Refined Asymptotics for the subcritical Keller-Segel system and Related Functional Inequalities

Vincent Calvez, José Antonio Carrillo

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Abstract

We analyze the rate of convergence towards self-similarity for the subcritical Keller-Segel system in the radially symmetric two-dimensional case and in the corresponding one-dimensional case for logarithmic interaction. We measure convergence in Wasserstein distance. The rate of convergence towards self-similarity does not degenerate as we approach the critical case. As a byproduct, we obtain a proof of the logarithmic Hardy-Littlewood-Sobolev inequality in the one dimensional and radially symmetric two dimensional case based on optimal transport arguments. In addition we prove that the one-dimensional equation is a contraction with respect to Fourier distance in the subcritical case.

1 Introduction

We will concentrate on seeking decay rates towards equilibria or self-similarity profiles for aggregation equations with linear diffusion in the fair competition regime. These models describe the evolution of a population of individuals which are diffusing by standard Brownian motion and attracting each other by a pairwise symmetric potential $W(x)$. We focus on a logarithmic interaction potential $W(x) = 2\chi \log |x|$, with $\chi > 0$. The Fokker-Planck equation governing the evolution of the probability density function $\rho(t,x)$ associated to this particle system reads as

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[\frac{1}{N} \nabla \rho + 2\chi \rho \left(\nabla \log |x| * \rho \right) \right] , \quad t > 0 , \quad x \in \mathbb{R}^N . \tag{1.1}$$

Due to translational invariance and mass conservation, in the rest of this work we restrict to zero center of mass probability densities,

$$\rho(t,x) \geq 0 , \quad \int_{\mathbb{R}^N} \rho(t,x)\, dx = 1 , \quad \int_{\mathbb{R}^N} x \rho(t,x)\, dx = 0 ,$$

By fair competition, we mean that the dynamics of (1.1) are driven by a simple dichotomy as in the classical Keller-Segel system in two dimensions [24, 30, 20, 12], the modified

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Keller-Segel system in one dimension [15, 7] or the Keller-Segel model with suitable non-linear diffusion in larger dimensions [9]. In all these examples there is a critical parameter which makes the distinction between global existence of solutions and finite-time blow-up. More precisely, we will discuss the modified one-dimensional Keller-Segel equation [15, 7]:

$$\partial_t \rho = \partial^2_{xx}\rho + 2\chi\partial_x\left(\rho\partial_x\left(\log|x| * \rho\right)\right), \quad t > 0, \quad x \in \mathbb{R}, \tag{1.2}$$

and the radially symmetric two-dimensional classical Keller-Segel equation:

$$\partial_t(r\rho(t,r)) = \frac{1}{2}\partial_r(r\partial_r\rho(t,r)) + 2\chi\partial_r\left[\rho(t,r)M[\rho(t)](r)\right], \quad t > 0, \quad r \in \mathbb{R}_+, \tag{1.3}$$

where $M[\rho]$ denotes the cumulated mass of ρ inside balls,

$$M[\rho](r) = 2\pi\int_0^r \rho(s)s\,ds\,.$$

Both equations (1.2) and (1.3) exhibit a transition depending on the sensitivity coefficient χ:

- **Subcritical Case.-** For any $0 < \chi < 1$ solutions exist globally-in-time and they approach a unique self-similar solution as $t \to \infty$, see [20, 12, 5, 15].
- **Critical Case.-** For $\chi = 1$ solutions exist globally-in-time. There are infinitely many stationary solutions with infinite second moment. Solutions having finite initial second moment concentrate in infinite time towards the Dirac mass δ_0 [10, 5, 23]. Solutions of infinite initial second moment close enough to a stationary solution converge to it as $t \to \infty$ [8].
- **Supercritical Case.-** For any $\chi > 1$ smooth fast-decaying solutions do not exist globally in time [28, 6, 12, 15].

The critical parameter $\chi = 1$ can be obtained from two formal computations at this stage. The evolution of the second moment satisfies in both cases the relation:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} |x|^2\rho(t,x)\,dx = 1 - \chi\,.$$

This implies that for $\chi > 1$ solutions will necessarily blow-up before the second moment touches zero. On the other hand, the Keller-Segel equation (1.1) is equipped with a free energy (entropy minus potential energy),

$$\mathcal{F}[\rho] = \frac{1}{N}\int_{\mathbb{R}^N} \rho(x)\log\rho(x)\,dx + \chi\iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho(x)\log(|x-y|)\rho(y)\,dxdy\,. \tag{1.4}$$

It is formally decreasing along the trajectories

$$\frac{d}{dt}\mathcal{F}[\rho(t)] = -\int_{\mathbb{R}^N} \rho(t,x)\left|\nabla\left(\frac{1}{N}\log\rho(t,x) + 2\chi\log|x| * \rho(t,x)\right)\right|^2 dx\,. \tag{1.5}$$

Moreover, it was shown in [20, 12] that for $\chi < 1$ the free energy estimate from above implies an *a priori* bound in the entropy part of the functional which is at the basis of the construction of global-in-time solutions. This was achieved by using the Logarithmic-HLS inequality [4, 17] which relates the entropy and the interaction part of the functional.

Nontrivial equilibrium profiles or critical profiles, only exist for the critical parameter $\chi = 1$. They are solutions to the following Euler-Lagrange equations:

$$\mu'(x) + 2\mu(x)\partial_x \left(\log |x| * \mu(x)\right) = 0\,, \tag{1.6}$$

$$\frac{1}{2} r\mu'(r) + 2\mu(r)M[\mu](r) = 0\,, \tag{1.7}$$

resp. in dimension $N = 1$ and in dimension $N = 2$ with radially symmetry. In fact, we have an explicit formulation of the stationary states,

$$\mu(x) = \frac{1}{\pi(1 + |x|^2)^N}\,, \quad N = 1, 2\,. \tag{1.8}$$

This coincides with the equality cases in the Logarithmic-HLS inequality.

In the subcritical case $\chi < 1$, solutions are known to converge to unique self-similar profiles [12]. For studying convergence towards self-similarity, it is generally useful to rescale the space and time variables in the subcritical regime $\chi < 1$. The Keller-Segel system rewrites as

$$\frac{\partial \rho}{\partial t} = \frac{1}{N}\Delta\rho + 2\chi\nabla \cdot \left[\rho\left(\nabla \log |x| * \rho\right)\right] + \nabla \cdot [x\rho]\ , \quad t > 0\,, \quad x \in \mathbb{R}^N\,, \tag{1.9}$$

and the free energy is complemented with a quadratic confinement potential:

$$\mathcal{F}_{\mathbf{resc}}[\rho] = \mathcal{F}[\rho] + \frac{1}{2}\int_{\mathbb{R}^N} |x|^2 \rho(x)\, dx\,. \tag{1.10}$$

Due to the change of variables, self-similar solutions correspond to equilibrium solutions of (1.9). The rate of convergence towards equilibrium for (1.9) in the subcritical case was recently studied in [11] where the same rate as for the heat equation was obtained for small mass.

Let us finally mention that both (1.1) and (1.9) are gradient flows of the free energy functionals (1.4) and (1.10) respectively, when the space of probability measures is endowed with the euclidean Wasserstein metric W_2. We refer to the seminal papers [22, 29] and to [1] for a general theory. For instance, we can write (1.1) in short as

$$\dot\rho(t) = -\nabla_{W_2}\mathcal{F}[\rho(t)]\,. \tag{1.11}$$

This assertion was made rigorous in [7], where the variational minimizing movement scheme [1] was shown to converge for (1.1). This fact allows us to consider a way of measuring the distance towards equilibrium or self-similarity intimately related to the evolution due to (1.11). In fact, we will show that optimal transport tools are key techniques to describe this behavior at least in the one dimensional case (1.2) and in the radial case in two dimensions (1.3).

In order to investigate further the bounds of the free energy functional leading to the dichotomy discussed above and the characterization of the critical profiles, the Logarithmic-HLS inequality proved in [4, 17] is essential. In Section 2, we show an alternative proof based on optimal transport tools in the one dimensional case and in the radial case in two dimensions. There is another recent proof of this inequality with sharp constants in the two dimensional case by fast diffusion flows [16]. The Logarithmic-HLS inequality can be restated with our notations as:

Theorem 1.1 (Logarithmic HLS inequality)**.** *Assume $N = 1, 2$. The functional $\mathcal{F}$ is bounded from below. The extremal functions are uniquely given by* (1.8) *up to dilations in the set of probability densities with zero center of mass.*

In short, we demonstrate that any critical point of the free energy is in fact a global minimizer. This is a property which holds true for convex functionals, although the functional $\mathcal{F}$ is not displacement convex in the sense of McCann [27].

The ideas behind the proof of the sharp Logarithmic-HLS inequality allow us to tackle the rate of convergence in W_2 by similar methods for the rescaled version (1.9) in one dimension and for radial densities in the two dimensional case provided $\chi < 1$. We prove in Section 3 the following result.

Theorem 1.2 (Long-time asymptotics)**.** *Assume that $N = 1, 2$ being the initial data ρ_0 radially symmetric if $N = 2$. In the subcritical case $\chi < 1$, solutions of* (1.9) *in the rescaled variables converge exponentially fast towards the unique equilibrium configuration ν. More precisely, the following estimate holds true*

$$\frac{d}{dt} W_2(\rho(t), \nu)^2 \leq -2W_2(\rho(t), \nu)^2 \, .$$

Surprisingly enough, the rate of convergence that we obtain does not depend on the parameter χ. Our estimate is uniform as long as χ remains subcritical and is equal to the rate of convergence towards self-similarity for the heat equation. This is due to the fact that entropy and interaction contributions cancel each other, and only the confinement contribution remains yielding a uniform estimate. Although convergence is likely to be uniform, notice that the asymptotic profile becomes more and more singular as $\chi \to 1^-$, as shown by the simple second moment identity

$$\int_{\mathbb{R}} |a|^2 \nu(a) \, da = 1 - \chi \, . \tag{1.12}$$

Finally, we devote Section 4 to propose an alternative method of measuring the distance towards self-similarity in the one dimensional case. We make a connection between the one dimensional modified Keller-Segel model (1.2) and certain Boltzmann-like equations used in granular gases and wealth-distribution models, see [18, 21] and the references therein. This connection is due to the fact that (1.2) can be written in Fourier variables like the referred Boltzmann equations. Following the ideas of [18] we prove that equation (1.2) is indeed a contraction for the so-called Fourier distances defined in Section 4.

Theorem 1.3. *Assume $\chi < 1$ and the initial data have finite second moments. The one-dimensional Keller-Segel system* (1.2) *is a contraction for the distance d_1. It is a uniformly strict contraction in the rescaled frame, with a contraction factor which does not depend on χ.*

2 An alternative proof of the logarithmic HLS inequality

2.1 Preliminaries on Optimal Transport Tools

Let μ and ρ be two density probabilities. According to [13, 26] there exists a convex function ψ whose gradient pushes forward the measure $\mu(a)\,da$ onto $\rho(x)\,dx$: $\nabla\psi\#\left(\mu(a)\,da\right) = \rho(x)\,dx$. This convex function satisfies the Monge-Ampre equation in the weak sense,

$$\mu(a) = \rho(\nabla\psi(a)) \det D^2\psi(a) \, .$$

Regularity of the transport map is a big issue in general. Here we will use the fact that the Hessian measure $\det{}_H D^2\psi(a)$ can be decomposed in an absolute continuous part $\det{}_A D^2\psi(a)$ and a positive singular measure [31, Chapter 4]. In particular we have $\det{}_H D^2\psi(a) \geq \det{}_A D^2\psi(a)$. The formula for the change of variables will be important when dealing with the entropy contribution. For any measurable function U, bounded below such that $U(0) = 0$ we have [27, 31]

$$\int_{\mathbb{R}^N} U(\rho(x))\, dx = \int_{\mathbb{R}^N} U\left(\frac{\mu(a)}{\det{}_A D^2\psi(a)}\right) \det{}_A D^2\psi(a)\, da\,. \tag{2.1}$$

In fact this paper will only be concerned with the one-dimensional case, and the two-dimensional radial case. The complexity of Brenier's transport problem dramatically reduces in both cases. In dimension one, the transport map ϕ' is explicitely given by: $\psi'(a) = X \circ A^{-1}(a)$ where X and A denote respectively the pseudo-inverse cumulative distribution function of the densities ρ and μ. The singular part of the positive measure ψ'' corresponds to having holes in the support of the density ρ.

In the two-dimensional radial case, the Brenier's map can be expressed as the one-dimensional transport between the densities $2\pi\mu(a)a^{d-1}\, da$ and $2\pi\rho(r)r^{d-1}\, dr$. The determinant of the Hessian is given by

$$\det{}_H D^2\psi(a) = \frac{1}{2a}\frac{d}{da}\left(\psi'\right)^2(a)\,,$$

where the derivative of $(\psi')^2$ has to be understood in the distributional sense.

The following Lemma will be used to estimate the interaction contribution in the free energy, and in the evolution of the Wasserstein distance. For notational convenience we denote the convex combination of a and b by $[a, b]_t = (1 - t)a + tb$.

Lemma 2.1. *Let $K : (0, \infty) \to \mathbb{R}$ be an increasing and concave function such that $\lim_{z\to 0} F(z) = -\infty$. Then*

$$K\left(\int_0^1 \psi''([a, b]_t)\, dt\right) \geq \int_0^1 K\left(\psi''_{\mathrm{ac}}([a, b]_t)\right)\, dt\,. \tag{2.2}$$

Equality is achieved in (2.2) *if and only if the distributional derivative of the transport map ψ'' is a constant function.*

Analogously in the two-dimensional radially symmetric case we deduce

$$K\left(\int_0^1 \det{}_H D^2\psi([a, b]_t)\, dt\right) \geq \int_0^1 K\left(\det{}_A D^2\psi([a, b]_t)\right)\, dt\,. \tag{2.3}$$

Equality is achieved in (2.3) *if and only if ψ' is a multiple of the identity.*

Proof. We have on the one hand $\psi'' \geq \psi''_{\mathrm{ac}}$. We next use the concavity of K to conclude. Equality occurs if ψ'' is absolutely continuous and if ψ''_{ac} is constant. In the two-dimensional case we use $\det{}_H D^2\psi(a) \geq \det{}_A D^2\psi(a)$. □

Optimal transport is a powerful tool for reducing functional inequalities onto pointwise inequalities (*e.g.* matrix inequalities). We highlight for example the seminal paper by McCann [27] where the displacement convexity issue for some energy functional is reduced to the concavity of $\det^{1/N}$. We also refer to the works of Barthe [2, 3] and Cordero-Erausquin *et al.* [19]. We require simple pointwise inequalities which are extensions of the classical Jensen's inequality.

Lemma 2.2. *We have the following convex-like inequality for some exponent $\gamma > 0$ and any positive u, v, α, β,*

$$\alpha \left(\frac{u+v}{2}\right)^{-\gamma} - \beta \left(\frac{u+v}{2}\right)^{\gamma} \leq (\alpha+\beta)\left(\frac{u^{-\gamma}+v^{-\gamma}}{2}\right) - 2\beta\,. \tag{2.4}$$

Equality occurs if and only if $u = v = 1$. The continuous version reads as follows. For any measurable function $u : (0,1) \to (0,+\infty)$:

$$\alpha \left(\int_0^1 u(t)\,dt\right)^{-\gamma} - \beta \left(\int_0^1 u(t)\,dt\right)^{\gamma} \leq (\alpha+\beta)\int_0^1 (u(t))^{-\gamma}\,dt - 2\beta\,, \tag{2.5}$$

Proof. We only prove (2.4). The continuous version (2.5) is obtained by an approximation procedure. We introduce the auxiliary function J defined as follows.

$$J(u,v) = (\alpha+\beta)\left(\frac{u^{-\gamma}+v^{-\gamma}}{2}\right) - \alpha\left(\frac{u+v}{2}\right)^{-\gamma} + \beta\left(\frac{u+v}{2}\right)^{\gamma}\,.$$

Clearly, J diverges towards $+\infty$ as $u \to 0$ or $v \to 0$, and as $u \to \infty$ or $v \to \infty$, and J is bounded below. Then there exists at least one critical point. Any critical point (u_0, v_0) satisfies

$$\begin{cases} -\gamma\dfrac{\alpha+\beta}{2}u_0^{-\gamma-1} + \gamma\dfrac{\alpha}{2}\left(\dfrac{u_0+v_0}{2}\right)^{-\gamma-1} + \gamma\dfrac{\beta}{2}\left(\dfrac{u_0+v_0}{2}\right)^{\gamma-1} = 0\,, \\ -\gamma\dfrac{\alpha+\beta}{2}v_0^{-\gamma-1} + \gamma\dfrac{\alpha}{2}\left(\dfrac{u_0+v_0}{2}\right)^{-\gamma-1} + \gamma\dfrac{\beta}{2}\left(\dfrac{u_0+v_0}{2}\right)^{\gamma-1} = 0\,. \end{cases}$$

Hence $u_0 = v_0$ and

$$-\frac{\alpha+\beta}{2}u_0^{-\gamma-1} + \frac{\alpha}{2}u_0^{-\gamma-1} + \frac{\beta}{2}u_0^{\gamma-1} = 0\,.$$

We conclude that $u_0^{-\gamma} = u_0^{\gamma}$. Therefore the unique critical point of J is $(1,1)$. □

2.2 The one-dimensional case

The novelty here is contained in the proof of the logarithmic HLS inequality. This brings no information by itself since the uniqueness of the extremal functions is already known [17]. We show below that the logarithmic HLS inequality is a simple consequence of the Jensen's inequality. However our proof relies on the existence of a critical point of the free energy $\mathcal{F}$. In short, we demonstrate that any critical point of the free energy is in fact a global minimizer. This is a property which holds true for convex functionals. However the functional here is not convex.

Our first Lemma is a reformulation of the Euler-Lagrange equation for the extremal function (1.6).

Lemma 2.3 (Characterization of extremal functions)**.** *The critical profiles satisfy the following identity,*

$$\mu(p) = \int_{\mathbb{R}}\int_0^1 \mu(p-tq)\mu(p-tq+q)\,dtdq\,. \tag{2.6}$$

In the subcritical regime $\chi < 1$, the equilibrium in the rescaled frame satisfies the following identity,

$$\nu(p) = \int_{q\in\mathbb{R}}\int_0^1 \left(\chi + \frac{|q|^2}{2}\right)\nu(p-tq)\nu(p-tq+q)\,dtdq\,. \tag{2.7}$$

Proof. The formulation (2.6) is equivalent to integrating once the equation for the critical profile. We integrate equation (1.6) against some test function φ.

$$\begin{aligned}
\int_{\mathbb{R}} \varphi'(p)\mu(p)\,dp &= 2\iint_{\mathbb{R}\times\mathbb{R}} \frac{\varphi(x)}{x-y}\mu(x)\mu(y)\,dxdy \\
&= \iint_{\mathbb{R}\times\mathbb{R}} \frac{\varphi(x)-\varphi(y)}{x-y}\mu(x)\mu(y)\,dxdy \\
&= \iint_{\mathbb{R}\times\mathbb{R}} \int_0^1 \varphi'\left([x,y]_t\right)\mu(x)\mu(y)\,dtdxdy \\
&= \int_{\mathbb{R}} \varphi'(p)\left\{\int_{\mathbb{R}}\int_0^1 \mu(p-tq)\mu(p-tq+q)\,dtdq\right\}\,dp\,,
\end{aligned}$$

where we have finally used the change of variables: $(x,y)\mapsto(p=[x,y]_t, q=y-x)$. This holds true for any derivative φ', so we obtain identity (2.6) up to a constant. Since both sides of (2.6) have mass one, the constant is zero. The identity (2.7) is obtained in a similar way. $\square$

Proof of Theorem 1.1. Applying the change of variables formula (2.1) for $x=\psi'(p)$, the functional $\mathcal{F}$ rewrites as follows,

$$\begin{aligned}
\mathcal{F}[\rho]-\mathcal{F}[\mu] &= \int_{\mathbb{R}} \log\left(\frac{\mu(a)}{\psi''_{\mathrm{ac}}(a)}\right)\mu(a)\,da + \iint_{\mathbb{R}\times\mathbb{R}} \log|\psi'(a)-\psi'(b)|\mu(a)\mu(b)\,dadb - \mathcal{F}[\mu] \\
&= -\int_{\mathbb{R}} \log\left(\psi''_{\mathrm{ac}}(a)\right)\mu(a)\,da + \iint_{\mathbb{R}\times\mathbb{R}} \log\left(\frac{\psi'(a)-\psi'(b)}{a-b}\right)\mu(a)\mu(b)\,dadb \\
&= -\int_{\mathbb{R}} \log\left(\psi''_{\mathrm{ac}}(a)\right)\mu(a)\,da + \iint_{\mathbb{R}\times\mathbb{R}} \log\left(\int_0^1 \psi''([a,b]_t)\,dt\right)\mu(a)\mu(b)\,dadb
\end{aligned}$$

Using Lemma 2.1 for $K=\log z$ which is increasing and concave, we deduce

$$\begin{aligned}
\mathcal{F}[\rho]-\mathcal{F}[\mu] &\geq -\int_{\mathbb{R}} \log\left(\psi''_{\mathrm{ac}}(p)\right)\mu(p)\,dp + \iint_{\mathbb{R}\times\mathbb{R}} \int_0^1 \log\left(\psi''_{\mathrm{ac}}([a,b]_t)\right)\mu(a)\mu(b)\,dtdadb \\
&= -\int_{\mathbb{R}} \log\left(\psi''_{\mathrm{ac}}(p)\right)\mu(p)\,dp \\
&\quad + \int_{\mathbb{R}} \log\left(\psi''_{\mathrm{ac}}(p)\right)\left\{\int_{q\in\mathbb{R}}\int_0^1 \mu(p-tq)\mu(p-tq+q)\,dtdq\right\}\,dp = 0\,.
\end{aligned}$$

Equality arises if and only if the transport map ψ'' is a constant function. Such a map corresponds exactly to the dilations of the critical profile μ. $\square$

It is possible to extend Theorem 1.1 to the rescaled energy $\mathcal{F}_{\mathbf{resc}}$ (1.10).

Theorem 2.4 (Logarithmic HLS inequality with a quadratic confinement)**.** *Assume $N=1,2$ and $\chi<1$. The functional $\mathcal{F}_{\mathbf{resc}}$ is bounded from below. The extremal functions are unique in the set of probability densities with zero center of mass.*

We give below the main lines of the proof following a direct argument analogous to the proof of Theorem 1.1. Note that the uniqueness of the extremal functions in dimension $N=1$ or in dimension $N=2$ in the class of radially symmetric densities is a consequence of Theorem 1.2.

Sketch of proof of Theorem 2.4. The key point consists in replacing the Jensen's inequality with the following convex-like inequality. For any positive u, v, α, β the following inequality holds true.

$$\alpha \log\left(\frac{u+v}{2}\right) + \beta\left(\frac{u+v}{2}\right)^2 \geq (\alpha + 2\beta)\left(\frac{\log u + \log v}{2}\right) + \beta\,, \tag{2.8}$$

Equality occurs if and only if $u = v = 1$. It reduces to the usual Jensen's inequality when $\beta = 0$. The proof of (2.8) is analogous to Lemma 2.2. The proof of uniqueness for the extremal functions of $\mathcal{F}_{\mathbf{resc}}$ is a mixture between the proofs of Theorem 1.1 and Theorem 1.2. □

2.3 The two-dimensional case

We restrict to radially symmetric functions in the two-dimensional case due to decreasing rearrangement [25, 4, 17]. We recall the Newton's theorem for Poisson potential: the field induced by a radially symmetric distribution of masses outside a given ball is equivalent to the field induced by a point at the center of the ball [25]. Equivalently it reads

$$\frac{1}{2}\int_{\theta=0}^{2\pi} \log\left(r^2 + s^2 - 2rs\cos(\theta)\right)\, d\theta = 2\pi \log\max(r,s)\,. \tag{2.9}$$

As a consequence we can rewrite the functional $\mathcal{F}$ simpler under radial symmetry:

$$\frac{1}{2\pi}\mathcal{F}[\rho] = \frac{1}{2}\int_{\mathbb{R}_+} \rho(r)\log(\rho(r))\, r dr + \chi\int_{\mathbb{R}_+} \rho(r) M[\rho](r)\log(r)\, r dr\,.$$

The following characterizations are direct consequences of (1.7) and (1.3).

Lemma 2.5 (Characterization of extremal functions under radial symmetry)**.** *The critical profiles satisfy the following identity*

$$\frac{1}{2}\mu(b) = 2\int_b^{+\infty} \mu(a) M[\mu](a)\frac{1}{a}\, da\,. \tag{2.10}$$

In the subcritical regime $\chi < 1$, the radially-symmetric equilibrium satisfies the following identity

$$\frac{1}{2}\nu(b) = \int_b^{+\infty} \nu(a)\left(2\chi M[\nu](a)\frac{1}{a} + a\right)\, da\,. \tag{2.11}$$

We are now ready to examinate the logarithmic Hardy-Littlewood-Sobolev inequality in the two-dimensional radial setting.

Proof of Theorem 1.1. We apply the change of variables formula (2.1) for $r = \psi'(a)$ to get:

$$\frac{1}{2\pi}\mathcal{F}[\rho] = \frac{1}{2}\int_{\mathbb{R}_+} \mu(a)\log\left(\frac{\mu(a)}{\det{}_A D^2\psi(a)}\right) a\, da + 2\int_{\mathbb{R}_+} \mu(a) M[\mu](a)\log\left(\psi'(a)\right)\, a da\,,$$

where we have used $M[\rho](r) = M[\mu](a)$. We have consequently,

$$\begin{aligned}\frac{1}{2\pi}\mathcal{F}[\rho] - \frac{1}{2\pi}\mathcal{F}[\mu] = &-\frac{1}{2}\int_{\mathbb{R}_+} \mu(a)\log\left(\det{}_A D^2\psi(a)\right) a\, da \\ &+ \int_{\mathbb{R}_+} \mu(a) M[\mu](a)\log\left(\frac{(\psi')^2(a)}{a^2}\right) a\, da\,. \end{aligned} \tag{2.12}$$

The last contribution of (2.12) can be evaluated using Lemma 2.1

$$\int_{\mathbb{R}_+} \mu(a) M[\mu](a) \log\left(\int_0^a \left(\det{}_H D^2\psi(b)\right) \frac{2b}{a^2}\, db\right) a\, da$$
$$\geq \int_{\mathbb{R}_+} \int_0^a \mu(a) M[\mu](a) \log\left(\det{}_A D^2\psi(b)\right) \frac{2b}{a}\, dbda$$
$$= \int_{\mathbb{R}_+} \log\left(\det{}_A D^2\psi(b)\right) \left\{ 2\int_b^{+\infty} \mu(a) M[\mu](a) \frac{1}{a}\, da \right\} b\, db\,.$$

We obtain from the characterization (2.10) $\mathcal{F}[\rho] \geq \mathcal{F}[\mu]$. Again equality occurs if and only if the transport map ψ' is a multiple of the identity. □

2.4 Obstruction in dimension higher than three

We explain in this Section why the above strategy fails to work in dimension higher than three, even in the radially-symmetric setting. A first remark is that Newton's Theorem is not valid, since the logarithm kernel is not the fundamental solution of the Poisson equation, although this is not essential as shown in dimension one. It turns out that our strategy works fine for any interaction kernel $W(x) = |x|^k/k$, for $k \in (-N, 2-N]$. The case $k = 0$ corresponds to $W = \log|x|$. The case $k = -N$ is critical for integrability reasons. The case $k = 2-N$ is exactly the harmonic case for which the Newton's Theorem holds true. We refer to [14] for details in the case $k \in (-N, 2-N]$. Hence the case $k = 0$ is out of range when $N \geq 3$. We sketch below where some obstruction appears when $N = 3$.

The identity which generalizes (2.9) reads as follows. If $r > s$ we have

$$\frac{1}{2}\int_{\theta=0}^{\pi} \log\left(r^2 + s^2 - 2rs\cos(\theta)\right) \sin(\theta)\, d\theta = \frac{1}{2} H\left(\frac{s}{r}\right) + 2\log r\,,$$

where H is defined as follows for $t \in (0,1)$,

$$H(t) = \frac{1}{2}\frac{(1+t)^2}{t}\log(1+t) - \frac{1}{2}\frac{(1-t)^2}{t}\log(1-t) - 2\,.$$

To continue our strategy, it is required to decouple the variables r and s, and more precisely to make the quantity $r^N - s^N$ appearing. As a matter of fact this is homogeneous to the determinant (under radial symmetry), which is the key quantity to look at in dimension higher than two. Therefore we seek a convex-like inequality

$$H(t) \geq \alpha + \beta \log\left(1 - t^N\right)\,,$$

where α and β are suitable constants determined by zero and first order conditions. If we denote $\varphi(t) = \log(1 - t^N)$, this is equivalent to say that $H \circ \varphi^{-1}$ is convex. However simple computations show that it is indeed a concave function. In the case of an interaction kernel having homogeneity $k \in (-N, 2-N]$ we show in [14] that the corresponding function $H \circ \varphi^{-1}$ is convex.

3 Exponential convergence towards the self-similar profile

3.1 The one-dimensional case

To illustrate the strategy of proof of Theorem 1.2, we show a formal computation in the critical case $\chi = 1$. Up to our knowledge, the regularity of solutions under very weak assumptions is still an open problem In particular it is not known whether the solutions satisfy the identity (1.5) or not. So the following computation is questionable because the velocity field $\partial_x(\log \rho(t,x) + 2\log|x| * \rho(t,x))$ is not clearly defined in $L^2(\rho(t,x)dx)$.

We compute formally the evolution of the Wasserstein distance to one of the equilibria (1.8) in the critical case $\chi = 1$. Notice that equilibria are infinitely far from each other with respect to the Wasserstein distance [8]. Using the gradient flow structure with respect to W_2, one obtains the following formula for the derivative of $F(t) = W_2(\rho(t),\mu)^2$, see [31, Chapter 8] and [1].

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}F(t) &= \int_{\mathbb{R}} (\phi'(t,x) - x)\left(\partial_x\left(\log\rho(t,x) + 2\log|x| * \rho(t,x)\right)\right)\rho(t,x)\,dx \\
&= -\int_{\mathbb{R}} \phi''(t,x)\rho(t,x)\,dx + \iint_{\mathbb{R}\times\mathbb{R}} \frac{\phi'(t,x) - \phi'(t,y)}{x-y}\rho(t,x)\rho(t,y)\,dxdy \\
&= -\int_{\mathbb{R}} \left(\psi''(t,a)\right)^{-1}\mu(a)\,da + \iint_{\mathbb{R}\times\mathbb{R}} \left(\frac{\psi'(t,a) - \psi'(t,b)}{a-b}\right)^{-1}\mu(a)\mu(b)\,dadb \\
&\le -\int_{\mathbb{R}} \left(\psi''(t,a)\right)^{-1}\mu(a)\,da + \iint_{\mathbb{R}\times\mathbb{R}} \int_0^1 \left(\psi''(t,[a,b]_s)\right)^{-1}\mu(a)\mu(b)\,dsdadb\,.
\end{aligned}
$$

We recognize the characterization (2.6). Hence, we have at least formally $F'(t) \le 0$. Observe that the Lemma 2.1 has been used with $K(z) = -z^{-1}$.

The same strategy is valid in the subcritical case $\chi < 1$ for which we know that solutions are regular enough to ensure the validity of the computations. As a matter of fact, the density $\rho(t,x)$ is everywhere positive and thus ψ'' is absolutely continuous. On the other hand the dissipation of energy is well-defined and the dissipation estimate (1.5) holds true [12].

Proof of Theorem 1.2. We compute the evolution of $F(t) = W_2(\rho(t),\nu)^2$:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}F(t) &= \int_{\mathbb{R}} (\phi'(t,x) - x)\left(\partial_x\left(\log\rho(t,x) + 2\chi\log|x| * \rho(t,x) + \frac{|x|^2}{2}\right)\right)\rho(t,x)\,dx \\
&= -\int_{\mathbb{R}} \phi''(t,x)\rho(t,x)\,dx + \chi\iint_{\mathbb{R}\times\mathbb{R}} \frac{\phi'(t,x) - \phi'(t,y)}{x-y}\rho(t,x)\rho(t,y)\,dxdy \\
&\quad - \frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}} (\phi'(t,x) - \phi'(t,y))(x-y)\rho(t,x)\rho(t,y)\,dxdy \\
&\quad + 2\int_{\mathbb{R}} \phi'(t,x)x\rho(t,x)\,dx + 1 - \chi - \int_{\mathbb{R}} |x|^2\rho(t,x)\,dx\,,
\end{aligned}
$$

where we have used the fact that the center of mass is zero to double the variables. We rewrite each contribution using the reverse transport map ψ':

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}F(t) = &-\int_{\mathbb{R}} \big(\psi''(t,a)\big)^{-1}\nu(a)\,da + \chi\iint_{\mathbb{R}\times\mathbb{R}} \left(\frac{\psi'(t,a)-\psi'(t,b)}{a-b}\right)^{-1}\nu(a)\nu(b)\,dadb \\
&-\frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}} |a-b|^2\frac{\psi'(t,a)-\psi'(t,b)}{a-b}\nu(a)\nu(b)\,dadb \\
&+1-\chi-\int_{\mathbb{R}} |\psi'(t,a)|^2\nu(a)\,dx + 2\int_{\mathbb{R}} a\psi'(t,a)\nu(t,a)\,da \\
\le &\iint_{\mathbb{R}\times\mathbb{R}} \left[\left(\chi+\frac{|a-b|^2}{2}\right)\int_0^1 \big(\psi''(t,[a,b]_s)\big)^{-1}\,ds - |a-b|^2\right]\nu(a)\nu(b)\,dadb \\
&-\int_{\mathbb{R}} \big(\psi''(t,a)\big)^{-1}\nu(a)\,da + 2\int_{\mathbb{R}} |a|^2\nu(a)\,da - \int_{\mathbb{R}} |\psi'(t,a)-a|^2\nu(a)\,da\,,
\end{aligned}
$$

where we have used that the second moment of the stationary state is explicitly given by (1.12). Applying now (2.5) for $\gamma=1$, $\alpha=\chi$ and $\beta=|a-b|^2/2$, we deduce

$$
\frac{1}{2}\frac{d}{dt}F(t) \le -\int_{\mathbb{R}} |\psi'(t,a)-a|^2\nu(a)\,da = -W_2(\rho(t),\nu)^2 = -F(t)\,,
$$

giving the desired inequality. ☐

3.2 The two-dimensional radially-symmetric case

Proving convergence towards a self-similar profile in the rescaled logarithmic case under radial symmetry goes as previously.

Proof of Theorem 1.2. The virial computation reads equivalently

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}_+} \rho(t,r)r^3\,dr &= -\int_{\mathbb{R}_+} r\left(\frac{1}{2}\partial_r \log\rho(t,r) + 2\chi M[\rho](t,r)\frac{1}{r} + r\right)\rho(t,r)r\,dr \\
&= \int_{\mathbb{R}_+} \rho(t,r)r\,dr - 2\chi\int_{\mathbb{R}_+} M[\rho](t,r)\rho(t,r)r\,dr - \int_{\mathbb{R}_+} \rho(t,r)r^3\,dr \\
&= 1-\chi-\int_{\mathbb{R}_+} \rho(t,r)r^3\,dr = \int_{\mathbb{R}_+} \nu(a)a^3\,da - \int_{\mathbb{R}_+} \rho(t,r)r^3\,dr
\end{aligned}
$$

We compute again the evolution of the Wasserstein distance $F(t) = W_2(\rho(t),\nu)^2$.

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}F(t) = &\int_{\mathbb{R}_+} (\phi'(r)-r)\left(\frac{1}{2}\partial_r \log\rho(t,r) + 2\chi M[\rho](t,r)\frac{1}{r} + r\right)\rho(t,r)r\,dr \\
=&\frac{1}{2}\int_{\mathbb{R}_+} r\phi'(r)\partial_r\rho(t,r)\,dr + 2\chi\int_{\mathbb{R}_+} \phi'(r)M[\rho](t,r)\rho(t,r)\,dr - \int_{\mathbb{R}_+} \phi'(r)\rho(t,r)r^2\,dr \\
&+\int_{\mathbb{R}_+} \nu(a)a^3\,da - \int_{\mathbb{R}_+} \rho(t,r)r^3\,dr + 2\int_{\mathbb{R}_+} \phi'(r)\rho(t,r)r^2\,dr \\
\le &2\chi\int_{\mathbb{R}_+} \left(\int_0^a \det D^2\psi(b)\frac{2b}{a^2}\,db\right)^{-1/2} M[\nu](a)\nu(a)a\,da \\
&-\int_{\mathbb{R}_+} \left(\frac{1}{\det D^2\psi(b)}\right)^{1/2}\nu(b)b\,db - \int_{\mathbb{R}_+} \left(\int_0^a \det D^2\psi(b)\frac{2b}{a^2}\,db\right)^{1/2} a^2\nu(a)ada \\
&+2\int_{\mathbb{R}_+} \nu(a)a^3\,da - \int_{\mathbb{R}_+} |\phi'(r)-r|^2\rho(t,r)r\,dr\,.
\end{aligned}
$$

The last step in the inequality is a consequence of the arithmetic and geometric means inequality: $-\partial_r(r\phi'(r))/r = -\phi'(r)/r - \phi''(r) \leq -2(\phi''(r)\phi'(r)/r)^{1/2}$. Next we use Lemma 2.2 to handle the interaction contribution. More precisely, we choose $\gamma = 1/2$, $\alpha = 2\chi M[\nu](a)$ and $\beta = a^2$. One gets finally:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}F(t) \leq &- F(t) - \int_{\mathbb{R}_+} \left(\det D^2\psi(b)\right)^{-1/2} \nu(b) b\, db \\ &+ \int_{\mathbb{R}_+}\int_b^{+\infty} \left(2\chi M[\nu](a) + a^2\right)\left(\det D^2\psi(b)\right)^{-1/2} \frac{2b}{a^2}\nu(a) a\, dbda\,.\end{aligned}$$

We conlude using characterization (2.11). □

4 Contraction in the one-dimensional case

The aim of this Section is to point out the peculiar structure of the modified one-dimensional Keller-Segel system (1.2).

Lemma 4.1. *Equation* (1.2) *rewrites in Fourier variables as:*

$$\partial_t\hat\rho(t,\xi) = |\xi|^2\left(-\hat\rho(t,\xi) + \chi\int_0^1 \hat\rho(t,\sigma\xi)\hat\rho(t,(1-\sigma)\xi)\, d\sigma\right)\,. \tag{4.1}$$

Proof. We test equation (1.2) against $\exp(i\xi x)$:

$$\begin{aligned}\frac{\partial}{\partial t}\widehat{\rho}(t,\xi) &= \int_{\mathbb{R}}\left(\partial_{xx}\rho(t,x) + 2\chi\partial_x\left(\rho(t,x)\left(\text{p.v.}\,\frac{1}{x}\right) * \rho(t,x)\right)\right)e^{i\xi x}\, dx \\ &= -|\xi|^2\hat\rho(t,\xi) - \chi i\xi\iint_{\mathbb{R}\times\mathbb{R}} \rho(t,x)\frac{e^{i\xi x}-e^{i\xi y}}{x-y}\rho(t,y)\, dxdy \\ &= -|\xi|^2\hat\rho(t,\xi) + \chi|\xi|^2\iint_{\mathbb{R}\times\mathbb{R}}\rho(t,x)\left(\int_0^1 e^{i\xi[x,y]_\sigma}\, d\sigma\right)\rho(t,y)\, dxdy \\ &= -|\xi|^2\hat\rho(t,\xi) + \chi|\xi|^2\int_0^1\left(\int_{\mathbb{R}}\rho(t,x)e^{i(1-\sigma)\xi x}\, dx\right)\left(\int_{\mathbb{R}}\rho(t,y)e^{i\sigma\xi y}\, dy\right)d\sigma\,,\end{aligned}$$

which gives the desired formulation. □

According to (4.1) the information propagates from lower to higher frequencies. The evolution of $\hat\rho(t,\xi)$ requires the knowledge of lower frequencies $|\xi'| < |\xi|$ due to the integral contribution. This is of particular importance for designing a numerical scheme. Indeed there is no loss of information after truncation of the frequency box.

Remark 4.2 (Analogy with 1D Boltzmann)**.** *It is worthy to metion that the integral operator in the right-hand-side of* (4.1) *is reminiscent of the homogeneous Boltzmann equations in 1D used for granular gases [18] or wealth distribution models [21] in Fourier variables.*

Remark 4.3 (Evidence for blow-up in the supercritical case)**.** *We can directly notice the occurence of blow-up when $\chi > 1$ from* (4.1)*. Observe that for $|\xi| \ll 1$, the right-hand-side is equivalent to:*

$$\partial_t\hat\rho(t,\xi) \sim |\xi|^2\left(-\hat\rho(t,0) + \chi\hat\rho(t,0)^2\right) = |\xi|^2(-1+\chi)\,. \tag{4.2}$$

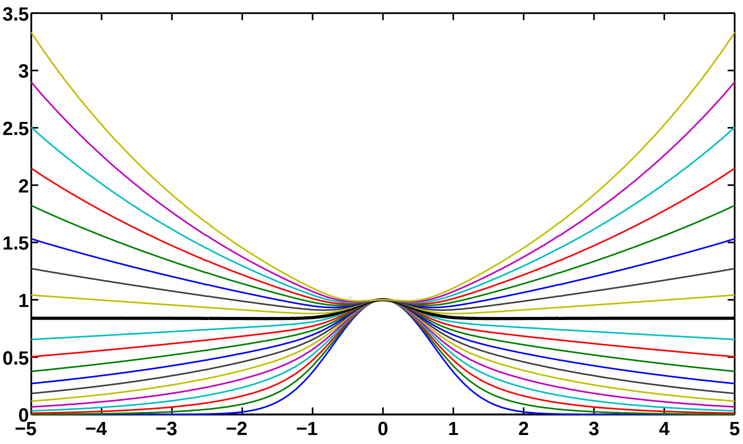

Figure 1: Simulation of (4.1) in the supercritical case $\chi > 1$ for successive times. The blow-up time is plotted in bolded dark.

This argues in favor of blow-up at low modes although misleadingly. We have plotted in Figure 1 numerical simulation of (4.1) *in the supercritical case. Observe that blow-up arises for* $|\xi| \gg 1$, *on the contrary to the misleading heuristics* (4.2). *The integro-differential equation* (4.1) *makes perfect sense even in the supercritical regime* $\chi > 1$ *after the first blow-up event. However the outcoming fonction* $\hat{\rho}(t,\xi)$ *is no longer the Fourier transform of a probability measure. In fact the blow-up time coincides with the formation of the first dirac mass, namely when the frequency distribution* $\widehat{\rho}(t,\xi)$ *is flat at infinity. This contradictory intuition is similar to the proof of blow-up based on the virial identity: the second momentum provides information at infinity but is used to prove blow-up which is a local behaviour.*

Recall the definition of Fourier distances [18] as they have been introduced for the analysis of the Boltzmann equation.

Definition 4.4 (Fourier distances)**.** *Let* ρ_1, ρ_2 *being two probability measures having the same center of mass. The* d_1-*distance is defined as follows:*

$$d_1\left(\rho_1,\rho_2\right) = \sup_{\xi\neq 0}\left\{|\xi|^{-1}\left|\hat{\rho}_1(\xi)-\hat{\rho}_2(\xi)\right|\right\}\,. \tag{4.3}$$

Proof of Theorem 1.3. First notice that supremum in (4.3) is attained in $\mathbb{R}\setminus\{0\}$. Clearly we have $|\hat{\rho}_1(\xi)-\hat{\rho}_2(\xi)|\leq 2$ and

$$\hat{\rho}_1(\xi)-\hat{\rho}_2(\xi) \sim \left(\int_{\mathbb{R}} |x|^2[\rho_1(x)-\rho_2(x)]\,dx\right)|\xi|^2/2 \text{ as } \xi\to 0.$$

We denote $F(t) = d_1(\rho_1(t),\rho_2(t))$ and $h(t,\xi) = |\xi|^{-1}(\hat{\rho}_1(t,\xi)-\hat{\rho}_2(t,\xi))$. We multiply the difference between the two equations (4.1) by $\mathrm{sign}(h(t,\xi))$,

$$\partial_t\left|h(t,\xi)\right| = |\xi|^2\left(-\left|h(t,\xi)\right| + \chi\mathrm{sign}\left(\hat{\rho}_1(t,\xi)-\hat{\rho}_2(t,\xi)\right)A(t,\xi)\right)\,,$$

where

$$A(t,\xi) = |\xi|^{-1} \int_0^1 \hat\rho_1(t,\sigma\xi)\hat\rho_1(t,(1-\sigma)\xi)\,d\sigma - |\xi|^{-1} \int_0^1 \hat\rho_2(t,\sigma\xi)\hat\rho_2(t,(1-\sigma)\xi)\,d\sigma\,.$$

The self-attraction contributions are handled as follows [18, Th. 6.3]:

$$\begin{aligned}
|A(t,\xi)| \leq & |\xi|^{-1} \int_0^1 |\hat\rho_1(t,\sigma\xi) - \hat\rho_2(t,\sigma\xi)|\,|\hat\rho_1(t,(1-\sigma)\xi)|\,d\sigma \\
& + |\xi|^{-1} \int_0^1 |\hat\rho_1(t,(1-\sigma)\xi) - \hat\rho_2(t,(1-\sigma)\xi)|\,|\hat\rho_2(t,\sigma\xi)|\,d\sigma \\
\leq & d_1\left(\rho_1(t),\rho_2(t)\right) \int_0^1 \left(\sigma + (1-\sigma)\right)\,d\sigma = F(t)\,.
\end{aligned}$$

We obtain finally

$$\partial_t |h(t,\xi)| \leq |\xi|^2 \left(-|h(t,\xi)| + \chi F(t)\right)\,.$$

We deduce

$$\begin{aligned}
|h(t+\epsilon,\xi)| \leq & e^{-\epsilon|\xi|^2}|h(t,\xi)| + \chi\left(1 - e^{-\epsilon|\xi|^2}\right) \sup_{s\in(0,\epsilon)} F(t+s)\,, \\
|h(t+\epsilon,\xi)| - F(t) \leq & \left(1 - e^{-\epsilon|\xi|^2}\right)\left(-F(t) + \chi \sup_{s\in(0,\epsilon)} F(t+s)\right)\,, \\
\limsup_{\epsilon\to 0^+} \frac{F(t+\epsilon)-F(t)}{\epsilon} \leq & (\chi - 1)\left(\liminf_{\epsilon\to 0^+} |\xi^*(t+\epsilon)|^2\right) F(t)\,,
\end{aligned}$$

where $|\xi^*(t)|$ denotes the lowest frequency moduli for which the supremum is attained in $F(t) = \sup|h(t,\xi)|$. We have used the continuity of F to pass to the limit. Therefore we get a contraction estimate as soon as $\chi < 1$. There is no explicit rate since we do not know how to control $|\xi^*(t)|$ from below.

We also obtain a uniform strict contractivity in self-similar variables. The Keller-Segel equation (1.9) writes as follows in Fourier variables:

$$\partial_t\hat\rho(t,\xi) = |\xi|^2\left(-\hat\rho(t,\xi) + \chi\int_0^1 \hat\rho(t,\sigma\xi)\hat\rho(t,(1-\sigma)\xi)\,d\sigma\right) - \xi\partial_\xi\hat\rho(t,\xi)\,.$$

We proceed as above to get:

$$\begin{aligned}
\partial_t |h(t,\xi)| = & |\xi|^2\left(-|h(t,\xi)| + \chi \mathrm{sign}\left(\hat\rho_1(t,\xi) - \hat\rho_2(t,\xi)\right) A(t,\xi)\right) \\
& - \xi\partial_\xi\left(|h(t,\xi)|\right) - |h(t,\xi)|\,.
\end{aligned}$$

We integrate along characteristics and argue as previously,

$$\begin{aligned}
|h(t+\epsilon,\xi)| - F(t) \leq & F(t)\left(\exp\left(-\epsilon + \frac{e^{-2\epsilon}-1}{2}|\xi|^2\right) - 1\right) \\
& + \chi\left(\int_0^\epsilon |e^{s-\epsilon}\xi|^2 \exp\left(s - \epsilon + \frac{e^{2(s-\epsilon)}-1}{2}|\xi|^2\right)\,ds\right) \sup_{s\in(0,\epsilon)} F(t+s)
\end{aligned}$$

We deduce the following contraction estimate,

$$\limsup_{\epsilon\to 0^+} \frac{F(t+\epsilon)-F(t)}{\epsilon} \leq -F(t) + (\chi-1)\left(\liminf_{\epsilon\to 0^+} |\xi^*(t+\epsilon)|^2\right) F(t)\,.$$

Hence the one-dimensional Keller-Segel equation (4.1) is a contraction with rate 1 with respect to the Fourier distance d_1. $\square$

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