



HAL
open science

Partial collapsing and the spectrum of the Hodge Laplacian

Colette Anné, Junya Takahashi

► **To cite this version:**

Colette Anné, Junya Takahashi. Partial collapsing and the spectrum of the Hodge Laplacian. 2010.
hal-00503230v2

HAL Id: hal-00503230

<https://hal.science/hal-00503230v2>

Preprint submitted on 21 Feb 2011 (v2), last revised 17 Jul 2014 (v5)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

PARTIAL COLLAPSING AND THE SPECTRUM OF THE HODGE-DE RHAM OPERATOR

COLETTE ANNÉ AND JUNYA TAKAHASHI

ABSTRACT. The goal of the present paper is to calculate the limit spectrum of the Hodge-de Rham operator under the perturbation of collapsing one part of a manifold obtained by gluing together two manifolds with the same boundary. It appears to take place in the general problem of blowing-up conical singularities as introduced in Mazzeo [Maz06] and Rowlett [Row06, Row08].

Résumé. Nous calculons la limite du spectre de l'opérateur de Hodge-de Rham sur les formes différentielles dans le cas d'effondrement d'une partie d'une variété construite par recollement de deux variétés bord isométrique. Ce résultat apporte un nouvel éclairage aux questions de *blowing up conical singularities* introduites par Mazzeo [Maz06] et Rowlett [Row06, Row08].

1. INTRODUCTION.

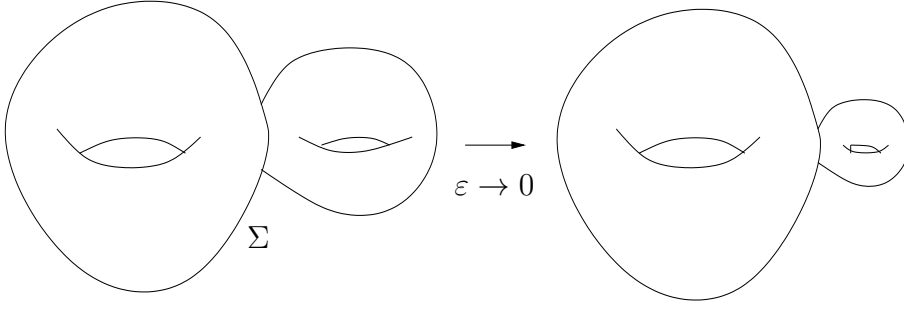
This work takes place in the general context of the spectral studies of singular perturbations of the metrics, as a manner to know what are the topological or metrical meanings carried by the spectrum of geometric operators. We can mention in this direction, without exhaustivity, studies on the adiabatic limits ([MM90],[Ru00]), on collapsing ([F87], [Lo02a, Lo02b]), on resolution blowups of conical singularities ([Maz06],[Row06, Row08]) and on shrinking handles ([AC95, ACP09]).

The present study can be considered as a generalization of the results of [AT09], where we studied the limit of the spectrum of the Hodge-de Rham (or the Hodge-Laplace) operator under collapsing of one part of a connected sum.

In our previous work, we restricted the submanifold Σ , used to glue the two parts, to be a sphere. In fact, this problem is quite related to resolution blowups of conical singularities: the point is to measure the influence of the topology of the part which disappears and of the conical singularity created at the limit of the 'big part'. If we look at the situation from the 'small part', we understand the importance of the *quasi-asymptotically conical space* obtained from rescaling the small part and gluing an infinite cone.

Date: February 21, 2011,

2000 *Mathematics Subject Classification.* Primary 58J50; Secondary 35P15, 53C23, 58J32. *Key Words and Phrases.* Laplacian, Hodge-de Rham operator, differential forms, eigenvalue, collapsing of Riemannian manifolds, manifold with conical singularity, Atiyah-Patodi-Singer type boundary condition.

FIGURE 1. partial collapsing of M_ε

When $\Sigma = \mathbb{S}^n$, the conical singularity is quite simple, and there is no *semi-bounded states*, called extended solutions in the sequel, on the quasi-asymptotically conical space, our result presented here takes care of this new possibilities and gives a general answer to the problem studied by Mazzeo and Rowlett. Indeed, in [Maz06, Row06, Row08], it is supposed that the spectrum of the operator on the quasi-asymptotically conical space does not meet 0. Our study relax this hypothesis. It is done only with the Hodge-de Rham operator, but can easily be generalized.

Let us fix some notations.

1.1. Set up. Let M_1 and M_2 be two connected manifolds with the same boundary Σ , a compact manifold of dimension $n \geq 2$. We denote by $m = n + 1$ the dimension of M_1 and M_2 .

We endow Σ with a fixed metric h .

Let \overline{M}_1 be the manifold with conical singularity obtained from M_1 by gluing M_1 to a cone $\mathcal{C} = [0, 1) \times \Sigma \ni (r, y)$: *there exists on $\overline{M}_1 = M_1 \cup \mathcal{C}$ a metric \overline{g}_1 which writes, on the smooth part $r > 0$ of the cone as $dr^2 + r^2h$.*

We choose on M_2 a metric g_2 which is ‘trumpet like’, *i.e. M_2 is isomorphic near the boundary to $[0, 1/2) \times \Sigma$ with the conical metric which writes $ds^2 + (1 - s)^2h$, if s is the coordinate defining the boundary by $s = 0$.*

For any ε , $0 < \varepsilon < 1$, we define

$$\mathcal{C}_\varepsilon = \{(r, y) \in \mathcal{C} \mid r > \varepsilon\} \quad \text{and} \quad M_1(\varepsilon) = M_1 \cup \mathcal{C}_\varepsilon.$$

The goal of the following calculus is to determine the limit spectrum of the Hodge-de Rham operator acting on the differential forms of the Riemannian manifold

$$M_\varepsilon = M_1(\varepsilon) \cup_{\varepsilon.\Sigma} \varepsilon.M_2$$

obtained by gluing together $(M_1(\varepsilon), g_1)$ and $(M_2, \varepsilon^2 g_2)$. We remark that, by construction, these two manifolds have isometric boundary and that the metric g_ε obtained on M_ε is smooth.

Remark 1. *The common boundary Σ of dimension n has some topological obstructions. In fact, since Σ is the boundary of oriented compact manifold M_1 , Σ is oriented cobordant to zero. So, by Thom’s cobordism theory, all the Stiefel-Whitney*

and all the Pontrjagin numbers vanish (cf. C. T. C. Wall [Wa60] or [MS74], §18, p.217). Furthermore, this condition is also sufficient, that is, the inverse does hold. Especially, it is impossible to take Σ^{4k} as the complex projective spaces $\mathbb{C}\mathbb{P}^{2k}$, ($k \geq 1$), because the Pontrjagin number $p_k(\mathbb{C}\mathbb{P}^{2k}) \neq 0$.

1.2. Results. We can describe the limit spectrum as follows: it has two parts. One comes from the big part, namely \bar{M}_1 , and is exprimed by the spectrum of a good extension of the Hodge-de Rham operator on this manifold with conical singularities. This extension is self-adjoint and comes from an extension of the Gauß-Bonnet operators. All these extensions are classified by subspaces W of the total eigenspace corresponding to eigenvalues within $(-1/2, 1/2)$ of an operator A acting on the boundary Σ , this point is developed below in Section 2.2. The other part comes from the collapsing part, namely M_2 , where the limit Gauß-Bonnet operator is taken with boundary conditions of the Atiyah-Patodi-Singer type. This point is developed below in Section 2.3. This operator, denoted \mathcal{D}_2 in the sequel, can also be seen on the quasi-asymptotically conical space \widetilde{M}_2 already mentioned, namely

$$\widetilde{M}_2 = M_2 \cup \left([1, \infty) \times \Sigma \right).$$

with the metric $dr^2 + r^2h$ on the conical part. Only the eigenvalue zero is concerned with this part. In fact, the manifolds M_ε has small eigenvalues, in the difference with [AT09], and the *multiplicity of 0* at the limit corresponds to the total eigenspace of these small, or null eigenvalues. Thus, our main theorem, which asserts the convergence of the spectrum, has two components.

Theorem A. *If the limit value $\lambda \neq 0$, then it belongs to the positive spectrum of the Hodge-de Rham operator $\Delta_{1,W}$ on \bar{M}_1 , with*

$$W = \bigoplus_{|\gamma| < \frac{1}{2}} \text{Ker}(A - \gamma).$$

Theorem B. *The multiplicity of 0 in the limit spectrum is given by the sum*

$$\dim \text{Ker } \Delta_{1,W} + \dim \text{Ker } \mathcal{D}_2 + i_{1/2},$$

where $i_{1/2}$ denotes the dimension of the vector space $\mathcal{I}_{1/2}$, see (3), of extended solutions ω on \widetilde{M}_2 introduced by Carron [C01a], admitting on restriction to $r = 1$ a non-trivial component in $\text{Ker}(A - \frac{1}{2})$.

1.3. Comments. We choose a simple metric to make explicit computations. This fact is not a restriction, as already explained in [AT09], because of the result of Dodziuk [D82] which assures uniform control of the eigenvalues of geometric operators with regard to variations of the metric.

2. GAUSS-BONNET OPERATOR.

On a Riemannian manifold, the Gauß-Bonnet operator is defined as the operator $D = d + d^*$ acting on differential forms. It is symmetric and can have few closed extensions on manifolds with boundary or with conical singularities. We review these extensions in the cases involved in our study.

2.1. Gauß-Bonnet operator on M_ε . We recall that, on M_ε , a Gauß-Bonnet operator D_ε , Sobolev spaces and also a Hodge-de Rham operator Δ_ε can be defined as a general construction on any manifold $X = X_1 \cup X_2$, which is the union of two Riemannian manifolds with isometric boundaries (the details are given in [AC95]): if D_1 and D_2 are the Gauß-Bonnet “ $d + d^*$ ” operators acting on the differential forms of each part, the quadratic form

$$q(\varphi) = \int_{X_1} |D_1(\varphi \upharpoonright_{X_1})|^2 d\mu_{X_1} + \int_{X_2} |D_2(\varphi \upharpoonright_{X_2})|^2 d\mu_{X_2}$$

is well-defined and closed on the domain

$$\mathcal{D}(q) = \{\varphi = (\varphi_1, \varphi_2) \in H^1(\Lambda T^* X_1) \times H^1(\Lambda T^* X_2), \varphi_1 \upharpoonright_{\partial X_1} \stackrel{L_2}{=} \varphi_2 \upharpoonright_{\partial X_2}\}$$

and on this space the total Gauß-Bonnet operator $D(\varphi) = (D_1(\varphi_1), D_2(\varphi_2))$ is defined and self-adjoint. For this definition, we have, in particular, to identify $(\Lambda T^* X_1) \upharpoonright_{\partial X_1}$ and $(\Lambda T^* X_2) \upharpoonright_{\partial X_2}$. This can be done by decomposing the forms in tangential and normal part (with inner normal), the equality above means then that the tangential parts are equal and the normal part opposite. This definition generalizes the definition in the smooth case.

The Hodge-de Rham operator $(d + d^*)^2$ of X is then defined as the operator obtained by the polarization of the quadratic form q . This gives compatibility conditions between φ_1 and φ_2 on the common boundary. We do not give details on these facts, because our manifold is smooth. But we shall use this presentation for the quadratic form.

2.2. Gauß-Bonnet operator on \overline{M}_1 . Let $D_{1,\min}$ be the closure of the Gauß-Bonnet operator defined on the smooth forms with compact support in the smooth part $M_1(0)$. For any such form φ_1 , we write on the cone

$$\varphi_1 = dr \wedge r^{-(n/2-p+1)} \beta_{1,\varepsilon} + r^{-(n/2-p)} \alpha_{1,\varepsilon}$$

and define $\sigma_1 = (\beta_1, \alpha_1) = U(\varphi_1)$. The operator has, on the cone \mathcal{C} , the expression

$$UD_1U^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left(\partial_r + \frac{1}{r}A \right) \quad \text{with} \quad A = \begin{pmatrix} \frac{n}{2} - P & -D_0 \\ -D_0 & P - \frac{n}{2} \end{pmatrix},$$

where P is the operator of degree which multiplies by p per a p -form, and D_0 is the Gauß-Bonnet operator on the manifold (Σ, h) .

While the Hodge-de Rham operator has, in these coordinates, the expression

$$U\Delta_1U^* = -\partial_r^2 + \frac{1}{r^2}A(A+1). \quad (1)$$

The closed extensions of the operator $D_1 = d + d^*$ on the manifold with conical singularities \overline{M}_1 has been studied in [BS88] and [Le97]. They are classified by the spectrum of its *Mellin symbol*, which is here the operator with parameter $A + z$.

Spectrum of A . — The spectrum of A was calculated in Brüning and Seeley [BS88], p.703. By their result, the spectrum of A is given by the values

$$\begin{cases} \pm(p - \frac{n}{2}) \text{ with multiplicity } \dim H^p(\Sigma) \text{ and} \\ \pm\frac{1}{2} \pm \sqrt{\mu^2 + \left(\frac{n-1}{2} - p\right)^2}, \end{cases} \quad (2)$$

where p is any integer, $0 \leq p \leq n$ and μ^2 runs over the spectrum of the Hodge-de Rham operator on (Σ, h) acting on the coexact p -forms.

Indeed, looking at the Gauß-Bonnet operator acting on even forms, they identify even forms on the cone with the sections $(\varphi_0, \dots, \varphi_n)$ of the total bundle $T^*(\Sigma)$ by $\varphi_0 + \varphi_1 \wedge dr + \varphi_2 + \varphi_3 \wedge dr + \dots$. These sections can as well represent odd forms on the cone by $\varphi_0 \wedge dr + \varphi_1 + \varphi_2 \wedge dr + \varphi_3 + \dots$. With these identifications, they have to study the spectrum of the following operator acting on sections of $\Lambda T^*(\Sigma)$

$$S_0 = \begin{pmatrix} c_0 & d_0^* & 0 & \cdots & 0 \\ d_0 & c_1 & d_0^* & \ddots & \vdots \\ 0 & d_0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & c_{n-1} & d_0^* \\ 0 & \cdots & 0 & d_0 & c_n \end{pmatrix},$$

if $c_p = (-1)^{p+1}(p - \frac{n}{2})$. With the same identification, if we introduce the operator \tilde{S}_0 having the same formula but with on the diagonal the terms $\tilde{c}_p = (-1)^p(p - \frac{n}{2}) = -c_p$, the operator A can be written as

$$A = - \left(S_0 \oplus \tilde{S}_0 \right).$$

The expression of the spectrum of A is then a direct consequence of the computations of [BS88].

Closed extensions of D_1 . — If $\text{spec}(A) \cap (-\frac{1}{2}, \frac{1}{2}) = \emptyset$, then $D_{1,\max} = D_{1,\min}$. In particular, D_1 is essentially self-adjoint on the space of smooth forms with compact support *away* from the conical singularities.

Otherwise, the quotient $\text{Dom}(D_{1,\max}) / \text{Dom}(D_{1,\min})$ is isomorphic to

$$B := \bigoplus_{|\gamma| < \frac{1}{2}} \text{Ker}(A - \gamma).$$

More precisely, by Lemma 3.2 of [BS88], there is a surjective linear map

$$\mathcal{L} : \text{Dom}(D_{1,\max}) \rightarrow B$$

with $\text{Ker } \mathcal{L} = \text{Dom}(D_{1,\min})$. Furthermore, we have the estimate

$$\|u(t) - t^{-A}\mathcal{L}(\varphi)\|_{L^2(\Sigma)}^2 \leq C(\varphi) |t \log t|$$

for $\varphi \in \text{Dom}(D_{1,\max})$ and $u = U(\varphi)$.

Now, for any subspace $W \subset B$, we can associate the operator $D_{1,W}$ with the domain $\text{Dom}(D_{1,W}) := \mathcal{L}^{-1}(W)$. As a result of [BS88], all closed extensions of $D_{1,\min}$ are obtained by this way. Remark that each $D_{1,W}$ defines a self-adjoint extension

$\Delta_{1,W} = (D_{1,W})^* \circ D_{1,W}$ of the Hodge-de Rham operator, and, as a result, we have $(D_{1,W})^* = D_{1,\mathbb{I}(W^\perp)}$, where

$$\mathbb{I} = \begin{pmatrix} 0 & \text{id} \\ -\text{id} & 0 \end{pmatrix}, \quad \text{i.e.} \quad \mathbb{I}(\beta, \alpha) = (\alpha, -\beta).$$

This extension is associated to the quadratic form $\varphi \mapsto \|D\varphi\|_{L^2}^2$ with the domain $\text{Dom}(D_{1,W})$.

Finally, we recall the results of Lesch [Le97]. The operators $D_{1,W}$, and in particular $D_{1,\min}$ and $D_{1,\max}$, are elliptic and satisfy the singular estimate (SE), see page 54 of [Le97], so by Proposition 1.4.6 and the compactness of \overline{M}_1 they satisfy the *Rellich property*: the inclusion of $\text{Dom}(D_{1,W})$ into $L^2(\overline{M}_1)$ is compact.

2.3. Gauß-Bonnet operator on M_2 . We know, by the works of Carron [C01a, C01b], following Atiyah-Patodi-Singer [APS75], that the operator D_2 admits a closed extension \mathcal{D}_2 with the domain defined by the global boundary condition

$$\Pi_{\leq 1/2} \circ U = 0,$$

if Π_I is the spectral projector of A relative to the interval I , and $\leq 1/2$ denotes the interval $(-\infty, 1/2]$. Moreover, this extension is elliptic in the sense that the H^1 -norm of elements of the domain is controlled by the norm of the graph. Indeed this boundary condition is related to a problem on a complete unbounded manifold as follows:

Let \widetilde{M}_2 denote the large manifold obtained from M_2 by gluing a conical cylinder $\mathcal{C}_{1,\infty} = [1, \infty) \times \Sigma$ with metric $dr^2 + r^2h$ and \widetilde{D}_2 its Gauß-Bonnet operator. A differential form on M_2 admits a harmonic L^2 extension on \widetilde{M}_2 precisely, when the restriction on the boundary satisfies $\Pi_{\leq 1/2} \circ U = 0$.

Indeed, from the harmonicity, these L^2 -forms must satisfy $(\partial_r + \frac{1}{r}A)\sigma = 0$ or, if we decompose the form associated with the eigenspaces of A as $\sigma = \sum_{\gamma \in \text{Spec}(A)} \sigma^\gamma$, then the equation imposes that for all $\gamma \in \text{Spec}(A)$ there exists $\sigma_0^\gamma \in \text{Ker}(A - \gamma)$ such that $\sigma^\gamma = r^{-\gamma}\sigma_0^\gamma$. This expression is in $L^2(\mathcal{C}_{1,\infty})$ if and only if $\gamma > 1/2$ or $\sigma_0^\gamma = 0$.

It will be convenient to introduce the harmonic L^2 extension operator

$$\begin{aligned} P_2 : \Pi_{> \frac{1}{2}} \left(H^{1/2}(\Sigma) \right) &\rightarrow L^2(\Lambda T^* \mathcal{C}_{1,\infty}) \\ \sigma = \sum_{\substack{\gamma \in \text{Spec}(A) \\ \gamma > \frac{1}{2}}} \sigma_\gamma &\mapsto P_2(\sigma) = U^* \left(\sum_{\substack{\gamma \in \text{Spec}(A) \\ \gamma > \frac{1}{2}}} r^{-\gamma} \sigma_\gamma \right). \end{aligned}$$

This limit problem is of the category *non-parabolic at infinity* in the terminology of Carron, see particularly Theorem 2.2 of [C01a] and Proposition 5.1 of [C01b], then as a consequence of Theorem 0.4 of [C01a], we know that the kernel of \mathcal{D}_2 is of finite dimension and that the graph norm of the operator controls the H^1 -norm (Theorem 2.1 of [C01a]).

Proposition 2. *There exists a constant $C > 0$ such that for each differential form $\varphi \in H^1(\Lambda T^* M_2)$ satisfying the boundary condition $\Pi_{\leq 1/2} \circ U(\varphi) = 0$, then*

$$\|\varphi\|_{H^1(M_2)}^2 \leq C \left\{ \|\varphi\|_{L^2(M_2)}^2 + \|D_2\varphi\|_{L^2(M_2)}^2 \right\}.$$

As a consequence, the kernel of \mathcal{D}_2 , which is isomorphic to $\text{Ker } \widetilde{D}_2$, is of finite dimension and can be sent in the total space $\sum_p H^p(M_2)$ of the absolute cohomology.

A proof of this proposition can be obtained by the same way as Proposition 5 in [AT09].

Recall that Carron defined, for this type of operators, behind the L^2 solutions of $\widetilde{D}_2(\varphi) = 0$ which corresponds to the solutions of the elliptic operator of Proposition 2, *extended* solutions which are included in the bigger space \mathcal{W} which is defined as the closure of the space of smooth p -forms with compact support $\Omega_0^p(\widetilde{M}_2)$ for the norm

$$\|\varphi\|_{\mathcal{W}}^2 := \|\varphi\|_{L^2(M_2)}^2 + \|D_2\varphi\|_{L^2(\widetilde{M}_2)}^2.$$

A Hardy-type inequality describes the growth at infinity of an extended solution.

Lemma 3. *For a function $v \in C_0^\infty(e, \infty)$ and a real number λ , we have*

- if $\lambda \neq -\frac{1}{2}$, $(\lambda + \frac{1}{2})^2 \int_e^\infty \frac{v^2}{r^2} dr \leq \int_e^\infty \frac{1}{r^{2\lambda}} |\partial_r(r^\lambda v)|^2 dr$,
- if $\lambda = -\frac{1}{2}$, $\int_e^\infty \frac{v^2}{|r \log r|^2 \log(\log r)} dr \leq \int_e^\infty r |\partial_r(r^{-1/2} v)|^2 dr$.

We remark now that, for $\varphi \in \Omega_0^p(\widetilde{M}_2)$ with support in the infinite cone $\mathcal{C}_{e, \infty}$, we can write

$$\begin{aligned} \|D_2\varphi\|_{L^2(\widetilde{M}_2)}^2 &= \sum_{\lambda \in \text{Spec}(A)} \int_e^\infty \|(\partial_r + \frac{\lambda}{r})\sigma_\lambda\|_{L^2(\Sigma)}^2 dr \\ &= \sum_{\lambda \in \text{Spec}(A)} \int_e^\infty \frac{1}{r^{2\lambda}} \|\partial_r(r^\lambda \sigma_\lambda)\|_{L^2(\Sigma)}^2 dr. \end{aligned}$$

Thus, by application of Lemma 3, we see that a solution of \widetilde{D}_2 , which must be $\sigma_\lambda(r) = r^{-\lambda}\sigma_\lambda(1)$ on the infinite cone, satisfies the condition of Lemma 3 for $\lambda = -1/2$: if $v = r^{1/2}v_0$ for r large then the integral $\int \frac{v^2}{|r \log r|^2 \log(\log r)} dr$ is convergent, and if we require that $\frac{1}{r}\varphi$ is in L^2 then for any $\lambda < -\frac{1}{2}$

$$\sigma_\lambda(1) = 0,$$

while the L^2 solutions correspond to the condition $\sigma_\lambda(1) = 0$ for any $\lambda \leq \frac{1}{2}$. As a consequence, the extended solutions which are not L^2 correspond to boundary terms with components in the total eigenspace related with eigenvalues of A in the interval $[-1/2, 1/2]$. In the case studied in [AT09], there was not such eigenvalues and we had not to take care of extended solutions.

To be more precise, we must introduce the operator (see **2.a** in [C01b])

$$\begin{aligned} T : H^{k+1/2}(\Sigma) &\rightarrow H^{k-1/2}(\Sigma) \\ \sigma &\mapsto D_2(\mathcal{E}(\sigma)) \upharpoonright_\Sigma, \end{aligned}$$

where $\mathcal{E}(\sigma)$ is the solution of the Poisson problem

$$(D_2)^2(\mathcal{E}(\sigma)) = 0 \text{ on } M_2 \text{ and } \mathcal{E}(\sigma) \upharpoonright_\Sigma = \sigma \text{ on } \partial M_2.$$

Carron proved that this operator is continuous for $k \geq 0$. The L^2 solutions correspond to boundary values in $\text{Ker}(T) \cap \text{Im}(\Pi_{>1/2})$, while extended solutions correspond to the space $\text{Ker}(T) \cap \text{Im}(\Pi_{\geq -1/2})$. Let us denote by

$$\mathcal{I}_{1/2} := \left(\text{Ker}(T) \cap \text{Im}(\Pi_{\geq 1/2}) \right) / \left(\text{Ker}(T) \cap \text{Im}(\Pi_{>1/2}) \right) \quad (3)$$

the subspace of extended solutions with non-trivial component on $\text{Ker}(A - 1/2)$.

Proof of Lemma 3. Let $u \in C_0^\infty(e, \infty)$ and ρ a positive function on the interval (e, ∞) . We calculate

$$\begin{aligned} \int_e^\infty \frac{1}{r^{2\lambda}} |\partial_r(r^\lambda \rho u)|^2 dr &= \int_e^\infty \frac{1}{r^{2\lambda}} \left| \rho' r^\lambda u + \rho \partial_r(r^\lambda u) \right|^2 dr \\ &= \int_e^\infty \left\{ |\rho' u|^2 + \frac{\rho^2}{r^{2\lambda}} |\partial_r(r^\lambda u)|^2 + \frac{\rho \rho'}{r^{2\lambda}} \partial_r(|r^\lambda u|^2) \right\} dr. \end{aligned}$$

By integrating by parts the last term, we obtain

$$\begin{aligned} \int_e^\infty \frac{1}{r^{2\lambda}} |\partial_r(r^\lambda \rho u)|^2 dr &= \int_e^\infty \left\{ \frac{\rho^2}{r^{2\lambda}} |\partial_r(r^\lambda u)|^2 - (\rho \rho'' - 2\lambda \frac{\rho \rho'}{r}) u^2 \right\} dr \\ &\geq \int_e^\infty \frac{2\lambda \rho' - r \rho''}{r \rho} |\rho u|^2 dr. \end{aligned}$$

We obtain finally the first result of the lemma by setting $\rho(r) = r^{\lambda+1/2}$ and $v = \rho u$.

The second one is obtained in the same way with $\lambda = -1/2$ and the choice of $\rho = \log(\log r)$. \square

3. A PRIORI ESTIMATES.

A good method to evaluate what the limit problem should be is to suppose a priori that we have a normalized family φ_ε of eigenforms on M_ε of degree p for the Hodge-de Rham operator:

$$\Delta_\varepsilon \varphi_\varepsilon = \lambda_\varepsilon \varphi_\varepsilon \text{ with } \lim_{\varepsilon \rightarrow 0} \lambda_\varepsilon = \lambda < +\infty$$

and to try to obtain all the consequences for the limit. Of course, for the moment we are not sure that this family exists but the min-max techniques will assure this, and it will be the subject of the next section.

3.1. Notations. As in [ACP09], we use the following change of variables : with

$$\varphi_{1,\varepsilon} := \varphi_\varepsilon \upharpoonright_{M_1(\varepsilon)} \quad \text{and} \quad \varphi_{2,\varepsilon} := \varepsilon^{m/2-p} \varphi_\varepsilon \upharpoonright_{M_2(1)}.$$

We write on the cone

$$\varphi_{1,\varepsilon} = dr \wedge r^{-(n/2-p+1)} \beta_{1,\varepsilon} + r^{-(n/2-p)} \alpha_{1,\varepsilon}$$

and define $\sigma_1 = (\beta_1, \alpha_1) = U(\varphi_1)$.

On the other part, it is more convenient to define $r = 1 - s$ for $s \in [0, 1/2]$ and write $\varphi_{2,\varepsilon} = (dr \wedge r^{-(n/2-p+1)} \beta_{2,\varepsilon} + r^{-(n/2-p)} \alpha_{2,\varepsilon})$ near the boundary. Then we can define, for $r \in [1/2, 1]$ (the boundary of $M_2(1)$ corresponds to $r = 1$)

$$\sigma_2(r) = (\beta_2(r), \alpha_2(r)) = U(\varphi_2)(r).$$

The L^2 -norm, for a form supported on M_1 in the cone $\mathcal{C}_{\varepsilon,1}$, has the expression

$$\|\varphi\|_{L^2(M_\varepsilon)}^2 = \int_{M_1} |\sigma_1|^2 d\mu_{g_1} + \int_{M_2} |\varphi_2|^2 d\mu_{g_2}$$

and the quadratic form on our study is

$$\begin{aligned} q(\varphi) &= \int_{M_\varepsilon} |(d + d^*)\varphi|_{g_\varepsilon}^2 d\mu_{g_\varepsilon} \\ &= \int_{M_1(\varepsilon)} |UD_1U^*(\sigma_1)|^2 d\mu_{g_1} + \frac{1}{\varepsilon^2} \int_{M_2(1)} |D_2(\varphi_2)|^2 d\mu_{g_2}. \end{aligned} \quad (4)$$

The compatibility condition is, for the quadratic form, $\varepsilon^{1/2}\alpha_1(\varepsilon) = \alpha_2(1)$ and $\varepsilon^{1/2}\beta_1(\varepsilon) = \beta_2(1)$ or

$$\sigma_2(1) = \varepsilon^{1/2}\sigma_1(\varepsilon). \quad (5)$$

The compatibility condition for the Hodge-de Rham operator, of first order, is obtained by expressing that $D\varphi \sim (UD_1U^*\sigma_1, \frac{1}{\varepsilon}UD_2U^*\sigma_2)$ belongs to the domain of D . In terms of σ , it gives

$$\sigma'_2(1) = \varepsilon^{3/2}\sigma'_1(\varepsilon). \quad (6)$$

To understand the limit problem, we proceed to several estimates.

3.2. On the regular part of M_1 . Let ξ_1 be a cut-off function on M_1 around the conical singularity:

$$\xi_1(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1/2, \\ 0 & \text{if } 1 \leq r. \end{cases} \quad (7)$$

Lemma 4. *For our given family φ_ε satisfying $\Delta\varphi_\varepsilon = \lambda_\varepsilon\varphi_\varepsilon$ with λ_ε bounded, the family $\{(1 - \xi_1) \cdot \varphi_{1,\varepsilon}\}_{\varepsilon>0}$ is bounded in $H^1(M_1)$.*

Then it remains to study $\xi_1 \cdot \varphi_{1,\varepsilon}$ which can be expressed with the polar coordinates. We remark that the quadratic form of these forms is uniformly bounded.

• *Expression of the quadratic form.* For any φ such that the component φ_1 is supported in the cone $\mathcal{C}_{1,\varepsilon}$, one has, with $\sigma_1 = U(\varphi_1)$ and by the same calculus as in [ACP09] :

$$\begin{aligned} \int_{\mathcal{C}_{\varepsilon,1}} |D_1\varphi|^2 d\mu_{g_\varepsilon} &= \int_\varepsilon^1 \left\| \left(\partial_r + \frac{1}{r}A \right) \sigma_1 \right\|_{L^2(\Sigma)}^2 dr \\ &= \int_\varepsilon^1 \left[\|\sigma'_1\|_{L^2(\Sigma)}^2 + \frac{2}{r} (\sigma'_1, A\sigma_1)_{L^2(\Sigma)} + \frac{1}{r^2} \|A\sigma_1\|_{L^2(\Sigma)}^2 \right] dr. \end{aligned}$$

3.3. Estimates of the boundary term. The expression above can be decomposed with respect to the eigenspaces of A ; in the following calculus, we suppose that

$\sigma_1(1) = 0$:

$$\begin{aligned} & \int_{\varepsilon}^1 \left[\|\sigma_1'\|_{L^2(\Sigma)}^2 + \frac{2}{r}(\sigma_1', A\sigma_1)_{L^2(\Sigma)} + \frac{1}{r^2}\|A\sigma_1\|_{L^2(\Sigma)}^2 \right] dr \\ &= \int_{\varepsilon}^1 \left[\|\sigma_1'\|_{L^2(\Sigma)}^2 + \partial_r \left(\frac{1}{r}(\sigma_1, A\sigma_1)_{L^2(\Sigma)} \right) + \frac{1}{r^2} \left\{ (\sigma_1, A\sigma_r)_{L^2(\Sigma)} + \|A\sigma_1\|_{L^2(\Sigma)}^2 \right\} \right] dr \\ &= \int_{\varepsilon}^1 \left[\|\sigma_1'\|_{L^2(\Sigma)}^2 + \frac{1}{r^2}(\sigma_1, (A + A^2)\sigma_1)_{L^2(\Sigma)} \right] dr - \frac{1}{\varepsilon}(\sigma_1(\varepsilon), A\sigma_1(\varepsilon))_{L^2(\Sigma)}. \end{aligned}$$

This shows that the quadratic form controls the boundary term, if the operator A is negative but $(A + A^2)$ is non-negative. This last condition is satisfied exactly on the orthogonal complement of the spectral space corresponding to the interval $(-1, 0)$. By applying $\xi_1 \cdot \varphi_{1,\varepsilon}$ to this fact, we obtain the following lemma:

Lemma 5. *Let $\Pi_{\leq -1}$ be the spectral projection of the operator A relative to the interval $(-\infty, -1]$. There exists a constant $C > 0$ such that, for any $\varepsilon > 0$*

$$\|\Pi_{\leq -1} \circ U(\varphi_{1,\varepsilon}(\varepsilon))\|_{H^{1/2}(\Sigma)} \leq C\sqrt{\varepsilon}.$$

In view of Proposition 2, we want also a control of the components of σ_1 along the eigenvalues of A in $(-1, 1/2]$. The number of these components are finite and we can work term by term. So we write, on $\mathcal{C}_{\varepsilon,1}$,

$$\sigma_1(r) = \sum_{\gamma \in \text{Spec}(A)} \sigma_1^\gamma(r) \text{ with } A\sigma_1^\gamma(r) = \gamma\sigma_1^\gamma(r)$$

and we suppose again $\sigma_1(1) = 0$. From the equation $(\partial_r + A/r)\sigma_1^\gamma = r^{-\gamma}\partial_r(r^\gamma\sigma_1^\gamma)$ and the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} \|\varepsilon^\gamma\sigma_1^\gamma(\varepsilon)\|_{L^2(\Sigma)}^2 &= \left\{ \int_{\varepsilon}^1 \|\partial_r(r^\gamma\sigma_1^\gamma)\|_{L^2(\Sigma)} dr \right\}^2 \\ &\leq \left\{ \int_{\varepsilon}^1 \left\| r^\gamma \cdot \left(\partial_r + \frac{1}{r}A \right) \sigma_1^\gamma(r) \right\|_{L^2(\Sigma)} dr \right\}^2 \\ &\leq \int_{\varepsilon}^1 r^{2\gamma} dr \cdot \int_{\varepsilon}^1 \left\| \partial_r(\sigma_1^\gamma) + \frac{\gamma}{r}(\sigma_1^\gamma) \right\|_{L^2(\Sigma)}^2 dr. \end{aligned}$$

Thus, if the quadratic form is bounded, there exists a constant $C > 0$ such that

$$\|\sigma_1^\gamma(\varepsilon)\|_{L^2(\Sigma)}^2 \leq \begin{cases} C\varepsilon^{-2\gamma} \frac{1 - \varepsilon^{2\gamma+1}}{2\gamma + 1} & \text{if } \gamma \neq -\frac{1}{2} \\ C\varepsilon |\log \varepsilon| & \text{if } \gamma = -\frac{1}{2}. \end{cases} \quad (8)$$

This gives

Lemma 6. *Let Π_I be the spectral projector of the operator A relative to the interval I . There exist constants $\alpha, C > 0$ such that, for any $\varepsilon > 0$*

$$\|\Pi_{(-1,0)} \circ U(\varphi_{1,\varepsilon}(\varepsilon))\|_{H^{1/2}(\Sigma)} \leq C\varepsilon^\alpha.$$

If $0 < \alpha < 1/2$, then $(-\alpha)$ is bigger than any negative eigenvalue of A . The estimate above gives also

Lemma 7. *With the same notation, there exist constants $\beta, C > 0$ such that, for any $\varepsilon > 0$*

$$\|\Pi_{[0,1/2)} \circ U(\varphi_{2,\varepsilon}(1))\|_{H^{1/2}(\Sigma)} \leq C\varepsilon^\beta.$$

Here, $(1/2 - \beta)$ is the biggest non-negative eigenvalue of A strictly smaller than $1/2$ (if there is no such eigenvalue, we put $\beta = 1/2$).

Finally, we study $\sigma_1^{1/2}$ for our family of forms (the parameter ε is omitted in the notation). It satisfies, for $\varepsilon < r < 1/2$, the equation

$$\left(-\partial_r^2 + \frac{3}{4r^2}\right)\sigma_1^{1/2} = \lambda_\varepsilon\sigma_1^{1/2}.$$

The solutions of this equation have expression in term of the Bessel functions: there exist entire functions F, G with $F(0) = G(0) = 1$ and differential forms in $\text{Ker}(A - 1/2)$ $c_\varepsilon, d_\varepsilon$ such that

$$\sigma_1^{1/2}(r) = c_\varepsilon r^{3/2} F(\lambda_\varepsilon r^2) + d_\varepsilon \left(r^{-1/2} G(\lambda_\varepsilon r^2) + \frac{2}{\pi} \log(r) r^{3/2} F(\lambda_\varepsilon r^2) \right). \quad (9)$$

The fact that the L^2 -norm is bounded gives that $c_\varepsilon^2 + |\log \varepsilon| d_\varepsilon^2$ is bounded and finally, reporting this estimate in the expression above, that

$$\|\sigma_1^{1/2}(\varepsilon)\|_{H^{1/2}(\Sigma)} = O\left(\frac{1}{\varepsilon |\log \varepsilon|}\right).$$

This gives on the other part

Lemma 8. *There exists a constant $C > 0$ such that, for any $\varepsilon > 0$*

$$\|\Pi_{\{1/2\}} \circ U(\varphi_{2,\varepsilon})(1)\|_{H^{1/2}(\Sigma)} \leq C \frac{1}{\sqrt{|\log \varepsilon|}}.$$

4. PROOF OF THE SPECTRAL CONVERGENCE.

The previous estimates show that $\Pi_{\leq 1/2} \circ U(\varphi_{2,\varepsilon})(1)$ goes to zero with ε and that the limit would satisfy the good APS-boundary conditions ; on the other hand, there is no restriction for the boundary term of $U(\varphi_{1,\varepsilon})(\varepsilon)$ and finally the control on $\Pi_{\{1/2\}} \circ U(\varphi_{2,\varepsilon})(1)$ does not permit to admit that this term will disappear.

It suggests that we have to introduce for the limit problem, as the Hilbert space

$$\mathcal{H}_\infty := L^2(\overline{M}_1) \oplus \text{Ker } \tilde{D}_2 \oplus \mathcal{I}_{1/2} \quad (10)$$

with the space $\mathcal{I}_{1/2}$ defined in (3), and as limit operator $\Delta_{1,W} \oplus 0 \oplus 0$ with W defined in Theorem A. Let us denote by λ_N ($N \geq 1$) its spectrum and also let us $\lambda_N(\varepsilon)$ ($N \geq 1$) be the total spectrum of the Hodge-de Rham operator on M_ε .

4.1. Upper bound : $\limsup \lambda_N(\varepsilon) \leq \lambda_N$. With the min-max formula, which says that

$$\lambda_N(\varepsilon) = \inf_{\substack{E \subset \text{dom}(D_\varepsilon) \\ \dim E = N}} \left\{ \sup_{\substack{\varphi \in E \\ \|\varphi\|=1}} \int_{M_\varepsilon} |D_\varepsilon \varphi|^2 d\mu_{g_\varepsilon} \right\},$$

we have to describe how transplanting eigenforms of the limit problem on M_ε .

We describe this transplantation term by term. For the first term, we use the same ideas as in [ACP09].

Any eigenform φ of $\Delta_{1,W}$ can be written, as any element of $\text{dom}(D_{1,\max})$,

$$\varphi = \varphi_0 + \bar{\varphi},$$

$$\varphi_0 \in \text{dom}(D_{1,\min}) \quad \text{and} \quad U(\bar{\varphi}) = \begin{cases} 0 & \text{for } r \geq 1, \\ \sum_{\gamma \in (-\frac{1}{2}, \frac{1}{2}) \cap \text{Spec}(A)} d_\gamma r^{-\gamma} \sigma_\gamma & \text{for } r \leq \frac{1}{2} \end{cases}$$

with $\sigma_\gamma \in \text{Ker}(A - \gamma)$ and $d_\gamma \in \mathbb{R}$. (This fact is a consequence of the expression of the Bessel functions, it is the same calculus as in (18).) By the definition of $D_{1,\min}$, φ_0 can be approached, with the operator norm, by a sequence of smooth forms with compact support in $M_1(0)$. On the other hand, $\bar{\varphi}$ is a finite sum, it is sufficient to prolongate each $r^{-\gamma} \sigma_\gamma$ on M_2 , the good candidate is

$$\psi_\gamma = U^*(\chi \cdot \varepsilon^{1/2-\gamma} r^{-\gamma} \sigma_\gamma)$$

for χ a cut-off function supported on $(1/2, 3/2)$ and equal to 1 on a neighbourhood of 1 (for instance, $\chi = 1 - \xi_1$).

Indeed we have

$$\|\psi_\gamma\|_{L^2(M_2)} + \|D_2(\psi_\gamma)\|_{L^2(M_2)} = O(\varepsilon^{1/2-\gamma}).$$

For the two last ones, we shrink the infinite cone on M_1 and cut with the function ξ_1 , already defined in (7).

Let us define

$$P_\varepsilon : \Pi_{>\frac{1}{2}}(H^{1/2}(\Sigma)) \rightarrow H^1(\mathcal{C}_{\varepsilon,1}) \quad (11)$$

$$\sigma = \sum_{\substack{\gamma \in \text{Spec}(A) \\ \gamma > \frac{1}{2}}} \sigma_\gamma \mapsto P_\varepsilon(\sigma) = U^* \left(\sum_{\substack{\gamma \in \text{Spec}(A) \\ \gamma > \frac{1}{2}}} \varepsilon^{\gamma-1/2} r^{-\gamma} \sigma_\gamma \right).$$

We remark that $P_\varepsilon(\sigma)$ is the transplanted on M_1 of $P_2(\sigma)$ then there exists a constant $C > 0$ such that

$$\|P_2(\sigma)\|_{L^2(\mathcal{C}_{1,1/\varepsilon})}^2 = \|P_\varepsilon(\sigma)\|_{L^2(\mathcal{C}_{\varepsilon,1})}^2 \leq C \sum \|\sigma_\gamma\|_{L^2(\Sigma)}^2 = C \|\Pi_{>\frac{1}{2}} \sigma_2(1)\|_{L^2(\Sigma)}^2 \quad (12)$$

and also that, if $\psi_2 \in \text{Dom}(D_2)$, then $(\xi_1 P_\varepsilon(U(\psi_2 \upharpoonright_\Sigma)), \psi_2)$ defines an element of $H^1(M_\varepsilon)$.

Finally, if $\text{Ker}(A - 1/2)$ is not empty, for each $\bar{\sigma}^{1/2} \in \text{Ker}(A - 1/2)$ such that there exists ψ_2 with $D_2(\psi_2) = 0$ on M_2 and boundary value $\bar{\sigma}^{1/2}$ modulo $\text{Im} \Pi_{>1/2}$, one can construct a *pseudomode* as follows:

$$\psi_\varepsilon := |\log \varepsilon|^{-1/2} \left(\xi_1 \cdot (r^{-1/2} U^*(\bar{\sigma}^{1/2}) + P_\varepsilon(U(\psi_2 \upharpoonright_\Sigma - \bar{\sigma}^{1/2}))), \psi_2 \right) \quad (13)$$

The L^2 norm of this element is bounded from above and below, and

$$\lim_{\varepsilon \rightarrow 0} \|\psi_\varepsilon\|_{L^2(M_\varepsilon)} = \|\bar{\sigma}^{1/2}\|_{L^2(\Sigma)}.$$

Moreover, it satisfies $q(\psi_\varepsilon) = O(|\log \varepsilon|^{-1})$ giving then a ‘small eigenvalue’, as well as the elements of $\ker \mathcal{D}_2$ and of $\text{Ker} \Delta_W$. [nb. It is remarkable that the same construction, for an extended solution with corresponding boundary value in $\text{Ker}(A - \gamma)$, $\gamma \in (-1/2, 1/2)$ does not give a quasimode: indeed if ψ_2 is such a

solution, the transplanted element will be, $\psi_\varepsilon = \left(\xi_1 \cdot (r^{-\gamma} U^*(\bar{\sigma}^\gamma) + \varepsilon^{(1/2-\gamma)} P_\varepsilon(U(\psi_2 \upharpoonright_\Sigma) - \bar{\sigma}^\gamma)), \varepsilon^{(1/2-\gamma)} \psi_2 \right)$ for which $q(\psi_\varepsilon)$ does not converge to 0 as ε .]

To conclude the estimate of upper bound, we have only to verify that this transplanted forms have a Rayleigh-Ritz quotient comparable to the initial one and that the orthogonality is fast conserved by transplantation.

4.2. Lower bound : $\liminf \lambda_N(\varepsilon) \geq \lambda_N$. We first proceed for one indice. We know, by the paragraph 4.1, that for each N , the family $\{\lambda_N(\varepsilon)\}_{\varepsilon>0}$ is bounded, set

$$\lambda := \liminf_{\varepsilon \rightarrow 0} \lambda_N(\varepsilon).$$

There exists a sequence $\varepsilon_m, m \in \mathbb{N}$ such that $\lim_{m \rightarrow \infty} \lambda_N(\varepsilon_m) = \lambda$. We shall apply the previous calculus to this family.

4.2.1. On the side of M_2 . Let χ be a cut-off function such that

$$\chi(r) := \begin{cases} 0 & \text{if } r \leq \frac{3}{4}, \\ 1 & \text{if } \frac{7}{8} \leq r < 1 \end{cases}$$

and $\sigma_{2,\varepsilon_m} = U(\varphi_{2,\varepsilon_m})$. We construct the family

$$\psi_{2,\varepsilon_m} := \varphi_{2,\varepsilon_m} - U^* \left(\Pi_{\leq \frac{1}{2}}(\chi \sigma_{2,\varepsilon_m}) \right),$$

which belongs to the domain of \mathcal{D}_2 , and then, by the ellipticity of this operator

Claim 1. *The family ψ_{2,ε_m} is bounded in $H^1(M_2)$, and as a consequence $\Pi_{> \frac{1}{2}} \sigma_2(1)$ is bounded in $H^{1/2}(\Sigma)$.*

Moreover, its proximity with $\varphi_{2,\varepsilon_m}$ is controlled:

Claim 2. *They satisfy*

$$\begin{aligned} (1) \quad & \lim_{\varepsilon_m \rightarrow 0} \|\psi_{2,\varepsilon_m} - \varphi_{2,\varepsilon_m}\|_{L^2} = 0, \\ (2) \quad & \|D_2(\psi_{2,\varepsilon_m} - \varphi_{2,\varepsilon_m})\|_{L^2(M_2)} = O(\sqrt{\varepsilon_m}). \end{aligned} \tag{14}$$

Indeed

$$\begin{aligned} \|\psi_{2,\varepsilon_m} - \varphi_{2,\varepsilon_m}\|^2 & \leq \int_{\frac{3}{4}}^1 \|\Pi_{\leq \frac{1}{2}} \sigma_2(r)\|_{L^2}^2 dr \\ & = \int_{\frac{3}{4}}^1 \left\{ \|\Pi_{\leq -1} \sigma_2(r)\|_{L^2(\Sigma)}^2 + \|\Pi_{(-1, \frac{1}{2}]} \sigma_2(r)\|_{L^2(\Sigma)}^2 \right\} dr. \end{aligned} \tag{15}$$

The first term is controlled as follows: because of the expression (4) of the quadratic form, we know that

$$\int_{\mathcal{C}_{\varepsilon_m,1}} \left\| \left(\partial_r + \frac{A}{r} \right) \Pi_{\leq -1} \sigma_1 \right\|_{L^2(\Sigma)}^2 dr + \frac{1}{\varepsilon_m^2} \int_R^1 \left\| \left(\partial_r + \frac{A}{r} \right) \Pi_{\leq -1} \sigma_2 \right\|_{L^2(\Sigma)}^2 dr$$

is bounded, independently on $R \geq 3/4$. Then, let Λ be a bound. Developing and making an integration by parts gives, for any κ_2

$$\begin{aligned} \int_R^1 \|(\partial_r + \frac{A}{r})\kappa_2\|^2 dr &= \int_R^1 \left[\|\kappa_2'\|^2 + \frac{1}{r^2} \langle \kappa_2, (A + A^2)\kappa_2 \rangle \right] dr \\ &+ \left(\kappa_2(1), A\kappa_2(1) \right)_{L^2(\Sigma)} - \left(\kappa_2(R), A\kappa_2(R) \right)_{L^2(\Sigma)}. \end{aligned} \quad (16)$$

We can make the same calculus on the cone $\mathcal{C}_{\varepsilon_m,1}$ so the boundary term at $R = 1$ is absorbed. If we apply this control to $\kappa_2 = \Pi_{\leq -1}\sigma_2$ for which $A(A+1)$ and $(-A)$ are non negative we obtain that

$$\int_R^1 \|\Pi_{\leq -1}\sigma_2'(t)\|_{L^2(\Sigma)}^2 dt + \|\Pi_{\leq -1}\sigma_2'(R)\|_{L^2(\Sigma)}^2 = O(\varepsilon_m^2).$$

The second term in (15) is in fact the sum of few terms, we can control each of them as follows: let γ be an eigenvalue of A contained in the interval $(-1, \frac{1}{2}]$ and σ_2^γ the component of σ_2 along this eigenspace. We know that $\int_R^1 \|r^{-\gamma}\partial_r(r^\gamma\sigma_2^\gamma)\|_{L^2(\Sigma)}^2 dr = O(\varepsilon_m^2)$.

$$\|R^\gamma\sigma_2^\gamma(R)\|_{L^2(\Sigma)}^2 = -2 \int_R^1 (\partial_r(r^\gamma\sigma_2^\gamma(t)), r^\gamma\sigma_2^\gamma(t))_{L^2(\Sigma)} dt + \|\sigma_2^\gamma(1)\|_{L^2(\Sigma)}^2. \quad (17)$$

But, as a consequence of Lemmas 5, 6 and 7, we have $\|\sigma_2^\gamma(1)\|_{L^2(\Sigma)}^2 = O(|\log \varepsilon_m|^{-1})$, on the other hand, by the Cauchy-Schwarz inequality, the fact that the L^2 -norm of φ_{ε_m} is 1 and the fact just recalled we have

$$\|\sigma_2^\gamma(R)\|_{L^2(\Sigma)}^2 = O(1/|\log(\varepsilon_m)|).$$

For the second part of the assertion, notice that

$$\begin{aligned} D_2(\varphi_{2,\varepsilon_m} - \psi_{2,\varepsilon_m}) &= D_2U^* \left(\Pi_{\leq \frac{1}{2}}(\chi\sigma_{2,\varepsilon_m}) \right) \\ &= \chi' \cdot U^* \Pi_{\leq \frac{1}{2}}(\sigma_{2,\varepsilon_m}) + \chi \cdot D_2U^* \Pi_{\leq \frac{1}{2}}(\sigma_{2,\varepsilon_m}) \end{aligned}$$

and the norm of the first term is controled by $\int_{3/4}^1 \|\Pi_{\leq \frac{1}{2}}\sigma_2(r)\|_{L^2(\Sigma)}^2 dr$, which is $O(1/|\log(\varepsilon_m)|)$ by the previous estimate, and the norm of the second term, by $\|D_2\varphi_2\|$ which is $O(\varepsilon_m)$, because $q_{\varepsilon_m}(\varphi_{\varepsilon_m})$ is uniformly bounded (remark that D_2 preserves the orthogonal decomposition following $\Pi_{\leq \frac{1}{2}}$ and $\Pi_{> \frac{1}{2}}$ on the cone).

Proposition 9. *There exists a subfamily of the family $\{\varphi_{2,\varepsilon_m}\}_m$ which converges, as $m \rightarrow \infty$ to a bounded form φ_2 on M_2 which satisfies*

$$\varphi_2 \in \text{dom } \mathcal{D}_2, \quad \|\varphi_2\| \leq 1 \text{ and } D_2(\varphi_2) = 0.$$

Proof. Indeed, we know by Claim 1 that we can extract from the family ψ_{2,ε_m} a subfamily which converge in L^2 -norm and weakly in $H^1(M_2)$, denote φ_2 the limit. We know by Claim 2 that the corresponding subfamily of $\varphi_{2,\varepsilon_m}$ has the same limit and also that $\lim_{m \rightarrow \infty} \|D_2\psi_{2,\varepsilon_m}\| = 0$, because this is true for $\varphi_{2,\varepsilon_m}$, the conclusion follows. \square

4.2.2. *On the side of M_1 .* We first recall the better prolongation of $\Pi_{> \frac{1}{2}} \sigma_2(1)$ on $M_1(\varepsilon_m)$ given by P_ε in (11) so that if $\psi_2 \in \text{Dom} \mathcal{D}_2$ and with the same cut-off function ξ_1 , which has value 1 for $0 \leq r \leq 1/2$ and 0 for $r \geq 1$, $(\xi_1 P_{\varepsilon_m}(U(\psi_2 \upharpoonright_\Sigma)), \psi_2)$ defines an element of $H^1(M_{\varepsilon_m})$. Let

$$\tilde{\psi}_{1,\varepsilon_m} := \xi_1 P_{\varepsilon_m}(U(\psi_{2,\varepsilon_m} \upharpoonright_\Sigma)).$$

Corollary 10. *By uniform continuity of P_{ε_m} , and the convergence property of the last proposition*

$$\lim_{m \rightarrow \infty} \|\tilde{\psi}_{1,\varepsilon_m} - \xi_1 P_{\varepsilon_m}(U(\varphi_2 \upharpoonright_\Sigma))\|_{L^2(M_1)} = 0.$$

On the other hand, $\xi_1 P_{\varepsilon_m}(U(\varphi_2 \upharpoonright_\Sigma))$ converges weakly to 0 on the open manifold $M_1(0)$, more precisely, for any fixed η , $0 < \eta < 1$

$$\lim_{m \rightarrow \infty} \|\xi_1 P_{\varepsilon_m}(U(\varphi_2 \upharpoonright_\Sigma))\|_{L^2(M_1(\eta))} = 0.$$

We now decompose $\varphi_{1,\varepsilon_m}$ near the singularity as follows: Let

$$\xi_1 \varphi_{1,\varepsilon_m} = \xi_1 (\varphi_{1,\varepsilon_m}^{\leq -1/2} + \varphi_{1,\varepsilon_m}^{(-1/2, 1/2]} + \varphi_{1,\varepsilon_m}^{> 1/2})$$

according to the decomposition, on the cone, of σ_1 along the eigenvalues of A respectively less than $-1/2$, in $(-1/2, 1/2]$ and larger than $1/2$, respectively.

We remark first that $\tilde{\psi}_1$ and $\varphi_{1,\varepsilon_m}^{> 1/2}$ have the same values on the boundary so the difference $\xi_1 \varphi_{1,\varepsilon_m}^{> 1/2} - \tilde{\psi}_1$ can be viewed in $H^1(M_1)$ by a prolongation by 0 on the complementary of the cone, while the boundary value of $\varphi_{1,\varepsilon_m}^{\leq -1/2}$ is small. We introduce for this term the cut-off function taken in [ACP09]

$$\xi_{\varepsilon_m}(r) = \begin{cases} 1 & \text{if } r \geq 2\sqrt{\varepsilon_m}, \\ \frac{1}{\log(\sqrt{\varepsilon_m})} \log\left(\frac{2\varepsilon_m}{r}\right) & \text{if } r \in [2\varepsilon_m, 2\sqrt{\varepsilon_m}], \\ 0 & \text{if } r \leq 2\varepsilon_m. \end{cases}$$

Claim 3. $\lim_{m \rightarrow \infty} \|(1 - \xi_{\varepsilon_m}) \xi_1 \varphi_{1,\varepsilon_m}^{\leq -1/2}\|_{L^2(M_1)} = 0$.

This is a consequence of the estimate of Lemma 5 and 6: we remark that by the same argument we obtain also $\|\xi_1 \varphi_{1,\varepsilon_m}^{\leq -1/2}(r)\|_{H^{1/2}(\Sigma)} \leq C\sqrt{r}$ so

$$\|(1 - \xi_{\varepsilon_m}) \xi_1 \varphi_{1,\varepsilon_m}^{\leq -1/2}\|_{L^2(M_1)} = O(\varepsilon_m^{1/4}).$$

Claim 4. *If we write, with evident notations,*

$$\varphi_{1,\varepsilon_m}^{(-1/2, 1/2]} = \sum_{\gamma \in (-1/2, 1/2]} U^* \sigma_1^\gamma(r) = \varphi_{1,\varepsilon_m}^{(-1/2, 1/2)} + U^* \sigma_1^{1/2}(r)$$

$\sigma_1^{1/2}$ has the expression given in (9) and then can be decomposed on $\sigma_{1,\varepsilon_m}^{1/2} = \sigma_{0,\varepsilon_m}^{1/2} + \bar{\sigma}_{1,\varepsilon_m}^{1/2}$, where $\sigma_{0,\varepsilon_m}^{1/2} = c_{\varepsilon_m} r^{3/2} F(\lambda_{\varepsilon_m} r^2)$ belongs to $\text{dom}(D_{1,\min})$ and the family c_{ε_m} is bounded. We can extract subsequences such that $\varphi_{1,\varepsilon_m}^{(-1/2, 1/2)}$ and $U^*(\sigma_0^{1/2})$ converge in L^2 -norm, while $\bar{\sigma}_{1,\varepsilon_m}^{1/2}$ is asymptotically equivalent to

$$\bar{\sigma}_{1,\varepsilon_m}^{1/2} \sim \frac{1}{\sqrt{|\log \varepsilon_m|}} r^{-1/2} \bar{\sigma}_{1/2} \text{ for some } \bar{\sigma}_{1/2} \in \ker(A - 1/2).$$

Thus, $U^* \sigma_{1/2}$ converges on each $M_1(\eta)$ to the same limit as $U^*(\sigma_{0,\varepsilon_m}^{1/2})$, i.e. $\bar{\sigma}_{1,\varepsilon_m}^{1/2}$ concentrate on the singularity .

Indeed, for each eigenvalue $\gamma \in (-1/2, 1/2)$ of A the vector $\sigma_\gamma(r)$ satisfies the equation $(-\partial_r^2 + \gamma(1 + \gamma)/r^2)\sigma_\gamma = \lambda_{\varepsilon_m} \sigma_\gamma$ and it can be written as

$$\sigma_\gamma(r) = c_{\varepsilon_m} r^{\gamma+1} F(\lambda_{\varepsilon_m} r^2) + d_{\varepsilon_m} \left(r^{-\gamma} G(\lambda_{\varepsilon_m} r^2) \right) \quad (18)$$

if $\gamma \neq 1/2$, while the expression for $\gamma = 1/2$ has been given in (9). Then, $\varphi_{1,\varepsilon_m}^{(-1/2,1/2)}$ belongs in the domain of $D_{1,max}$, so, as a consequence of the Rellich property, we can extract subsequences which converge in L^2 . While, as already mentioned for $\gamma = 1/2$ the expression $c_{\varepsilon_m}^2 + |\log \varepsilon_m| d_{\varepsilon_m}^2$ is bounded so $\lim d_{\varepsilon_m} = 0$ and we can extract from c_{ε_m} and $\sqrt{|\log \varepsilon_m|} d_{\varepsilon_m}$ convergent subsequences. We define then

$$\bar{\sigma}_1^{1/2} = \lim_{m \rightarrow \infty} \sqrt{|\log \varepsilon_m|} d_{\varepsilon_m}. \quad (19)$$

Proposition 11. *The forms*

$$\psi_{1,\varepsilon_m} = (1 - \xi_1)\varphi_{1,\varepsilon_m} + (\xi_1\varphi_{1,\varepsilon_m}^{>1/2} - \tilde{\psi}_{1,\varepsilon_m}) + \xi_{\varepsilon_m}\xi_1\varphi_{1,\varepsilon_m}^{\leq-1/2} + \xi_1 U^*(\sigma_{0,\varepsilon_m}^{1/2})$$

belong to $\text{Dom}(D_{1,\min})$ and define a bounded family.

Proof. We will show that each term is bounded. For the last one, it was already mentioned in Claim 4. For the first one it is already done in Lemma 4. For the second one, we remark that

$$\begin{aligned} f_{\varepsilon_m} &:= \left(\partial_r + \frac{A}{r} \right) (\xi_1 \varphi_{1,\varepsilon_m}^{>1/2} - \tilde{\psi}_{1,\varepsilon_m}) \\ &= \xi_1 \left(\partial_r + \frac{A}{r} \right) (\varphi_{1,\varepsilon_m}^{>1/2}) + \partial_r (\xi_1) \left(\varphi_{1,\varepsilon_m}^{>1/2} - P_{\varepsilon_m}(\psi_{2,\varepsilon_m}|\Sigma) \right) \end{aligned} \quad (20)$$

is uniformly bounded in $L^2(M_1)$, because of (12). This estimate (12) shows also that the L_2 -norm of $(\varphi_{1,\varepsilon_m}^{>1/2} - \tilde{\psi}_{1,\varepsilon_m})$ is bounded.

For the third one, we use the estimate due to the expression of the quadratic form. Expriming that $\int_{C_{r,1}} |D_1(\xi_1 \varphi^{\leq-1/2})|^2 d\mu$ is bounded by Λ gives that

$$\left(\sigma_1^{\leq-1/2}(r), \sigma_1^{\leq-1/2}(r) \right)_{L^2(\Sigma)} \leq \Lambda r |\log r| \quad (21)$$

by the same argument as in Lemmas 5 and 6. Now

$$\begin{aligned} \|D_1(\xi_{\varepsilon_m} \xi_1 \varphi_{1,\varepsilon_m}^{\leq-1/2})\| &\leq \| \xi_{\varepsilon_m} D_1(\xi_1 \varphi_{1,\varepsilon_m}^{\leq-1/2}) \| + \| |d\xi_{\varepsilon_m}| \cdot \xi_1 \varphi_{1,\varepsilon_m}^{\leq-1/2} \| \\ &\leq \| D_1(\xi_1 \varphi_{1,\varepsilon_m}^{\leq-1/2}) \| + \| |d\xi_{\varepsilon_m}| \cdot \xi_1 \varphi_{1,\varepsilon_m}^{\leq-1/2} \| \end{aligned}$$

the first term is bounded and, with $|A| \geq 1/2$ for this term, and the estimate (21), we have

$$\begin{aligned} \| |d\xi_{\varepsilon_m}| \xi_1 \varphi_{1,\varepsilon_m}^{\leq-1/2} \|^2 &\leq \frac{4\Lambda}{\log^2 \varepsilon_m} \int_{\varepsilon_m}^{\sqrt{\varepsilon_m}} \frac{\log(r)}{r} dr \\ &\leq \frac{3\Lambda}{2}. \end{aligned}$$

This completes the proof. \square

In fact, the decomposition used here is almost orthogonal:

Lemma 12. *There exists $\beta > 0$ such that*

$$(\varphi_{1,\varepsilon_m}^{>1/2} - \tilde{\psi}_{1,\varepsilon_m}, \tilde{\psi}_{1,\varepsilon_m})_{L^2} = O(\varepsilon_m^\beta).$$

Proof of Lemma 12. — If we decompose the terms under the eigenspaces of A , we see that only the eigenvalues in $(1/2, +\infty)$ are involved and, with $f_{\varepsilon_m} = \sum_{\gamma>1/2} f^\gamma$ and $(\varphi_{1,\varepsilon_m}^{>1/2} - \tilde{\psi}_{1,\varepsilon_m}) = \sum_{\gamma>1/2} \varphi_0^\gamma$, the equation (20) and the fact that $(\varphi_{1,\varepsilon_m}^{>1/2} - \tilde{\psi}_{1,\varepsilon_m})(\varepsilon_m) = 0$ give

$$\varphi_0^\gamma(r) = r^{-\gamma} \int_{\varepsilon_m}^r \rho^\gamma f^\gamma(\rho) d\rho.$$

Then for each eigenvalue $\gamma > 1/2$ of A

$$\begin{aligned} (\varphi_0^\gamma, \tilde{\psi}_{1,\varepsilon_m}^\gamma)_{L^2(\mathcal{C}_{\varepsilon_m,1})} &= \varepsilon_m^{\gamma-1/2} \int_{\varepsilon_m}^1 r^{-2\gamma} \int_{\varepsilon_m}^r \rho^\gamma (\sigma_\gamma, f^\gamma(\rho))_{L^2(\Sigma)} d\rho \\ &= \varepsilon_m^{\gamma-1/2} \int_{\varepsilon_m}^1 \frac{r^{-2\gamma+1}}{2\gamma-1} \cdot r^\gamma \cdot (\sigma_\gamma, f^\gamma(r))_{L^2(\Sigma)} dr \\ &\quad + \frac{\varepsilon_m^{\gamma-1/2}}{2\gamma-1} \int_{\varepsilon_m}^1 \rho^\gamma (\sigma_\gamma, f^\gamma(\rho))_{L^2(\Sigma)} d\rho. \end{aligned}$$

Thus, if $\gamma > 3/2$ we have the majoration

$$\begin{aligned} |(\varphi_0^\gamma, \tilde{\psi}_{1,\varepsilon_m}^\gamma)_{L^2(\mathcal{C}_{\varepsilon_m,1})}| &\leq \varepsilon_m^{\gamma-1/2} \int_{\varepsilon_m}^1 \frac{r^{-\gamma+1}}{2\gamma-1} |(\sigma_\gamma, f^\gamma(r))_{L^2(\Sigma)}| dr \\ &\quad + \frac{\varepsilon_m^{\gamma-1/2}}{(2\gamma-1)\sqrt{2\gamma+1}} \|\sigma_\gamma\|_{L^2(\Sigma)} \cdot \|f^\gamma\|_{L^2(\mathcal{C}_{\varepsilon_m,1})} \\ &\leq C\varepsilon_m^{\gamma-1/2} \|\sigma_\gamma\|_{L^2(\Sigma)} \frac{\varepsilon_m^{(-2\gamma+3)/2}}{(2\gamma-1)\sqrt{2\gamma-3}} \|f^\gamma\|_{L^2(\mathcal{C}_{\varepsilon_m,1})} \\ &\quad + \frac{\varepsilon_m^{\gamma-1/2}}{(2\gamma-1)\sqrt{2\gamma+1}} \|\sigma_\gamma\|_{L^2(\Sigma)} \cdot \|f^\gamma\|_{L^2(\mathcal{C}_{\varepsilon_m,1})}, \end{aligned}$$

while, for $\gamma = 3/2$ the first term is $O(\varepsilon_m \sqrt{|\log \varepsilon_m|})$ and for $1/2 < \gamma < 3/2$, it is $O(\varepsilon_m^{\gamma-1/2})$. In short, we have

$$|(\varphi_0^\gamma, \tilde{\psi}_{1,\varepsilon_m}^\gamma)_{L^2(\mathcal{C}_{\varepsilon_m,1})}| \leq C\varepsilon_m^\beta \|\sigma_\gamma\| \cdot \|f^\gamma\|_{L^2(\mathcal{C}_{\varepsilon_m,1})},$$

if $\beta > 0$ satisfies $\gamma \geq \beta + 1/2$ for all eigenvalue γ of A in $]1/2, +\infty[$. This estimate gives the lemma.

Corollary 13. *There exists from $\psi_{1,\varepsilon_m} + \varphi_{1,\varepsilon_m}^{(-1/2,1/2)}$ a subfamily which converges in L^2 to a form φ_1 which satisfies on the open manifold $M_1(0)$ the equation $\Delta\varphi_1 = \lambda\varphi_1$. Moreover,*

$$\|\varphi_1\|_{L^2(M_1)}^2 + \|\widetilde{\varphi}_2\|_{L^2(\widetilde{M}_2)}^2 + \|\bar{\sigma}^{1/2}\|_{L^2(\Sigma)}^2 = 1. \quad (22)$$

If $\widetilde{\varphi}_2$ is the prolongation of φ_2 by $P_2(\varphi_2 \upharpoonright_\Sigma)$ on \widetilde{M}_2 , and

$$\bar{\sigma}^{1/2} = \lim_{m \rightarrow \text{inf ty}} \frac{1}{\sqrt{|\log \varepsilon_m|}} d_{\varepsilon_m}$$

with the expression of (9).

Proof. Indeed, the family $\psi_{1,\varepsilon_m} + \varphi_{1,\varepsilon_m}^{(-1/2,1/2)}$ is bounded in $\text{dom } D_{1,\max}$, one can then extract a subfamily which converges in L^2 but we know that $\tilde{\psi}_{1,\varepsilon_m}$ converges to 0 in any $M_1(\eta)$, the conclusion follows. We obtain also, with the help of Lemma 12 that

$$1 - \left\{ \|\varphi_1\|^2 + \|\varphi_2\|^2 \right\} = \lim \left\{ \|\tilde{\psi}_{1,\varepsilon_m}\|^2 + \|\xi_1 U^* \left(\frac{1}{\sqrt{|\log \varepsilon_m|}} r^{-1/2} \bar{\sigma}_{1/2} \right)\|^2 \right\}.$$

We remark that, by Corollary 10, $\varphi_2 = 0$ implies $\lim_{m \rightarrow \infty} \|\tilde{\psi}_{1,\varepsilon_m}\|_{L^2} = 0$.

In fact, one has by (12)

$$\lim_{m \rightarrow \infty} \|\tilde{\psi}_{1,\varepsilon_m}\| = \|P_2(\varphi_2 \upharpoonright \Sigma)\|_{L^2}. \quad (23)$$

Finally, one has

$$\lim_{m \rightarrow \infty} \|\xi_1 U^* \left(\frac{1}{\sqrt{|\log \varepsilon_m|}} r^{-1/2} \bar{\sigma}_{1/2} \right)\|_{L^2(M_{\varepsilon_m})} = \|\bar{\sigma}^{1/2}\|_{L^2(\Sigma)}. \quad (24)$$

□

4.3. Lower bound, the end. Let us now $\{\varphi_1(\varepsilon), \dots, \varphi_N(\varepsilon)\}$ be an orthonormal family of eigenforms of the Hodge-de Rham operator, associated to the eigenvalues $\lambda_1(\varepsilon), \dots, \lambda_N(\varepsilon)$. We can make the same procedure of extraction for the all family. This gives, in the limit domain, a family $(\varphi_1^j, \varphi_2^j, \bar{\sigma}_{1/2}^j)_{1 \leq j \leq N}$. We already know by Corollary 13 that each element has norm 1, if we show that they are orthogonal, we are done, by applying the min-max formula to the limit problem (10).

Lemma 14. *The limit family is orthonormal in \mathcal{H}_∞ .*

Proof. If we follow the procedure for one indice, up to terms converging to zero, we had decomposed the eigenforms $\varphi_j(\varepsilon)$ on M_1 on three terms

$$\begin{aligned} \Phi_\varepsilon^j &= \psi_{1,\varepsilon} + \varphi_{1,\varepsilon}^{(-1/2,1/2)}, \\ \tilde{\Phi}_\varepsilon^j &= \tilde{\psi}_{1,\varepsilon_m}, \\ \bar{\Phi}_\varepsilon^j &= U^* \left(\frac{1}{\sqrt{|\log \varepsilon|}} r^{-1/2} \bar{\sigma}_{1/2}^j \right). \end{aligned}$$

Let $a \neq b$ be two indices. If we apply Lemma 12 to any linear combination of $\varphi_a(\varepsilon)$ and $\varphi_b(\varepsilon)$, we obtain that

$$\lim_{m \rightarrow \infty} \left\{ (\Phi_{\varepsilon_m}^a, \tilde{\Phi}_{\varepsilon_m}^b)_{L^2(M_1(\varepsilon_m))} + (\Phi_{\varepsilon_m}^b, \tilde{\Phi}_{\varepsilon_m}^a)_{L^2(M_1(\varepsilon_m))} \right\} = 0.$$

If we apply (23), we obtain

$$\lim_{m \rightarrow \infty} \left\{ (\tilde{\Phi}_{\varepsilon_m}^a, \tilde{\Phi}_{\varepsilon_m}^b)_{L^2(M_{\varepsilon_m})} + (\varphi_{2,\varepsilon}^a, \varphi_{2,\varepsilon}^b)_{L^2(M_2)} \right\} = (\tilde{\varphi}_{2,\infty}^a, \tilde{\varphi}_{2,\infty}^b)_{L^2(\widetilde{M}_2)}.$$

Then finally, from $(\varphi_a(\varepsilon), \varphi_b(\varepsilon)) = 0$, we conclude that

$$(\varphi_1^a, \varphi_1^b)_{L^2(\overline{M}_1)} + (\varphi_2^a, \varphi_2^b)_{L^2(\widetilde{M}_2)} + (\bar{\sigma}_{1/2}^a, \bar{\sigma}_{1/2}^b)_{L^2(\Sigma)} = 0.$$

□

Proposition 15. *The multiplicity of 0 in the limit spectrum is given by the sum*

$$\dim \text{Ker } \Delta_{1,W} + \dim \text{Ker } \mathcal{D}_2 + i_{1/2},$$

where $i_{1/2}$ denotes the dimension of the vector space $\mathcal{I}_{1/2}$, see (3), of extended solutions ω on \widetilde{M}_2 introduced by Carron [C01a], corresponding to a boundary term on restriction to $r = 1$ with non-trivial component in $\text{Ker}(A - 1/2)$.

If the limit value $\lambda \neq 0$, then it belongs to the positive spectrum of the Hodge-de Rham operator $\Delta_{1,W}$ on \overline{M}_1 , with

$$W = \bigoplus_{|\gamma| < \frac{1}{2}} \text{Ker}(A - \gamma).$$

Proof. The last process, with in particular (23) and (13), constructs in fact an element in the limit Hilbert space

$$\mathcal{H}_\infty := L^2(\overline{M}_1) \oplus \text{Ker } \widetilde{D}_2 \oplus \mathcal{I}_{1/2}$$

and this process is clearly *isometric* in the sense that if we have an orthonormal family $\{\varphi_{k,\varepsilon_m}\}$, ($1 \leq k \leq n$), we obtain at the limit an orthonormal family, if \mathcal{H}_∞ is defined as an orthonormal space of the Hilbert spaces. And if we begin with eigenforms of D_{ε_m} , we obtain at the limit eigenforms of $\Delta_{1,W} \oplus \{0\} \oplus \{0\}$. The last calculus implies that $\liminf \lambda_N(\varepsilon_m) \geq \lambda_N$. \square

Remark 16. *In order to understand this result, it is important to remember when occurs the eigenvalue $1/2$ in the spectrum of A . By the expression (2), we find that it occurs exactly*

- for n even, if $3/4$ is an eigenvalue of the Hodge-de Rham operator Δ_Σ acting on the coexact forms of degree $n/2$ or $n/2 - 1$ of the submanifold Σ .
- for n odd, if 0 is an eigenvalue of Δ_Σ for the $(n-1)/2$, $(n+1)/2$ forms, but also if 1 is eigenvalue of the coexact forms of degree $(n-1)/2$ on Σ .

A dilation of the metric on Σ permits to avoid positive eigenvalues, but if harmonic forms of degree $(n-1)/2$ or $(n+1)/2$ create half-bounded states, then small eigenvalues will always appear.

5. HARMONIC FORMS AND SMALL EIGENVALUES.

It would be interesting to know how many small (but not zero) eigenvalues appear. For this purpose, we can use the topological meaning of harmonic forms.

5.1. Cohomology groups. The topology of M_ε is independant of $\varepsilon \neq 0$ and can be apprehended by the Mayer Vietoris sequence:

$$\dots \rightarrow H^p(M_\varepsilon) \xrightarrow{res} H^p(M_1(\varepsilon)) \oplus H^p(M_2) \xrightarrow{dif} H^p(\Sigma) \xrightarrow{ext} H^{p+1}(M_\varepsilon) \rightarrow \dots$$

As already mentionned, the space $\text{Ker } \mathcal{D}_2 \oplus \mathcal{I}_{1/2}$ can be sent in $H^*(M_2)$. More precisely, Hausel, Hunsicker and Mazzeo have proved in [HHM04] that the L^2 -cohomology of \widetilde{M}_2 is isomorphic for the degree $k < (n+1)/2$ to the relative cohomology group $H^k(M_2, \Sigma)$ and for $k > (n+1)/2$ to the absolute cohomology group

$H^k(M_2)$, while for $k = (n+1)/2$ it is isomorphic to the image of $H^{(n+1)/2}(M_2, \Sigma)$ in $H^{(n+1)/2}(M_2)$.

For \overline{M}_1 we can use the results of Cheeger. Following [Ch80] and [Ch83], we know that the intersection cohomology groups of \overline{M}_1 coincide with $\ker(D_{1,\max} \circ D_{1,\min})$, if $H^{n/2}(\Sigma) = 0$.

And we know also that

$$IH^p(\overline{M}_1) = \begin{cases} H^p(M_1(\varepsilon)) & \text{if } p \leq \frac{n}{2}, \\ H_c^p(M_1(\varepsilon)) & \text{if } p \geq \frac{n}{2} + 1 \end{cases} \quad (25)$$

These results can be used for our study only if $D_{1,\max}$ and $D_{1,\min}$ coincide. This appears if and only if A has no eigenvalues in the interval $(-1/2, 1/2)$. As a consequence of the expression of the eigenvalues of A , recalled in (2), this is the case if and only if

- **for n odd**, the operator Δ_Σ has no eigenvalues in $(0, 1)$ on coexact $(n-1)/2$ -forms,
- **for n even**, the operator Δ_Σ has no eigenvalues in $(0, 3/4)$ on $n/2$ or $(n/2)-1$ coexact forms and $H^{n/2}(\Sigma) = 0$.

Thus, if $D_{1,\max} = D_{1,\min}$, and this implies that $H^{n/2}(\Sigma) = 0$ in the case where n is even, then the map

$$H^{n/2}(M_\varepsilon) \xrightarrow{res} H^{n/2}(M_1(\varepsilon)) \oplus H^{n/2}(M_2)$$

is surjective and then any small eigenvalue in this degree must come from an element of $\ker \mathcal{D}_2 \oplus \mathcal{I}_{1/2}$ sent to 0 in $H^{n/2}(M_2)$. In this case also the map

$$H^{n/2+1}(M_\varepsilon) \xrightarrow{res} H^{n/2+1}(M_1(\varepsilon)) \oplus H^{n/2+1}(M_2)$$

is injective so there may exist small eigenvalues in this degree.

5.2. Some examples. We exhibit a general procedure to construct new examples as follows: Let $W_i, i = 1, 2$ be two compact Riemannian manifolds with boundary Σ_i and dimension $(n_i + 1)$ such that $n_1 + n_2 = n \geq 2$. We can apply our result to $M_1 := W_1 \times \Sigma_2$ and $M_2 := \Sigma_1 \times W_2$. The manifold M_ε is always diffeomorphic to $M = M_1 \cup M_2$.

For instance, let v_2 be the volume form of (Σ_2, h_2) . It defines a harmonic form on M_1 and this form will appear for the limit spectrum if, transplanted on \overline{M}_1 , it defines an element of the domain of the operator $\Delta_{1,W}$.

In the writing introduced in Section 2.2, this element corresponds to $\beta = 0$ and $\alpha = r^{n/2-n_2}v_2$ and the expression of A gives that

$$A(\beta, \alpha) = \left(n_2 - \frac{n}{2}\right) (\beta, \alpha).$$

If $\frac{n}{2} - n_2 > 0$, then (β, α) is in the domain of $D_{1,\max} \circ D_{1,\min}$ and if $n_2 = \frac{n}{2}$, it is in the domain of $\Delta_{1,W}$ for the eigenvalue 0 of A .

So, if we know that $H^{n_2}(M) = 0$, or more generally that $\dim H^{n_2}(M) < \dim H^{n_2}(\Sigma_2)$ in the case where Σ_2 is not connected, this element will create a small eigenvalue on

M_ε . This is the case, if D^k denotes the unit ball in \mathbb{R}^k , for

$$W_1 = D^{n_1+1}, \text{ and } W_2 = D^{n_2+1} \text{ for } n_2 \leq n_1.$$

Then, $M = \mathbb{S}^{n_1+n_2+1}$ and we obtain

Corollary 17. *For any degree k and any $\varepsilon > 0$, there exists a metric on \mathbb{S}^m such that the Hodge-de Rham operator acting on k -forms admits an eigenvalue smaller than ε . We can see that, for $k < \frac{m}{2}$, it is in the spectrum of co-exact forms, and by duality, for $k \geq \frac{m}{2}$ in the spectrum of exact k -forms.*

Indeed, the case $k < \frac{m}{2}$ is a direct application, as explained above. We see that our pseudomode is coclosed. Thus, in the case when m is even, if ω is an eigenform of degree $(\frac{1}{2} - 1)$ with small eigenvalue, then $d\omega$ is a closed eigenform with the same eigenvalue and degree $\frac{m}{2}$. Finally, the case $k > \frac{m}{2}$ is obtained by the Hodge duality. We remark that, in the case $k = 0$ we recover *Cheeger' dumbbell*, and also that this result has been proved by Guerini in [Gu04] with another deformation, although he did not give the convergence of the spectrum.

By the surgery of the precedent case, we obtain, for

$$W_1 := \mathbb{S}^{n_1} \times (0, 1), \text{ and } W_2 := D^{n_2+1} \text{ for } 0 \leq n_2 < n_1, \text{ and } n = n_1 + n_2 \geq 2$$

that $\Sigma_1 = \mathbb{S}^{n_1} \sqcup \mathbb{S}^{n_1}$, $\Sigma_2 = \mathbb{S}^{n_2}$ and $M = \mathbb{S}^{n_1} \times \mathbb{S}^{n_2+1}$. The volume form $v_2 \in H^{n_2}(\Sigma_2)$ defines again a harmonic form on \overline{M}_1 and, since $H^{n_2}(\mathbb{S}^{n_1} \times \mathbb{S}^{n_2+1}) = 0$, if $n_2 < n_1$, then v_2 defines a small eigenvalue on the n_2 -forms of M_ε .

Thus, by duality, we obtain

Corollary 18. *For any $k, l \geq 0$ with $0 \leq k - 1 < l$ and any $\varepsilon > 0$, there exists a metric on $\mathbb{S}^l \times \mathbb{S}^k$ such that the Hodge-de Rham operator acting on $(k - 1)$ -forms and on $(l + 1)$ -forms admits an eigenvalue smaller than ε .*

This corollary is also a consequence of the previous one: we know that there exists a metric on \mathbb{S}^k whose Hodge-de Rham operator admits a small eigenvalue on $(k - 1)$ -forms, this property is maintained on $\mathbb{S}^l \times \mathbb{S}^{k+1}$.

With the same construction, we can exchange the roles of M_1 and M_2 : the two volume forms of $\mathbb{S}^{n_1} \sqcup \mathbb{S}^{n_1}$ create one n_1 -form with small but non-zero eigenvalue on $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2+1}$, if $n_1 \leq (n_2 + 1)$. By duality, we obtain an $(n_2 + 1)$ -form with small eigenvalue. So, with new notations, we have obtained

Corollary 19. *For any $k < l$ with $k + l \geq 3$ and any $\varepsilon > 0$, there exists a metric on $\mathbb{S}^l \times \mathbb{S}^k$ such that the Hodge-de Rham operator acting on l -forms and on k -forms admits a positive eigenvalue smaller than ε .*

More generally, by repeating the $(k - 1)$ -dimensional surgery by L -times, we obtain the following:

Proposition 20 (Sha-Yang [SY91]). *The connected sum of the L -copies of the product spheres $\mathop{\#}_{i=1}^L (\mathbb{S}^k \times \mathbb{S}^l)$ can be decomposed as follows:*

$$\mathop{\#}_{i=1}^L (\mathbb{S}^k \times \mathbb{S}^l) \cong \left(\mathbb{S}^{k-1} \times \left(\mathbb{S}^{l+1} \setminus \prod_{i=0}^L D_i^{l+1} \right) \right) \bigcup_{\partial} \left(D^k \times \prod_{i=0}^L \mathbb{S}_i^l \right).$$

Remark 21. *J.-P. Sha and D.-G. Yang [SY91] constructed a Riemannian metric of positive Ricci curvature on this manifold. More generally, see also Wraith [Wr07].*

As similar way using Proposition 20, we can obtain the small positive eigenvalues on the connected sum of the L -copies of the product spheres $\mathop{\#}_{i=1}^L (\mathbb{S}^k \times \mathbb{S}^l)$.

All these examples use the spectrum of \overline{M}_1 . We can obtain also examples using the L^2 -cohomology of \widetilde{M}_2 . As already mentioned, it is proved by Hausel, Hunsicker and Mazzeo in [HHM04] that the L^2 -cohomology of \widetilde{M}_2 is isomorphic for the degree $k < (n+1)/2$ to the relative cohomology group $H^k(M_2, \Sigma)$ and for $k > (n+1)/2$ to the absolute cohomology group $H^k(M_2)$, while for $k = (n+1)/2$ it is isomorphic to the image of $H^{(n+1)/2}(M_2, \Sigma)$ in $H^{(n+1)/2}(M_2)$.

Suppose now that $n = \dim \Sigma$ is odd, we have the long exact sequence

$$\cdots \rightarrow H^k(M_2, \Sigma) \rightarrow H^k(M_2) \rightarrow H^k(\Sigma) \rightarrow H^{k+1}(M_2, \Sigma) \rightarrow \dots$$

For $k = (n-1)/2$, the space $H^k(M_2, \Sigma)$ is isomorphic to the L^2 -cohomology of M_2 , then if $H^{(n-1)/2}(\Sigma)$ is any not-trivial harmonic form on Σ of this degree. will create an extended solution, corresponding to an eigenvector of A with eigenvalue $1/2$.

REFERENCES

- [ACP09] C. Anné, G. Carron and O. Post, *Gaps in the spectrum of Dirac type operators on non-compact manifolds*, Math. Zeit. **262** no. 1 (2009), 57–90.
- [AC93] C. Anné, B. Colbois, *Opérateur de Hodge-Laplace sur des variétés compactes privées d'un nombre fini de boules*, J. Funct. Anal. **115** n° 1 (1993), 190–211.
- [AC95] C. Anné, B. Colbois, *Spectre du Laplacien agissant sur les p -formes différentielles et écrasement d'anses*, Math. Ann. **303** n° 3 (1995), 545–573.
- [AT09] C. Anné, J. Takahashi, *P -spectrum and collapsing of connected sums*, to appear in Trans. AMS, arXiv:0807.0760v2 [math.DG].
- [APS75] M. F. Atiyah, V. K. Patodi and I. M. Singer, *Spectral asymmetry and Riemannian geometry. I*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69.
- [BS88] J. Brüning and R. Seeley, *An index theorem for first order regular singular operators*, Amer. J. Math. **110** (1988), no. 4, 659–714.
- [C01a] G. Carron, *Un théorème de l'indice relatif*, Pac. J. Math. **198** no. 1 (2001), 81–107.
- [C01b] G. Carron, *Théorèmes de l'indice sur des variétés non-compactes* J. reine angew. Math. **541** (2001), 81–115.
- [Ch80] J. Cheeger, *On the Hodge theory of Riemannian pseudomanifolds*, Proc. Symp. Pure. Math. **36**, AMS Providence, RI, (1980), 91–145.
- [Ch83] J. Cheeger, *Spectral geometry of singular Riemannian spaces*, J. Diff. Geom. **18** (1983), 575–657.
- [D82] J. Dodziuk, *Eigenvalues of the Laplacian on forms*, Proc. Amer. Math. Soc. **85** (1982), 437–443.

- [F87] K. Fukaya, *Collapsing of Riemannian manifolds and eigenvalues of Laplace operator*, Invent. Math. **87** (1987), 517–547.
- [GP95] G. Gentile and V. Pagliara, *Riemannian metrics with large first eigenvalue on forms of degree p* , Proc. Am. Math. Soc. **123** 12 (1995), 3855–3858.
- [Gu04] P. Guerini, *Prescription du spectre du Laplacien de Hodge-de Rham*, Ann. Scient. Éc. Norm. Sup., 4^{ième} serie **37** (2004), 270–303.
- [HHM04] T. Hausel, E. Hunsicker and R. Mazzeo, *Hodge cohomology and gravitational instantons*, Duke Math. J. **122** n° 3 (2004), 485–548.
- [McG93] J. McGowan, *The p -spectrum of the Laplacian on compact hyperbolic three manifolds*, Math. Ann. **279** (1993), 729–745.
- [Le97] M. Lesch, *Operators of Fuchs type, conical singularities, and asymptotic methods*, Teubner-Texte zur Mathematik 136, Stuttgart (1997).
- [Lo02a] J. Lott, *Collapsing and Dirac-type operators*, Geom. Dedicata **91** (2002), 175–196.
- [Lo02b] J. Lott, *Collapsing and the differential form Laplacian: the case of a smooth limit space*, Duke Math. J. **114** n°. 2 (2002), 267–306.
- [Lo04] J. Lott, *Remark about the spectrum of the p -form Laplacian under a collapse with curvature bounded below*, Proc. Amer. Math. Soc. **132** (2004), no. 3, 911–918.
- [MM90] R. Mazzeo and R. Melrose, *The adiabatic limit, Hodge cohomology and Leray’s spectral sequence for a fibration*, J. Diff. Geom. **31** no. 1 (1990), 185–213.
- [Maz06] R. Mazzeo, *Resolution blowups, spectral convergence and quasi-asymptotically conical spaces*, Actes Colloque EDP Evian-les-Bains, (2006).
- [MS74] J. W. Milnor, J. D. Stasheff, *Lectures on Characteristic Classes*, Ann. Math. Studies **76**, Princeton Univ. Press, (1974).
- [Ru00] M. Rumin, *Sub-Riemannian limit of the differential form spectrum of contact manifolds*, Geom. Funct. Anal., **10**, no. 2, (2000), 407–452.
- [Row06] J. Rowlett, *Spectral geometry and asymptotically conic convergence*, Thesis, Stanford (2006).
- [Row08] J. Rowlett, *Spectral geometry and asymptotically conic convergence*, Comm. Anal. Geom. **16** no. 4 (2008), 735–798. arXiv:math/0701383.
- [SY91] J-P. Sha and D-G. Yang, *Positive Ricci curvature on the connected sums of $S^n \times S^m$* , J. Diff. Geom. **33** (1991), 127–137.
- [Wa60] C. T. C. Wall, *Determination of the cobordism ring*, Ann. of Math. **72** n°2, (1960), 292–311.
- [Wr07] D. J. Wraith, *New connected sums with positive Ricci curvature*, Ann. Global Anal. Geom. **32** (2007), 343–360.

LABORATOIRE DE MATHÉMATIQUES JEAN LERAY, UNIVERSITÉ DE NANTES, CNRS, FACULTÉ DES SCIENCES, BP 92208, 44322 NANTES, FRANCE
E-mail address: colette.anne@univ-nantes.fr

DIVISION OF MATHEMATICS, GRADUATE SCHOOL OF INFORMATION SCIENCES, TÔHOKU UNIVERSITY, AOBA 6-3-09, SENDAI, 980-8579, JAPAN
E-mail address: t-junya@math.is.tohoku.ac.jp