neighborhood of a state of rest (i.e. when all velocity fields and their gradients are vanishing) the aforementioned inertia terms do not produce, in the resulting boundary conditions, any term additional to those appearing in Beavers-Joseph-Saffman conditions. However, when the linearization procedure is performed in the neighborhood of a state in which some velocity fields are not vanishing, then inertia terms cannot be neglected.

Moreover, one should remark that in Ochoa-Tapia and Whitaker (1998) some inertial effects at the interface are considered. However, they deduce no-slip conditions for tangential velocity and a normal-to-the-interface boundary condition by means of an averaging procedure involving “excess quantities”. Their assumptions produce inertia terms in which only the tangential part of the fluid velocity appears.

Finally, we remark that the dissipation Rayleigh functional which we assumed to be a quadratic form in the relative and fluid velocities, does not need to be restricted to this particular form and can be generalized to any power dissipation law. This generalization would be of interest in the study of particular fluids (as polymeric fluids and slurries) in which dissipation effects cannot be adequately described by the classical linearly viscous fluid model (see Dunn and Rajagopal (1995); Rajagopal and Srinivasa (2000), Srinivasa (2000)). In this paper we preferred to limit the range of applicability of our model to the dissipation phenomena encompassed by the classical approach of Rayleigh and to postpone a more complex modeling to further investigations.

Future investigations will be aimed to get a generalization of the newly found boundary conditions to the case of shock waves.

## A. Proof of the Properties of the Gradients of $C^2$ Vector Fields

We prove here the properties of smooth tensor fields which are the gradient of some vector field.

**Notation 1.** For any differentiable  $k$ -th order tensor field  $\mathbf{t} = \mathbf{t}_{i_1 \dots i_k}$  defined on  $\mathbb{R}^n$ , we denote by  $\nabla \mathbf{t}$  its gradient and, when  $k \geq 1$  and  $i_k$  varies in  $\{1, 2, \dots, n\}$ , by  $\text{DIV } \mathbf{t}$  its divergence.

The components of these tensors are given by <sup>2</sup>

$$(\nabla \mathbf{t})_{i_1 i_2 \dots i_{k+1}} := \partial_{i_{k+1}} (\mathbf{t}_{i_1 i_2 \dots i_k}); \quad (\text{DIV } \mathbf{t})_{i_1 i_2 \dots i_{k-1}} := \sum_{i_k=1}^n \partial_{i_k} (\mathbf{t}_{i_1 i_2 \dots i_k}).$$

We assume here that  $\mathcal{X}$  is a  $C^2$ -diffeomorphism from  $\mathbb{B}_a \subset \mathbb{R}^n$  onto  $\mathbb{B}_b \subset \mathbb{R}^n$  and we recall that  $\mathbb{F} := \nabla \mathcal{X}$  and  $\mathbb{J} := \det \mathbb{F}$ .

Next Proposition gives a transport formula for the divergence operator and states an important property for the gradient of a diffeomorphism.

**Proposition 1.** *Let  $\mathcal{X}$  be a  $C^2$ -diffeomorphism between the domains  $\mathbb{B}_a$  and  $\mathbb{B}_b$ . For any differentiable tensor field  $\mathbf{z}$  (of order  $\geq 1$ ) defined on  $\mathbb{B}_b$  the following equation holds*

$$\text{DIV} (\mathbb{J} \mathbf{z}^{\textcircled{a}} \cdot \mathbb{F}^{-T}) = \mathbb{J} (\text{DIV } \mathbf{z})^{\textcircled{a}}. \quad (44)$$

In particular

$$\text{DIV} (\mathbb{J} \mathbb{F}^{-T}) = 0. \quad (45)$$

*Proof.* Let us consider a differentiable scalar field  $\psi$  with compact support included in  $\mathbb{B}_a$ . Owing to the regularity assumptions on  $\mathcal{X}$ , the corresponding scalar field  $\psi^{\textcircled{b}}$  on  $\mathbb{B}_b$  has compact support and is differentiable on  $\mathbb{B}_b$ . A simple change of variables gives

$$\int_{\mathbb{B}_b} \psi^{\textcircled{b}} \text{DIV } \mathbf{z} = \int_{\mathbb{B}_a} \psi (\text{DIV } \mathbf{z})^{\textcircled{a}} \mathbb{J}. \quad (46)$$

On the other hand, using the divergence theorem and recalling that  $\psi^{\textcircled{b}}$  has compact support

$$\int_{\mathbb{B}_b} \psi^{\textcircled{b}} \text{DIV } \mathbf{z} = - \int_{\mathbb{B}_b} \mathbf{z} \cdot \nabla \psi^{\textcircled{b}}. \quad (47)$$

Starting from Eq. (47), using successively formula (2) for  $\psi^{\textcircled{b}}$ , a change of variables, the divergence theorem and the fact that  $\psi$  has compact support we get the following equalities

$$\int_{\mathbb{B}_b} \psi^{\textcircled{b}} \text{DIV } \mathbf{z} = - \int_{\mathbb{B}_b} \mathbf{z} \cdot (\nabla \psi \cdot \mathbb{F}^{-1})^{\textcircled{b}} = - \int_{\mathbb{B}_a} \mathbf{z}^{\textcircled{a}} \cdot \mathbb{F}^{-T} \cdot \nabla \psi \mathbb{J} = \int_{\mathbb{B}_a} \psi \text{DIV} (\mathbb{J} \mathbf{z}^{\textcircled{a}} \cdot \mathbb{F}^{-T}). \quad (48)$$

The comparison between Eq. (46) and the last term of Eq.(48) gives

$$\int_{\mathbb{B}_a} \psi (\text{DIV } \mathbf{z})^{\textcircled{a}} \mathbb{J} = \int_{\mathbb{B}_a} \psi \text{DIV} (\mathbb{J} \mathbf{z}^{\textcircled{a}} \cdot \mathbb{F}^{-T}).$$

The fact that this last equality is satisfied for any  $\psi$  with compact support included in  $\mathbb{B}_a$  proves (44). It is enough to apply (44) choosing for  $\mathbf{z}$  the identity tensor to get (45).  $\square$

Note that the previous proposition can be applied to  $\mathcal{X}^{-1}$  so that, for any differentiable tensor field  $\mathbf{t}$  (of order  $\geq 1$ ) defined on  $\mathbb{B}_a$  one gets

$$\text{DIV} (\mathbb{J}^{-1} \mathbf{t} \cdot \mathbb{F}^T)^{\textcircled{b}} = (\mathbb{J}^{-1} \text{DIV } \mathbf{t})^{\textcircled{b}} \quad \text{and} \quad \text{DIV} (\mathbb{J}^{-1} \mathbb{F}^T)^{\textcircled{b}} = 0. \quad (49)$$

<sup>2</sup>The symbol  $\partial_j$  indicates the partial derivative of a function with respect to the  $j$ -th component of its argument. Moreover in order to lighten notations, we adopt the Einstein summation convention on repeated indices dropping the summation symbol.

**Corollary 1.** Let  $\mathbf{U}$  be a  $C^1$  vector field defined in  $\mathbb{B}_b$  and let  $\mathbf{U}^\oplus$  be its corresponding vector field on  $\mathbb{B}_a$ . Let  $\mathbf{N}_a$  and  $\mathbf{N}_b$  be the outward unit normal vectors to  $\partial\mathbb{B}_a$  and  $\partial\mathbb{B}_b$  respectively, then

$$\int_{\partial\mathbb{B}_b} \mathbf{U} \cdot \mathbf{N}_b = \int_{\partial\mathbb{B}_a} \mathbf{U}^\oplus \cdot (\mathbf{J} \mathbf{F}^{-T} \cdot \mathbf{N}_a), \quad (50)$$

*Proof.* Recalling Eq. (44), one gets

$$\int_{\partial\mathbb{B}_b} \mathbf{U} \cdot \mathbf{N}_b = \int_{\mathbb{B}_b} \text{DIV}(\mathbf{U}) = \int_{\mathbb{B}_a} (\text{DIV}(\mathbf{U}))^\oplus \mathbf{J} = \int_{\mathbb{B}_a} \text{DIV}(\mathbf{J} \mathbf{U}^\oplus \cdot \mathbf{F}^{-T}) = \int_{\partial\mathbb{B}_a} \mathbf{U}^\oplus \cdot (\mathbf{J} \mathbf{F}^{-T} \cdot \mathbf{N}_a).$$

□

Note that the last corollary applied to  $\mathbf{x}^{-1}$  reads

$$\int_{\partial\mathbb{B}_a} \mathbf{U}^\oplus \cdot \mathbf{N}_a = \int_{\partial\mathbb{B}_b} \mathbf{U} \cdot ((\mathbf{J}^{-1} \mathbf{F}^T)^\oplus \cdot \mathbf{N}_b).$$

## B. Proof of the Properties of the Gradients of Piecewise $C^1$ Vector Fields

We now precisely define piecewise differentiable vector fields and prove some properties which hold in their singular points.

Let  $\mathbb{S}_a$  be a smooth codimension-one hypersurface in  $\mathbb{B}_a$ ; this means that, at least locally, there exists a parametric representation of  $\mathbb{S}_a$ , i.e. an open subset  $\Omega$  of  $\mathbb{R}^{n-1}$  and a smooth embedding  $\varphi \in C^1(\Omega, \mathbb{B}_a \subset \mathbb{R}^n)$  such that  $\mathbb{S}_a := \varphi(\Omega)$ . By definition of an embedding, for any  $\mathbf{x} = \varphi(\mathbf{s}) \in \mathbb{S}_a$  the vectors

$$\mathbb{T}_i(\mathbf{x}) := \partial_i \varphi|_{\mathbf{s}=\varphi^{-1}(\mathbf{x})}, \quad i = 1, 2, \dots, n-1$$

make a basis spanning the tangent space  $\mathcal{T}_{\mathbf{x}}(\mathbb{S}_a)$  of  $\mathbb{S}_a$  at  $\mathbf{x}$ . The orthogonal space to  $\mathcal{T}_{\mathbf{x}}(\mathbb{S}_a)$  is one-dimensional: there exists a unique unit vector  $\mathbf{N}_a(\mathbf{x})$  in this space which completes  $\{\mathbb{T}_i(\mathbf{x})\}$  in a direct basis of  $\mathbb{R}^n$ . This vector  $\mathbf{N}_a$  locally provides an orientation for  $\mathbb{S}_a$  and we call it the *normal* to  $\mathbb{S}_a$ .

**Notation 2.** Let  $\mathbf{t}$  be a tensor field defined on  $\mathbb{B}_a$  (and consequently on  $\mathbb{S}_a$ ). We say that  $\mathbf{t}$  is differentiable on  $\mathbb{S}_a$  if  $\mathbf{t} \circ \varphi \in C^1(\Omega)$ . The surface gradient  $\nabla^{\mathbb{S}_a} \mathbf{t}(\mathbf{x})$  at point  $\mathbf{x}$  is the linear operator which, to any tangent vector  $\mathbb{T} = \sum_{i=1}^{n-1} v_i \mathbb{T}_i$ , associates the derivative of  $\mathbf{t}$  in the direction  $\mathbb{T}$  defined by

$$\nabla^{\mathbb{S}_a} \mathbf{t}(\mathbf{x}) \cdot \mathbb{T} := \sum_{i=1}^{n-1} v_i \partial_i (\mathbf{t} \circ \varphi).$$

Recall that, even if the basis  $\mathbb{T}_i$  depends on the choice of the parametrization  $\varphi$ , the surface gradient (regarded as a linear operator) does not.

**Notation 3.** We say that a tensor field  $\mathbf{t}$  defined on  $\mathbb{B}_a$  is piecewise continuous (or briefly  $C_{pw}^0$ ) if there exists a smooth codimension-one  $C^1$  hypersurface  $\mathbb{S}_a$  in  $\mathbb{B}_a$  (or a finite union of such hypersurfaces) such that  $\mathbf{t}$  belongs to  $C^0(\mathbb{B}_a \setminus \mathbb{S}_a, \mathbb{R}^p)$  and admits continuous traces  $\mathbf{t}^+$  and  $\mathbf{t}^-$  on both sides of  $\mathbb{S}_a$ . The quantity

$$[[\mathbf{t}(\mathbf{x})]] := \mathbf{t}^+(\mathbf{x}) - \mathbf{t}^-(\mathbf{x})$$

is called *jump* of  $\mathbf{t}$  through the surface  $\mathbb{S}_a$  at point  $\mathbf{x}$ . Moreover, the surface  $\mathbb{S}_a$  is said to be the singularity surface of the field  $\mathbf{t}$ . When  $\mathbf{t}$  has vanishing jump across the singularity surface, we simply indicate by  $\mathbf{t}$  the common value  $\mathbf{t}^+ = \mathbf{t}^-$ .

Indeed, at least locally, the normal  $\mathbf{N}_a$  to  $\mathbb{S}_a$  defines the “upper” part  $\mathbb{B}_a^+$  of  $\mathbb{B}_a$  toward which the normal is pointing and the “lower” part  $\mathbb{B}_a^-$  of  $\mathbb{B}_a$  in the opposite direction. Then, for any  $\mathbf{x} \in \mathbb{S}_a$

$$\mathbf{t}^+(\mathbf{x}) = \lim_{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{t}(\mathbf{y}), \quad \mathbf{y} \in \mathbb{B}_a^+; \quad \mathbf{t}^-(\mathbf{x}) = \lim_{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{t}(\mathbf{y}), \quad \mathbf{y} \in \mathbb{B}_a^-.$$

**Notation 4.** We say that a tensor field  $\mathbf{t}$  defined on  $\mathbb{B}_a$  is piecewise differentiable (or briefly  $C_{pw}^1$ ) if it is continuous and if its gradient  $\nabla \mathbf{t}$  is  $C_{pw}^0$ .

**Property 1.** The well-known Hadamard property (see e.g. Kosiński (1986)) states that the jump of the gradient of a  $C_{pw}^1$  tensor field  $\mathbf{t}$  is a rank-one matrix field in the form

$$[[\nabla \mathbf{t}]] = [[\nabla \mathbf{t}]] \cdot \mathbf{N}_a \otimes \mathbf{N}_a = ((\nabla \mathbf{t})^+ \cdot \mathbf{N}_a - (\nabla \mathbf{t})^- \cdot \mathbf{N}_a) \otimes \mathbf{N}_a = \left[ \left[ \frac{\partial \mathbf{t}}{\partial \mathbf{N}_a} \right] \right] \otimes \mathbf{N}_a \quad (51)$$

In other words, for any  $\mathbb{T} \in \mathcal{T}_{\mathbf{x}}(\mathbb{S}_a)$

$$[[\nabla \mathbf{t}(\mathbf{x}) \cdot \mathbb{T}]] = 0. \quad (52)$$

This property simply reflects the fact that  $\mathbf{t}$ , when restricted to  $\mathbb{S}_a$ , reduces to a differentiable field and

$$\nabla^{\mathbb{S}_a} \mathbf{t} \cdot \mathbb{T} = (\nabla \mathbf{t})^+ \cdot \mathbb{T} = (\nabla \mathbf{t})^- \cdot \mathbb{T} \quad (53)$$

The following proposition states some important consequences of Property 1.

**Proposition 2.** Let  $\mathbb{B}_a, \mathbb{B}_b$  be two regular open subsets of  $\mathbb{R}^n$  respectively, and let  $\mathcal{X} \in C_{pw}^1(\mathbb{B}_a, \mathbb{B}_b)$  with singularity surface  $\mathbb{S}_a \subset \mathbb{B}_a$ . Assume that  $\mathbb{J} \neq 0$  everywhere on  $\mathbb{S}_a$ , then

- (i) For any  $\mathbb{T}$  tangent to  $\mathbb{S}_a$ ,  $[[\mathbb{F} \cdot \mathbb{T}]] = 0$  on  $\mathbb{S}_a$ .
- (ii) The surface  $\mathbb{S}_b := \mathcal{X}(\mathbb{S}_a)$  is a smooth codimension-one  $C^1$  surface in  $\mathbb{B}_b$  with tangent vectors  $(\mathbb{F} \cdot \mathbb{T}_i)^{\oplus}$ ,  $i \in \{1, 2, \dots, n-1\}$ .
- (iii) For any normal vector field  $\mathbb{M}_b$  to  $\mathbb{S}_b$ , the following jump condition holds on  $\mathbb{S}_a$

$$[[\mathbb{J}^{-1} \mathbb{F}^T]] \cdot \mathbb{M}_b^{\oplus} = 0, \quad (54)$$

- (iv) Moreover, the quantity  $\mathbb{J}^{-1} \mathbb{F}^T \cdot \mathbb{M}_b^{\oplus}$  which is continuous through the surface is orthogonal to  $\mathbb{S}_a$ .

*Proof.* Point (i) is an immediate consequence of (52) if we recall that  $\mathbb{F}$  denotes  $\nabla \mathcal{X}$ . To prove point (ii) we note that, as  $\mathcal{X}$  is  $C_{pw}^1$ , its restriction to  $\mathbb{S}_a$  is differentiable and so is  $\xi := \mathcal{X} \circ \varphi$  which makes a parametrization for  $\mathbb{S}_b := \mathcal{X}(\mathbb{S}_a)$ . Moreover, for any  $\mathbf{y} = \xi(\mathbf{s}) \in \mathbb{S}_b$  and for any  $i = 1, 2, \dots, n-1$ , the vectors

$$\partial_i \xi|_{\mathbf{s}=\xi^{-1}(\mathbf{y})} = \mathbb{F}^{\pm}|_{\mathcal{X}^{-1}(\mathbf{y})} \cdot \partial_i \varphi|_{\varphi^{-1}(\mathcal{X}^{-1}(\mathbf{y}))} = (\mathbb{F}^{\pm} \cdot \mathbb{T}_i)^{\oplus} = (\mathbb{F} \cdot \mathbb{T}_i)^{\oplus}$$

make a basis spanning the tangent space  $\mathcal{T}_{\mathbf{y}}(\mathbb{S}_b)$  of  $\mathbb{S}_b$  at  $\mathbf{y}$ .

To prove points (iii) and (iv) we consider, at any point  $\mathbf{x}$  of  $\mathbb{S}_a$ , the three linear applications  $L_a$  and  $L_b^{\pm}$  respectively defined on  $\mathbb{R}^n$  by

$$L_a(\mathbb{U}) := \det(\mathbb{T}_1, \mathbb{T}_2, \dots, \mathbb{T}_{n-1}, \mathbb{U}), \quad (55)$$

$$L_b^{\pm}(\mathbb{V}) := \det\left((\mathbb{F}^{\pm} \cdot \mathbb{T}_1)^{\oplus}, (\mathbb{F}^{\pm} \cdot \mathbb{T}_2)^{\oplus}, \dots, (\mathbb{F}^{\pm} \cdot \mathbb{T}_{n-1})^{\oplus}, \mathbb{V}\right), \quad (56)$$

Owing to point (i) one easily gets  $L_b^+(\mathbb{V}) = L_b^-(\mathbb{V})$ . In virtue of the Rietz theorem there exist unique vectors  $\tilde{\mathbb{M}}_a$  and  $\tilde{\mathbb{M}}_b$  such that

$$L_a(\mathbb{U}) = \tilde{\mathbb{M}}_a \cdot \mathbb{U}, \quad \forall \mathbb{U} \in \mathbb{R}^n, \quad L_b(\mathbb{V}) = \tilde{\mathbb{M}}_b \cdot \mathbb{V} \quad \forall \mathbb{V} \in \mathbb{R}^n. \quad (57)$$

Hence

$$\tilde{\mathbb{M}}_a \cdot \mathbb{U} = \det(\mathbb{T}_1, \mathbb{T}_2, \dots, \mathbb{T}_{n-1}, \mathbb{U}), \quad (58)$$

and

$$\begin{aligned} \tilde{\mathbb{M}}_b \cdot (\mathbb{F}^{\pm} \cdot \mathbb{U})^{\oplus} &= \det\left((\mathbb{F}^{\pm} \cdot \mathbb{T}_1)^{\oplus}, (\mathbb{F}^{\pm} \cdot \mathbb{T}_2)^{\oplus}, \dots, (\mathbb{F}^{\pm} \cdot \mathbb{T}_{n-1})^{\oplus}, (\mathbb{F}^{\pm} \cdot \mathbb{U})^{\oplus}\right) \\ &= (\mathbb{J}^{\pm})^{\oplus} \det\left((\mathbb{T}_1)^{\oplus}, (\mathbb{T}_2)^{\oplus}, \dots, (\mathbb{T}_{n-1})^{\oplus}, \mathbb{U}^{\oplus}\right). \end{aligned} \quad (59)$$

From (58) and (59) we get

$$\tilde{\mathbb{M}}_b^{\oplus} \cdot (\mathbb{F}^{\pm} \cdot \mathbb{U}) = \mathbb{J}^{\pm} \det(\mathbb{T}_1, \mathbb{T}_2, \dots, \mathbb{T}_{n-1}, \mathbb{U})$$

and so

$$\tilde{\mathbb{M}}_b^{\oplus} \cdot (\mathbb{F}^{\pm} \cdot \mathbb{U}) = \mathbb{J}^{\pm} \tilde{\mathbb{M}}_a \cdot \mathbb{U}$$

Since this last identity is satisfied for any  $\mathbb{U} \in \mathbb{R}^n$  then

$$\tilde{\mathbb{M}}_a = (\mathbb{J}^+)^{-1} (\mathbb{F}^+)^T \cdot \tilde{\mathbb{M}}_b^{\oplus} = (\mathbb{J}^-)^{-1} (\mathbb{F}^-)^T \cdot \tilde{\mathbb{M}}_b^{\oplus}, \quad (60)$$

thus

$$[[\mathbb{J}^{-1} \mathbb{F}^T]] \cdot \tilde{\mathbb{M}}_b^{\oplus} = 0. \quad (61)$$

From (58), (59) we also get that for any  $i \in \{1, 2, \dots, n-1\}$ ,

$$\tilde{\mathbb{M}}_a \cdot \mathbb{T}_i = 0, \quad \text{and} \quad \tilde{\mathbb{M}}_b \cdot (\mathbb{F}^{\pm} \cdot \mathbb{T}_i)^{\oplus} = 0.$$

It follows that  $\tilde{\mathbb{M}}_a$  and  $\tilde{\mathbb{M}}_b$  belong to the one dimensional orthogonal spaces to  $\mathbb{S}_a$  and  $\mathbb{S}_b$  respectively. As they clearly are non vanishing, Eq. (7) remains valid for any  $\mathbb{M}_b$  normal to  $\mathbb{S}_b$ .  $\square$

**Definition 1.** We call piecewise diffeomorphism a  $C_{pw}^1$  homeomorphism  $\mathfrak{X}$  from  $\mathbb{B}_a$  onto  $\mathbb{B}_b$  such that  $\mathfrak{X}^{-1} \in C_{pw}^1(\mathbb{B}_b, \mathbb{B}_a)$ .

Note that if  $\mathfrak{X}$  is a piecewise diffeomorphism with singularity surface  $\mathbb{S}_a$  then the previous proposition can be applied to both  $\mathfrak{X}$  and  $\mathfrak{X}^{-1}$ . Thus we also have, for any  $\mathbb{M}_a \perp \mathbb{S}_a$  the following jump condition on  $\mathbb{S}_b = \mathfrak{X}(\mathbb{S}_a)$

$$[[\left(\mathbb{J} \mathbb{F}^{-T}\right)^{\oplus}]] \cdot \mathbb{M}_a^{\oplus} = 0, \quad (62)$$

and the quantity  $\left(\mathbb{J} \mathbb{F}^{-T} \cdot \mathbb{M}_a\right)^{\oplus}$  is orthogonal to  $\mathbb{S}_b$ . If  $\mathfrak{t}$  is a differentiable tensor field defined on  $\mathbb{B}_a$  then its corresponding tensor field  $\mathfrak{t}^{\oplus}$  on  $\mathbb{B}_b$  may be not differentiable on  $\mathbb{S}_b$ . However,  $\mathfrak{t}^{\oplus}$  is  $C_{pw}^1$  and on  $\mathbb{S}_b$  we have

$$(\nabla \mathfrak{t}^{\oplus})^+ = \left(\nabla \mathfrak{t} \cdot (\mathbb{F}^+)^{-1}\right)^{\oplus}; \quad (\nabla \mathfrak{t}^{\oplus})^- = \left(\nabla \mathfrak{t} \cdot (\mathbb{F}^-)^{-1}\right)^{\oplus}. \quad (63)$$

### C. Variation of the Action and Rayleigh Functionals

We focus on the singularity  $S_s$  which we assumed to be fixed in  $B_s$ . We assume that all the physical fields introduced in our model are smooth out of this singularity. As for the test fields,  $\delta\phi_s = \mathbf{\Gamma}\xi$  they may be singular on a surface  $S_f^*$  fixed in  $B_f$ . Indeed, recall that  $\mathbf{\Gamma} = \nabla \mathcal{F}$  where  $\mathcal{F}$  is a time-independent  $C_{pw}^1$  diffeomorphism on  $B_f$ . Hence,  $S^* := \phi_s^{-1}(S_f^*)$  is a moving singularity surface in  $B_s$ ; we denote  $\mathbb{S}^* := \cup_{t \in (0, T)} S^*(t) \times \{t\}$ . Out of the singularity surfaces we have

$$\begin{aligned} \delta\epsilon &= \frac{1}{2} \delta \left( \mathbf{F}_s^T \cdot \mathbf{F}_s - \mathbf{I} \right) = \frac{1}{2} \left( \delta \mathbf{F}_s^T \cdot \mathbf{F}_s + \mathbf{F}_s^T \cdot \delta \mathbf{F}_s \right) = \left( \mathbf{F}_s^T \cdot \nabla (\delta \chi_s) \right)^{sym}, \\ \delta\eta_s &= 0, \quad \delta \mathbf{v}_s = \delta \dot{\chi}_s \end{aligned} \quad (64)$$

$$\delta I_s = \delta \left( \det(\nabla \phi_s) \right) = I_s \mathbf{G}_s^{-T} \cdot \nabla (\delta \phi_s); \quad (65)$$

$$\delta \eta_f^{\oplus} = \delta (\eta_f \circ \phi_s) = (\nabla \eta_f)^{\oplus} \cdot \delta \phi_s = \nabla \eta_f^{\oplus} \cdot \mathbf{G}_s^{-1} \cdot \delta \phi_s, \quad (66)$$

$$\delta m_f = \delta \left( I_s \eta_f^{\oplus} \right) = \eta_f^{\oplus} \delta I_s + I_s \delta \eta_f^{\oplus} = \text{div} \left( \eta_f^{\oplus} I_s \mathbf{G}_s^{-1} \cdot \delta \phi_s \right) = \text{div} \left( m_f \mathbf{G}_s^{-1} \cdot \delta \phi_s \right).$$

Recalling Eq. (18), the variation of the solid Lagrangian fluid velocity  $\mathbf{v}_f^{\oplus}$  is now computed

$$\begin{aligned} \delta \mathbf{v}_f^{\oplus} &= \delta \mathbf{v}_s - \delta \mathbf{F}_s \cdot \mathbf{G}_s^{-1} \cdot \mathbf{u}_s + \mathbf{F}_s \cdot \mathbf{G}_s^{-1} \cdot \delta \mathbf{G}_s \cdot \mathbf{G}_s^{-1} \cdot \mathbf{u}_s - \mathbf{F}_s \cdot \mathbf{G}_s^{-1} \cdot \delta \mathbf{u}_s, \\ \delta \mathbf{v}_f^{\oplus} &= \delta \dot{\chi}_s - \nabla (\delta \chi_s) \cdot \mathbf{G}_s^{-1} \cdot \mathbf{u}_s + \mathbf{F}_s \cdot \mathbf{G}_s^{-1} \cdot \nabla (\delta \phi_s) \cdot \mathbf{G}_s^{-1} \cdot \mathbf{u}_s - \mathbf{F}_s \cdot \mathbf{G}_s^{-1} \cdot \delta \dot{\phi}_s. \end{aligned}$$

Let us now perform the solid-Lagrangian variation  $\delta\mathcal{A}$  of the action functional. We have

$$\delta\Psi = \frac{\partial\Psi}{\partial\varepsilon} \delta\varepsilon + \frac{\partial\Psi}{\partial\chi_s} \delta\chi_s + \frac{\partial\Psi}{\partial m_f} \delta m_f = \frac{\partial\Psi}{\partial\varepsilon} \left( \mathbf{F}_s^T \cdot \nabla (\delta\chi_s) \right) + \frac{\partial\Psi}{\partial\chi_s} \cdot \delta\chi_s + \frac{\partial\Psi}{\partial m_f} \operatorname{div} (m_f \mathbf{G}_s^{-1} \cdot \delta\phi_s) \quad (67)$$

and

$$\delta\Lambda = \eta_s \mathbf{v}_s \cdot \delta\mathbf{v}_s + m_f \mathbf{v}_f^{\otimes} \cdot \delta\mathbf{v}_f^{\otimes} + \frac{1}{2} \mathbf{v}_s^2 \delta\eta_s + \frac{1}{2} \left( \mathbf{v}_f^{\otimes} \right)^2 \delta m_f,$$

which can be written  $\delta\Lambda = \delta\Lambda_s + \delta\Lambda_f$  with

$$\begin{aligned} \delta\Lambda_s &:= \left( \eta_s \mathbf{v}_s + m_f \mathbf{v}_f^{\otimes} \right) \cdot \delta\dot{\chi}_s - m_f \mathbf{v}_f^{\otimes} \cdot \nabla (\delta\chi_s) \cdot \mathbf{G}_s^{-1} \cdot \mathbf{u}_s, \\ \delta\Lambda_f &:= \frac{1}{2} \left( \mathbf{v}_f^{\otimes} \right)^2 \operatorname{div} (m_f \mathbf{G}_s^{-1} \cdot \delta\phi_s) + m_f \mathbf{v}_f^{\otimes} \cdot \mathbf{F}_s \cdot \mathbf{G}_s^{-1} \cdot \nabla (\delta\phi_s) \cdot \mathbf{G}_s^{-1} \cdot \mathbf{u}_s - m_f \mathbf{v}_f^{\otimes} \cdot \mathbf{F}_s \cdot \mathbf{G}_s^{-1} \cdot \delta\dot{\phi}_s. \end{aligned}$$

We first compute the variation of the potential energy by integrating Eq. (67). Integrating by parts in space we get

$$\begin{aligned} \int_{\mathbb{B}_s} \delta\Psi &= \int_{\mathbb{B}_s \setminus \mathbb{S}_s} \left[ \frac{\partial\Psi}{\partial\chi_s} - \operatorname{div} \left( \mathbf{F}_s \cdot \frac{\partial\Psi}{\partial\varepsilon} \right) \right] \cdot \delta\chi_s - \int_{\mathbb{B}_s \setminus (\mathbb{S}_s \cup \mathbb{S}^*)} \left[ m_f \mathbf{G}_s^{-T} \cdot \nabla \left( \frac{\partial\Psi}{\partial m_f} \right) \right] \cdot \delta\phi_s + \\ &\int_{\mathbb{S}_s} \left[ \mathbf{F}_s \cdot \frac{\partial\Psi}{\partial\varepsilon} \mathbf{N}_s \right] \cdot \delta\chi_s + \int_{\mathbb{S}_s \cup \mathbb{S}^*} \left[ m_f \mathbf{G}_s^{-T} \cdot \frac{\partial\Psi}{\partial m_f} \mathbf{N}_s \cdot \delta\phi_s \right]. \end{aligned}$$

We integrate separately the two parts of the variation of the kinetic energy. Integrating by parts in space and time the first term, recalling that  $\eta_s$  is constant in space and time, that  $\mathbf{D} = -m_f \mathbf{G}_s^{-1} \cdot \mathbf{u}_s$  and using expressions (18)-(19) for  $\mathbf{v}_f^{\otimes}$  and  $\dot{\mathbf{v}}_f^{\otimes}$ , the balance of mass (21) for  $\dot{m}_f$ , we have

$$\begin{aligned} \int_{\mathbb{B}_s} \delta\Lambda_s &= - \int_{\mathbb{B}_s \setminus \mathbb{S}_s} \left[ \eta_s \gamma_s + m_f \dot{\mathbf{v}}_f^{\otimes} + \dot{m}_f \mathbf{v}_f^{\otimes} + \operatorname{div} \left( \mathbf{v}_f^{\otimes} \otimes \mathbf{D} \right) \right] \cdot \delta\chi_s + \int_{\mathbb{S}_s} \left( \left[ \mathbf{v}_f^{\otimes} \otimes \mathbf{D} \right] \cdot \mathbf{N}_s \right) \cdot \delta\chi_s, \\ &= - \int_{\mathbb{B}_s \setminus \mathbb{S}_s} \left( \eta_s \gamma_s + m_f \gamma_f^{\otimes} \right) \cdot \delta\chi_s + \int_{\mathbb{S}_s} \left( \left[ \mathbf{v}_f^{\otimes} \otimes \mathbf{D} \right] \cdot \mathbf{N}_s \right) \cdot \delta\chi_s, \\ \int_{\mathbb{B}_s} \delta\Lambda_f &= \int_{\mathbb{B}_s \setminus (\mathbb{S}_s \cup \mathbb{S}^*)} \left[ -\frac{1}{2} m_f \nabla \left( \left( \mathbf{v}_f^{\otimes} \right)^2 \right) \cdot \mathbf{G}_s^{-1} + \operatorname{div} \left( \mathbf{v}_f^{\otimes} \cdot \mathbf{F}_s \cdot \mathbf{G}_s^{-1} \otimes \mathbf{D} \right) + \frac{\partial}{\partial t} \left( m_f \mathbf{v}_f^{\otimes} \cdot \mathbf{F}_s \cdot \mathbf{G}_s^{-1} \right) \right] \cdot \delta\phi_s \\ &\quad + \int_{\mathbb{S}_s \cup \mathbb{S}^*} \left[ \left( \frac{1}{2} m_f \left( \mathbf{v}_f^{\otimes} \right)^2 \mathbf{G}_s^{-T} \cdot \mathbf{N}_s - \mathbf{v}_f^{\otimes} \cdot \mathbf{F}_s \cdot \mathbf{G}_s^{-1} \otimes \mathbf{D} \cdot \mathbf{N}_s \right) \cdot \delta\phi_s \right] \\ &= \int_{\mathbb{B}_s \setminus (\mathbb{S}_s \cup \mathbb{S}^*)} \left( m_f \gamma_f^{\otimes} \cdot \mathbf{F}_s \cdot \mathbf{G}_s^{-1} \right) \cdot \delta\phi_s + \int_{\mathbb{S}_s \cup \mathbb{S}^*} \left[ \left( \frac{1}{2} m_f \mathbf{G}_s^{-T} \left( \mathbf{v}_f^{\otimes} \right)^2 \cdot \mathbf{N}_s - \mathbf{G}_s^{-T} \cdot \mathbf{F}_s^T \cdot \mathbf{v}_f^{\otimes} \otimes \mathbf{D} \cdot \mathbf{N}_s \right) \cdot \delta\phi_s \right]. \end{aligned}$$

where in the last equality we also used expression (20) for  $\partial (\mathbf{F}_s \cdot \mathbf{G}_s^{-1}) / \partial t$ .

We now compute the Rayleigh dissipation term. We start by recalling that, owing to (24), (25) and (2), (18), the pull back of the dissipation densities reads

$$\begin{aligned} \mathcal{D}^{\otimes} &= (\mathbf{F}_s \cdot \mathbf{G}_s^{-1} \cdot \mathbf{u}_s) \cdot \mathbf{K}^{\otimes} \cdot (\mathbf{F}_s \cdot \mathbf{G}_s^{-1} \cdot \mathbf{u}_s) + \left( \nabla \mathbf{v}_f^{\otimes} \cdot \mathbf{F}_s^{-1} \right) : \mathbf{M}^{\otimes} : \left( \nabla \mathbf{v}_f^{\otimes} \cdot \mathbf{F}_s^{-1} \right) \\ &\quad + \left( \nabla (\mathbf{F}_s \cdot \mathbf{G}_s^{-1} \cdot \mathbf{u}_s) \cdot \mathbf{F}_s^{-1} \right) : \mathbf{B}^{\otimes} : \left( \nabla (\mathbf{F}_s \cdot \mathbf{G}_s^{-1} \cdot \mathbf{u}_s) \cdot \mathbf{F}_s^{-1} \right), \\ \mathcal{D}_S^{\otimes} &= \left[ \left[ \mathbf{v}_f^{\otimes} \right] \right] \cdot \mathbf{S}^{\otimes} \cdot \left[ \left[ \mathbf{v}_f^{\otimes} \right] \right]. \end{aligned}$$

Using these expressions and then integrating by parts in space we get

$$\begin{aligned}
 \int_0^T \left( \frac{\partial \mathcal{R}}{\partial \mathbf{q}_t} \mid \delta \mathbf{q}_t \right) dt &= \int_{\mathbb{B}_s \setminus (\mathbb{S}_s \cup \mathbb{S}^*)} J_s(\mathbf{F}_s \cdot \mathbf{G}_s^{-1} \cdot \delta \phi_s) \cdot \kappa^{\otimes} + \int_{\mathbb{B}_s \setminus (\mathbb{S}_s \cup \mathbb{S}^*)} J_s(\nabla(\delta \chi_s - \mathbf{F}_s \cdot \mathbf{G}_s^{-1} \cdot \delta \phi_s) \cdot \mathbf{F}_s^{-1}) : \Pi_f^{\otimes} \\
 &+ \int_{\mathbb{B}_s \setminus (\mathbb{S}_s \cup \mathbb{S}^*)} J_s(\nabla(\mathbf{F}_s \cdot \mathbf{G}_s^{-1} \cdot \delta \phi_s) \cdot \mathbf{F}_s^{-1}) : \Pi^{\otimes} + \int_{(\mathbb{S}_s \cup \mathbb{S}^*)} \|J_s \mathbf{F}_s^{-T} \cdot \mathbf{N}_s\| [\delta \chi_s - \mathbf{F}_s \cdot \mathbf{G}_s^{-1} \cdot \delta \phi_s] \cdot \sigma^{\otimes} \\
 &= \int_{\mathbb{B}_s \setminus \mathbb{S}_s} -\operatorname{div} \left( J_s(\Pi_f^{\otimes})^T \cdot \mathbf{F}_s^{-T} \right) \cdot \delta \chi_s + \mathbf{G}_s^{-T} \cdot \mathbf{F}_s^T \cdot \left( J_s \kappa^{\otimes} - \operatorname{div} \left( J_s(\Pi^{\otimes} - \Pi_f^{\otimes})^T \cdot \mathbf{F}_s^{-T} \right) \right) \cdot \delta \phi_s \\
 &+ \int_{\mathbb{S}_s} \left[ \left[ J_s(\Pi_f^{\otimes})^T \cdot \mathbf{F}_s^{-T} \cdot \mathbf{N}_s \right] \cdot \delta \chi_s + \left[ \mathbf{G}_s^{-T} \cdot \mathbf{F}_s^T \cdot \left( J_s \left( (\Pi^{\otimes} - \Pi_f^{\otimes})^T \right) \cdot \mathbf{F}_s^{-T} \cdot \mathbf{N}_s - \|J_s \mathbf{F}_s^{-T} \cdot \mathbf{N}_s\| \sigma^{\otimes} \right) \cdot \delta \phi_s \right] \right].
 \end{aligned} \tag{68}$$

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