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PARTIAL COLLAPSING AND THE SPECTRUM OF THE HODGE-DE RHAM OPERATOR

COLETTE ANNÉ AND JUNYA TAKAHASHI

ABSTRACT. The goal of the present paper is to calculate the limit spectrum of the Hodge-de Rham operator under the perturbation of collapsing one part of a manifold obtained by gluing together two manifolds with the same boundary. It appears to take place in the general problem of blowing-up conical singularities as introduced in Mazzeo [Ma06] and Rowlett [Ro06, Ro08].

Résumé. Nous calculons la limite du spectre de l'opérateur de Hodge-de Rham sur les formes différentielles dans le cas d'effondrement d'une partie d'une variété. Ce calcul généralise le travail précédent sur les sommes connexes puisque la sphère qui sert de joint entre la partie stable et celle effondrée est remplacée par une sous-variété quelconque. Ce résultat apporte un nouvel éclairage aux questions de blowing up conical singularities introduites par Mazzeo [Ma06] et Rowlett [Ro06, Ro08].

1. INTRODUCTION.

This work takes place in the general context of the spectral studies of singular perturbations of the metrics, as a manner to know what are the topological or metrical meanings carried by the spectrum of geometric operators. We can mention in this direction, without exhaustivity, studies on the adiabatic limits ([MM90],[Ru00]), on collapsing ([F87], [Lo02a, Lo02b]), on resolution blowups of conical singularities ([Ma06],[Ro06, Ro08]) and on shrinking handles ([AC95, ACP09]).

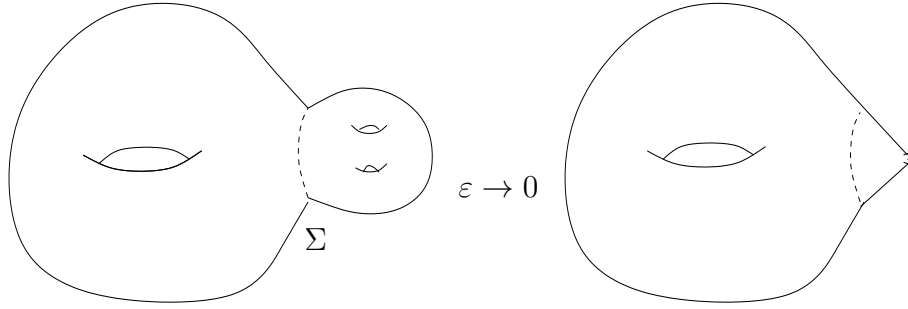
The present study can be considered as a generalization of the results of [AT09], where we studied the limit of the spectrum of the Hodge-de Rham (or the Hodge-Laplace) operator under collapsing of one part of a connected sum.

In our previous work, we restricted the submanifold Σ , used to glue the two parts, to be a sphere. In fact, this problem is quite related to resolution blowups of conical singularities: the point is to measure the influence of the topology of the part which disappears and of the conical singularity created at the limit of the 'big part'. If we look at the situation from the 'small part', we understand the importance of the *quasi-asymptotically conical space* obtained from rescaling the small part and gluing an infinite cone, see the definition in (1).

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fig:conn-sum

FIGURE 1. partial collapsing of M_ε

When $\Sigma = \mathbb{S}^n$, the conical singularity is quite simple, and there are no *semi-bounded states*, called extended solutions in the sequel, on the quasi-asymptotically conical space, our result presented here takes care of this new possibilities and gives a general answer to the problem studied by Mazzeo and Rowlett. Indeed, in [Ma06, Ro06, Ro08], it is supposed that the spectrum of the operator on the quasi-asymptotically conical space does not meet 0. Our study relaxes this hypothesis. It is done only with the Hodge-de Rham operator, but can easily be generalized.

Let us fix some notations.

1.1. Set up. Let M_1 and M_2 be two connected manifolds with the same boundary Σ , a compact manifold of dimension $n \geq 2$. We denote by $m = n + 1$ the dimension of M_1 and M_2 .

We endow Σ with a fixed metric h .

Let \overline{M}_1 be the manifold with conical singularity obtained from M_1 by gluing M_1 to a cone $\mathcal{C} = [0, 1) \times \Sigma \ni (r, y)$: *there exists on $\overline{M}_1 = M_1 \cup \mathcal{C}$ a metric \overline{g}_1 which writes, on the smooth part $r > 0$ of the cone as $dr^2 + r^2h$.*

We choose on M_2 a metric g_2 which is ‘trumpet like’, *i.e. M_2 is isomorphic near the boundary to $[0, \frac{1}{2}) \times \Sigma$ with the conical metric which writes $ds^2 + (1 - s)^2h$, if s is the coordinate defining the boundary by $s = 0$.*

For any ε , $0 \leq \varepsilon < 1$, we define

$$\mathcal{C}_\varepsilon = \{(r, y) \in \mathcal{C} \mid r > \varepsilon\} \quad \text{and} \quad M_1(\varepsilon) = M_1 \cup \mathcal{C}_\varepsilon.$$

The goal of the following calculus is to determine the limit spectrum of the Hodge-de Rham operator acting on the differential forms of the Riemannian manifold

$$M_\varepsilon = M_1(\varepsilon) \cup_{\varepsilon.\Sigma} \varepsilon.M_2$$

obtained by gluing together $(M_1(\varepsilon), g_1)$ and $(M_2, \varepsilon^2 g_2)$. We remark that, by construction, these two manifolds have isometric boundary and that the metric g_ε obtained on M_ε is smooth.

Remark 1. *The common boundary Σ of dimension n has some topological obstructions. In fact, since Σ is the boundary of oriented compact manifold M_1 , Σ is oriented cobordant to zero. So, by Thom’s cobordism theory, all the Stiefel-Whitney*

and all the Pontrjagin numbers vanish (cf. C. T. C. Wall [Wa60] or [MS74], §18, p.217). Furthermore, this condition is also sufficient, that is, the inverse does hold. Especially, it is impossible to take Σ^{4k} as the complex projective spaces $\mathbb{C}\mathbb{P}^{2k}$, ($k \geq 1$), because the Pontrjagin number $p_k(\mathbb{C}\mathbb{P}^{2k}) \neq 0$.

1.2. Results. We can describe the limit spectrum as follows: it has two parts. One comes from the big part, namely $\overline{M_1}$, and is exprimed by the spectrum of a good extension of the Hodge-de Rham operator on this manifold with conical singularities. This extension is self-adjoint and comes from an extension of the Gauß-Bonnet operators. All these extensions are classified by subspaces W of the total eigenspace corresponding to eigenvalues within $(-\frac{1}{2}, \frac{1}{2})$ of an operator A acting on the boundary Σ , this point is developed below in Section 2.2. The other part comes from the collapsing part, namely M_2 , where the limit Gauß-Bonnet operator is taken with boundary conditions of the Atiyah-Patodi-Singer type. This point is developed below in Section 2.3. This operator, denoted \mathcal{D}_2 in the sequel, can also be seen on the quasi-asymptotically conical space \widetilde{M}_2 already mentioned, namely

$$\widetilde{M}_2 = M_2 \cup \left([1, \infty) \times \Sigma \right) \tag{1}$$

M2tilde

with the metric $dr^2 + r^2h$ on the conical part. Only the eigenvalue zero is concerned with this part. The notion of *extended solutions* was introduced by Carron [Ca01a], these are solution of $\widetilde{D}_2(\varphi) = 0$ with certain growth at infinity. The details are given in section 2.3 below.

In fact, the manifolds M_ε has small eigenvalues, in the difference with [AT09], and the *multiplicity of 0* at the limit corresponds to the total eigenspace of these small, or null eigenvalues. Thus, our main theorem, which asserts the convergence of the spectrum, has two components.

[A] Theorem A. *If the limit value $\lambda \neq 0$, then it belongs to the positive spectrum of the Hodge-de Rham operator $\Delta_{1,W}$ on $\overline{M_1}$, with*

$$W \subset \bigoplus_{|\gamma| < \frac{1}{2}} \text{Ker}(A - \gamma)$$

is the space of the elements that generate extended solutions on M_2 . A precise definition is given below in (8).

[B] Theorem B. *The multiplicity of 0 in the limit spectrum is given by the sum*

$$\dim \text{Ker } \Delta_{1,W} + \dim \text{Ker } \mathcal{D}_2 + i_{\frac{1}{2}},$$

where $i_{\frac{1}{2}}$ denotes the dimension of the vector space $\mathcal{I}_{\frac{1}{2}}$, see (9), of extended solutions ω on \widetilde{M}_2 admitting on restriction to $r = 1$ a non-trivial component in $\text{Ker}(A - \frac{1}{2})$.

1.3. Comments. We choose a simple metric to make explicit computations. This fact is not a restriction, as already explained in [AT09], because of the result of Dodziuk [D82] which assures uniform control of the eigenvalues of geometric operators with regard to variations of the metric.

Examples are given in the last section of the present paper.

2. GAUSS-BONNET OPERATOR.

On a Riemannian manifold, the Gauß-Bonnet operator is defined as the operator $D = d + d^*$ acting on differential forms. It is symmetric and can have some closed extensions on manifolds with boundary or with conical singularities. We review these extensions in the cases involved in our study.

2.1. Gauß-Bonnet operator on M_ε . We recall that, on M_ε , a Gauß-Bonnet operator D_ε , Sobolev spaces and also a Hodge-de Rham operator Δ_ε can be defined as a general construction on any manifold $X = X_1 \cup X_2$, which is the union of two Riemannian manifolds with isometric boundaries (the details are given in [AC95]): if D_1 and D_2 are the Gauß-Bonnet “ $d + d^*$ ” operators acting on the differential forms of each part, the quadratic form

$$q(\varphi) = \int_{X_1} |D_1(\varphi \upharpoonright_{X_1})|^2 d\mu_{X_1} + \int_{X_2} |D_2(\varphi \upharpoonright_{X_2})|^2 d\mu_{X_2} \quad (2) \quad \boxed{\text{defq}}$$

is well-defined and closed on the domain

$$\mathcal{D}(q) = \{\varphi = (\varphi_1, \varphi_2) \in H^1(\Lambda T^* X_1) \times H^1(\Lambda T^* X_2), \varphi_1 \upharpoonright_{\partial X_1} \stackrel{L_2}{=} \varphi_2 \upharpoonright_{\partial X_2}\}$$

and on this space the total Gauß-Bonnet operator $D(\varphi) = (D_1(\varphi_1), D_2(\varphi_2))$ is defined and self-adjoint. For this definition, we have, in particular, to identify $(\Lambda T^* X_1) \upharpoonright_{\partial X_1}$ and $(\Lambda T^* X_2) \upharpoonright_{\partial X_2}$. This can be done by decomposing the forms in tangential and normal part (with inner normal), the equality above means then that the tangential parts are equal and the normal part opposite. This definition generalizes the definition in the smooth case.

The Hodge-de Rham operator $(d + d^*)^2$ of X is then defined as the operator obtained by the polarization of the quadratic form q . This gives compatibility conditions between φ_1 and φ_2 on the common boundary. We do not give details on these facts, because our manifold is smooth. But we shall use this presentation for the quadratic form.

GBconic

2.2. Gauß-Bonnet operator on \overline{M}_1 . Let $D_{1,\min}$ be the closure of the Gauß-Bonnet operator defined on the smooth forms with compact support in the smooth part $M_1(0)$. For any such form φ_1 , we write on the cone

$$\varphi_1 = dr \wedge r^{-(\frac{n}{2}-p+1)} \beta_{1,\varepsilon} + r^{-(\frac{n}{2}-p)} \alpha_{1,\varepsilon}$$

and define $\sigma_1 = (\beta_1, \alpha_1) = U(\varphi_1)$. The operator has, on the cone \mathcal{C} , the expression

$$UD_1U^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left(\partial_r + \frac{1}{r} A \right) \quad \text{with} \quad A = \begin{pmatrix} \frac{n}{2} - P & -D_0 \\ -D_0 & P - \frac{n}{2} \end{pmatrix},$$

where P is the operator of degree which multiplies by p per a p -form, and $D_0 = d_0 + d_0^*$ is the Gauß-Bonnet operator on the manifold (Σ, h) .

While the Hodge-de Rham operator has, in these coordinates, the expression

$$U\Delta_1U^* = -\partial_r^2 + \frac{1}{r^2} A(A+1). \quad (3) \quad \boxed{\text{laplace}}$$

The closed extensions of the operator $D_1 = d + d^*$ on the manifold with conical singularities \overline{M}_1 has been studied in [BS88] and [Le97]. They are classified by the spectrum of its *Mellin symbol*, which is here the operator with parameter $A + z$.

spectA

Spectrum of A. — The spectrum of A was calculated in Brüning and Seeley [BS88], p.703. By their result, the spectrum of A is given by the values

$$\begin{cases} \pm(p - \frac{n}{2}) \text{ with multiplicity } \dim H^p(\Sigma) \text{ and} \\ \pm\frac{1}{2} \pm \sqrt{\mu^2 + \left(\frac{n-1}{2} - p\right)^2}, \end{cases} \quad (4) \quad \text{vpA}$$

where p is any integer, $0 \leq p \leq n$ and μ^2 runs over the spectrum of the Hodge-de Rham operator on (Σ, h) acting on the coexact p -forms.

Indeed, looking at the Gauß-Bonnet operator acting on even forms, they identify even forms on the cone with the sections $(\varphi_0, \dots, \varphi_n)$ of the total bundle $T^*(\Sigma)$ by $\varphi_0 + \varphi_1 \wedge dr + \varphi_2 + \varphi_3 \wedge dr + \dots$. These sections can as well represent odd forms on the cone by $\varphi_0 \wedge dr + \varphi_1 + \varphi_2 \wedge dr + \varphi_3 + \dots$. With these identifications, they have to study the spectrum of the following operator acting on sections of $\Lambda T^*(\Sigma)$

$$S_0 = \begin{pmatrix} c_0 & d_0^* & 0 & \cdots & 0 \\ d_0 & c_1 & d_0^* & \ddots & \vdots \\ 0 & d_0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & c_{n-1} & d_0^* \\ 0 & \cdots & 0 & d_0 & c_n \end{pmatrix},$$

if $c_p = (-1)^{p+1}(p - \frac{n}{2})$. With the same identification, if we introduce the operator \tilde{S}_0 having the same formula but with on the diagonal the terms $\tilde{c}_p = (-1)^p(p - \frac{n}{2}) = -c_p$, the operator A can be written as

$$A = - \left(S_0 \oplus \tilde{S}_0 \right).$$

The expression of the spectrum of A is then a direct consequence of the computations of [BS88].

Closed extensions of D_1 . — If $\text{spec}(A) \cap (-\frac{1}{2}, \frac{1}{2}) = \emptyset$, then $D_{1,\max} = D_{1,\min}$. In particular, D_1 is essentially self-adjoint on the space of smooth forms with compact support *away* from the conical singularities.

Otherwise, the quotient $\text{Dom}(D_{1,\max}) / \text{Dom}(D_{1,\min})$ is isomorphic to

$$B := \bigoplus_{|\gamma| < \frac{1}{2}} \text{Ker}(A - \gamma).$$

More precisely, by Lemma 3.2 of [BS88], there exists a surjective linear map

$$\mathcal{L} : \text{Dom}(D_{1,\max}) \rightarrow B$$

with $\text{Ker } \mathcal{L} = \text{Dom}(D_{1,\min})$. Furthermore, we have the estimate

$$\|u(t) - t^{-A}\mathcal{L}(\varphi)\|_{L^2(\Sigma)}^2 \leq C(\varphi) |t \log t|$$

for $\varphi \in \text{Dom}(D_{1,\max})$ and $u = U(\varphi)$.

Now, for any subspace $W \subset B$, we can associate the operator $D_{1,W}$ with the domain $\text{Dom}(D_{1,W}) := \mathcal{L}^{-1}(W)$. As a result of [BS88], all closed extensions of $D_{1,\min}$ are obtained by this way. Remark that each $D_{1,W}$ defines a self-adjoint extension $\Delta_{1,W} = (D_{1,W})^* \circ D_{1,W}$ of the Hodge-de Rham operator, and, as a result, we have $(D_{1,W})^* = D_{1,\mathbb{I}(W^\perp)}$, where

$$\mathbb{I} = \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix}, \quad \text{i.e.} \quad \mathbb{I}(\beta, \alpha) = (\alpha, -\beta).$$

This extension is associated to the quadratic form $\varphi \mapsto \|D\varphi\|_{L^2}^2$ with the domain $\text{Dom}(D_{1,W})$.

Finally, we recall the results of Lesch [Le97]. The operators $D_{1,W}$, and in particular $D_{1,\min}$ and $D_{1,\max}$, are elliptic and satisfy the singular estimate (SE), see page 54 of [Le97], so by Proposition 1.4.6 and the compactness of \overline{M}_1 , they satisfy the *Rellich property*: the inclusion of $\text{Dom}(D_{1,W})$ into $L^2(\overline{M}_1, \overline{g}_1)$ is compact.

GBaps

2.3. Gauß-Bonnet operator on M_2 . We know, by the works of Carron [Ca01a, Ca01b], following Atiyah-Patodi-Singer [APS75], that the operator D_2 admits a closed extension \mathcal{D}_2 with the domain defined by the global boundary condition

$$\Pi_{\leq \frac{1}{2}} \circ U = 0,$$

if Π_I is the spectral projector of A relative to the interval I , and $\leq 1/2$ denotes the interval $(-\infty, 1/2]$. Moreover, this extension is elliptic in the sense that the H^1 -norm of elements of the domain is controlled by the norm of the graph. Indeed this boundary condition is related to a problem on a complete unbounded manifold as follows:

Let \widetilde{M}_2 denote the large manifold obtained from M_2 by gluing a conical cylinder $\mathcal{C}_{1,\infty} = [1, \infty) \times \Sigma$ with metric $dr^2 + r^2h$ and \widetilde{D}_2 its Gauß-Bonnet operator. A differential form on M_2 admits a harmonic L^2 extension on \widetilde{M}_2 precisely, when the restriction on the boundary satisfies $\Pi_{\leq \frac{1}{2}} \circ U = 0$.

Indeed, from the harmonicity, these L^2 -forms must satisfy $(\partial_r + \frac{1}{r}A)\sigma = 0$ or, if we decompose the form associated with the eigenspaces of A as $\sigma = \sum_{\gamma \in \text{Spec}(A)} \sigma^\gamma$, then the equation imposes that for all $\gamma \in \text{Spec}(A)$ there exists $\sigma_0^\gamma \in \text{Ker}(A - \gamma)$ such that $\sigma^\gamma = r^{-\gamma} \sigma_0^\gamma$. This expression is in $L^2(\mathcal{C}_{1,\infty})$ if and only if $\gamma > \frac{1}{2}$ or $\sigma_0^\gamma = 0$.

It will be convenient to introduce the harmonic L^2 extension operator

$$\begin{aligned} P_2 : \Pi_{> \frac{1}{2}} \left(H^{1/2}(\Sigma) \right) &\rightarrow L^2(\Lambda T^* \mathcal{C}_{1,\infty}) \\ \sigma = \sum_{\substack{\gamma \in \text{Spec}(A) \\ \gamma > \frac{1}{2}}} \sigma_\gamma &\mapsto P_2(\sigma) = U^* \left(\sum_{\substack{\gamma \in \text{Spec}(A) \\ \gamma > \frac{1}{2}}} r^{-\gamma} \sigma_\gamma \right). \end{aligned} \quad (5) \quad \boxed{\text{P2}}$$

This limit problem is of the category *non-parabolic at infinity* in the terminology of Carron, see particularly Theorem 2.2 of [Ca01a] and Proposition 5.1 of [Ca01b], then as a consequence of Theorem 0.4 of [Ca01a], we know that the kernel of \mathcal{D}_2 is of finite dimension and that the graph norm of the operator controls the H^1 -norm (Theorem 2.1 of [Ca01a]).

D2 **Proposition 2.** *There exists a constant $C > 0$ such that for each differential form $\varphi \in H^1(\Lambda T^*M_2)$ satisfying the boundary condition $\Pi_{\leq \frac{1}{2}} \circ U(\varphi) = 0$, then*

$$\|\varphi\|_{H^1(M_2)}^2 \leq C \left\{ \|\varphi\|_{L^2(M_2)}^2 + \|D_2\varphi\|_{L^2(M_2)}^2 \right\}.$$

As a consequence, the kernel of \mathcal{D}_2 , which is isomorphic to $\text{Ker } \widetilde{D}_2$, is of finite dimension and can be sent in the total space $\sum_p H^p(M_2)$ of the absolute cohomology.

A proof of this proposition can be obtained by the same way as Proposition 5 in [AT09].

Extended solutions. — Recall that Carron defined, for this type of operators, behind the L^2 solutions of $\widetilde{D}_2(\varphi) = 0$ which correspond to the solutions of the elliptic operator of Proposition 2, *extended solutions* which are included in the bigger space \mathcal{W} which is defined as the closure of the space of smooth p -forms with compact support $\Omega_0^p(\widetilde{M}_2)$ for the norm

$$\|\varphi\|_{\mathcal{W}}^2 := \|\varphi\|_{L^2(M_2)}^2 + \|D_2\varphi\|_{L^2(\widetilde{M}_2)}^2.$$

A Hardy-type inequality describes the growth at infinity of an extended solution.

extend **Lemma 3.** *For a function $v \in C_0^\infty(e, \infty)$ and a real number λ , we have*

- if $\lambda \neq -\frac{1}{2}$, $(\lambda + \frac{1}{2})^2 \int_e^\infty \frac{v^2}{r^2} dr \leq \int_e^\infty \frac{1}{r^{2\lambda}} |\partial_r(r^\lambda v)|^2 dr$,
- if $\lambda = -\frac{1}{2}$, $\int_e^\infty \frac{v^2}{|r \log r|^2 \log(\log r)} dr \leq \int_e^\infty r |\partial_r(r^{-\frac{1}{2}} v)|^2 dr$.

We remark now that, for $\varphi \in \Omega_0^p(\widetilde{M}_2)$ with support in the infinite cone $\mathcal{C}_{e, \infty}$, we can write

$$\begin{aligned} \|D_2\varphi\|_{L^2(\widetilde{M}_2)}^2 &= \sum_{\lambda \in \text{Spec}(A)} \int_e^\infty \|(\partial_r + \frac{\lambda}{r})\sigma_\lambda\|_{L^2(\Sigma)}^2 dr \\ &= \sum_{\lambda \in \text{Spec}(A)} \int_e^\infty \frac{1}{r^{2\lambda}} \|\partial_r(r^\lambda \sigma_\lambda)\|_{L^2(\Sigma)}^2 dr. \end{aligned}$$

Thus, by application of Lemma 3, we see that a solution of \widetilde{D}_2 , which must be $\sigma_\lambda(r) = r^{-\lambda}\sigma_\lambda(1)$ on the infinite cone, satisfies the condition of Lemma 3 if and only if

$$\forall \lambda < -\frac{1}{2}, \quad \sigma_\lambda(1) = 0.$$

Indeed, for $\lambda = -\frac{1}{2}$: if $v = r^{\frac{1}{2}}v_0$ for r large then the integral $\int \frac{v^2}{|r \log r|^2 \log(\log r)} dr$ is convergent, and for $\lambda \neq -\frac{1}{2}$, if $v = r^{-\lambda}v_0$, the requirement that $\frac{1}{r}v$ is in L^2 imposes that $v_0 = 0$ if $\lambda < -\frac{1}{2}$.

While the L^2 solutions correspond to the condition $\sigma_\lambda(1) = 0$ for any $\lambda \leq \frac{1}{2}$. As a consequence, the extended solutions which are not L^2 correspond to boundary terms with components in the total eigenspace related with eigenvalues of A in the interval

$[-\frac{1}{2}, \frac{1}{2}]$. In the case studied in [AT09], there do not exist such eigenvalues and we had not to take care of extended solutions.

More precisely, we must introduce the operator (see **2.a** in [Ca01b])

$$\begin{aligned} T : H^{k+\frac{1}{2}}(\Sigma) &\rightarrow H^{k-\frac{1}{2}}(\Sigma) \\ \sigma &\mapsto U \circ D_2(\mathcal{E}(\sigma)) \upharpoonright_{\Sigma}, \end{aligned} \quad (6) \quad \boxed{\text{T}}$$

where $\mathcal{E}(\sigma)$ is the solution of the Poisson problem

$$(D_2)^2(\mathcal{E}(\sigma)) = 0 \text{ on } M_2 \text{ and } U \circ \mathcal{E}(\sigma) \upharpoonright_{\Sigma} = \sigma \text{ on } \partial M_2.$$

In the same way one can define

$$\begin{aligned} T_{\mathcal{C}} : H^{k+\frac{1}{2}}(\Sigma) &\rightarrow H^{k-\frac{1}{2}}(\Sigma) \\ \sigma &\mapsto U \circ D_2(\tilde{\mathcal{E}}(\sigma)) \upharpoonright_{\Sigma}, \end{aligned} \quad (7) \quad \boxed{\text{TC}}$$

where $\tilde{\mathcal{E}}(\sigma)$ is the solution of the Poisson problem

$$(D_2)^2(\tilde{\mathcal{E}}(\sigma)) = 0 \text{ on } \mathcal{C}_{1,\infty} \text{ and } U \circ \mathcal{E}(\sigma) \upharpoonright_{\Sigma} = \sigma \text{ on } \Sigma.$$

Then $\text{Im } T_{\mathcal{C}} = \text{Im } \Pi_{>1/2}$ is a subspace of $\text{Ker } T_{\mathcal{C}} = \text{Im } \Pi_{\geq -1/2}$. Carron proved that this operator is continuous for $k \geq 0$. The L^2 solutions correspond to boundary values in $\text{Im}(T) \cap \text{Im}(\Pi_{>\frac{1}{2}})$, while extended solutions correspond to the space $\text{Ker}(T) \cap \text{Im}(\Pi_{\geq -\frac{1}{2}})$. Carron proved also that in the compact case, $\text{Ker}(T) = \text{Im}(T)$. We can now define the space W entering in theorem A :

$$W = \bigoplus_{|\gamma| < 1/2} W_{\gamma} \quad (8) \quad \boxed{\text{defW}}$$

where $W_{\gamma} = \{\varphi \in \text{Ker}(A - \gamma); \exists \eta \in \text{Im}(\Pi_{>\gamma}), T(\varphi + \eta) = 0\}$.

Let us denote by

$$\mathcal{I}_{\frac{1}{2}} := \left(\text{Ker}(T) \cap \text{Im}(\Pi_{\geq \frac{1}{2}}) \right) / \left(\text{Ker}(T) \cap \text{Im}(\Pi_{>\frac{1}{2}}) \right) \quad (9) \quad \boxed{\text{I1/2}}$$

the space of extended solutions with non-trivial component on $\text{Ker}(A - \frac{1}{2})$.

Proof of Lemma 3. Let $v \in C_0^\infty(e, \infty)$, with one integration by parts and application of the Cauchy-Schwarz inequality, we obtain, for $\lambda \neq -\frac{1}{2}$

$$\begin{aligned} \int_e^\infty \frac{v^2}{r^2} dr &= \int_e^\infty \frac{1}{r^{2\lambda+2}} |r^\lambda v|^2 dr = \int_e^\infty \partial_r \left\{ \frac{-1}{(2\lambda+1)r^{2\lambda+1}} \right\} |r^\lambda v|^2 dr \\ &= \int_e^\infty \left\{ \frac{1}{(2\lambda+1)r^{2\lambda+1}} \right\} 2(r^\lambda v) \partial_r(r^\lambda v) dr = \int_e^\infty \frac{2}{(2\lambda+1)r} v \cdot r^{-\lambda} \partial_r(r^\lambda v) dr \\ &\leq \frac{2}{|2\lambda+1|} \sqrt{\int_e^\infty \frac{v^2}{r^2} dr} \cdot \sqrt{\int_e^\infty |r^{-\lambda} \partial_r(r^\lambda v)|^2 dr}, \end{aligned}$$

which gives directly the first result of the lemma.

The second one is obtained in the same way:

$$\begin{aligned}
\int_e^\infty \frac{v^2}{r^2 \log^2 r} dr &= \int_e^\infty \left(\frac{v}{\sqrt{r}} \right)^2 \frac{1}{r \log^2 r} dr = \int_e^\infty \left(\frac{v}{\sqrt{r}} \right)^2 \partial_r \left(\frac{-1}{\log r} \right) dr \\
&= \int_e^\infty \frac{1}{\log r} \frac{2v}{\sqrt{r}} \partial_r \left(\frac{v}{\sqrt{r}} \right) dr = \int_e^\infty \frac{2v}{r \log r} \cdot \sqrt{r} \partial_r \left(\frac{v}{\sqrt{r}} \right) dr \\
&\leq 2 \sqrt{\int_e^\infty \frac{v^2}{r^2 \log^2 r} dr} \cdot \sqrt{\int_e^\infty \left| \sqrt{r} \partial_r \left(\frac{v}{\sqrt{r}} \right) \right|^2 dr}.
\end{aligned}$$

□

3. NOTATIONS AND TOOLS.

As in [ACP09], we use, to write the forms of M_ε the following change of variables : let q_ε be the quadratic form defined on M_ε by the formula (2), a form $\varphi \in \mathcal{D}(q_\varepsilon)$, defines

$$\varphi_1 := \varphi_\varepsilon \upharpoonright_{M_1(\varepsilon)} \quad \text{and} \quad \varphi_2 := \varepsilon^{\frac{m}{2}-p} \varphi_\varepsilon \upharpoonright_{M_2}.$$

We write on the conical part of $M_1(\varepsilon)$

$$\varphi_1 = dr \wedge r^{-(\frac{n}{2}-p+1)} \beta_1 + r^{-(\frac{n}{2}-p)} \alpha_1$$

and define $\sigma_1 = (\beta_1, \alpha_1) = U(\varphi_1)$. This formula defines an isometry U .

On M_2 , it is more convenient to define $r = 1 - s$ for $s \in [0, 1/2]$ representing the distance to the boundary. We write $\varphi_2 = (dr \wedge r^{-(\frac{n}{2}-p+1)} \beta_{2,\varepsilon} + r^{-(\frac{n}{2}-p)} \alpha_{2,\varepsilon})$ near the boundary. Then we can define, for $r \in [\frac{1}{2}, 1]$ (the boundary of M_2 corresponds to $r = 1$)

$$\sigma_2(r) = (\beta_2(r), \alpha_2(r)) = U(\varphi_2)(r).$$

The L^2 -norm, for a form supported on M_1 in the cone $\mathcal{C}_{\varepsilon,1}$, has the expression

$$\|\varphi\|_{L^2(M_\varepsilon)}^2 = \int_{M_1} |\sigma_1|^2 d\mu_{g_1} + \int_{M_2} |\varphi_2|^2 d\mu_{g_2}$$

and the quadratic form on our study is

$$\begin{aligned}
q(\varphi) &= \int_{M_\varepsilon} |(d + d^*)\varphi|_{g_\varepsilon}^2 d\mu_{g_\varepsilon} \\
&= \int_{M_1(\varepsilon)} |UD_1U^*(\sigma_1)|^2 d\mu_{g_1} + \frac{1}{\varepsilon^2} \int_{M_2} |D_2(\varphi_2)|^2 d\mu_{g_2}.
\end{aligned} \tag{10} \quad \boxed{\text{quadform}}$$

The compatibility condition is, for the quadratic form, $\varepsilon^{\frac{1}{2}}\alpha_1(\varepsilon) = \alpha_2(1)$ and $\varepsilon^{\frac{1}{2}}\beta_1(\varepsilon) = \beta_2(1)$, or

$$\sigma_2(1) = \varepsilon^{\frac{1}{2}}\sigma_1(\varepsilon). \tag{11} \quad \boxed{\text{recol0}}$$

The compatibility condition for the Hodge-de Rham operator, of first order, is obtained by expressing that $D\varphi \sim (UD_1U^*\sigma_1, \frac{1}{\varepsilon}UD_2U^*\sigma_2)$ belongs to the domain of D . In terms of σ , it gives

$$\sigma'_2(1) = \varepsilon^{\frac{3}{2}}\sigma'_1(\varepsilon). \tag{12} \quad \boxed{\text{recol1}}$$

3.1. Expression of the quadratic form. For any φ such that the component φ_1 is supported in the cone $\mathcal{C}_{\varepsilon,1}$, one has, with $\sigma_1 = U(\varphi_1)$ and by the same calculus as in [ACP09] :

$$\begin{aligned} \int_{\mathcal{C}_{\varepsilon,1}} |D_1\varphi|^2 d\mu_{g_\varepsilon} &= \int_\varepsilon^1 \left\| \left(\partial_r + \frac{1}{r}A \right) \sigma_1 \right\|_{L^2(\Sigma)}^2 dr \\ &= \int_\varepsilon^1 \left[\|\sigma_1'\|_{L^2(\Sigma)}^2 + \frac{2}{r} (\sigma_1', A\sigma_1)_{L^2(\Sigma)} + \frac{1}{r^2} \|A\sigma_1\|_{L^2(\Sigma)}^2 \right] dr. \end{aligned}$$

limprob

3.2. Limit problem. As Hilbert space, we introduce

$$\mathcal{H}_\infty := L^2(\overline{M}_1) \oplus \text{Ker } \tilde{D}_2 \oplus \mathcal{I}_{\frac{1}{2}} \quad (13) \quad \text{limitpb}$$

with the space $\mathcal{I}_{1/2}$ defined in (9), and as limit operator

$$\Delta_{1,W} \oplus 0 \oplus 0$$

with W defined in (8). Let us denote by λ_N ($N \geq 1$) its spectrum and also let us $\lambda_N(\varepsilon)$ ($N \geq 1$) be the total spectrum of the Hodge-de Rham operator on M_ε .

Finally, let us define

- a cut-off function ξ_1 on M_1 around the conical singularity:

$$\xi_1(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{1}{2}, \\ 0 & \text{if } 1 \leq r. \end{cases} \quad (14) \quad \text{coupure}$$

- the prolongation operator

$$\begin{aligned} P_\varepsilon : \Pi_{>-\frac{1}{2}} \left(H^{\frac{1}{2}}(\Sigma) \right) &\rightarrow H^1(\mathcal{C}_{\varepsilon,1}) \quad (15) \quad \text{Peps} \\ \sigma &= \sum_{\substack{\gamma \in \text{Spec}(A) \\ \gamma > -\frac{1}{2}}} \sigma_\gamma \mapsto P_\varepsilon(\sigma) = U^* \left(\sum_{\substack{\gamma \in \text{Spec}(A) \\ \gamma > -\frac{1}{2}}} \varepsilon^{\gamma-\frac{1}{2}} r^{-\gamma} \sigma_\gamma \right). \end{aligned}$$

We remark that, if $\sigma \in \Pi_{>-\frac{1}{2}} \left(H^{\frac{1}{2}}(\Sigma) \right)$ is the restriction of some $\psi_2 \in H^1(\Lambda T^* M_2)$, then $(\xi_1 P_\varepsilon(U(\psi_2 \upharpoonright_\Sigma)), \psi_2)$ defines an element of $H^1(M_\varepsilon)$.

Moreover, for $\sigma \in \Pi_{>\frac{1}{2}} \left(H^{\frac{1}{2}}(\Sigma) \right)$ the element $P_\varepsilon(\sigma)$ is the transplanted on M_1 of $P_2(\sigma)$ (see 2.3), then there exists a constant $C > 0$ such that

$$\begin{aligned} \forall \sigma \in \Pi_{>\frac{1}{2}} \left(H^{\frac{1}{2}}(\Sigma) \right) \\ \|P_\varepsilon(\sigma)\|_{L^2(\mathcal{C}_{\varepsilon,1})}^2 = \|P_2(\sigma)\|_{L^2(\mathcal{C}_{1,1/\varepsilon})}^2 \leq C \|\Pi_{>\frac{1}{2}} \sigma_2(1)\|_{H^{\frac{1}{2}}(\Sigma)}^2. \quad (16) \quad \text{contL2} \end{aligned}$$

4. PROOF OF THE SPECTRAL CONVERGENCE.

We denote by $\lambda_k(\varepsilon)$, $k \geq 1$ the spectrum of the total Hodge-Laplace operator of M_ε and by λ_k , $k \geq 1$ the spectrum of the limit operator defined in the section 3.2.

upper

4.1. **Upper bound** : $\limsup \lambda_N(\varepsilon) \leq \lambda_N$. With the min-max formula, which says that

$$\lambda_N(\varepsilon) = \inf_{\substack{E \subset \text{Dom}(D_\varepsilon) \\ \dim E = N}} \left\{ \sup_{\substack{\varphi \in E \\ \|\varphi\|=1}} \int_{M_\varepsilon} |D_\varepsilon \varphi|^2 d\mu_{g_\varepsilon} \right\},$$

we have to describe how transplanting eigenforms of the limit problem on M_ε .

We describe this transplantation term by term. For the first term, we use the same ideas as in [ACP09].

For an eigenform φ of $\Delta_{1,W}$ corresponding to the eigenvalue λ , $U(\varphi)$ can be decomposed on an orthonormal base σ_γ of eigenforms of A and each component can be expressed by the Bessel functions. For $\gamma \in (-1/2, 1/2)$ it has the form $cr^{\gamma+1}F(\lambda r^2) + dr^{-\gamma}G(\lambda r^2)$ for F, G entire functions satisfying $F(0) = G(0) = 1$ and c, d constants.

We remark that $cr^{\gamma+1}F(\lambda r^2) \in \text{Dom}(D_{1,\min})$ and also that $dr^{-\gamma}(G(\lambda r^2) - G(0)) \in \text{Dom}(D_{1,\min})$, so we can write $\varphi = \varphi_0 + \bar{\varphi}$ with

$$\begin{aligned} \varphi_0 &\in \text{Dom}(D_{1,\min}) \quad \text{and} \\ U(\bar{\varphi})(r) &= \xi_1(r) \sum_{\substack{\gamma \in \text{Spec}(A) \\ |\gamma| < \frac{1}{2}}} (d_\gamma r^{-\gamma} \sigma_\gamma). \end{aligned}$$

By the definition of $D_{1,\min}$, φ_0 can be approached, with the operator norm, by a sequence of smooth forms $\varphi_{0,\varepsilon}$ with compact support in $M_1(\varepsilon)$.

By definition of W we know that $\sum_{\substack{\gamma \in \text{Spec}(A) \\ |\gamma| < \frac{1}{2}}} d_\gamma \sigma_\gamma \in W \Rightarrow$ there exist $\varphi_{2,\gamma} \in \text{Ker } D_2$

such that $U\varphi_{2,\gamma}(1) - d_\gamma \sigma_\gamma \in \text{Im}(\Pi_{>\gamma})$. We remark finally that, by the definition (15), we can write $U(\bar{\varphi})(r) = \xi_1(r) \sum_{\substack{\gamma \in \text{Spec}(A) \\ |\gamma| < \frac{1}{2}}} \varepsilon^{1/2-\gamma} P_\varepsilon(d_\gamma \sigma_\gamma)$.

Let $\varphi_{2,\varepsilon} = \sum_{|\gamma| < 1/2} \varepsilon^{1/2-\gamma} \varphi_{2,\gamma}$ and

$$\varphi_\varepsilon = (\varphi_{0,\varepsilon} + \xi_1 P_\varepsilon \left(\sum_{\substack{\gamma \in \text{Spec}(A) \\ |\gamma| < \frac{1}{2}}} \varepsilon^{1/2-\gamma} U\varphi_{2,\gamma}(1) \right), \varphi_{2,\varepsilon}) \in H^1(M_\varepsilon).$$

It is a good transplantation: $\|\varphi_{2,\varepsilon}\| \rightarrow 0$ as the term added on M_1 (indeed, a term of the sum $\xi_1 P_\varepsilon(\varepsilon^{1/2-\gamma}(U\varphi_{2,\gamma}(1) - d_\gamma \sigma_\gamma))$ corresponds to some $\gamma' > \gamma$, if $\gamma' > 1/2$, by (16), it is $O(\varepsilon^{1/2-\gamma'})$, if $\gamma' < 1/2$ it is $O(\varepsilon^{\gamma'-\gamma})$ and if $\gamma' = 1/2$ it is $O(\varepsilon^{1/2-\gamma} \sqrt{|\log \varepsilon|})$. Moreover they are harmonic, up to ξ_1 .

For the two last ones, we shrink the infinite cone on M_1 and cut with the function ξ_1 , already defined in (14).

Finally, if $\text{Ker}(A - \frac{1}{2})$ is not empty, for each $\bar{\sigma}^{\frac{1}{2}} \in \text{Ker}(A - \frac{1}{2})$ such that there exists ψ_2 with $D_2(\psi_2) = 0$ on M_2 and which boundary value gives (through U) $\bar{\sigma}^{\frac{1}{2}}$ modulo $\text{Im } \Pi_{>\frac{1}{2}}$, one can construct a *pseudomode* as follows:

$$\psi_\varepsilon := |\log \varepsilon|^{-\frac{1}{2}} \left(\xi_1 \cdot (r^{-\frac{1}{2}} U^*(\bar{\sigma}^{\frac{1}{2}}) + P_\varepsilon(U(\psi_2) \upharpoonright_\Sigma - \bar{\sigma}^{\frac{1}{2}})), \psi_2 \right) \quad (17) \quad \boxed{1/2}$$

The L^2 norm of this element is uniformly bounded from above and below, and

$$\lim_{\varepsilon \rightarrow 0} \|\psi_\varepsilon\|_{L^2(M_\varepsilon)} = \|\bar{\sigma}^{\frac{1}{2}}\|_{L^2(\Sigma)}.$$

Moreover, it satisfies $q(\psi_\varepsilon) = O(|\log \varepsilon|^{-1})$ giving then a ‘small eigenvalue’, as well as the elements of $\text{Ker } \mathcal{D}_2$ and of $\text{Ker } \Delta_W$.

[*nb.* It is remarkable that the same construction, for an extended solution with corresponding boundary value in $\text{Ker}(A - \gamma)$, $\gamma \in (-\frac{1}{2}, \frac{1}{2})$ does not give a quasimode: indeed if ψ_2 is such a solution, the transplanted element will be,

$$\psi_\varepsilon = \left(\xi_1 \cdot (r^{-\gamma} U^*(\bar{\sigma}^\gamma) + \varepsilon^{\frac{1}{2}-\gamma}) P_\varepsilon(U(\psi_2 \upharpoonright_\Sigma) - \bar{\sigma}^\gamma), \varepsilon^{\frac{1}{2}-\gamma} \psi_2 \right)$$

for which $q(\psi_\varepsilon)$ does not converge to 0 as ε .]

To conclude the estimate of upper bounds, we have only to verify that this transplanted forms have a Rayleigh-Ritz quotient comparable to the initial one and that the orthogonality is fast conserved by transplantation.

4.2. Lower bound : $\liminf \lambda_N(\varepsilon) \geq \lambda_N$. We first proceed for one indice. We know, by the paragraph 4.1, that for each N , the family $\{\lambda_N(\varepsilon)\}_{\varepsilon > 0}$ is bounded, set

$$\lambda := \liminf_{\varepsilon \rightarrow 0} \lambda_N(\varepsilon).$$

There exists a sequence $\varepsilon_m, m \in \mathbb{N}$ such that $\lim_{m \rightarrow \infty} \lambda_N(\varepsilon_m) = \lambda$. Let, for each m , φ_m be a normed eigenform relative to $\lambda_m = \lambda_N(\varepsilon_m)$.

4.2.1. *On the regular part of \mathbf{M}_1 .*

10 **Lemma 4.** *For our given family φ_m the family $\{(1 - \xi_1) \cdot \varphi_{1,m}\}_{m \in \mathbb{N}}$ is bounded in $H^1(M_1, g_1)$.*

Then it remains to study $\xi_1 \cdot \varphi_{1,m}$ which can be expressed with the polar coordinates. We remark that the quadratic form of these forms is uniformly bounded.

4.2.2. *Estimates of the boundary term.* The expression of the quadratic form can be decomposed with respect to the eigenspaces of A ; in the following calculus, we suppose that $\sigma_1(1) = 0$:

$$\begin{aligned} & \int_\varepsilon^1 \left[\|\sigma'_1\|_{L^2(\Sigma)}^2 + \frac{2}{r} (\sigma'_1, A\sigma_1)_{L^2(\Sigma)} + \frac{1}{r^2} \|A\sigma_1\|_{L^2(\Sigma)}^2 \right] dr \\ &= \int_\varepsilon^1 \left[\|\sigma'_1\|_{L^2(\Sigma)}^2 + \partial_r \left(\frac{1}{r} (\sigma_1, A\sigma_1)_{L^2(\Sigma)} \right) + \frac{1}{r^2} \left\{ (\sigma_1, A\sigma_1)_{L^2(\Sigma)} + \|A\sigma_1\|_{L^2(\Sigma)}^2 \right\} \right] dr \\ &= \int_\varepsilon^1 \left[\|\sigma'_1\|_{L^2(\Sigma)}^2 + \frac{1}{r^2} (\sigma_1, (A + A^2)\sigma_1)_{L^2(\Sigma)} \right] dr - \frac{1}{\varepsilon} (\sigma_1(\varepsilon), A\sigma_1(\varepsilon))_{L^2(\Sigma)}. \end{aligned}$$

This shows that the quadratic form controls the boundary term, if the operator A is negative but $(A + A^2)$ is non-negative. The latter condition is satisfied exactly on the orthogonal complement of the spectral space corresponding to the interval $(-1, 0)$. By applying $\xi_1 \cdot \varphi_{1,m}$ to this fact, we obtain the following lemma:

[11] Lemma 5. *Let $\Pi_{\leq -1}$ be the spectral projection of the operator A relative to the interval $(-\infty, -1]$. There exists a constant $C > 0$ such that, for any $m \in \mathbb{N}$*

$$\|\Pi_{\leq -1} \circ U(\varphi_{1,m}(\varepsilon_m))\|_{H^{1/2}(\Sigma)} \leq C\sqrt{\varepsilon_m}.$$

In view of Proposition 2, we want also a control of the components of σ_1 associated with the eigenvalues of A in $(-1, 1/2]$. The number of these components is finite and we can work term by term. So we write, on $\mathcal{C}_{\varepsilon_m,1}$,

$$\sigma_1(r) = \sum_{\gamma \in \text{Spec}(A)} \sigma_1^\gamma(r) \text{ with } A\sigma_1^\gamma(r) = \gamma\sigma_1^\gamma(r)$$

and we suppose again $\sigma_1(1) = 0$. From the equation $(\partial_r + A/r)\sigma_1^\gamma = r^{-\gamma}\partial_r(r^\gamma\sigma_1^\gamma)$ and the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} \|\varepsilon^\gamma\sigma_1^\gamma(\varepsilon)\|_{L^2(\Sigma)}^2 &\leq \left\{ \int_\varepsilon^1 \|\partial_r(r^\gamma\sigma_1^\gamma)\|_{L^2(\Sigma)} dr \right\}^2 \\ &\leq \left\{ \int_\varepsilon^1 \left\| r^\gamma \cdot \left(\partial_r + \frac{1}{r}A \right) \sigma_1^\gamma(r) \right\|_{L^2(\Sigma)} dr \right\}^2 \\ &\leq \int_\varepsilon^1 r^{2\gamma} dr \cdot \int_\varepsilon^1 \left\| \partial_r(\sigma_1^\gamma) + \frac{\gamma}{r}(\sigma_1^\gamma) \right\|_{L^2(\Sigma)}^2 dr. \end{aligned}$$

Thus, if the quadratic form is bounded, there exists a constant $C > 0$ such that

$$\|\sigma_1^\gamma(\varepsilon)\|_{L^2(\Sigma)}^2 \leq \begin{cases} C\varepsilon^{-2\gamma} \frac{1 - \varepsilon^{2\gamma+1}}{2\gamma + 1} & \text{if } \gamma \neq -\frac{1}{2} \\ C\varepsilon |\log \varepsilon| & \text{if } \gamma = -\frac{1}{2}. \end{cases} \quad (18)$$

This gives

[12] Lemma 6. *Let Π_I be the spectral projector of the operator A relative to the interval I . There exist constants $\alpha, C > 0$ such that, for any $m \in \mathbb{N}$*

$$\|\Pi_{(-1,0)} \circ U(\varphi_{1,m}(\varepsilon_m))\|_{H^{1/2}(\Sigma)} \leq C\varepsilon_m^\alpha.$$

If $0 < \alpha < \frac{1}{2}$, then $(-\alpha)$ is bigger than any negative eigenvalue of A . With the compatibility condition (11), the estimate above gives

[13] Lemma 7. *With the same notation, there exist constants $\beta, C > 0$ such that, for any $m \in \mathbb{N}$*

$$\|\Pi_{[0, \frac{1}{2})} \circ U(\varphi_{2,\varepsilon_m}(1))\|_{H^{1/2}(\Sigma)} \leq C\varepsilon_m^\beta.$$

Here, $(\frac{1}{2} - \beta)$ is the biggest non-negative eigenvalue of A strictly smaller than $\frac{1}{2}$ (if there is no such eigenvalue, we put $\beta = \frac{1}{2}$).

Finally, we study $\sigma_1^{\frac{1}{2}}$ for our family of forms (the parameter ε is omitted in the notation). It satisfies, for $\varepsilon_m < r < \frac{1}{2}$, the equation

$$\left(-\partial_r^2 + \frac{3}{4r^2} \right) \sigma_1^{\frac{1}{2}} = \lambda_\varepsilon \sigma_1^{\frac{1}{2}}.$$

The solutions of this equation have expression in term of the Bessel functions: there exist entire functions F, G with $F(0) = G(0) = 1$ and differential forms $c_\varepsilon, d_\varepsilon$ in $\text{Ker}(A - \frac{1}{2})$ such that

$$\sigma_1^{\frac{1}{2}}(r) = c_\varepsilon r^{\frac{3}{2}} F(\lambda_\varepsilon r^2) + d_\varepsilon \left(r^{-\frac{1}{2}} G(\lambda_\varepsilon r^2) + \frac{2}{\pi} \log(r) r^{\frac{3}{2}} F(\lambda_\varepsilon r^2) \right). \quad (19)$$

sigma1/2

The fact that the L^2 -norm is bounded gives that $c_\varepsilon^2 + |\log \varepsilon| d_\varepsilon^2$ is bounded and finally, reporting this estimate in the expression above, that

$$\|\sigma_1^{\frac{1}{2}}(\varepsilon)\|_{L^2(\Sigma)}^2 = O\left(\frac{1}{\sqrt{|\log \varepsilon|}}\right).$$

This gives on the other part

13b **Lemma 8.** *There exists a constant $C > 0$ such that, for any $m \in \mathbb{N}$*

$$\|\Pi_{\{\frac{1}{2}\}} \circ U(\varphi_{2,\varepsilon_m})(1)\|_{H^{1/2}(\Sigma)} \leq C \frac{1}{\sqrt{|\log \varepsilon_m|}}.$$

onM2

4.2.3. *Convergence of $\varphi_{2,\varepsilon_m}$.* Let's now define $\tilde{\varphi}_{2,\varepsilon}$ to be the form obtained by prolongation of $\varphi_{2,\varepsilon}$ by $\sqrt{\varepsilon} \xi_1(\varepsilon r) \varphi_{1,\varepsilon}(\varepsilon r)$ on the infinite cone $\mathcal{C}_{1,\infty}$. A change of variables gives that

$$\|\tilde{\varphi}_{2,\varepsilon}\|_{L^2(\mathcal{C}_{1,\infty})} = \|\xi_1 \varphi_{1,\varepsilon}\|_{L^2(\mathcal{C}_{\varepsilon,1})}$$

while

$$\int |\tilde{D}_2(\tilde{\varphi}_{2,\varepsilon})|^2 = \varepsilon^2 \int_{\mathcal{C}_{\varepsilon,1}} |D_1(\xi_1 \varphi_{1,\varepsilon})|^2 + \int_{M_2} |D_2(\varphi_{2,\varepsilon})|^2.$$

Thus, by hypothesis on φ_m , the family $(\tilde{\varphi}_{2,m})_{m \in \mathbb{N}}$ is bounded in the space \mathcal{W} and $\int |\tilde{D}_2(\tilde{\varphi}_{2,m})|^2 = O(\varepsilon_m^2)$. The works of Carron give us that $\|\tilde{\varphi}_{2,m}(1)\|_{H^{1/2}(\Sigma)}$ is bounded and the following

inkerD2

Proposition 9. *There exists a subfamily of the family $(\tilde{\varphi}_{2,m})_{m \in \mathbb{N}}$ which converge in $L^2(M_2)$, the limit $\tilde{\varphi}_2$ defines an extended solution on \tilde{M}_2 , ie. $\tilde{D}_2(\tilde{\varphi}_2) = 0$ and $(\tilde{\varphi}_2)|_\Sigma \in \text{Ker}(T) \cap \text{Im}(\Pi_{\geq -\frac{1}{2}})$.*

We still denote $\tilde{\varphi}_{2,m}$ the subfamily obtained.

4.2.4. *Convergence near the singularity.* Now we use the fact that we deal with eigenforms, they satisfy an equation which imposes a local form. We concentrate on $\gamma \in [-1/2, 1/2]$. If we write

$$\varphi_{1,m}^{[-\frac{1}{2}, \frac{1}{2}]} = \sum_{\gamma \in [-\frac{1}{2}, \frac{1}{2}]} U^* \sigma_1^\gamma(r)$$

the terms σ_1^γ satisfy the equations

$$\left(-\partial_r^2 + \frac{\gamma(1+\gamma)}{r^2} \right) \sigma_1^\gamma = \lambda_\varepsilon \sigma_1^\gamma.$$

The solutions of this equation have expression in term of the Bessel functions: there exist entire functions F, G with $F(0) = G(0) = 1$ and differential forms $c_{\gamma,m}, d_{\gamma,m}$ in

$\text{Ker}(A - \gamma)$ such that

$$\sigma_1^\gamma(r) = \begin{cases} c_{\gamma,m} r^{\gamma+1} F_\gamma(\lambda_m r^2) + d_{\gamma,m} \left(r^{-\gamma} G_\gamma(\lambda_m r^2) \right) & \text{if } |\gamma| < 1/2 \\ c_{1/2,m} r^{\frac{3}{2}} F_{1/2}(\lambda_m r^2) + d_{1/2,m} \left(r^{-\frac{1}{2}} G_{1/2}(\lambda_m r^2) + \frac{2}{\pi} \log(r) r^{\frac{3}{2}} F_{1/2}(\lambda_m r^2) \right) \\ c_{-1/2,m} r^{\frac{1}{2}} F_{-1/2}(\lambda_m r^2) + d_{-1/2,m} \left(r^{\frac{1}{2}} \log(r) G_{-1/2}(\lambda_m r^2) \right) \end{cases} \quad (20)$$

sigma-gamma

The lemmata of the precedent section gives us the result that the families $c_{\gamma,m}$ and $d_{\gamma,m}$ are bounded and by extraction we can suppose they converge. In the case where $\gamma = 1/2$ we have more : $|d_{1/2,m}| = O(|\log \varepsilon_m|^{-1/2})$.

But we know also, turning back to the family of the last proposition, that the family $\sqrt{\varepsilon_m} \xi_1(\varepsilon_m r) \varphi_{1,m}(\varepsilon_m r)$ converges on any sector $1 \leq r \leq R$ the explicite form of $\sigma_1^\gamma(r)$ gives that this limit is 0. *As a consequence, the form $\tilde{\varphi}_2$ has no component for $\gamma \in [-1/2, 1/2]$ and is indeed a L^2 solution.* We have proved

phi2

Proposition 10. *The form $\tilde{\varphi}_2$ of the proposition 9 has no component for $\gamma \in [-1/2, 1/2]$. Let $\varphi_2 = \tilde{\varphi}_2 \upharpoonright_{M_2}$, there exists a subfamily of the family $\{\varphi_{2,m}\}_m$ which converges, as $m \rightarrow \infty$ to φ_2 and it satisfies*

$$\varphi_2 \in \text{Dom } \mathcal{D}_2, \quad \|\varphi_2\|_{L^2(M_2, g_2)} \leq 1 \text{ and } D_2(\varphi_2) = 0.$$

Moreover, the harmonic prolongation of $\sqrt{\varepsilon_m} \varphi_{1,m}(\varepsilon_m)$

$$\bar{\varphi}_{2,m} = \mathcal{E}(\sqrt{\varepsilon_m} \varphi_{1,m}(\varepsilon_m))$$

minimizes the norm of $D_2(\varphi_2)$, as a consequence

$$\|D_2(\bar{\varphi}_{2,m})\|_{L^2(M_2)} = O(\varepsilon_m) \Rightarrow \|T(\sqrt{\varepsilon_m} U \varphi_{1,m}(\varepsilon_m))\|_{H^{-1/2}(\Sigma)} = O(\varepsilon_m)$$

with the operator T defined in (6).

But, by Lemma 5 and 6 we know that $\|\Pi_{<-1/2}(U \varphi_{1,\varepsilon}(\varepsilon))\|_{H^{1/2}(\Sigma)} = O(\sqrt{\varepsilon})$. The continuity of T gives hence $\|T \circ \Pi_{\geq -1/2} \circ U(\varphi_{1,m}(\varepsilon_m))\|_{H^{-1/2}(\Sigma)} = O(\sqrt{\varepsilon_m})$. To use this result as a consequence on $\Pi_{[-1/2, 1/2]} \circ U(\varphi_{1,m}(\varepsilon_m))$ we must make sens to the possibility of working modulo $\text{Im}(T)$.

Bphi

Proposition 11. *The space $T(\text{Im } \Pi_{>1/2} \cap H^{1/2}(\Sigma))$ is closed in $H^{-1/2}(\Sigma)$, as a consequence of the works of Carron. Let's define, for $\sigma \in \text{Im } \Pi_{[-1/2, 1/2]}$, $B(\sigma)$ as the orthonormal projection of $T(\sigma)$ on the orthogonal of this space $T(\text{Im } \Pi_{>1/2} \cap H^{1/2}(\Sigma))$. Then B is linear and satisfies*

$$\|B(\sigma)\| \leq \|T(\sigma)\| \text{ and } B(\sigma) = 0 \Rightarrow \exists \eta \in \text{Im } \Pi_{>1/2} T(\sigma + \eta) = 0.$$

Proof. To prove that $T(\text{Im } \Pi_{>1/2} \cap H^{1/2}(\Sigma))$ is closed we must recall some facts contained in [Ca01b]. Let us recall the operator T_C defined in (7).

Then $\text{Im } T_C = \text{Im } \Pi_{>1/2}$ is a subspace of $\text{Ker } T_C = \text{Im } \Pi_{\geq -1/2}$. We know that $T + T_C$ is an elliptic operator of order 1 on Σ which is compact. As a consequence $\text{Ker}(T + T_C)$ is finite dimensional and $(T + T_C)(H^{1/2}(\Sigma))$ is a closed subspace of $H^{-1/2}(\Sigma)$ and $T + T_C$ admits a continuous parametrix $Q : H^{-1/2}(\Sigma) \rightarrow H^{1/2}(\Sigma)$ such that

$$Q \circ (T + T_C) = \text{Id} - \Pi_{\text{Ker}(T+T_C)}^\perp$$

if $\Pi_{\text{Ker}(T+T_C)}^\perp$ denote the projector, orthonormal for the scalar product of $H^{1/2}(\Sigma)$, on $\text{Ker}(T + T_C)$. We can now prove that $T(\text{Im } \Pi_{>1/2} \cap H^{1/2}(\Sigma))$ is closed.

Let f_n be a sequence of elements of $\text{Im } \Pi_{>1/2} \cap H^{1/2}(\Sigma)$ such that $T(f_n)$ converges, and let $\psi = \lim_{n \rightarrow \infty} T(f_n)$. We can suppose that

$$f_n \in \left(\text{Ker } T \cap \text{Im } \Pi_{>1/2} \cap H^{1/2}(\Sigma) \right)^\perp.$$

We have $\text{Im } \Pi_{>1/2} \cap H^{1/2}(\Sigma) \subset \text{Ker } T_C$, it means that $(T + T_C)f_n = T(f_n)$ converges and $g_n = Q \circ (T + T_C)f_n$ converges, let $g = \lim_{n \rightarrow \infty} g_n$. Thus

$$f_n = g_n + e_n, g_n \in (\text{Ker}(T + T_C))^\perp, g = \lim_{n \rightarrow \infty} g_n, e_n \in \text{Ker}(T + T_C).$$

The sequence e_n must be bounded, unless we can extract a subsequence $\|e_n\| \rightarrow \infty$, so it is true also for $\|f_n\|$ and by extraction we can suppose that the bounded sequence $e_n/\|f_n\|$ converges, because it leaves in a finite dimensional space. Let e' be this limit, then $e' = \lim f_n/\|f_n\|$ also and $e' \in \text{Im } \Pi_{>1/2} \cap H^{1/2}(\Sigma)$.

Finally e' satisfies

$$\|e'\| = 1, e' \in \text{Ker}(T + T_C) \text{ as } e_n, e' \in \text{Ker } T_C \text{ as } f_n \Rightarrow T(e') = 0.$$

Thus, $e' = \lim f_n/\|f_n\| \in \text{Im } \Pi_{>1/2} \cap H^{1/2}(\Sigma) \cap \text{Ker } T$ but, by hypothesis on f_n , e' must be orthonormal to this space! There is a contradiction.

So, e_n is a bounded sequence in a finite dimensional space, by extraction we can suppose that it converges, then f_n admits a convergent subsequence, let f denote its limit,

$$f \in \text{Im } \Pi_{>1/2} \cap H^{1/2}(\Sigma) \text{ and } \psi = T(f).$$

□

By application of this proposition we have

$$\|B \circ \Pi_{[-1/2, 1/2]} \circ U(\varphi_{1,m}(\varepsilon_m))\|_{H^{-1/2}(\Sigma)} = O(\sqrt{\varepsilon_m}).$$

This is the sum of few terms. We remark that the term with $c_{\gamma,m}$ is in fact always $O(\sqrt{\varepsilon_m})$. For the same reason we can freeze the function G at 0, where its value is 1. So we can say

$$\|\varepsilon_m^{\frac{1}{2}} \log(\varepsilon_m) B(d_{-1/2,m}) + \sum_{|\gamma| < 1/2} \varepsilon_m^{-\gamma} B(d_{\gamma,m}) + \varepsilon_m^{-\frac{1}{2}} B(d_{1/2,m})\| = O(\sqrt{\varepsilon_m}). \quad (21) \quad \boxed{\text{e18}}$$

We have to gather the terms concerning the same eigenvalue and still denote $d_{\gamma,m}$ the sum of all the terms with the same eigenvalue. Let $-1/2 \leq \gamma_p < \dots < \gamma_0 \leq 1/2$ be the eigenvalues of A in $[-1/2, 1/2]$ and d_γ the limit of $d_{\gamma,m}$, respectively the limit of $|\log \varepsilon_m|^{1/2} d_{1/2,m}$ for $\gamma = 1/2$.

If this limit is not 0, we denote $\sigma_\gamma = d_\gamma/\|d_\gamma\|$. Then we can conclude, beginning by the right that, in the case where $\gamma_0 = 1/2$,

$$B(\sigma_{1/2}) \neq 0 \Rightarrow d_{1/2,m} = |\log \varepsilon_m|^{1/2} \varepsilon_m^{1/2-\gamma_1} d_{1/2,m}^{(0)}$$

and by extraction we can suppose that $d_{1/2,m}^{(0)}$ converge to $d_{1/2}^{(0)}$, idem if $d_{1/2}^{(0)} \neq 0$ let $\sigma_{1/2}^{(0)} = d_{1/2}^{(0)} / \|d_{1/2}^{(0)}\|$. Reporting this result in (21) we find by the same processus that

$$B(d_{\gamma_1,m} + |\log \varepsilon_m|^{1/2} d_{1/2,m}^{(0)}) = O(\varepsilon^{\gamma_1 - \gamma_2}), \text{ so}$$

$$B(\sigma_{1/2}^{(0)}) \neq 0 \Rightarrow d_{1/2,m}^{(0)} = O(|\log \varepsilon_m|^{-1/2})$$

and we can write $d_{1/2,m}^{(0)} = |\log \varepsilon_m|^{-1/2} d_{1/2,m}^{(0')}$ with $d_{1/2,m}^{(0')}$ converging. Then by extraction we can suppose that $d_{\gamma_1,m} + d_{1/2,m}^{(0')}$ is convergent, and idem if we denote, in the case where the limit is not 0, by $\sigma_{\gamma_1}^{(1)}$ the normed limit, then

$$B(\sigma_{\gamma_1}^{(1)}) \neq 0 \Rightarrow d_{\gamma_1,m} + d_{1/2,m}^{(0')} = O(\varepsilon^{\gamma_1 - \gamma_2})$$

We continue step by step

$$B(\sigma_{\gamma_j}^{(j)}) \neq 0 \Rightarrow \forall 0 \leq k \leq j \ d_{\gamma_k,m} = \varepsilon_m^{\gamma_k - \gamma_{j+1}} d_{\gamma_j,\varepsilon}^{(j)} \dots$$

It means that there exist elements $\bar{\sigma}_\gamma \in \text{Ker}(A - \gamma)$, $|\gamma| \leq 1/2$ such that

$$\exists \eta_\gamma \in \text{Im } \Pi_{>\gamma}, \ T(\bar{\sigma}_\gamma + \eta_\gamma) = 0 \text{ and if } \Pi_{(\gamma, 1/2]}(\eta_\gamma) = \sum_{\mu > \gamma} \eta_\gamma^\mu$$

$$\Rightarrow \Pi_{(-1/2, 1/2]} \circ U(\varphi_{1,m}(r)) \sim$$

$$\sum_{-\frac{1}{2} \leq \mu < \gamma < \frac{1}{2}} r^{-\gamma} (\bar{\sigma}_\gamma + \varepsilon_m^{\gamma - \mu} \eta_\mu^\gamma) + r^{-1/2} (|\log \varepsilon_m|^{-1/2} \bar{\sigma}_{1/2} + \sum_{-\frac{1}{2} \leq \mu < \frac{1}{2}} \varepsilon_m^{1/2 - \mu} \eta_\mu^{1/2}). \quad (22) \quad \boxed{\text{asymptot}}$$

Here the term $\varepsilon^{-\mu}$ has to be replaced by $\varepsilon_m^{1/2} \log \varepsilon_m$ in the case where $\mu = -1/2$. We have proved in particular

phi1W

Proposition 12. *One can write $\Pi_{(-1/2, 1/2]} \circ U(\xi_1 \varphi_{1,m}) = \bar{\sigma}_{1,m} + \sigma_{0,m}$ with $U^* \sigma_{0,m} \in \text{Dom}(D_{1,\min})$ bounded and $\bar{\sigma}_{1,m} = \bar{\sigma}_{1,m}^{< \frac{1}{2}} + \bar{\sigma}_{1,m}^{\frac{1}{2}}$ satisfies that there exists a subfamily of $\bar{\sigma}_{1,m}^{< \frac{1}{2}}$ which converges, as $m \rightarrow \infty$ to $\sum_{\gamma \in (-\frac{1}{2}, \frac{1}{2})} r^{-\gamma} \sigma_\gamma$ with*

$$\sum_{\gamma \in (-\frac{1}{2}, \frac{1}{2})} \sigma_\gamma \in W.$$

while

$$\bar{\sigma}_{1,\varepsilon_m}^{\frac{1}{2}} \sim \frac{1}{\sqrt{|\log \varepsilon_m|}} r^{-\frac{1}{2}} \bar{\sigma}_{\frac{1}{2}} \text{ for some } \bar{\sigma}_{\frac{1}{2}} \in \text{Ker}(A - \frac{1}{2}).$$

Thus, $\bar{\sigma}_{1,\varepsilon_m}^{\frac{1}{2}}$ concentrate on the singularity.

Indeed, we can, at each step, decompose $d_{\gamma,m}$ on a part in $\text{Ker}(B \circ U^*)$ and a part which appears on a smaller behaviour in ε_m .

4.2.5. *Conclusions on the side of \mathbf{M}_1 .* We now decompose $\varphi_{1,m}$ near the singularity as follows: Let

$$\xi_1 \varphi_{1,m} = \xi_1 \left\{ \varphi_{1,m}^{\leq -\frac{1}{2}} + \varphi_{1,m}^{(-\frac{1}{2}, \frac{1}{2}]} + \varphi_{1,m}^{> \frac{1}{2}} \right\}$$

according to the decomposition, on the cone, of σ_1 along the eigenvalues of A respectively less than $-\frac{1}{2}$, in $(-\frac{1}{2}, \frac{1}{2}]$ and larger than $\frac{1}{2}$.

We first remark that the expression and the convergence of $\varphi_{1,m}^{(-\frac{1}{2}, \frac{1}{2}]}$ is given by the preceding proposition 12.

Now $\varphi_{1,m}^{> \frac{1}{2}}$ and $\tilde{\psi}_{1,m} = \xi_1 P_{\varepsilon_m} (\Pi_{>1/2} \circ U(\varphi_{2,\varepsilon_m}(1)))$ have the same boundary value. But, by propositions 9 and 10

$$\lim_{m \rightarrow \infty} U\varphi_{2,\varepsilon_m}(1) = U\varphi_2(1) \in \text{Im}(\Pi_{>\frac{1}{2}}) \text{ for the norm of } H^{1/2}(\Sigma).$$

So, $\varphi_{1,m}^{> \frac{1}{2}} - \tilde{\psi}_{1,m}$ can be viewed in $H^1(M_1(0))$ by a prolongation by 0 and

tild1 **Proposition 13.** *By uniform continuity of P_{ε_m} , and the convergence property just recalled*

$$\lim_{m \rightarrow \infty} \|\tilde{\psi}_{1,\varepsilon_m} - \xi_1 P_{\varepsilon_m}(U(\varphi_2 \upharpoonright_{\Sigma}))\|_{L^2(M_1)} = 0.$$

On the other hand, $\xi_1 P_{\varepsilon_m}(U(\varphi_2 \upharpoonright_{\Sigma}))$ converges weakly to 0 on the open manifold $M_1(0)$, more precisely, for any fixed η , $0 < \eta < 1$

$$\lim_{m \rightarrow \infty} \|\xi_1 P_{\varepsilon_m}(U(\varphi_2 \upharpoonright_{\Sigma}))\|_{L^2(M_1(\eta))} = 0.$$

We remark finally that the boundary value of $\varphi_{1,\varepsilon_m}^{\leq -\frac{1}{2}}$ is small. We introduce for this term the cut-off function taken in [ACP09]:

$$\xi_{\varepsilon_m}(r) = \begin{cases} 1 & \text{if } 2\sqrt{\varepsilon_m} \leq r, \\ \frac{1}{\log(\sqrt{\varepsilon_m})} \log\left(\frac{2\varepsilon_m}{r}\right) & \text{if } 2\varepsilon_m \leq r \leq 2\sqrt{\varepsilon_m}, \\ 0 & \text{if } r \leq 2\varepsilon_m. \end{cases}$$

Proposition 14. $\lim_{m \rightarrow \infty} \left\| (1 - \xi_{\varepsilon_m}) \xi_1 \varphi_{1,\varepsilon_m}^{\leq -\frac{1}{2}} \right\|_{L^2(M_1, g_1)} = 0.$

This is a consequence of the estimates of Lemmas 5 and 6: we remark that by the same argument, we obtain also $\|\xi_1 \varphi_{1,\varepsilon_m}^{\leq -1/2}(r)\|_{L^2(\Sigma)} \leq C\sqrt{r}$ so

$$\left\| (1 - \xi_{\varepsilon_m}) \xi_1 \varphi_{1,\varepsilon_m}^{\leq -\frac{1}{2}} \right\|_{L^2(M_1)} = O(\varepsilon_m^{\frac{1}{4}}).$$

psi1 **Proposition 15.** *The forms*

$$\psi_{1,\varepsilon_m} = (1 - \xi_1) \varphi_{1,\varepsilon_m} + \left(\xi_1 \varphi_{1,\varepsilon_m}^{>1/2} - \tilde{\psi}_{1,\varepsilon_m} \right) + \xi_{\varepsilon_m} \xi_1 \varphi_{1,\varepsilon_m}^{\leq -1/2} + \xi_1 U^* \sigma_{0,m}^{\frac{1}{2}}$$

belong to $\text{Dom}(D_{1,\min})$ and define a bounded family.

Proof. We will show that each term is bounded. For the last one, it is a consequence of proposition 12. For the first one, it is already done in Lemma 4. For the second one, we remark that

$$\begin{aligned} f_{\varepsilon_m} &:= \left(\partial_r + \frac{A}{r}\right)U \left(\xi_1 \varphi_{1,\varepsilon_m}^{\frac{1}{2}} - \tilde{\psi}_{1,\varepsilon_m}\right) \\ &= \xi_1 \left(\partial_r + \frac{A}{r}\right)(U \varphi_{1,\varepsilon_m}^{\frac{1}{2}}) + \partial_r(\xi_1)U \left(\varphi_{1,\varepsilon_m}^{\frac{1}{2}} - P_{\varepsilon_m}(\Pi_{>1/2}\varphi_{2,\varepsilon_m}(1))\right) \end{aligned} \quad (23) \quad \boxed{\text{feps}}$$

is uniformly bounded in $L^2(M_1)$, because of (16). This estimate (16) shows also that the L_2 -norm of $(\varphi_{1,\varepsilon_m}^{\frac{1}{2}} - \tilde{\psi}_{1,\varepsilon_m})$ is bounded.

For the third one, we use the estimate due to the expression of the quadratic form. Expriming that $\int_{C_{r,1}} |D_1(\xi_1 \varphi^{\leq -\frac{1}{2}})|^2 d\mu$ is bounded by Λ gives that

$$\left\| \sigma_1^{\leq -\frac{1}{2}}(r) \right\|_{L^2(\Sigma)}^2 \leq \Lambda r |\log r| \quad (24) \quad \boxed{\text{val-}}$$

by the same argument as in Lemmas 5 and 6. Now

$$\begin{aligned} \left\| D_1(\xi_{\varepsilon_m} \xi_1 \varphi_{1,\varepsilon_m}^{\leq -\frac{1}{2}}) \right\|_{L^2(M_1)} &\leq \left\| \xi_{\varepsilon_m} D_1(\xi_1 \varphi_{1,\varepsilon_m}^{\leq -\frac{1}{2}}) \right\|_{L^2(M_1)} + \left\| |d\xi_{\varepsilon_m}| \cdot \xi_1 \varphi_{1,\varepsilon_m}^{\leq -\frac{1}{2}} \right\|_{L^2(M_1)} \\ &\leq \left\| D_1(\xi_1 \varphi_{1,\varepsilon_m}^{\leq -\frac{1}{2}}) \right\|_{L^2(M_1)} + \left\| |d\xi_{\varepsilon_m}| \cdot \xi_1 \varphi_{1,\varepsilon_m}^{\leq -\frac{1}{2}} \right\|_{L^2(M_1)} \end{aligned}$$

the first term is bounded and, with $|A| \geq \frac{1}{2}$ for this term, and the estimate (24), we have

$$\left\| |d\xi_{\varepsilon_m}| \xi_1 \varphi_{1,\varepsilon_m}^{\leq -\frac{1}{2}} \right\|_{L^2(M_1)}^2 \leq \frac{4\Lambda}{\log^2 \varepsilon_m} \int_{\varepsilon_m}^{\sqrt{\varepsilon_m}} \frac{\log(r)}{r} dr \leq \frac{3}{2}\Lambda.$$

This completes the proof. \square

In fact, the decomposition used here is almost orthogonal:

$\boxed{14}$ **Lemma 16.** *There exists $\beta > 0$ such that*

$$(\varphi_{1,\varepsilon_m}^{\frac{1}{2}} - \tilde{\psi}_{1,\varepsilon_m}, \tilde{\psi}_{1,\varepsilon_m})_{L^2(M_1)} = O(\varepsilon_m^\beta).$$

Proof of Lemma 16. — If we decompose the terms under the eigenspaces of A , we see that only the eigenvalues in $(\frac{1}{2}, +\infty)$ are involved and, with $f_{\varepsilon_m} = \sum_{\gamma > \frac{1}{2}} f^\gamma$ and $U(\varphi_{1,\varepsilon_m}^{\frac{1}{2}} - \tilde{\psi}_{1,\varepsilon_m}) = \sum_{\gamma > \frac{1}{2}} \varphi_0^\gamma$, the equation (23) and the fact that $(\varphi_{1,\varepsilon_m}^{\frac{1}{2}} - \tilde{\psi}_{1,\varepsilon_m})(\varepsilon_m) = 0$ give

$$\varphi_0^\gamma(r) = r^{-\gamma} \int_{\varepsilon_m}^r \rho^\gamma f^\gamma(\rho) d\rho.$$

Then for each eigenvalue $\gamma > \frac{1}{2}$ of A

$$\begin{aligned} (\varphi_0^\gamma, \tilde{\psi}_{1,\varepsilon_m}^\gamma)_{L^2(C_{\varepsilon_m,1})} &= \varepsilon_m^{\gamma-\frac{1}{2}} \int_{\varepsilon_m}^1 r^{-2\gamma} \int_{\varepsilon_m}^r \rho^\gamma (\sigma_\gamma, f^\gamma(\rho))_{L^2(\Sigma)} d\rho \\ &= \varepsilon_m^{\gamma-\frac{1}{2}} \int_{\varepsilon_m}^1 \frac{r^{-2\gamma+1}}{2\gamma-1} \cdot r^\gamma \cdot (\sigma_\gamma, f^\gamma(r))_{L^2(\Sigma)} dr \\ &\quad + \frac{\varepsilon_m^{\gamma-1/2}}{2\gamma-1} \int_{\varepsilon_m}^1 \rho^\gamma (\sigma_\gamma, f^\gamma(\rho))_{L^2(\Sigma)} d\rho. \end{aligned}$$

Thus, if $\gamma > \frac{3}{2}$, we have the majoration

$$\begin{aligned} |(\varphi_0^\gamma, \tilde{\psi}_{1,\varepsilon_m}^\gamma)_{L^2(\mathcal{C}_{\varepsilon_m,1})}| &\leq \varepsilon_m^{\gamma-\frac{1}{2}} \int_{\varepsilon_m}^1 \frac{r^{-\gamma+1}}{2\gamma-1} |(\sigma_\gamma, f^\gamma(r))_{L^2(\Sigma)}| dr \\ &\quad + \frac{\varepsilon_m^{\gamma-\frac{1}{2}}}{(2\gamma-1)\sqrt{2\gamma+1}} \|\sigma_\gamma\|_{L^2(\Sigma)} \cdot \|f^\gamma\|_{L^2(\mathcal{C}_{\varepsilon_m,1})} \\ &\leq C\varepsilon_m^{\gamma-\frac{1}{2}} \|\sigma_\gamma\|_{L^2(\Sigma)} \frac{\varepsilon_m^{\frac{-2\gamma+3}{2}}}{(2\gamma-1)\sqrt{2\gamma-3}} \|f^\gamma\|_{L^2(\mathcal{C}_{\varepsilon_m,1})} \\ &\quad + \frac{\varepsilon_m^{\gamma-\frac{1}{2}}}{(2\gamma-1)\sqrt{2\gamma+1}} \|\sigma_\gamma\|_{L^2(\Sigma)} \cdot \|f^\gamma\|_{L^2(\mathcal{C}_{\varepsilon_m,1})}, \end{aligned}$$

while, for $\gamma = \frac{3}{2}$ the first term is $O(\varepsilon_m \sqrt{|\log \varepsilon_m|})$ and for $\frac{1}{2} < \gamma < \frac{3}{2}$, it is $O(\varepsilon_m^{\gamma-\frac{1}{2}})$. In short, we have

$$|(\varphi_0^\gamma, \tilde{\psi}_{1,\varepsilon_m}^\gamma)_{L^2(\mathcal{C}_{\varepsilon_m,1})}| \leq C\varepsilon_m^\beta \|\sigma_\gamma\|_{L^2(\Sigma)} \cdot \|f^\gamma\|_{L^2(\mathcal{C}_{\varepsilon_m,1})},$$

if $\beta > 0$ satisfies $\gamma \geq \beta + \frac{1}{2}$ for all eigenvalue γ of A in $(\frac{1}{2}, +\infty)$. This estimate gives the lemma.

conc

Corollary 17. *There exists from $\psi_{1,\varepsilon_m} + \varphi_{1,m}^{(-\frac{1}{2}, \frac{1}{2})}$ a subfamily which converges in L^2 to a form $\varphi_1 \in \text{Dom}(D_{1,W})$ which satisfies on the open manifold $M_1(0)$ the equation $\Delta\varphi_1 = \lambda\varphi_1$. Moreover,*

$$\|\varphi_1\|_{L^2(M_1(0))}^2 + \|\widetilde{\varphi}_2\|_{L^2(\widetilde{M}_2)}^2 + \|\overline{\sigma}^{\frac{1}{2}}\|_{L^2(\Sigma)}^2 = 1, \quad (25)$$

decompose

where $\widetilde{\varphi}_2$ is the prolongation of φ_2 by $P_2(\varphi_2 \upharpoonright_\Sigma)$ on \widetilde{M}_2 , and $\overline{\sigma}^{\frac{1}{2}}$ given by the proposition 12.

Proof. Indeed, the family $\{\psi_{1,\varepsilon_m} + \varphi_{1,m}^{(-\frac{1}{2}, \frac{1}{2})}\}_m$ is bounded in $\text{Dom } D_{1,max}$, one can then extract a subfamily which converges in $L^2(M_1)$ but we know that $\tilde{\psi}_{1,\varepsilon_m}$ converges to 0 in any $M_1(\eta)$, the conclusion follows. We obtain also, with the help of Lemma 16 that

$$\begin{aligned} &1 - \{\|\varphi_1\|_{L^2(M_1(0))}^2 + \|\varphi_2\|_{L^2(M_2)}^2\} \\ &= \lim_{m \rightarrow \infty} \left\{ \|\tilde{\psi}_{1,\varepsilon_m}\|_{L^2(M_1)}^2 + \|\xi_1 U^* \left(\frac{1}{\sqrt{|\log \varepsilon_m|}} r^{-\frac{1}{2}} \overline{\sigma}^{\frac{1}{2}} \right)\|_{L^2(M_1)}^2 \right\}. \end{aligned}$$

We remark that, by Corollary 13, $\varphi_2 = 0$ implies $\lim_{m \rightarrow \infty} \|\tilde{\psi}_{1,\varepsilon_m}\|_{L^2(M_1)} = 0$.

In fact, one has by (16)

$$\lim_{m \rightarrow \infty} \|\tilde{\psi}_{1,\varepsilon_m}\|_{L^2(M_1)} = \|P_2(U\varphi_2 \upharpoonright_\Sigma)\|_{L^2(\widetilde{M}_2)}. \quad (26)$$

p2phi2

Finally, one has

$$\lim_{m \rightarrow \infty} \|\xi_1 U^* \left(\frac{1}{\sqrt{|\log \varepsilon_m|}} r^{-\frac{1}{2}} \overline{\sigma}^{\frac{1}{2}} \right)\|_{L^2(M_{\varepsilon_m})} = \|\overline{\sigma}^{\frac{1}{2}}\|_{L^2(\Sigma)}. \quad (27)$$

□

4.3. Lower bound, the end. Let us now $\{\varphi_1(\varepsilon), \dots, \varphi_N(\varepsilon)\}$ be an orthonormal family of eigenforms of the Hodge-de Rham operator, associated to the eigenvalues $\lambda_1(\varepsilon), \dots, \lambda_N(\varepsilon)$. We can make the same procedure of extraction for the all family. This gives, in the limit domain, a family $(\varphi_1^j, \varphi_2^j, \bar{\sigma}_{\frac{1}{2}}^j)_{1 \leq j \leq N}$. We already know by Corollary 17 that each element has norm 1, if we show that they are orthogonal, we are done, by applying the min-max formula to the limit problem (13).

Lemma 18. *The limit family is orthonormal in \mathcal{H}_∞ .*

Proof. If we follow the procedure for one indice, up to terms converging to zero, we had decomposed the eigenforms $\varphi_j(\varepsilon)$ on M_1 on three terms

$$\begin{aligned}\Phi_\varepsilon^j &= \psi_{1,\varepsilon} + \varphi_{1,\varepsilon}^{(-\frac{1}{2}, \frac{1}{2})}, \\ \tilde{\Phi}_\varepsilon^j &= \tilde{\psi}_{1,\varepsilon_m}, \\ \bar{\Phi}_\varepsilon^j &= U^* \left(\frac{1}{\sqrt{|\log \varepsilon|}} r^{-\frac{1}{2}} \bar{\sigma}_{\frac{1}{2}}^j \right).\end{aligned}$$

Let $a \neq b$ be two indices. If we apply Lemma 16 to any linear combination of $\varphi_a(\varepsilon)$ and $\varphi_b(\varepsilon)$, we obtain that

$$\lim_{m \rightarrow \infty} \left\{ (\Phi_{\varepsilon_m}^a, \tilde{\Phi}_{\varepsilon_m}^b)_{L^2(M_1(\varepsilon_m))} + (\Phi_{\varepsilon_m}^b, \tilde{\Phi}_{\varepsilon_m}^a)_{L^2(M_1(\varepsilon_m))} \right\} = 0.$$

If we apply (26), we obtain

$$\lim_{m \rightarrow \infty} \left\{ (\tilde{\Phi}_{\varepsilon_m}^a, \tilde{\Phi}_{\varepsilon_m}^b)_{L^2(M_{\varepsilon_m})} + (\varphi_{2,\varepsilon}^a, \varphi_{2,\varepsilon}^b)_{L^2(M_2)} \right\} = (\tilde{\varphi}_2^a, \tilde{\varphi}_2^b)_{L^2(\widetilde{M}_2)}.$$

Then finally, from $(\varphi_a(\varepsilon), \varphi_b(\varepsilon)) = 0$, we conclude that

$$(\varphi_1^a, \varphi_1^b)_{L^2(\overline{M}_1)} + (\varphi_2^a, \varphi_2^b)_{L^2(\widetilde{M}_2)} + (\bar{\sigma}_{\frac{1}{2}}^a, \bar{\sigma}_{\frac{1}{2}}^b)_{L^2(\Sigma)} = 0.$$

□

limit **Proposition 19.** *The multiplicity of 0 in the limit spectrum is given by the sum*

$$\dim \text{Ker } \Delta_{1,W} + \dim \text{Ker } \mathcal{D}_2 + i_{\frac{1}{2}},$$

where $i_{\frac{1}{2}}$ denotes the dimension of the vector space $\mathcal{I}_{\frac{1}{2}}$, see (9), of extended solutions ω on \widetilde{M}_2 introduced by Carron [Ca01a], corresponding to a boundary term on restriction to $r = 1$ with non-trivial component in $\text{Ker}(A - 1/2)$.

If the limit value $\lambda \neq 0$, then it belongs to the positive spectrum of the Hodge-de Rham operator $\Delta_{1,W}$ on \overline{M}_1 , with the space W defined in (8).

Proof. The last process, with in particular (26) and (17), constructs in fact an element in the limit Hilbert space

$$\mathcal{H}_\infty := L^2(\overline{M}_1) \oplus \text{Ker } \widetilde{D}_2 \oplus \mathcal{I}_{\frac{1}{2}}$$

and this process is clearly *isometric* in the sense that if we have an orthonormal family $\{\varphi_{k,\varepsilon_m}\}$, ($1 \leq k \leq n$), we obtain at the limit an orthonormal family, if \mathcal{H}_∞ is defined as an orthonormal space of the Hilbert spaces. And if we begin with eigenforms of Δ_{ε_m} , we obtain at the limit eigenforms of $\Delta_{1,W} \oplus \{0\} \oplus \{0\}$. The last calculus implies that $\liminf \lambda_N(\varepsilon_m) \geq \lambda_N$. □

Remark 20. *In order to understand this result, it is important to remember when occurs the eigenvalue $\frac{1}{2}$ in the spectrum of A . By the expression (4), we find that it occurs exactly*

- for n even, if $\frac{3}{4}$ is an eigenvalue of the Hodge-de Rham operator Δ_Σ acting on the coexact forms of degree $\frac{n}{2}$ or $\frac{n}{2} - 1$ of the submanifold Σ .
- for n odd, if 0 is an eigenvalue of Δ_Σ for the $\frac{n-1}{2}, \frac{n+1}{2}$ forms, but also if 1 is eigenvalue of the coexact forms of degree $\frac{n-1}{2}$ on Σ .

A dilation of the metric on Σ permits to avoid positive eigenvalues, but if harmonic forms of degree $\frac{n-1}{2}$ or $\frac{n+1}{2}$ create half-bounded states, then small eigenvalues will always appear.

5. HARMONIC FORMS AND SMALL EIGENVALUES.

It would be interesting to know how many small (but not zero) eigenvalues appear. For this purpose, we can use the topological meaning of harmonic forms.

5.1. Cohomology groups. The topology of M_ε is independant of $\varepsilon \neq 0$ and can be apprehended by the Mayer-Vietoris exact sequence:

$$\cdots \rightarrow H^p(M_\varepsilon) \xrightarrow{res} H^p(M_1(\varepsilon)) \oplus H^p(M_2) \xrightarrow{dif} H^p(\Sigma) \xrightarrow{ext} H^{p+1}(M_\varepsilon) \rightarrow \cdots$$

As already mentionned, the space $\text{Ker } \mathcal{D}_2 \oplus \mathcal{I}_{\frac{1}{2}}$ can be sent in $H^*(M_2)$. More precisely, Hausel, Hunsicker and Mazzeo in [HHM04], Theorem 1.A, p.490, have proved that the space of the L^2 -harmonic forms $\mathcal{H}_{L^2}^k(\widetilde{M}_2)$ on \widetilde{M}_2 is given by:

$$\mathcal{H}_{L^2}^k(\widetilde{M}_2) \cong \begin{cases} H^k(M_2, \Sigma) & \text{if } k < \frac{n+1}{2}, \\ \text{Im} \left(H^{\frac{n+1}{2}}(M_2, \Sigma) \rightarrow H^{\frac{n+1}{2}}(M_2) \right) & \text{if } k = \frac{n+1}{2}, \\ H^k(M_2) & \text{if } k > \frac{n+1}{2}. \end{cases} \quad (28)$$

L2-cohomolog

We note that the space of L^2 -harmonic forms is equal to that of L^2 -harmonic fields, or the Hodge cohomology group, since \widetilde{M}_2 is complete.

For \overline{M}_1 we can use the results of Cheeger. Following [Ch80] and [Ch83], we know that the intersection cohomology groups of \overline{M}_1 coincide with $\text{Ker}(D_{1,\max} \circ D_{1,\min})$, if $H^{\frac{n}{2}}(\Sigma) = 0$. And we know also that

$$IH^p(\overline{M}_1) = \begin{cases} H^p(M_1(\varepsilon)) & \text{if } p \leq \frac{n}{2}, \\ H_c^p(M_1(\varepsilon)) & \text{if } p \geq \frac{n}{2} + 1. \end{cases} \quad (29)$$

These results can be used for our study only if $D_{1,\max}$ and $D_{1,\min}$ coincide. This occurs if and only if A has no eigenvalues in the interval $(-\frac{1}{2}, \frac{1}{2})$. As a consequence of the expression of the eigenvalues of A , recalled in (4), this is the case if and only if

- **for n odd**, the operator Δ_Σ has no eigenvalues in $(0, 1)$ on coexact $\frac{n-1}{2}$ -forms,
- **for n even**, the operator Δ_Σ has no eigenvalues in $(0, \frac{3}{4})$ on $\frac{n}{2}$ or $(\frac{n}{2} - 1)$ coexact forms and $H^{\frac{n}{2}}(\Sigma) = 0$.

Thus, if $D_{1,\max} = D_{1,\min}$, which implies $H^{\frac{n}{2}}(\Sigma) = 0$ in the case where n is even, then the map

$$H^{\frac{n}{2}}(M_\varepsilon) \xrightarrow{res} H^{\frac{n}{2}}(M_1(\varepsilon)) \oplus H^{\frac{n}{2}}(M_2)$$

is surjective and then any small eigenvalue in this degree must come from an element of $\text{Ker } \mathcal{D}_2 \oplus \mathcal{I}_{\frac{1}{2}}$ sent to 0 in $H^{\frac{n}{2}}(M_2)$. In this case also the map

$$H^{\frac{n}{2}+1}(M_\varepsilon) \xrightarrow{res} H^{\frac{n}{2}+1}(M_1(\varepsilon)) \oplus H^{\frac{n}{2}+1}(M_2)$$

is injective, so there may exist small eigenvalues in this degree.

5.2. Some examples. We exhibit a general procedure to construct new examples as follows: Let W_i , $i = 1, 2$, be two compact Riemannian manifolds with boundary Σ_i and dimension $(n_i + 1)$ such that $n_1 + n_2 = n \geq 2$. We can apply our result to $M_1 := W_1 \times \Sigma_2$ and $M_2 := \Sigma_1 \times W_2$. The manifold M_ε is always diffeomorphic to $M = M_1 \cup M_2$.

For instance, let v_2 be the volume form of (Σ_2, h_2) . It defines a harmonic form on M_1 and this form will appear for the limit spectrum if, transplanted on $\overline{M_1}$, it defines an element of the domain of the operator $\Delta_{1,W}$.

In the writing introduced in Section 2.2, this element corresponds to $\beta = 0$ and $\alpha = r^{\frac{n}{2}-n_2}v_2$ and the expression of A gives that

$$A(\beta, \alpha) = \left(n_2 - \frac{n}{2}\right) (\beta, \alpha).$$

If $\frac{n}{2} - n_2 > 0$, then (β, α) is in the domain of $D_{1,\max} \circ D_{1,\min}$ and if $n_2 = \frac{n}{2}$, it is in the domain of $\Delta_{1,W}$ for the eigenvalue 0 of A .

So, if we know that $H^{n_2}(M) = 0$, or more generally that $\dim H^{n_2}(M) < \dim H^{n_2}(\Sigma_2)$ in the case where Σ_2 is not connected, this element will create a small eigenvalue on M_ε . This is the case, if D^k denotes the unit ball in \mathbb{R}^k , for

$$W_1 = D^{n_1+1} \text{ and } W_2 = D^{n_2+1} \text{ for } n_2 \leq n_1.$$

Then, $M = \mathbb{S}^{n_1+n_2+1}$ and we obtain

Corollary 21. *For any degree k and any $\varepsilon > 0$, there exists a metric on \mathbb{S}^m such that the Hodge-de Rham operator acting on k -forms admits an eigenvalue smaller than ε . We can see that, for $k < \frac{m}{2}$, it is in the spectrum of coexact forms, and by duality, for $k \geq \frac{m}{2}$ in the spectrum of exact k -forms.*

Indeed, the case $k < \frac{m}{2}$ is a direct application, as explained above. We see that our pseudomode is coclosed. Thus, in the case when m is even, if ω is an eigenform of degree $(\frac{m}{2} - 1)$ with small eigenvalue, then $d\omega$ is a closed eigenform with the same eigenvalue and degree $\frac{m}{2}$. Finally, the case $k > \frac{m}{2}$ is obtained by the Hodge duality. We remark that, in the case $k = 0$ we recover *Cheeger's dumbbell*, and also that this result has been proved by Guerini in [G04] with another deformation, although he did not give the convergence of the spectrum.

By the surgery of the previous case, we obtain, for

$$W_1 := \mathbb{S}^{n_1} \times [0, 1] \text{ and } W_2 := D^{n_2+1} \text{ for } 0 \leq n_2 < n_1, \text{ and } n = n_1 + n_2 \geq 2$$

that $\Sigma_1 = \mathbb{S}^{n_1} \sqcup \mathbb{S}^{n_1}$, $\Sigma_2 = \mathbb{S}^{n_2}$ and $M = \mathbb{S}^{n_1} \times \mathbb{S}^{n_2+1}$. The volume form $v_2 \in H^{n_2}(\Sigma_2)$ defines again a harmonic form on \overline{M}_1 and, since $H^{n_2}(\mathbb{S}^{n_1} \times \mathbb{S}^{n_2+1}) = 0$, if $n_2 < n_1$, then v_2 defines a small eigenvalue on the n_2 -forms of M_ε .

Thus, by duality, we obtain

Corollary 22. *For any $k, l \geq 0$ with $0 \leq k - 1 < l$ and any $\varepsilon > 0$, there exists a metric on $\mathbb{S}^l \times \mathbb{S}^k$ such that the Hodge-de Rham operator acting on $(k - 1)$ -forms and on $(l + 1)$ -forms admits an eigenvalue smaller than ε .*

This corollary is also a consequence of the previous one: we know that there exists a metric on \mathbb{S}^k whose Hodge-de Rham operator admits a small eigenvalue on $(k - 1)$ -forms, this property is maintained on $\mathbb{S}^l \times \mathbb{S}^{k+1}$.

With the same construction, we can exchange the roles of M_1 and M_2 : the two volume forms of $\mathbb{S}^{n_1} \sqcup \mathbb{S}^{n_1}$ create one n_1 -form with small but non-zero eigenvalue on $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2+1}$, if $n_1 \leq (n_2 + 1)$. By duality, we obtain an $(n_2 + 1)$ -form with small eigenvalue. So, with new notations, we have obtained

Corollary 23. *For any $k < l$ with $k + l \geq 3$ and any $\varepsilon > 0$, there exists a metric on $\mathbb{S}^l \times \mathbb{S}^k$ such that the Hodge-de Rham operator acting on l -forms and on k -forms admits a positive eigenvalue smaller than ε .*

More generally, by repeating the $(k - 1)$ -dimensional surgery by L -times, we obtain the following:

Proposition 24 ([SY91]). *The connected sum of the L -copies of the product spheres $\mathbb{S}^k \times \mathbb{S}^l$ can be decomposed as follows:*

$$\mathbb{S}^k \times \mathbb{S}^l \cong \left(\mathbb{S}^{k-1} \times \left(\mathbb{S}^{l+1} \setminus \prod_{i=0}^L D_i^{l+1} \right) \right) \cup_{\partial} \left(D^k \times \prod_{i=0}^L \mathbb{S}_i^l \right).$$

Remark 25. *J-P. Sha and D-G. Yang [SY91] constructed a Riemannian metric of positive Ricci curvature on this manifold. More generally, see also Wraith [Wr07].*

As similar way using Proposition 24, we can obtain the small positive eigenvalues on the connected sum of the L -copies of the product spheres $\mathbb{S}^k \times \mathbb{S}^l$.

All these examples use the spectrum of \overline{M}_1 . We can obtain also examples using the L^2 -cohomology group of \widetilde{M}_2 , which is given by (28) (Hausel, Hunsicker and Mazzeo [HHM04]).

Suppose now that $n = \dim \Sigma$ is odd. Then, we have the long exact sequence

$$\cdots \rightarrow H^k(M_2, \Sigma) \rightarrow H^k(M_2) \rightarrow H^k(\Sigma) \rightarrow H^{k+1}(M_2, \Sigma) \rightarrow \cdots$$

For $k = \frac{n-1}{2}$, the space $H^k(M_2, \Sigma)$ is isomorphic to the L^2 -cohomology group of \widetilde{M}_2 . If $H^{\frac{n-1}{2}}(\Sigma)$ is non-trivial, then any non-trivial harmonic k -form on Σ will create an extended solution, corresponding to an eigenvector of A with eigenvalue $\frac{1}{2}$.

For example, take $\Sigma = \mathbb{S}^k \times \mathbb{S}^{k+1}$ for $k = \frac{n-1}{2}$, then $H^k(\Sigma)$ is non-trivial. Any non-trivial form $\omega \in H^k(\Sigma)$ sent to $0 \in H^{k+1}(M_2, \Sigma)$ comes from an element $\tilde{\omega} \in H^k(M_2)$ which is not in the L^2 -cohomology group of \widetilde{M}_2 by (28).

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LABORATOIRE DE MATHÉMATIQUES JEAN LERAY, UNIVERSITÉ DE NANTES, CNRS, FACULTÉ DES SCIENCES, BP 92208, 44322 NANTES, FRANCE

E-mail address: `colette.anne@univ-nantes.fr`

DIVISION OF MATHEMATICS, GRADUATE SCHOOL OF INFORMATION SCIENCES, TÔHOKU UNIVERSITY, AOBA 6-3-09, SENDAI, 980-8579, JAPAN

E-mail address: `t-junya@math.is.tohoku.ac.jp`