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PARTIAL COLLAPSING AND THE SPECTRUM OF THE HODGE LAPLACIAN

COLETTE ANNÉ AND JUNYA TAKAHASHI

ABSTRACT. The goal of the present paper is to calculate the limit spectrum of the Hodge-de Rham operator under the perturbation of collapsing one part of a manifold obtained by gluing together two manifolds with the same boundary. It appears to take place in the general problem of blowing up conical singularities as introduced in [Maz06] and [Row06, Row08].

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1. INTRODUCTION

This work takes place in the general context of the spectral studies of singular perturbations of the metrics, as a manner to know what are the topological or metrical meanings carried by the spectrum of geometric operators. We can mention in this direction, without exhaustivity, studies on the adiabatic limit ([MM90],[Rum00]), on collapsing ([F87], [Lo02a, Lo02b]), on resolution blowups of conical singularities ([Maz06],[Row06, Row08]) and on shrinking handles ([AC95, ACP07]).

The present study can be considered as a generalization of the results of [AT09] where we studied the limit of the spectrum of the Hodge deRham operator under collapsing of one part of a connected sum.

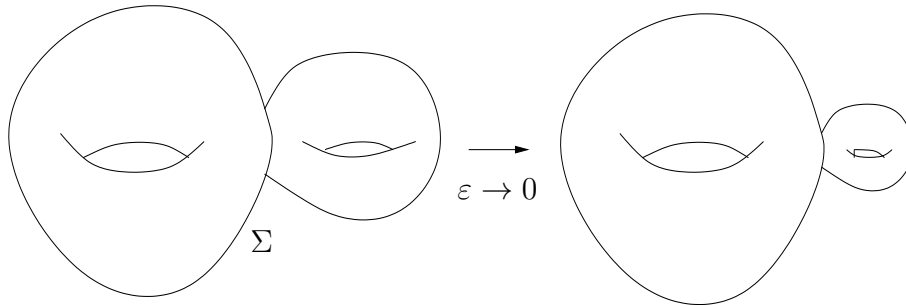


FIGURE 1. partial collapsing of M_ε

In our previous work we restricted the submanifold Σ , used to glue the two parts, to be a sphere. In fact this problem is quite related to resolution blowups of conical singularities: the point is to measure the influence of the topology of the part which disappears and of the conical singularity created at the limit of the 'big part'. If we look at the situation from the 'small part' we understand the importance of the *quasi-asymptotically conical space* obtained by rescaling the small part and adding an infinite cone.

When $\Sigma = \mathbb{S}^n$ the conical singularity is quite simple, and there is no *semi-bounded states*, called extended solutions in the sequel, on the quasi-asymptotically conical space, our result presented here takes care of this new possibilities and gives a general answer of the problem studied by Mazzeo and Rowlett. Indeed, in [Maz06, Row06, Row08] it is supposed that the spectrum of the operator on the quasi-asymptotically conical space does not meet 0. Our study relax this hypothesis. It is done only with the Hodge-de Rham operator but can easily be generalised.

Let's fix some notations.

1.1. **set up.** Let M_1 and M_2 be two connected manifolds with the same boundary Σ , a compact manifold of dimension $n \geq 2$. We denote by $m = n+1$ the dimension of M_1 and M_2 .

We endow Σ with a fixed metric h .

Let \overline{M}_1 be the manifold with conical singularity obtained from M_1 by adding to M_1 a cone $\mathcal{C} = [0, 1[\times \Sigma \ni (r, y)$: there exists on $\overline{M}_1 = M_1 \cup \mathcal{C}$ a metric \overline{g}_1 which writes, on the smooth part $r > 0$ of the cone as $dr^2 + r^2h$.

We choose on M_2 a metric g_2 which is 'trumpet like', ie. M_2 is isomorphic near the boundary to $[0, 1/2[\times \Sigma$ with the conical metric which writes $ds^2 + (1-s)^2h$ if s is the coordinate defining the boundary by $s = 0$.

For any ε , $0 < \varepsilon < 1$ we define

$$\mathcal{C}_\varepsilon = \{(r, y) \in \mathcal{C}, r > \varepsilon\} \quad \text{and} \quad M_1(\varepsilon) = M_1 \cup \mathcal{C}_\varepsilon.$$

The goal of the following calculus is to determine the limit spectrum of the Hodge-Rham operator acting on the differential forms of the riemannian manifold

$$M_\varepsilon = M_1(\varepsilon) \cup_{\varepsilon, \Sigma} \varepsilon.M_2$$

obtained by gluing together $M_1(\varepsilon)$ and (M_2, ε^2g_2) . We remark that, by construction, these two manifolds have isometric boundary and that the metric g_ε obtained on M_ε is smooth.

Remark 1. *The common boundary Σ of dimension n has some topological obstructions. In fact, since Σ is the boundary of oriented compact manifold M_1 (or M_2), Σ is oriented cobordant to zero. So, by Thom's cobordism theory, all of the Stiefel-Whitney and the Pontrjagin numbers vanish (cf. C. T. C. Wall [W60] or [MS74], §18, p.217). Futhermore, this condition is also sufficient, that is, the inverse does hold. Especially, it is impossible to take Σ^{2k} as the complex projective spaces $\mathbb{C}\mathbb{P}^k$, ($k \geq 1$).*

1.2. **results.** We can describe the limit spectrum as follows: it has two parts one comes from the big part, namely \overline{M}_1 , and is exprimed by the spectrum of a good extension of the Hodge Laplacian on this manifold with conical singularities, This extension is self adjoint and comes from an extension of the Gauß-Bonnet operators. All these extensions are classified by subspaces W of the total eigenspace corresponding to eigenvalues belonging to $] - 1/2, 1/2[$ of an operator A acting on the boundary Σ , this point is developed below in Section 2.2. The other part comes from the collapsed part, namely M_2 , where the limit Gauß-Bonnet operator is taken with boundary conditions of Atiyah-Patodi-Singer type, this point is developed below in Section 2.3. This operator, denoted $\widetilde{\mathcal{D}}_2$ in the sequel, can also be seen on the quasi-asymptotically conical space \widetilde{M}_2 already mentionned, namely

$$\widetilde{M}_2 = M_2 \cup \left([1, \infty[\times \Sigma \right).$$

with the metric $dr^2 + r^2h$ on the conical part. Only the eigenvalue zero is concerned with this part. In fact the manifolds M_ε has small eigenvalues, in the

difference with [AT09], and the *multiplicity of 0* at the limit correspond to the total eigenspace of these small, or null, eigenvalues. Thus, our main theorem, which asserts the convergence of the spectrum, has two components.

Proposition A. *If the limit value $\lambda \neq 0$ then it belongs to the positive spectrum of the Hodge-de Rham operator $\Delta_{1,W}$ on \overline{M}_1 , with*

$$W = \bigoplus_{\gamma \in]-\frac{1}{2}, \frac{1}{2}[} \text{Ker}(A - \gamma).$$

Proposition B. *The multiplicity of 0 in the limit spectrum is given by the sum*

$$\dim \ker \Delta_{1,W} + \dim \ker \mathcal{D}_2 + i_{1/2}$$

where $i_{1/2}$ denote the dimension of the vector space $\mathcal{I}_{1/2}$, see (4), of extended solutions ω on \widetilde{M}_2 introduced by Carron [C01], such that $A\omega = (1/2)\omega$ on restriction to $r = 1$.

1.3. comments. We choose a simple metric to make explicit computations. This fact is not a restriction, as already explained in [AT09], because of the result of Dodziuk [D82] which assure uniform control of the eigenvalues of geometric operators with regard to variations of the metric.

1.4. examples. We exhibit some examples which bear small positive eigenvalues of the Hodge Laplacian action on p -forms.

Let W_i ($i = 1, 2$) be a connected oriented compact manifold of dimension n_i with boundary Σ_i ($i = 1, 2$), respectively. Then, we define a closed manifold M as

$$M := W_1 \times \Sigma_2 \cup_{\Sigma_1 \times \Sigma_2} \Sigma_1 \times W_2 = M_1 \cup_{\Sigma} M_2, \quad (1)$$

where $M_1 := W_1 \times \Sigma_2$, $M_2 := \Sigma_1 \times W_2$ and $\Sigma := \Sigma_1 \times \Sigma_2$.

This example is considered as a higher dimensional analogue of a simple graph manifold M_3 of dimension 3:

$$M^3 := \Sigma_1^* \times S^1 \cup_{T^2} S^1 \times \Sigma_2^*,$$

where Σ_i^* ($i = 1, 2$) is a Riemann surface removed a disk respectively.

1.4.1. Blowup of complex surfaces. — We recall here an example exhibited in [AT09]: a blow-up at a one point of the complex projective surface $\mathbb{C}\mathbb{P}^2$. This manifold is diffeomorphic to $\mathbb{C}\mathbb{P}^2 \sharp \overline{\mathbb{C}\mathbb{P}^2}$, where $\overline{\mathbb{C}\mathbb{P}^2}$ is a reversed orientation of $\mathbb{C}\mathbb{P}^2$.

Then, by our main theorem, there exists a Riemannian metric g_ε on $\mathbb{C}\mathbb{P}^2 \sharp \overline{\mathbb{C}\mathbb{P}^2}$ whose positive eigenvalues are close to those of $\mathbb{C}\mathbb{P}^2$, as desired.

Next, we consider the blowup of a closed complex surface (M, J) of real dimension 4, where J is a complex structure.

If we blowup (M, J) at one point $p \in M$, then the resulting manifold \widehat{M} is also compact complex surface which is diffeomorphic to $M \sharp \overline{\mathbb{C}\mathbb{P}^2}$, that is, the connected sum of M and $\overline{\mathbb{C}\mathbb{P}^2}$. Note that M^* is not biholomorphic to $M \sharp \overline{\mathbb{C}\mathbb{P}^2}$.

Take any Riemannian metric g_1 on M which is flat around a point p . Then, from Theorem C of [AT09], we can construct a family of C^∞ metrics g_ε on $\widehat{M} \cong M \sharp \overline{\mathbb{C}\mathbb{P}^2}$ such that the spectrum of the blowup manifold $(\widehat{M}, g_\varepsilon)$ is as closed as that of the original manifold (M, g_1) .

In particular, all the positive spectrum of $(\widehat{M}, g_\varepsilon)$ converge to those of (M_1, g_1) , as $\varepsilon \rightarrow 0$. This is one expression of blow-down from $(\widehat{M}, g_\varepsilon)$ to (M, g_1) . Indeed, the collapsing part $\overline{\mathbb{C}\mathbb{P}^2}$ containing an exceptional (-1) -curve $\mathbb{C}\mathbb{P}^1$ shrinks to a point.

Note that Perel'man [P97] constructed a sequence of closed Riemannian manifolds with positive Ricci curvature whose second Betti number is as large as desired. This example, roughly speaking, glued complex projective surfaces $\mathbb{C}\mathbb{P}^2$ more and more, so the limit space is glued by infinity many complex projective spaces which is no longer a manifold.

2. GAUSS-BONNET OPERATOR

On a Riemannian manifold, the Gauß-Bonnet operator is defined as the operator $D = d + d^*$ acting on differential forms. It is symmetric and can have few closed extensions on manifolds with boundary or with conical singularities. We review these extensions on the cases involved in our study.

2.1. Gauß-Bonnet operator on M_ε . We recall that, on M_ε , a Gauß-Bonnet operator D_ε , Sobolev spaces and also a Hodge-de Rham operator Δ_ε can be defined by a general construction on any manifold $X = X_1 \cup X_2$, which is the union of two Riemannian manifolds with isometric boundaries (the details are given in [AC95]): if D_1 and D_2 are the Gauß-Bonnet “ $d + d^*$ ” operators acting on the differential forms of each part, the quadratic form $q(\varphi) = \int_{X_1} |D_1(\varphi|_{X_1})|^2 + \int_{X_2} |D_2(\varphi|_{X_2})|^2$ is well defined and closed on the domain

$$\mathcal{D}(q) = \{\varphi = (\varphi_1, \varphi_2) \in H^1(\Lambda T^* X_1) \times H^1(\Lambda T^* X_2), (\varphi_1)|_{\partial X_1} \stackrel{L_2}{=} (\varphi_2)|_{\partial X_2}\}$$

and on this space the total Gauß-Bonnet operator $D(\varphi) = (D_1(\varphi_1), D_2(\varphi_2))$ is defined and selfadjoint. For this definition, we have, in particular, to identify $(\Lambda T^* X_1)|_{\partial X_1}$ and $(\Lambda T^* X_2)|_{\partial X_2}$. This can be done by decomposing the forms in tangential and normal part (with inner normal), the equality above means then that the tangential parts are equal and the normal part opposite. This definition generalizes the definition in the smooth case.

The Hodge-de Rham operator $(d + d^*)^2$ of X is then defined as the operator obtained by the polarization of the quadratic form q . This gives compatibility conditions between φ_1 and φ_2 on the common boundary. We don't give details

on these facts because our manifold is smooth. But we shall use this presentation for the quadratic form.

2.2. Gauß-Bonnet operator on $\overline{M_1}$. Let $D_{1,\min}$ be the closure of the Gauß-Bonnet operator defined on the smooth forms with compact support in the smooth part $M_1(0)$. For any such form φ_1 we write on the cone

$$\varphi_1 = dr \wedge r^{-(n/2-p+1)}\beta_{1,\varepsilon} + r^{-(n/2-p)}\alpha_{1,\varepsilon}$$

and define $\sigma_1 = (\beta_1, \alpha_1) = U(\varphi_1)$. The operator has, on the cone, the expression

$$UD_1U^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left(\partial_r + \frac{1}{r}A \right) \quad \text{with} \quad A = \begin{pmatrix} \frac{n}{2} - P & -D_0 \\ -D_0 & P - \frac{n}{2} \end{pmatrix}$$

where P is the operator of degree which multiplies by p a p -form, and D_0 is the Gauß-Bonnet operator on the manifold Σ .

While the Hodge-de Rham operator has, in these coordinates, the expression

$$U\Delta_1U^* = -\partial_r^2 + \frac{1}{r^2}A(A+1). \quad (2)$$

The closed extensions of the operator $D_1 = d + d^*$ on the manifold with conical singularities $\overline{M_1}$ has been studied in [BS88] and [L97] They are classified by the spectrum of its *Mellin symbol*, which is here the operator with parameter $A + z$.

Spectrum of A . — The spectrum of A was calculated in Brüning and Seeley [BS88], p.703. By their result, it holds that the spectrum of A is given by the values

$$\gamma = \pm \frac{1}{2} \pm \sqrt{\mu^2 + \left(\frac{n-1}{2} - p \right)^2}, \quad (3)$$

where μ^2 runs over the spectrum of the Hodge-de Rham operator on (Σ, h) acting on the coclosed p -forms.

Indeed, looking at the Gauß-Bonnet operator acting on even forms, they identify even forms on the cone with the sections $(\varphi_0, \dots, \varphi_n)$ of the total bundle $T^*(\Sigma)$ by $\varphi_0 + \varphi_1 \wedge dr + \varphi_2 + \varphi_3 \wedge dr + \dots$. These sections can as well represent odd forms on the cone by $\varphi_0 \wedge dr + \varphi_1 + \varphi_2 \wedge dr + \varphi_3 + \dots$. With these identifications, they have to study the spectrum of the following operator acting on sections of $\Lambda T^*(\Sigma)$

$$S_0 = \begin{pmatrix} c_0 & d_0^* & 0 & \cdots & 0 \\ d_0 & c_1 & d_0^* & \ddots & \vdots \\ 0 & d_0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & c_{n-1} & d_0^* \\ 0 & \cdots & 0 & d_0 & c_n \end{pmatrix}$$

if $c_p = (-1)^{p+1}(p - \frac{n}{2})$. With the same identification, if we introduce the operator \tilde{S}_0 having the same formula but with on the diagonal the terms $\tilde{c}_p = (-1)^p(p - \frac{n}{2}) = -c_p$, the operator A can be written

$$A = -(S_0 \oplus \tilde{S}_0).$$

The expression of the spectrum of A is then a direct consequence of the computations of [BS88].

Closed extensions of D_1 . — If $\text{spec}(A) \cap]-\frac{1}{2}, \frac{1}{2}[$ is empty then $D_{1,\max} = D_{1,\min}$. In particular, D_1 is essentially selfadjoint on the space of smooth forms with compact support *away* from the conical singularities.

Otherwise, the quotient $\text{dom}(D_{1,\max})/\text{dom}(D_{1,\min})$ is isomorphic to

$$B := \bigoplus_{\gamma \in]-\frac{1}{2}, \frac{1}{2}[} \text{Ker}(A - \gamma).$$

More precisely, by Lemma 3.2 of [BS88], there is a surjective linear map

$$\mathcal{L} : \text{dom}(D_{1,\max}) \rightarrow B$$

with $\ker \mathcal{L} = \text{dom}(D_{1,\min})$. Furthermore, we have the estimate

$$\|u(t) - t^{-A}\mathcal{L}(\varphi)\|_{L^2(\Sigma)} \leq C(\varphi)|t \log t|^{1/2}$$

for $\varphi \in \text{dom}(D_{1,\max})$ and $u = U(\varphi)$.

Now, to any subspace $W \subset B$, we can associate the operator $D_{1,W}$ with domain $\text{dom}(D_{1,W}) := \mathcal{L}^{-1}(W)$. As a result of [BS88], all closed extensions of $D_{1,\min}$ are obtained by this way. Remark that each $D_{1,W}$ defines a selfadjoint extension $\Delta_{1,W} = (D_{1,W})^* \circ D_{1,W}$ of the Hodge-de Rham operator, and, as a result, we have $(D_{1,W})^* = D_{1,\mathbb{I}(W^\perp)}$, where

$$\mathbb{I} = \begin{pmatrix} 0 & \text{id} \\ -\text{id} & 0 \end{pmatrix}, \quad \text{ie.,} \quad \mathbb{I}(\beta, \alpha) = (\alpha, -\beta).$$

This extension is associated to the quadratic form $\varphi \mapsto \|D\varphi\|^2$ with domain $\text{dom}(D_{1,W})$.

Finally we recall the results of Lesch [L97]. The operators $D_{1,W}$, and in particular $D_{1,\min}$ and $D_{1,\max}$, are elliptic and satisfy the singular estimate (SE), see page 54 of [L97], so by proposition 1.4.6 and the compactness of $\overline{M_1}$ they satisfy the *Rellich property*: the inclusion of $\text{dom}(D_{1,W})$ in L_2 is compact.

2.3. Gauß-Bonnet operator on M_2 . We know by the works of Carron [C01], following Atiyah-Patodi-Singer [APS75], that the operator D_2 admits a closed extension \mathcal{D}_2 with domain defined by a global boundary condition

$$\Pi_{\leq 1/2} \circ U = 0$$

if Π_I is the spectral projector of A relative to the interval I , and $\leq 1/2$ represents the interval $] -\infty, 1/2]$. Moreover this extension is elliptic in the sens that the H_1

norm of elements of the domain is controlled by the norm of the graph. Indeed this boundary condition is related to a problem on a complete non bounded manifold as follows:

Let \widetilde{M}_2 denote the large manifold obtained from M_2 by gluing a conical cylinder $\mathcal{C}_{1,\infty} = [1, \infty[\times \Sigma$ with metric $dr^2 + r^2h$ and \widetilde{D}_2 its Gauß-Bonnet operator. A differential form on M_2 admits an harmonic L_2 -extension on \widetilde{M}_2 precisely when the restriction on the boundary satisfies $\Pi_{\leq 1/2} \circ U = 0$.

Indeed, these L_2 forms must satisfy $(\partial_r + \frac{1}{r}A)\sigma = 0$ or, if we decompose the form along the eigenspaces of A as $\sigma = \sum_{\gamma \in \text{Spec}(A)} \sigma_\gamma$ then the equation imposes $\forall \gamma \in \text{Spec}(A), \exists \sigma_0^\gamma \in \ker(A - \gamma)$ such that $\sigma_\gamma = r^{-\gamma} \sigma_0^\gamma$. This expression is in $L_2(\mathcal{C}_{1,\infty})$ if and only if $\gamma > 1/2$ or $\sigma_0^\gamma = 0$.

It will be convenient to introduce the operator

$$P_2 : \Pi_{> \frac{1}{2}}(H^{1/2}(\Sigma)) \rightarrow L_2(\Lambda T^* \mathcal{C}_{1,\infty})$$

$$\sigma = \sum_{\gamma \in \text{Spec}(A), \gamma > \frac{1}{2}} \sigma_\gamma \mapsto P_2(\sigma) = U^* \left(\sum_{\gamma \in \text{Spec}(A), \gamma > \frac{1}{2}} r^{-\gamma} \sigma_\gamma \right).$$

This limit problem is of the category *non parabolic at infinity* in the terminology of Carron [C01], see particularly the theorem 2.1 there, then as a consequence of theorem 0.4 of the same paper we know that its kernel is finite dimensional.

Proposition 2. *There exists a constant $C > 0$ such that for each differential form $\varphi \in H^1(\Lambda T^* M_2)$ satisfying $\Pi_{\leq 1/2} \circ U(\varphi) = 0$, then*

$$\|\varphi\|_{H^1} \leq C(\|\varphi\|_{L_2} + \|D_2(\varphi)\|_{L_2}).$$

As a consequence, the kernel of \mathcal{D}_2 , which is isomorphic to $\ker \widetilde{D}_2$, is of finite dimension and can be sent in the total space $\sum_p H^p(M_2)$ of absolute cohomology.

A proof of this proposition can be obtain by the same way as proposition 5 in [AT09].

Recall that Carron proved that a Dirac type operator on a manifold euclidean at infinity is non parabolic at infinity and it is still true for a manifold like \widetilde{M}_2 , as a consequence of Theorem 2.2 of [C01]. For this type of operators Carron defined L_2 solutions which are in correspondance with the solutions of the elliptic operator of the proposition 2, he defined also *extended* solutions which belong to the bigger space \mathcal{W} which is defined as the closure of $C_0^\infty(\widetilde{M}_2)$ for the norm

$$\|\varphi\|_{\mathcal{W}}^2 = \|\varphi\|_{L_2(M_2)}^2 + \|D\varphi\|_{L_2(\widetilde{M}_2)}^2.$$

An inequality of Hardy's type describe the growth at infinity of an extended solution.

Lemma 3. *Let $v \in C_0^\infty(]e, \infty[)$ and λ a real, then*

$$\begin{aligned} \text{if } \lambda \neq -1/2 & \quad (\lambda + \frac{1}{2})^2 \int \frac{v^2}{r^2} dr \leq \int \frac{1}{r^{2\lambda}} (\partial_r(r^\lambda v))^2 dr \\ \text{if } \lambda = -1/2 & \quad \int \frac{v^2}{(r \log r)^2 \log \log r} dr \leq \int r (\partial_r(r^{-1/2} v))^2 dr. \end{aligned}$$

We remark now that, for $\varphi \in C_0^\infty(\widetilde{M}_2)$ with support on the infinite cone $r > e$ we can write

$$\|D_2\varphi\|^2 = \sum_{\lambda \in \text{Spec}(A)} \int |(\partial_r + \frac{\lambda}{r}\sigma_\lambda)|^2 = \sum_{\lambda \in \text{Spec}(A)} \int \frac{1}{r^{2\lambda}} |\partial_r(r^\lambda)\sigma_\lambda|^2 dr.$$

Thus, by application of the lemma, we see that a solution of \widetilde{D}_2 , which must be $\sigma_\lambda(r) = r^{-\lambda}\sigma_\lambda(1)$ on the infinite cone, and such that $\frac{1}{r}\varphi$ is L_2 satisfies

$$\forall \lambda \leq \frac{-1}{2}, \quad \sigma_\lambda(1) = 0$$

while the L_2 solutions corresponds to the condition $\sigma_\lambda(1) = 0 \quad \forall \lambda \leq \frac{1}{2}$. As a consequence, the extended solutions which are not L_2 correspond to boundary terms belonging to the total eigenspace related with eigenvalues of A in the interval $] -1/2, 1/2]$. In the case studied in [AT09] there was not such eigenvalues and we had not to take care of extended solutions.

Let denote, for $\gamma \in] -1/2, 1/2]$ by

$$\mathcal{I}_\gamma = \{\varphi, \widetilde{D}_2\varphi = 0, \varphi(r=1) \in \ker(A - \gamma)\} \quad (4)$$

the total space of extended solutions corresponding to the value γ of the spectrum of A .

proof of the lemma 3. Let $u \in C_0^\infty(]e, \infty[)$ and ρ a positive function on the interval $]e, \infty[$. We calculate

$$\begin{aligned} \int \frac{1}{r^{2\lambda}} (\partial_r(r^\lambda \rho u))^2 dr &= \\ &= \int \frac{1}{r^{2\lambda}} \left(\rho' r^\lambda u + \rho \partial_r(r^\lambda u) \right)^2 dr = \\ &= \int \left((\rho' u)^2 + \frac{\rho^2}{r^{2\lambda}} (\partial_r(r^\lambda u))^2 + \frac{\rho \rho'}{r^{2\lambda}} \partial_r((r^\lambda u)^2) \right) dr \end{aligned}$$

integrating by part the last term, we obtain

$$\begin{aligned} \int \frac{1}{r^{2\lambda}} (\partial_r(r^\lambda \rho u))^2 dr &= \\ &= \int \left(\frac{\rho^2}{r^{2\lambda}} (\partial_r(r^\lambda u))^2 - (\rho \rho'' - 2\lambda \frac{\rho \rho'}{r}) u^2 \right) dr \\ &\geq \int \left(\frac{2\lambda \rho' - r \rho''}{r \rho} (\rho u)^2 \right) dr \end{aligned}$$

We obtain finally the first result of the lemma by putting $\rho(r) = r^{\lambda+1/2}$ and $v = \rho u$.

The second one is obtained in the same way with $\lambda = -1/2$ and the choice of $\rho = \log r$. \square

3. A PRIORI ESTIMATES

A good method to evaluate what the limit problem should be is to suppose a priori that we have a normed family φ_ε of eigenforms on M_ε of degree p for the Hodge-de Rham operator:

$$\Delta_\varepsilon \varphi_\varepsilon = \lambda_\varepsilon \varphi_\varepsilon \text{ with } \lim_{\varepsilon \rightarrow 0} \lambda_\varepsilon = \lambda < +\infty$$

and to try to obtain all the consequences for the limit. Of course for the moment we are not sure that this family exists but the mini-max technics will assure this, and it will be the subject of the next section.

3.1. notations. As in [ACP07] we use the following change of variables : with

$$\varphi_\varepsilon|_{M_1(\varepsilon)} = \varphi_{1,\varepsilon} \text{ and } \varphi_\varepsilon|_{M_2(1)} = \varepsilon^{p-m/2} \varphi_{2,\varepsilon}$$

we write on the cone

$$\varphi_{1,\varepsilon} = dr \wedge r^{-(n/2-p+1)} \beta_{1,\varepsilon} + r^{-(n/2-p)} \alpha_{1,\varepsilon}$$

and define $\sigma_1 = (\beta_1, \alpha_1) = U(\varphi_1)$.

On the other part, it is more convenient to define $r = 1 - s$ for $s \in [0, 1/2]$ and write $\varphi_{2,\varepsilon} = (dr \wedge r^{-(n/2-p+1)} \beta_{2,\varepsilon} + r^{-(n/2-p)} \alpha_{2,\varepsilon})$ near the boundary. Then we can define, for $r \in [1/2, 1]$ (the boundary of $M_2(1)$ corresponds to $r = 1$)

$$\sigma_2(r) = (\beta_2(r), \alpha_2(r)) = U(\varphi_2).$$

The L_2 norm, for a form supported on M_1 in the cone $\mathcal{C}_{\varepsilon,1}$, has the expression

$$\|\varphi\|^2 = \int_{M_1} |\sigma_1|^2 dr \wedge d\text{vol}_\Sigma + \int_{M_2} |\varphi_2|^2 d\text{vol}_{M_2}$$

and the quadratic form on study is

$$q(\varphi) = \int_{M_\varepsilon} |(d + d^*)\varphi|^2 = \int_{M_1(\varepsilon)} |UD_1U^*(\sigma_1)|^2 + \frac{1}{\varepsilon^2} \int_{M_2(1)} |D_2(\varphi_2)|^2 \quad (5)$$

The compatibility condition is, for the quadratic form, $\varepsilon^{1/2}\alpha_1(\varepsilon) = \alpha_2(1)$ and $\varepsilon^{1/2}\beta_1 = \beta_2(1)$ or

$$\sigma_2(1) = \varepsilon^{1/2}\sigma_1(\varepsilon). \quad (6)$$

The compatibility condition for the Hodge-de Rham operator, of first order, is obtained by expressing that $D\varphi \sim (UD_1U^*\sigma_1, \frac{1}{\varepsilon}UD_2U^*\sigma_2)$ belongs to the domain of D . In terms of σ it gives

$$\sigma'_2(1) = \varepsilon^{3/2}\sigma'_1(\varepsilon). \quad (7)$$

To understand the limit problem we proceed to several estimates.

3.2. on the regular part of M_1 . Let ξ_1 be a cut-off function on M_1 around the conical singularity:

$$0 \leq r \leq 1/2 \Rightarrow \xi_1(r) = 1 \text{ and } r \geq 1 \Rightarrow \xi_1(r) = 0. \quad (8)$$

Lemma 4. *For our given family φ_ε satisfying $\Delta(\varphi_\varepsilon) = \lambda_\varepsilon \varphi_\varepsilon$ with λ_ε bounded, the family $(1 - \xi_1) \cdot \varphi_{1,\varepsilon}$ is bounded in $H^1(M_1)$.*

Then it remains to study $\xi_1 \cdot \varphi_{1,\varepsilon}$ which can be expressed with the polar coordinates. We remark that the quadratic form of these forms is uniformly bounded.

• *Expression of the quadratic form.* For any φ such that the component φ_1 is supported in the cone $\mathcal{C}_{1,\varepsilon}$, one has, with $\sigma_1 = U\varphi_1$ and by the same calculus as in [ACP07] :

$$\begin{aligned} \int_{\mathcal{C}_{\varepsilon,1}} |D_1\varphi|^2 d\text{vol}_{g_\varepsilon} &= \int_\varepsilon^1 \left| \left(\partial_r + \frac{1}{r}A \right) \sigma_1 \right|^2 dr \\ &= \int_\varepsilon^1 \left[|\sigma_1'|^2 + \frac{2}{r} \langle \sigma_1', A\sigma_1 \rangle + \frac{1}{r^2} |A\sigma_1|^2 \right] dr \end{aligned}$$

3.3. estimates of the boundary term. The expression above can be decomposed with respect to the eigenspaces of A ; in the following calculus, we suppose that $\sigma_1(1) = 0$:

$$\begin{aligned} \int_\varepsilon^1 \left[|\sigma_1'|^2 + \frac{2}{r} \langle \sigma_1', A\sigma_1 \rangle + \frac{1}{r^2} |A\sigma_1|^2 \right] dr &= \\ = \int_\varepsilon^1 \left[|\sigma_1'|^2 + \partial_r \left(\frac{1}{r} \langle \sigma_1, A\sigma_1 \rangle \right) + \frac{1}{r^2} (\langle \sigma_1, A\sigma_r \rangle + |A\sigma_1|^2) \right] dr \\ = \int_\varepsilon^1 \left[|\sigma_1'|^2 + \frac{1}{r^2} \langle \sigma_1, (A + A^2)\sigma_1 \rangle \right] dr - \frac{1}{\varepsilon} \langle \sigma_1(\varepsilon), A\sigma_1(\varepsilon) \rangle. \end{aligned}$$

This shows that a control of the quadratic form gives a control of the boundary term if the operator A is negative but $(A + A^2)$ is non negative. This last condition is satisfied exactly on the orthogonal complement of the spectral space corresponding to the interval $] -1, 0[$. Applied to $\xi_1 \cdot \varphi_{1,\varepsilon}$ it gives the following lemma

Lemma 5. *Let $\Pi_{\leq -1}$ be the spectral projector of the operator A relative to the interval $] -\infty, -1[$. There exists a constant $C > 0$ such that, for any $\varepsilon > 0$*

$$\|\Pi_{\leq -1} \circ U(\varphi_{1,\varepsilon}(\varepsilon))\|_{H^{1/2}(\Sigma)} \leq C\sqrt{\varepsilon}.$$

In view of Proposition 2, we want also a control of the components of σ_1 along the eigenvalues of A in $] -1, 1/2[$, these components are in finite number and we can work term by term. So we write, on $\mathcal{C}_{\varepsilon,1}$,

$$\sigma_1(r) = \sum_{\gamma \in \text{spec } A} \sigma_1^\gamma(r), \quad A\sigma_1^\gamma(r) = \gamma\sigma_1^\gamma(r)$$

and we suppose again $\sigma_1(1) = 0$. We have

$$\left(\partial_r + \frac{1}{r}A\right)\sigma_1^\gamma = r^{-\gamma}\partial_r(r^\gamma\sigma) \quad (9)$$

$$(\varepsilon^\gamma\sigma_1^\gamma(\varepsilon))^2 = \left(\int_\varepsilon^1 \partial_r(r^\gamma\sigma_1^\gamma)\right)^2 \leq \int_\varepsilon^1 r^{2\gamma} \int_\varepsilon^1 |\partial_r(\sigma_1^\gamma) + \frac{\gamma}{r}(\sigma_1^\gamma)|^2 \quad (10)$$

then, if the quadratic form is bounded, there exists a constant $C > 0$ such that

$$(\sigma_1^\gamma(\varepsilon))^2 \leq \begin{cases} C\varepsilon^{-2\gamma}\frac{1-\varepsilon^{2\gamma+1}}{2\gamma+1} & \text{if } \gamma \neq -1/2 \\ C\varepsilon|\log \varepsilon| & \text{if } \gamma = -1/2. \end{cases} \quad (11)$$

This gives

Lemma 6. *Let Π_I be the spectral projector of the operator A relative to the interval I . There exist constants $\alpha, C > 0$ such that, for any $\varepsilon > 0$*

$$\|\Pi_{]-1,0[} \circ U(\varphi_{1,\varepsilon}(\varepsilon))\|_{H^{1/2}(\Sigma)} \leq C\varepsilon^\alpha.$$

If $0 < \alpha < 1/2$ is such that $-\alpha$ is bigger than any negative eigenvalue of A . The estimate above gives also

Lemma 7. *With the same notation, there exist constants $\beta, C > 0$ such that, for any $\varepsilon > 0$*

$$\|\Pi_{[0,1/2[} \circ U(\varphi_{2,\varepsilon}(1))\|_{H^{1/2}(\Sigma)} \leq C\varepsilon^\beta.$$

If $1/2 - \beta$ is the biggest non negative eigenvalue of A strictly smaller than $1/2$ (if there is no such eigenvalue we put $\beta = 1/2$).

Finally we study $\sigma_1^{1/2}$ for our family of forms (the parameter ε is omitted in the notation). It satisfies, for $\varepsilon < r < 1/2$, the equation

$$\left(-\partial_r^2 + \frac{3}{4r^2}\right)\sigma_1^{1/2} = \lambda_\varepsilon\sigma_1^{1/2}.$$

The solutions of this equation have expression in term of Bessel functions: there exist entire functions F, G with $F(0) = G(0) = 1$ and differential forms in $\ker(A - 1/2)$ $c_\varepsilon, d_\varepsilon$ such that

$$\sigma_1^{1/2}(r) = c_\varepsilon r^{3/2}F(\lambda_\varepsilon r^2) + d_\varepsilon \left(r^{-1/2}G(\lambda_\varepsilon r^2) + \frac{2}{\pi} \log(r)r^{3/2}F(\lambda_\varepsilon r^2) \right). \quad (12)$$

The fact that the L_2 norm is bounded gives that $c_\varepsilon^2 + |\log \varepsilon|d_\varepsilon^2$ is bounded and finally, reporting this estimate in the expression above, that

$$\|\sigma_1^{1/2}(\varepsilon)\|_{H^{1/2}(\Sigma)} = O\left((\varepsilon|\log \varepsilon|)^{-1/2}\right).$$

This gives on the other part

Lemma 8. *There exists a constant $C > 0$ such that, for any $\varepsilon > 0$*

$$\|\Pi_{\{1/2\}} \circ U(\varphi_{2,\varepsilon})(1)\|_{H^{1/2}(\Sigma)} \leq C \frac{1}{\sqrt{|\log \varepsilon|}}$$

4. PROOF OF THE SPECTRAL CONVERGENCE

The previous estimates shows that $\Pi_{\leq 1/2} \circ U(\varphi_{2,\varepsilon})(1)$ goes to zero with ε and that the limit would satisfy the good APS-boundary conditions ; on the other hand there is no restriction for the boundary term of $U(\varphi_{1,\varepsilon})(\varepsilon)$ and finally the control on $\Pi_{\{1/2\}} \circ U(\varphi_{2,\varepsilon})(1)$ does not permit to admit that this term will disappear.

Its suggests that we have to introduce for the limit problem, as Hilbert space

$$\mathcal{H}_\infty = L_2(\overline{M_1}) \oplus \ker \widetilde{D}_2 \oplus \ker(A - \frac{1}{2}). \quad (13)$$

with the limit operator $\Delta_{1,W} \oplus 0 \oplus 0$ with W defined as in Proposition A. acting on this space. Let's denote by λ_N , $N \geq 1$ its spectrum and also let's $\lambda_N(\varepsilon)$, $N \geq 1$ be the total spectrum of the Hodge Laplacian on M_ε .

4.1. upper bound : $\limsup \lambda_N(\varepsilon) \leq \lambda_N$. With the minimax formula, which says that

$$\lambda_N(\varepsilon) = \text{Inf}_{\substack{E \subset \text{dom}(D_\varepsilon) \\ \dim E = N}} \left(\text{Sup}_{\substack{\varphi \in E \\ \|\varphi\|=1}} \int_{M_\varepsilon} |D_\varepsilon \varphi|^2 \right)$$

we have to describe how transplanting eigenforms of the limit problem on M_ε .

We describe this transplantation term by term. For the first we use the same ideas as in [ACP07].

Any eigenform φ of $\Delta_{1,W}$ can be written, as any element of $\text{dom}(D_{1,\max})$,

$$\begin{aligned} \varphi &= \varphi_0 + \bar{\varphi} \\ \varphi_0 \in \text{dom}(D_{1,\min}) \quad \text{and} \quad U(\bar{\varphi}) &= \begin{cases} 0 & \text{for } r \geq 1 \\ \sum_{\gamma \in]-1/2, 1/2[\cap \text{spec}(A)} d_\gamma r^{-\gamma} \sigma_\gamma & \text{for } r \leq \frac{1}{2} \end{cases} \end{aligned}$$

with $\sigma_\gamma \in \text{Ker}(A - \gamma)$ and $d_\gamma \in \mathbb{R}$. (This fact is a consequence of the expression of the Bessel functions, it is the same calculus as in (21).) By definition of $D_{1,\min}$, φ_0 can be approached, with the operator norm, by a sequence of smooth forms with compact support in $M_1(0)$. On the other hand, $\bar{\varphi}$ is a finite sum, it is sufficient to prolongate each $r^{-\gamma} \sigma_\gamma$ on M_2 , the good candidate is

$$\psi_\gamma = U^*(\chi \cdot \varepsilon^{1/2-\gamma} r^{-\gamma} \sigma_\gamma)$$

for χ a cut-off function supported on $]1/2, 3/2[$ and equal to 1 on a neighbourhood of 1 (for instance $\chi = 1 - \xi_1$).

Indeed one has

$$\|\psi_\gamma\|_{L_2(M_2)} + \|D_2(\psi_\gamma)\|_{L_2(M_2)} = O(\varepsilon^{1/2-\gamma}).$$

For the two last ones, we shrink the infinite cone on M_1 and cut with the function ξ_1 , already defined in (8).

Let's define

$$P_\varepsilon : \Pi_{>\frac{1}{2}}\left(H^{1/2}(\Sigma)\right) \rightarrow H^1(\mathcal{C}_{\varepsilon,1}) \quad (14)$$

$$\sigma = \sum_{\gamma \in \text{Spec}(A), \gamma > \frac{1}{2}} \sigma_\gamma \mapsto P_\varepsilon(\sigma) = U^* \left(\sum_{\gamma \in \text{Spec}(A), \gamma > \frac{1}{2}} \varepsilon^{\gamma-1/2} r^{-\gamma} \sigma_\gamma \right).$$

We remark that $P_\varepsilon(\sigma)$ is the transplanted on M_1 of $P_2(\sigma)$ then there exists a constant C such that

$$\|P_2(\sigma)\|_{L_2(\mathcal{C}_{1,1/\varepsilon})}^2 = \|P_\varepsilon(\sigma)\|_{L_2(\mathcal{C}_{\varepsilon,1})}^2 \leq C \sum |\sigma_\gamma|^2 = C \|\Pi_{>\frac{1}{2}}\sigma_2(1)\|_{L_2(\Sigma)}^2 \quad (15)$$

and also that, if $\psi_2 \in \text{Dom}\mathcal{D}_2$, then $(\xi_1 P_\varepsilon(U(\psi_2)|_\Sigma), \psi_2)$ defines an element of $H^1(M_\varepsilon)$.

Finally, if $\ker(A - 1/2)$ is not empty, for each $\bar{\sigma}^{1/2} \in \ker(A - 1/2)$ non zero one can construct a *pseudomode* as follows

$$\psi_\varepsilon = (\xi_1 \cdot (|\log \varepsilon| r)^{-1/2} U^*(\bar{\sigma}^{1/2}), |\log \varepsilon|^{-1/2} \psi_2) \quad (16)$$

where ψ_2 is the element of $\ker D_2$ on M_2 with boundary value $\bar{\sigma}^{1/2}$. The L_2 norm of this element is bounded from above and below, and

$$\lim_{\varepsilon \rightarrow 0} \|\psi_\varepsilon\|_{L_2(M_\varepsilon)} = \|\bar{\sigma}^{1/2}\|_{L_2(\Sigma)}.$$

Moreover it satisfies $q(\psi_\varepsilon) = O(|\log \varepsilon|^{-1})$ giving then a 'small eigenvalue', as well as the elements of $\ker \mathcal{D}_2$ and of $\ker \Delta_W$.

To conclude it suffices to verify that this transplanted forms have a Rayleigh-Ritz quotient comparable to the initial one and that the orthogonality is fast concerved by transplantation.

4.2. lower bound : $\liminf \lambda_N(\varepsilon) \geq \lambda_N$. We first proceed for one indice. We know, by the paragraph 4.1, that for each N , the family $(\lambda_N(\varepsilon), \varepsilon > 0)$ is bounded, let

$$\lambda = \liminf \lambda_N(\varepsilon).$$

There exists a sequence $\varepsilon_m, m \in \mathbb{N}$ such that $\lim_{m \rightarrow \infty} \lambda_N(\varepsilon_m) = \lambda$. We shall apply the previous calculus to this family.

4.2.1. on the side of M_2 . Let χ be a cut-off function supported in $[3/4, 1[$ equal to 1 on $[7/8, 1[$ and $\sigma_{2,\varepsilon_m} = U(\varphi_{2,\varepsilon_m})$. We construct the family

$$\psi_{2,\varepsilon_m} = \varphi_{2,\varepsilon_m} - U^* \left(\Pi_{\leq \frac{1}{2}}(\chi \sigma_{2,\varepsilon_m}) \right)$$

which belongs to the domain of \mathcal{D}_2 , and then, by ellipticity of this operator

claim 1. The family ψ_{2,ε_m} is bounded in $H^1(M_2)$, and as a consequence $\Pi_{>\frac{1}{2}}\sigma_2(1)$ is bounded in $H^{1/2}(\Sigma)$.

Moreover, its proximity with $\varphi_{2,\varepsilon_m}$ is controled:

claim 2. *They satisfy $\lim_{\varepsilon_m \rightarrow 0} \|\psi_{2,\varepsilon_m} - \varphi_{2,\varepsilon_m}\| = 0$ and*

$$\lim_{\varepsilon_m \rightarrow 0} \|D_2(\psi_{2,\varepsilon_m} - \varphi_{2,\varepsilon_m})\| = O(\sqrt{\varepsilon_m}). \quad (17)$$

Indeed

$$\|\psi_{2,\varepsilon_m} - \varphi_{2,\varepsilon_m}\|^2 \leq \int_{3/4}^1 |\Pi_{\leq \frac{1}{2}} \sigma_2(r)|^2 dr = \int_{3/4}^1 \left(|\Pi_{\leq -1} \sigma_2(r)|^2 + |\Pi_{]-1, \frac{1}{2}] \sigma_2(r)|^2 \right) dr. \quad (18)$$

The first term is controled as follows: because of the expression (5) of the quadratic form, we know that

$$\int_{\mathcal{C}_{\varepsilon_m,1}} \left| \left(\partial_r + \frac{A}{r} \right) \Pi_{\leq -1} \sigma_1 \right|^2 + \frac{1}{\varepsilon_m^2} \int_R^1 \left| \left(\partial_r + \frac{A}{r} \right) \Pi_{\leq -1} \sigma_2 \right|^2$$

is bounded, independently on $R \geq 3/4$, let Λ be a bound. Developping and making an integration by parts gives, for any κ_2

$$\begin{aligned} & \int_R^1 \left| \left(\partial_r + \frac{A}{r} \right) \kappa_2 \right|^2 = \\ & \int_R^1 \left[|\kappa_2'|^2 + \frac{1}{r^2} \langle \kappa_2, (A + A^2) \kappa_2 \rangle \right] dr + \langle \kappa_2(1), A \kappa_2(1) \rangle - \langle \kappa_2(R), A \kappa_2(R) \rangle. \end{aligned} \quad (19)$$

We can make the same calculus on the cone $\mathcal{C}_{\varepsilon_m,1}$ so the boundary term at $R = 1$ is absorbed. If we apply this control to $\kappa_2 = \Pi_{\leq -1} \sigma_2$ for which $A(A + 1)$ and $(-A)$ are non negative we obtain that

$$\int_R^1 |\Pi_{\leq -1} \sigma_2'(t)|^2 + \|\Pi_{\leq -1} \sigma_2'(R)\|^2 = O(\varepsilon_m^2).$$

The second term in (18) is in fact the sum of few terms, we can control each of them as follows. let γ be an eigenvalue of A contained in the interval $]-1, \frac{1}{2}]$ and σ_2^γ the component of σ_2 along this eigenspace. We know that $\int_R^1 |r^{-\gamma} \partial_r (r^\gamma \sigma_2^\gamma)|^2 = O(\varepsilon_m^2)$.

$$|R^\gamma \sigma_2^\gamma(R)|^2 = -2 \int_R^1 \langle \partial_r (r^\gamma \sigma_2^\gamma(t)), r^\gamma \sigma_2^\gamma(t) \rangle dt + |\sigma_2^\gamma(1)|^2 \quad (20)$$

But, as a consequence of lemma 5, 6 and 7, one has $\|\sigma_2^\gamma(1)\|^2 = O(1/|\log(\varepsilon_m)|)$, on the other hand, using the inequality of Cauchy-Schwarz, the fact that the L_2 -norm of φ_{ε_m} is 1 and the fact just recalled we have that

$$|\sigma_2^\gamma(R)|^2 = O(1/|\log(\varepsilon_m)|).$$

For the second part of the assertion, notice that

$$D_2(\varphi_{2,\varepsilon_m} - \psi_{2,\varepsilon_m}) = D_2 U^* \left(\Pi_{\leq \frac{1}{2}} (\chi \sigma_{2,\varepsilon_m}) \right) = \chi' U^* \Pi_{\leq \frac{1}{2}} (\sigma_{2,\varepsilon_m}) + \chi D_2 U^* \Pi_{\leq \frac{1}{2}} (\sigma_{2,\varepsilon_m})$$

and the norm of the first term is controled by $\int_{3/4}^1 |\Pi_{\leq \frac{1}{2}} \sigma_2(r)|^2 dr$ which is $O(1/|\log(\varepsilon_m)|)$ by the previous estimate, and the norm of the second term, by $\|D_2(\varphi_2)\|$ which is $O(\varepsilon_m)$ because $q_{\varepsilon_m}(\varphi_{\varepsilon_m})$ is uniformly bounded (remark that D_2 preserves the orthogonal decomposition following $\Pi_{\leq \frac{1}{2}}$ and $\Pi_{> \frac{1}{2}}$ on the cone).

Proposition 9. *There exists a subfamily of the family $\varphi_{2,\varepsilon_m}$ which converges, as $m \rightarrow \infty$ to a bounded form φ_2 on M_2 which satisfies*

$$\varphi_2 \in \text{dom } \mathcal{D}_2, \|\varphi_2\| \leq 1 \text{ and } D_2(\varphi_2) = 0.$$

Proof. Indeed, we know by the claim 1 that we can extract from the family ψ_{2,ε_m} a subfamily which converge in L_2 -norm and weakly in $H^1(M_2)$, denote φ_2 the limit. We know by the claim 2 that the corresponding subfamily of $\varphi_{2,\varepsilon_m}$ has the same limit and also that $\lim_{m \rightarrow \infty} \|D_2(\psi_{2,\varepsilon_m})\| = 0$ because this is true for $\varphi_{2,\varepsilon_m}$, the conclusion follows. \square

4.2.2. *on the side of M_1 .* We first recall the better prolongation of $\Pi_{> \frac{1}{2}} \sigma_2(1)$ on $M_1(\varepsilon_m)$ given by P_ε in (14) so that if $\psi_2 \in \text{Dom } \mathcal{D}_2$ and with the same cut-off function ξ_1 , which has value 1 for $0 \leq r \leq 1/2$ and 0 for $r \geq 1$, $(\xi_1 P_{\varepsilon_m}(U(\psi_2)|_\Sigma), \psi_2)$ defines an element of $H^1(M_{\varepsilon_m})$. Let

$$\tilde{\psi}_{1,\varepsilon_m} := \xi_1 P_{\varepsilon_m}(U(\psi_{2,\varepsilon_m})|_\Sigma).$$

Corollary 10. *By uniform continuity of P_{ε_m} , and the convergence property of the last proposition*

$$\lim_{m \rightarrow \infty} \|\tilde{\psi}_{1,\varepsilon_m} - \xi_1 P_{\varepsilon_m}(U(\varphi_2)|_\Sigma)\|_{L_2(M_1)} = 0.$$

On the other hand $\xi_1 P_{\varepsilon_m}(U(\varphi_2)|_\Sigma)$ converge weakly to 0 on the open manifold $M_1(0)$, more precisely, for any fixed η , $0 < \eta < 1$

$$\lim_{m \rightarrow \infty} \|\xi_1 P_{\varepsilon_m}(U(\varphi_2)|_\Sigma)\|_{L_2(M_1(\eta))} = 0.$$

We now decompose $\varphi_{1,\varepsilon_m}$ near the singularity as follows. Let

$$\xi_1 \varphi_{1,\varepsilon_m} = \xi_1 (\varphi_{1,\varepsilon_m}^{\leq -1/2} + \varphi_{1,\varepsilon_m}^{]-1/2, 1/2]} + \varphi_{1,\varepsilon_m}^{> 1/2})$$

according to the decomposition, on the cone, of σ_1 along the eigenvalues of A respectively less than $-1/2$, in $]-1/2, 1/2]$ and larger than $1/2$.

We remark first that $\tilde{\psi}_1$ and $\varphi_{1,\varepsilon_m}^{> 1/2}$ have the same values on the boundary so the difference $\xi_1 \varphi_{1,\varepsilon_m}^{> 1/2} - \tilde{\psi}_1$ can be viewed in $H^1(M_1)$ by a prolongation by 0 on the complementary of the cone, while the boundary value of $\varphi_{1,\varepsilon_m}^{\leq -1/2}$ is small. We introduce for this term the cut-off function taken in [ACP07]

$$\xi_{\varepsilon_m}(r) = \begin{cases} 1 & \text{if } r \geq 2\sqrt{\varepsilon_m}, \\ \frac{\log(2\varepsilon_m) - \log r}{\log(\sqrt{\varepsilon_m})} & \text{if } r \in [2\varepsilon_m, 2\sqrt{\varepsilon_m}], \\ 0 & \text{if } r \leq 2\varepsilon_m. \end{cases}$$

claim 3. $\lim_{m \rightarrow \infty} \|(1 - \xi_{\varepsilon_m})\xi_1 \varphi_{1,\varepsilon_m}^{\leq -1/2}\|_{L_2} = 0.$

This is a consequence of the estimate of the Lemma 5 and 6: we remark that by the same argument we obtain also $\|\xi_1 \varphi_{1,\varepsilon_m}^{\leq -1/2}(r)\|_{H^{1/2}(\Sigma)} \leq Cr^{1/2}$ so

$$\|(1 - \xi_{\varepsilon_m})\xi_1 \varphi_{1,\varepsilon_m}^{\leq -1/2}\|_{L_2} = O(\varepsilon_m^{1/4}).$$

claim 4. *If we write, with evident notations,*

$$\varphi_{1,\varepsilon_m}^{]-1/2,1/2[} = \sum_{\gamma \in]-1/2,1/2[} U^* \sigma_1^\gamma(r) = \varphi_{1,\varepsilon_m}^{]-1/2,1/2[} + U^* \sigma_1^{1/2}(r)$$

$\sigma_1^{1/2}$ has the expression given in (12) and then can be decomposed on $\sigma_{1,\varepsilon_m}^{1/2} = \sigma_{0,\varepsilon_m}^{1/2} + \bar{\sigma}_{1,\varepsilon_m}^{1/2}$ where $\sigma_{0,\varepsilon_m}^{1/2} = c_{\varepsilon_m} r^{3/2} F(\lambda_{\varepsilon_m} r^2)$ belongs to $\text{dom}(D_{1,\min})$ and the family c_{ε_m} is bounded. One can extract subsequences such that $\varphi_{1,\varepsilon_m}^{]-1/2,1/2[}$ and $U^*(\sigma_0^{1/2})$ converge in L_2 -norm while $\bar{\sigma}_{1,\varepsilon_m}^{1/2}$ is asymptotically equivalent to

$$\bar{\sigma}_{1,\varepsilon_m}^{1/2} \sim \frac{1}{\sqrt{|\log \varepsilon_m|}} r^{-1/2} \bar{\sigma}_{1/2} \text{ for some } \bar{\sigma}_{1/2} \in \ker(A - 1/2).$$

Thuse $U^* \sigma_{1/2}$ converge on each $M_1(\eta)$ to the same limit as $U^*(\sigma_{0,\varepsilon_m}^{1/2})$, ie. $\bar{\sigma}_{1,\varepsilon_m}^{1/2}$ concentrate on the singularity .

Indeed, for each eigenvalue $\gamma \in]-1/2, 1/2[$ of A the vector $\sigma_\gamma(r)$ satisfies the equation $(-\partial_r^2 + \frac{\gamma(1+\gamma)}{r^2})\sigma_\gamma = \lambda_{\varepsilon_m} \sigma_\gamma$ and it can be written as

$$\sigma_\gamma(r) = c_{\varepsilon_m} r^{\gamma+1} F(\lambda_{\varepsilon_m} r^2) + d_{\varepsilon_m} \left(r^{-\gamma} G(\lambda_{\varepsilon_m} r^2) \right) \quad (21)$$

if $\gamma \neq 1/2$, while the expression for $\gamma = 1/2$ has been given in (12). Then, $\varphi_{1,\varepsilon_m}^{]-1/2,1/2[}$ belongs in the domain of $D_{1,\max}$, so, as a consequence of the Rellich property, we can extract subsequences which converge in L_2 . While, as already mentionned for $\gamma = 1/2$ the expression $c_{\varepsilon_m}^2 + |\log \varepsilon_m| d_{\varepsilon_m}^2$ is bounded so $\lim d_{\varepsilon_m} = 0$ and we can extract from $c_{\varepsilon_m}, \sqrt{|\log \varepsilon_m|} d_{\varepsilon_m}$ convergent subsequences. We define then

$$\bar{\sigma}_1^{1/2} = \lim_{m \rightarrow \infty} \sqrt{|\log \varepsilon_m|} d_{\varepsilon_m}. \quad (22)$$

Proposition 11. *The forms $\psi_{1,\varepsilon_m} = (1 - \xi_1)\varphi_{1,\varepsilon_m} + (\xi_1 \varphi_{1,\varepsilon_m}^{>1/2} - \tilde{\psi}_{1,\varepsilon_m}) + \xi_{\varepsilon_m} \xi_1 \varphi_{1,\varepsilon_m}^{\leq -1/2} + \xi_1 U^*(\sigma_{0,\varepsilon_m}^{1/2})$ belong to $\text{dom}(D_{1,\min})$ and define a bounded family.*

Proof. We will show that each term is bounded. For the last one, it was already mentionned in claim 4. For the first one it is already done in Lemma 4. For the

second one, we remark that

$$\begin{aligned} f_{\varepsilon_m} &:= \left(\partial_r + \frac{A}{r}\right)(\xi_1 \varphi_{1,\varepsilon_m}^{>1/2} - \tilde{\psi}_{1,\varepsilon_m}) = \\ &\quad \xi_1 \left(\partial_r + \frac{A}{r}\right)(\varphi_{1,\varepsilon_m}^{>1/2}) + \partial_r(\xi_1) \left(\varphi_{1,\varepsilon_m}^{>1/2} - P_{\varepsilon_m}(\psi_{2,\varepsilon_m}|\Sigma)\right) \end{aligned} \quad (23)$$

is uniformly bounded in $L_2(M_1)$ because of (15). This estimate (15) shows also that the L_2 -norm of $(\varphi_{1,\varepsilon_m}^{>1/2} - \tilde{\psi}_{1,\varepsilon_m})$ is bounded.

For the third one we use the estimate due to the expression of the quadratic form. Expriming that $\int_{\mathcal{C}_{r,1}} |D_1(\xi_1 \varphi^{\leq -1/2})|^2$ is bounded by Λ gives that

$$\langle \sigma_1^{\leq -1/2}(r), \sigma_1^{\leq -1/2}(r) \rangle \leq \Lambda r |\log r| \quad (24)$$

by the same argument as used for the lemma 5 and 6. Now

$$\begin{aligned} \|D_1(\xi_{\varepsilon_m} \xi_1 \varphi_{1,\varepsilon_m}^{\leq -1/2})\| &\leq \|\xi_{\varepsilon_m} D_1(\xi_1 \varphi_{1,\varepsilon_m}^{\leq -1/2})\| + \|d\xi_{\varepsilon_m} |\xi_1 \varphi_{1,\varepsilon_m}^{\leq -1/2}\| \\ &\leq \|D_1(\xi_1 \varphi_{1,\varepsilon_m}^{\leq -1/2})\| + \|d\xi_{\varepsilon_m} |\xi_1 \varphi_{1,\varepsilon_m}^{\leq -1/2}\| \end{aligned}$$

the first term is bounded and, with $|A| \geq 1/2$ for this term, and the estimate (24), we have

$$\begin{aligned} \|d\xi_{\varepsilon_m} |\xi_1 \varphi_{1,\varepsilon_m}^{\leq -1/2}\|^2 &\leq \frac{4\Lambda}{\log^2 \varepsilon_m} \int_{\varepsilon_m}^{\sqrt{\varepsilon_m}} \frac{\log(r)}{r} dr \\ &\leq \frac{3\Lambda}{2}. \end{aligned}$$

This complete the proof. \square

In fact the decomposition used here is almost orthogonal:

Lemma 12. *There exists $\beta > 0$ such that*

$$\langle (\varphi_{1,\varepsilon_m}^{>1/2} - \tilde{\psi}_{1,\varepsilon_m}), \tilde{\psi}_{1,\varepsilon_m} \rangle = O(\varepsilon_m^\beta).$$

proof of lemma 12. — If we decompose the terms under the eigenspaces of A we see that only the eigenvalues in $]1/2, +\infty[$ are involved and, with $f_{\varepsilon_m} = \sum_{\gamma > 1/2} f^\gamma$ and $(\varphi_{1,\varepsilon_m}^{>1/2} - \tilde{\psi}_{1,\varepsilon_m}) = \sum_{\gamma > 1/2} \varphi_0^\gamma$, the equation (23) and the fact that $(\varphi_{1,\varepsilon_m}^{>1/2} - \tilde{\psi}_{1,\varepsilon_m})(\varepsilon_m) = 0$ give

$$\varphi_0^\gamma(r) = r^{-\gamma} \int_{\varepsilon_m}^r \rho^\gamma f^\gamma(\rho) d\rho.$$

Then for each eigenvalue $\gamma > 1/2$ of A

$$\begin{aligned} \langle (\varphi_0^\gamma, \tilde{\psi}_{1,\varepsilon_m}^\gamma) \rangle &= \varepsilon_m^{\gamma-1/2} \int_{\varepsilon_m}^1 r^{-2\gamma} \int_{\varepsilon_m}^r \rho^\gamma \langle \sigma_\gamma, f^\gamma(\rho) \rangle_{L_2(\Sigma)} d\rho \\ &= \varepsilon_m^{\gamma-1/2} \int_{\varepsilon_m}^1 \frac{r^{-2\gamma+1}}{2\gamma-1} r^\gamma \langle \sigma_\gamma, f^\gamma(r) \rangle_{L_2(\Sigma)} dr + \\ &\quad \frac{\varepsilon_m^{\gamma-1/2}}{2\gamma-1} \int_{\varepsilon_m}^1 \rho^\gamma \langle \sigma_\gamma, f^\gamma(\rho) \rangle_{L_2(\Sigma)} d\rho \end{aligned}$$

Thuse, if $\gamma > 3/2$ we have the majoration

$$\begin{aligned} |\langle (\varphi_0^\gamma, \tilde{\psi}_{1,\varepsilon_m}^\gamma) \rangle| &\leq \varepsilon_m^{\gamma-1/2} \int_{\varepsilon_m}^1 \frac{r^{-\gamma+1}}{2\gamma-1} |\langle \sigma_\gamma, f^\gamma(r) \rangle_{L_2(\mathbb{S}^n)}| dr + \\ &\quad \frac{\varepsilon_m^{\gamma-1/2}}{(2\gamma-1)\sqrt{2\gamma+1}} \|\sigma_\gamma\| \|f^\gamma\|_{L_2(\mathcal{C}_{\varepsilon_m,1})} \\ &\leq C \varepsilon_m^{\gamma-1/2} \|\sigma_\gamma\| \frac{\varepsilon_m^{(-2\gamma+3)/2}}{(2\gamma-1)\sqrt{2\gamma-3}} \|f^\gamma\|_{L_2(\mathcal{C}_{\varepsilon_m,1})} + \\ &\quad \frac{\varepsilon_m^{\gamma-1/2}}{(2\gamma-1)\sqrt{2\gamma+1}} \|\sigma_\gamma\| \|f^\gamma\|_{L_2(\mathcal{C}_{\varepsilon_m,1})} \end{aligned}$$

while, for $\gamma = 3/2$ the first term is $O(\varepsilon_m \sqrt{|\log \varepsilon_m|})$ and for $1/2 < \gamma < 3/2$ it is $O(\varepsilon_m^{\gamma-1/2})$. In short, we have

$$|\langle (\varphi_0^\gamma, \tilde{\psi}_{1,\varepsilon_m}^\gamma) \rangle| \leq C \varepsilon_m^\beta \|\sigma_\gamma\| \|f^\gamma\|_{L_2(\mathcal{C}_{\varepsilon_m,1})}$$

if $\beta > 0$ satisfies $\gamma \geq \beta + 1/2$ for all eigenvalue γ of A in $]1/2, +\infty[$. This estimate gives the lemma.

Corollary 13. *There exists from $\psi_{1,\varepsilon_m} + \varphi_{1,\varepsilon_m}^{]-1/2,1/2[}$ a subfamily which converges in L_2 to a form φ_1 which satisfies on the open manifold $M_1(0)$ the equation $\Delta\varphi_1 = \lambda\varphi_1$. Moreover*

$$\|\varphi_1\|^2 + \|\widetilde{\varphi}_2\|^2 + \|\bar{\sigma}^{1/2}\|_{L_2(\Sigma)}^2 = 1. \quad (25)$$

if $\widetilde{\varphi}_2$ is the prolongation of φ_2 by $P_2(\varphi_{2|\Sigma})$ on \widetilde{M}_2 , and

$$\bar{\sigma}^{1/2} = \lim_{m \rightarrow \infty} \frac{1}{\sqrt{|\log \varepsilon_m|}} d_{\varepsilon_m}$$

with the expression of (12).

Proof. Indeed, the family $\psi_{1,\varepsilon_m} + \varphi_{1,\varepsilon_m}^{]-1/2,1/2[}$ is bounded in $\text{dom } D_{1,max}$, one can then extract a subfamily which converges in L_2 but we know that $\tilde{\psi}_{1,\varepsilon_m}$ converges

to 0 in any $M_1(\eta)$, the conclusion follows. We obtain also, with the help of Lemma 12 that

$$1 - (\|\varphi_1\|^2 + \|\varphi_2\|^2) = \lim(\|\tilde{\psi}_{1,\varepsilon_m}\|^2 + \|\xi_1 U^*(\frac{1}{\sqrt{|\log \varepsilon_m|}} r^{-1/2} \bar{\sigma}_{1/2})\|^2)$$

We remark that, by Corollary 10, $\varphi_2 = 0 \Rightarrow \lim_{m \rightarrow \infty} \|\tilde{\psi}_{1,\varepsilon_m}\| = 0$. In fact one has by (15)

$$\lim_{m \rightarrow \infty} \|\tilde{\psi}_{1,\varepsilon_m}\| = \|P_2(\varphi_2|_\Sigma)\|. \quad (26)$$

Finally, one has

$$\lim_{m \rightarrow \infty} \|\xi_1 U^*(\frac{1}{\sqrt{|\log \varepsilon_m|}} r^{-1/2} \bar{\sigma}_{1/2})\|_{L_2(M_{\varepsilon_m})} = \|\bar{\sigma}^{1/2}\|_{L_2(\Sigma)}. \quad (27)$$

□

4.3. lower bound, the end. Let's now $\varphi_1(\varepsilon), \dots, \varphi_N(\varepsilon)$ be an orthonormal family of eigenforms of the Hodge-de Rham operator, for the eigenvalues $\lambda_1(\varepsilon), \dots, \lambda_N(\varepsilon)$. We can make the same procedure of extraction for the all family. This gives, in the limit domain, a family $(\varphi_1^j, \varphi_2^j, \bar{\sigma}_{1/2}^j)_{1 \leq j \leq N}$. We already know by Corollary 13 that each element has norm 1, if we show that they are orthogonal we are done, by the application of the minimax formula to the limit problem (13).

Lemma 14. *The limit family is orthonormal in \mathcal{H}_∞ .*

Proof. If we follow the procedure for one indice, up to terms converging to zero, we had decomposed the eigenforms $\varphi_j(\varepsilon)$ on M_1 on three terms

$$\begin{aligned} \Phi_\varepsilon^j &= \psi_{1,\varepsilon} + \varphi_{1,\varepsilon}^{j,-1/2,1/2} \\ \tilde{\Phi}_\varepsilon^j &= \tilde{\psi}_{1,\varepsilon_m} \\ \bar{\Phi}_\varepsilon^j &= U^*(\frac{1}{\sqrt{|\log \varepsilon|}} r^{-1/2} \bar{\sigma}_{1/2}^j) \end{aligned}$$

Let $a \neq b$ be two indices. If we apply Lemma 12 to any linear combination of $\varphi_a(\varepsilon)$ and $\varphi_b(\varepsilon)$ we obtain that

$$\lim_{m \rightarrow \infty} \langle \Phi_{\varepsilon_m}^a, \tilde{\Phi}_{\varepsilon_m}^b \rangle_{L_2(M_1(\varepsilon_m))} + \langle \Phi_{\varepsilon_m}^b, \tilde{\Phi}_{\varepsilon_m}^a \rangle_{L_2(M_1(\varepsilon_m))} = 0.$$

If we apply (26) we obtain

$$\lim_{m \rightarrow \infty} \langle \tilde{\Phi}_{\varepsilon_m}^a, \tilde{\Phi}_{\varepsilon_m}^b \rangle_{L_2(M_{\varepsilon_m})} + \langle \varphi_{2,\varepsilon}^a, \varphi_{2,\varepsilon}^b \rangle_{L_2(M_2)} = \langle \tilde{\varphi}_2^a, \tilde{\varphi}_2^b \rangle_{L_2(\widetilde{M_2})}$$

Then finally, from $\langle \varphi_a(\varepsilon), \varphi_b(\varepsilon) \rangle = 0$ we conclude

$$\langle \varphi_1^a, \varphi_1^b \rangle_{L_2(\widetilde{M_1})} + \langle \varphi_2^a, \varphi_2^b \rangle_{L_2(\widetilde{M_2})} + \langle \bar{\sigma}_{1/2}^a, \bar{\sigma}_{1/2}^b \rangle_{L_2(\Sigma)} = 0.$$

□

Proposition 15. *The multiplicity of 0 in the limit spectrum is given by the sum*

$$\dim \ker \Delta_{1,W} + \dim \ker \mathcal{D}_2 + i_{1/2}$$

where $i_{1/2}$ denote the dimension of the vector space $\mathcal{I}_{1/2}$, see (4), of extended solutions ω on \widetilde{M}_2 introduced by Carron [C01], such that $A\omega = (1/2)\omega$ on restriction to $r = 1$. If the limit value $\lambda \neq 0$ then it belongs to the positive spectrum of the Hodge-de Rham operator $\Delta_{1,W}$ on \overline{M}_1 , with

$$W = \bigoplus_{\gamma \in]-\frac{1}{2}, \frac{1}{2}[} \text{Ker}(A - \gamma).$$

Proof. The last process, with in particular (26) and (16), construct in fact an element in the limit Hilbert space

$$\mathcal{H}_\infty = L_2(\overline{M}_1) \oplus \ker \widetilde{D}_2 \oplus \ker(A - \frac{1}{2}).$$

and this process is clearly *isometric* in the sense that if we have an orthonormal family $\varphi_{k,\varepsilon_m}, 1 \leq k \leq n$ we obtain at the limit an orthonormal family if \mathcal{H}_∞ is defined as an orthonormal space of the Hilbert spaces. And if we begin with eigenforms of D_{ε_m} we obtain at the limit eigenforms of $\Delta_{1,W} \oplus 0 \oplus 0$. The last calculus has shown that $\liminf \lambda_N(\varepsilon_m) \geq \lambda_N$. □

Remark 16. *To understand this result it is important to remember when occurs the eigenvalue 1/2 in the spectrum of A. By the expression (3) we find that it occurs exactly*

- for n even, if $3/4$ is an eigenvalue of the Hodge Laplacian Δ_Σ acting on the coclosed forms of degree $n/2$ or $n/2 - 1$ of the submanifold Σ .
- for n odd, if 0 is an eigenvalue of Δ_Σ for the $(n - 1)/2, (n + 1)/2$ or $(n - 3)/2$ forms but also if 1 is eigenvalue of the coclosed forms of degree $(n - 1)/2$ on Σ .

5. HARMONIC FORMS AND SMALL EIGENVALUES

It would be interesting to know how many small (but not zero) eigenvalues occur. For this purpose, we can use the topological meaning of harmonic forms.

The topology of M_ε is independant of $\varepsilon \neq 0$ and can be apprehended by the Mayer Vietoris sequence:

$$\dots \rightarrow H^p(M_\varepsilon) \xrightarrow{res} H^p(M_1(\varepsilon)) \oplus H^p(M_2) \xrightarrow{dif} H^p(\Sigma) \xrightarrow{ext} H^{p+1}(M_\varepsilon) \rightarrow \dots$$

As already mentionned, the space $\ker \mathcal{D}_2 \oplus \mathcal{I}_{1/2}$ can be sent in $H^*(M_2)$.

For \overline{M}_1 we can use the results of Cheeger. Following [Ch83], we know that the intersection cohomology of \overline{M}_1 coincide with $\ker(D_{1,\max} \circ D_{1,\min})$ if $H^{n/2}(\Sigma) = 0$.

And we know also that

$$IH^p(\overline{M_1}) = \begin{cases} H^p(M_1(\varepsilon)) & \text{if } p \leq n/2 \\ H_c^p(M_1(\varepsilon)) & \text{if } p \geq 1 + n/2. \end{cases} \quad (28)$$

These results can be used for our study only if $D_{1,\max}$ and $D_{1,\min}$ coincide. This appears if and only if A has no eigenvalues in the interval $] - 1/2, 1/2[$. As a consequence of the expression of the eigenvalues of A , recalled in (3), this is the case if and only if

- **for n odd**, the operator Δ_Σ has no eigenvalues in $]0, 1[$ on coexact $(n - 1)/2$ forms,
- **for n even**, the operator Δ_Σ has no eigenvalues in $[0, 3/4[$ on $n/2$ or $(n/2) - 1$ coclosed forms.

Thuse, if $D_{1,\max} = D_{1,\min}$, and this implies that $H^{n/2}(\Sigma) = 0$ in the case where n is even, then the map

$$H^{n/2}(M_\varepsilon) \xrightarrow{res} H^{n/2}(M_1(\varepsilon)) \oplus H^{n/2}(M_2)$$

is onto and then any small eigenvalue in this degree must come from an element of $\ker \mathcal{D}_2 \oplus \mathcal{I}_{1/2}$ sent to 0. In this case also the map

$$H^{n/2+1}(M_\varepsilon) \xrightarrow{res} H^{n/2+1}(M_1(\varepsilon)) \oplus H^{n/2+1}(M_2)$$

in one to one so there may exist small eigenvalues in this degree.

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