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Generalized Poynting Effects in Predeformed Prismatic Bars

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Abstract. We use the Signorini expansion method to determine the second-order Saint-Venant solution for an infinitesimally bent and stretched bar. The bar in the unstressed reference configuration is straight, prismatic, isotropic, homogeneous and made of a second-order elastic material. These solutions and those found earlier for a pretwisted bar give generalized Poynting effects. A bar when bent stretches and the elongation is determined by the first and second-order elasticities, area of cross-section, torsional rigidity, bending vector and the inertia tensor. When an infinitesimally twisted bar is deformed, there is a second-order bending deformation even when there is no resultant bending moment applied on the end faces.

Mathematics Subject Classifications (1991): 73C99, 73B99.

Key words: Signorini's expansion, plane elliptic problems, Saint-Venant's problem, semi-inverse method.

1. Introduction

Poynting [1] observed that an elastic prismatic bar when twisted by torques applied at the end faces also elongates, and the elongation is proportional to the square of the twist. Rivlin [2] used the second-order elasticity theory to compute the elongation as a function of the angular twist and showed that the elongation was proportional to second-order elasticities. The work has been extended to general nonlinear elastic solids by Green and Shield [3]. Green and Adkins [4] noted that when the displacements and infinitesimal rotations of the centroid of one end-face vanish, then the compatibility condition for the loads in the Signorini's expansion method [5] is automatically satisfied. Signorini's method reduces the solution of a nonlinear elastic problem to that of a series of linear elastic problems with body forces and surface tractions determined by the solution of the previous linear elastic problem. Truesdell and Noll [6] and Wang and Truesdell [7] have reviewed

the historical background and the relevant literature on the Poynting effect and the Signorini expansion method. They have pointed out that Signorini's method delivers only those solutions that are in the neighborhood of solutions of the linear elastic problem with the same loads as for the nonlinear problem.

In [8] we used the Signorini expansion method to analyse the Saint-Venant problem [9,10] for a straight, isotropic, stress-free and homogeneous prismatic bar made of a second-order elastic material with the first term in the expansion for the displacement field corresponding to an infinitesimal twist of the body. Here we investigate the problem when the first term in Signorini's expansion corresponds to an infinitesimal bending and stretching of the bar. The solution of the problem yields generalized Poynting effects such as second-order extension of the bar in the absence of external resultant normal tractions at the end faces. This second-order extension is found to depend upon the first and second-order elasticities, initial bending, and other properties of the cross-section. Further such effects are deduced for a pretwisted bar by using results from [8].

The analysis of the problem is simplified by decomposing, as suggested by DiCarlo [11], and Davi and Tiero [12], the displacement field and other tensorial quantities into components along the axis of the bar and those in the cross-section.

Iesan [13–16] has studied the Saint-Venant problem for inhomogeneous and anisotropic linear elastic bodies, elastic dielectrics, and microstretch elastic solids. Dell'Isola and Rosa [17,18] and Davi [19] have studied the problem for linear piezoelectric bodies, and dell'Isola and Batra [20] for linear elastic porous solids.

2. Formulation of the Problem

We consider a second-order elastic, homogeneous, and isotropic prismatic bar occupying the region $\mathcal{A} \times [0, \ell]$ in the unstressed reference configuration. Thus \mathcal{A} is the cross-section of the bar and ℓ its length. Let the bar be infinitesimally bent and stretched by tractions applied only at its end faces. Following Signorini [5], we expand the displacement field \mathbf{u} in terms of the infinitesimal elongation ε and write it as

$$\mathbf{u} = \varepsilon \dot{\mathbf{u}} + \varepsilon^2 \ddot{\mathbf{u}} + \varepsilon^3 \dddot{\mathbf{u}} + \dots \quad (1)$$

Thus a superimposed dot on \mathbf{u} indicates the order of the displacement field. Our goal is to find $\ddot{\mathbf{u}}$ for a second-order elastic material when $\dot{\mathbf{u}}$ corresponds to infinitesimal elongation and/or bending of the bar. Saint-Venant's solution for $\dot{\mathbf{u}}$ is

$$\dot{\mathbf{u}}(\mathbf{r}, \zeta) = \zeta(\varepsilon - \mathbf{c} \cdot \mathbf{r})\mathbf{e} + \frac{\zeta^2}{2}\mathbf{c} + \mathbf{K}\mathbf{c} - \nu\varepsilon\mathbf{r}, \quad (2)$$

where ζ is the axial coordinate of a point when the origin of the co-ordinate system is at the centroid of one end face, \mathbf{e} is a unit vector along the centroidal axis of

the bar in the unstressed configuration, \mathbf{r} is the position vector of a point in the plane perpendicular to the centroidal axis, $\mathbf{c} = \mathbf{v}'$ is the curvature characterizing the bending of the bar, a prime denotes differentiation with respect to ζ , $*\mathbf{r}$ equals $\mathbf{e} \times \mathbf{r}$, and

$$\mathbf{u}(\mathbf{r}, \zeta) := \mathbf{v}(\mathbf{r}, \zeta) + w(\mathbf{r}, \zeta)\mathbf{e}, \quad (3)$$

$$\mathbf{K} := \nu[\mathbf{r} \otimes \mathbf{r} - (*\mathbf{r}) \otimes (*\mathbf{r})]/2, \quad (4)$$

$$\nu := \lambda/(2(\lambda + \mu)), \quad (5)$$

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{d} = (\mathbf{a} \cdot \mathbf{d})\mathbf{b} \quad (6)$$

for arbitrary vectors \mathbf{a} , \mathbf{b} and \mathbf{d} . Thus \mathbf{v} equals the displacement of a point within the cross-section and w its displacement in the axial direction, \mathbf{K} is the anticlastic curvature operator, ν is Poisson's ratio, and λ and μ are the Lamé constants. Resultant forces and moments to be applied to the end faces of the bar can be computed from the displacement field (2) and the constitutive relation for the bar (e.g. see Sokolnikoff [21]). We require that the centroid of one end face of the bar and the skew symmetric part of the displacement gradient \mathbf{H} vanish there. That is,

$$\mathbf{u}(\mathbf{0}, 0) = \mathbf{0}, \quad \mathbf{H}(\mathbf{0}, 0) = \mathbf{H}^T(\mathbf{0}, 0). \quad (7)$$

Green and Adkins [4] have shown that under conditions (7), there is no compatibility condition required by the first-order loads in the Signorini's series expansion of the solution. We assume that the bending curvature and extension are of the same order of magnitude; i.e.

$$\mathbf{c} = \varepsilon \dot{\mathbf{c}} + \mathbf{O}(\varepsilon^2). \quad (8)$$

Thus with

$$\delta := \dot{\mathbf{c}} \cdot \mathbf{r}, \quad (9)$$

$$\dot{\mathbf{u}}(\mathbf{r}, \zeta) = \varepsilon \left[\zeta(1 - \delta)\mathbf{e} + \frac{\zeta^2}{2}\dot{\mathbf{c}} + \mathbf{K}\dot{\mathbf{c}} - \nu\mathbf{r} \right].$$

Due to the bending of the bar, the axial displacement δ of a point is proportional to its distance from the centroidal axis. From (9) we derive the following expressions for the first-order displacement gradient $\dot{\mathbf{H}}$, the infinitesimal strain tensor $\dot{\mathbf{E}}$, and other deformation measures needed in the expression for the first Piola–Kirchhoff stress tensor \mathbf{S} .

$$\dot{\mathbf{H}} = \varepsilon[(1 - \delta)\mathbf{e} \otimes \mathbf{e} + \zeta\mathbf{e} \wedge \dot{\mathbf{c}} - \nu(1 - \delta)\hat{\mathbf{I}} + \nu\varphi*], \quad (10)$$

$$\varphi := (*\dot{\mathbf{c}}) \cdot \mathbf{r}, \quad (11)$$

$$\dot{\mathbf{E}} := (\dot{\mathbf{H}} + \dot{\mathbf{H}}^T)/2 = \varepsilon(1 - \delta)(\mathbf{e} \otimes \mathbf{e} - \nu \hat{\mathbf{I}}), \quad (12)$$

$$\begin{aligned} \dot{\mathbf{H}}\dot{\mathbf{H}}^T &= \varepsilon^2\{[(1 - \delta)^2 + \zeta^2\|\dot{\mathbf{c}}\|^2]\mathbf{e} \otimes \mathbf{e} \\ &\quad + (1 + \nu)(1 - \delta)\zeta\mathbf{e} \vee \dot{\mathbf{c}} - \nu\varphi\zeta\mathbf{e} \vee (*\dot{\mathbf{c}}) \\ &\quad + \zeta^2\dot{\mathbf{c}} \otimes \dot{\mathbf{c}} + \nu^2[\varphi^2 + (1 - \delta)^2]\hat{\mathbf{I}}\}, \end{aligned} \quad (13)$$

$$\dot{\mathbf{E}}^2 = \varepsilon^2(1 - \delta)^2(\mathbf{e} \otimes \mathbf{e} + \nu^2\hat{\mathbf{I}}), \quad (14)$$

$$\begin{aligned} (\dot{\mathbf{H}}^T)^2 &= \varepsilon^2\{[(1 - \delta)^2 - \zeta^2\|\dot{\mathbf{c}}\|^2]\mathbf{e} \otimes \mathbf{e} \\ &\quad + \zeta(1 - \delta)(1 - \nu)\dot{\mathbf{c}} \wedge \mathbf{e} + \nu\varphi\zeta(*\dot{\mathbf{c}}) \vee \mathbf{e} \\ &\quad - \zeta^2\dot{\mathbf{c}} \otimes \dot{\mathbf{c}} + \nu^2[(1 - \delta)^2 - \varphi^2]\hat{\mathbf{I}} + 2\nu^2(1 - \delta)\varphi*\}. \end{aligned} \quad (15)$$

Here

$$\mathbf{a} \wedge \mathbf{b} = \mathbf{a} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a}, \quad (16)$$

$$\mathbf{a} \vee \mathbf{b} = \mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a}, \quad (17)$$

for arbitrary vectors \mathbf{a} and \mathbf{b} , $\hat{\mathbf{I}}$ is the two-dimensional identity matrix, $\|\mathbf{a}\|$ equals the length of vector \mathbf{a} , and φ is the twist caused by the bending of the bar. The two-dimensional identity matrix is extended to a three-dimensional matrix by adding zeros in the third row and third column, and the rotation operator $*$ is also similarly extended. In a rectangular Cartesian coordinate system with \mathbf{e} pointing along the 3-axis, the elements of the first, second and third rows of the 3×3 matrix corresponding to the $*$ operator are 0, -1 , 0; 1, 0, 0; 0, 0, 0 respectively.

For a second-order elastic material, the first Piola–Kirchhoff stress tensor, \mathbf{S} , is given by (e.g. see Wang and Truesdell [7])

$$\begin{aligned} \mathbf{S} &= \mu \left[\left(\alpha_1 I_{\dot{\mathbf{E}}} + 2\dot{\mathbf{E}} + \frac{\alpha_1}{2}(I_{\dot{\mathbf{H}}\dot{\mathbf{H}}^T} + 2I_{\dot{\mathbf{E}}^2} + \alpha_3 I_{\dot{\mathbf{E}}}^2 + \alpha_4(I_{\dot{\mathbf{E}}^2} - (I_{\dot{\mathbf{E}}})^2)/2) \right) \mathbf{1} \right. \\ &\quad \left. + (\alpha_5 + 2)I_{\dot{\mathbf{E}}}\dot{\mathbf{E}} - \alpha_1 I_{\dot{\mathbf{E}}}\dot{\mathbf{H}}^T - (\dot{\mathbf{H}}^T)^2 + \alpha_6 \dot{\mathbf{E}}^2 \right], \end{aligned} \quad (18)$$

where μ is the shear modulus, $\mu\alpha_1$ is a Lamé constant, α_3 , α_4 , α_5 and α_6 are non-dimensional material constants, and $I_{\dot{\mathbf{E}}} = \text{tr}(\dot{\mathbf{E}})$. Substitution from (10)–(15) into (18), collecting terms of different orders of ε , and expanding \mathbf{S} as a power series in ε , viz.

$$\mathbf{S} = \varepsilon \dot{\mathbf{S}} + \varepsilon^2 \ddot{\mathbf{S}} + \dots \quad (19)$$

we can derive expressions for $\dot{\mathbf{S}}$, $\ddot{\mathbf{S}}$ etc. Because of the assumption of stress free reference configuration, \mathbf{S} vanishes for $\varepsilon = 0$. Also $\dot{\mathbf{u}}$ is assumed to be known,

therefore, explicit relation for $\dot{\mathbf{S}}$ in terms of $\dot{\mathbf{H}}$ etc. is omitted. With the objective of finding $\ddot{\mathbf{u}}$, we list below the expression for $\ddot{\mathbf{S}}$ in terms of $\ddot{\mathbf{u}}$ and other known quantities

$$\begin{aligned} \ddot{\mathbf{S}} = \ddot{\mathbf{S}} + \varepsilon^2 \left\{ \left[\lambda \left(\left(\frac{1-4\nu+10\nu^2}{2} \right) (1-\delta)^2 + \nu^2 \varphi^2 \right) \right. \right. \\ \left. \left. + (\lambda + \mu) \zeta^2 \|\dot{\mathbf{c}}\|^2 + \mu\beta(1-\delta)^2 \right] \mathbf{e} \otimes \mathbf{e} \right. \\ \left. + \mu[\zeta^2 \dot{\mathbf{c}} \otimes \dot{\mathbf{c}} - \nu\varphi\zeta(*\dot{\mathbf{c}}) \vee \mathbf{e}] \right. \\ \left. - \mu(1+\nu)(1-\delta)\zeta\dot{\mathbf{c}} \wedge \mathbf{e} + k\hat{\mathbf{I}} \right\}, \end{aligned} \quad (20)$$

where

$$\ddot{\mathbf{S}} = \lambda \operatorname{tr}(\ddot{\mathbf{E}}) \mathbf{1} + 2\mu \ddot{\mathbf{E}}, \quad (21)$$

$$2\ddot{\mathbf{E}} = \operatorname{Grad} \ddot{\mathbf{u}} + (\operatorname{Grad} \ddot{\mathbf{u}})^T, \quad (22)$$

$$\beta = 1 - 4\nu + \alpha_3(1 - 2\nu)^2 + (\alpha_4\nu + \alpha_5)(1 - 2\nu) + \alpha_6, \quad (23)$$

$$k = \lambda \left[\left(\frac{3 - 6\nu + 6\nu^2}{2} \right) (1 - \delta)^2 + \zeta^2 \|\dot{\mathbf{c}}\|^2 \right] + (\lambda + \mu)\nu^2\varphi^2 + \mu\gamma, \quad (24)$$

$$\begin{aligned} \gamma = (3\nu^2 - 2\nu + \alpha_3(1 - 2\nu)^2 + \alpha_4(\nu^2 - 2\nu) \\ + \alpha_5(2\nu^2 - \nu) + \alpha_6\nu^2)(1 - \delta)^2, \end{aligned} \quad (25)$$

and Grad (Div) is the three-dimensional gradient (divergence) operator in the reference configuration. The second-order stress tensor $\ddot{\mathbf{S}}$ must satisfy the following equilibrium equations and boundary conditions

$$\operatorname{Div} \ddot{\mathbf{S}} = \mathbf{0} \quad \text{in } \mathcal{A} \times [0, \ell], \quad (26)$$

$$\ddot{\mathbf{S}}\mathbf{N} = \mathbf{0} \quad \text{on } \partial\mathcal{A} \times [0, \ell]. \quad (27)$$

Here \mathbf{N} is an outward unit normal on $\partial\mathcal{A} \times [0, \ell]$. Decomposing $\ddot{\mathbf{S}}$ into components as

$$\ddot{\mathbf{S}} = \ddot{\sigma} \mathbf{e} \otimes \mathbf{e} + \ddot{\mathbf{t}} \vee \mathbf{e} + \ddot{\mathbf{S}}, \quad (28)$$

and recalling (21) and the decomposition (3) of the displacement into its axial and in-plane components, we obtain

$$\ddot{\sigma} = (\lambda + 2\mu)\ddot{w}' + \lambda \operatorname{tr} \ddot{\mathbf{E}}, \quad (29)$$

$$\ddot{t} = \mu(\ddot{\mathbf{v}}' + \text{grad } \ddot{w}), \quad (30)$$

$$\ddot{\mathbf{S}} = \lambda(\text{tr } \ddot{\mathbf{E}})\hat{\mathbf{I}} + 2\mu\ddot{\mathbf{E}}, \quad (31)$$

where

$$2\ddot{\mathbf{E}} = \text{grad } \ddot{\mathbf{v}} + (\text{grad } \ddot{\mathbf{v}})^T, \quad (32)$$

and $\text{grad}(\text{div})$ is the two-dimensional gradient (divergence) operator in plane \mathcal{A} . Substitution from (20) and (28) into (26) and (27) yields

$$\ddot{\sigma}' + \text{div} \ddot{\mathbf{t}} = -(2\lambda + 3\mu)\zeta \varepsilon^2 \|\dot{\mathbf{c}}\|^2 \quad \text{in } \mathcal{A} \times [0, \ell], \quad (33)$$

$$\begin{aligned} \ddot{\mathbf{t}} + \text{div} \ddot{\mathbf{S}} = \varepsilon^2 \{ & (\mu - \lambda)\varphi(*\dot{\mathbf{c}}) + [\lambda(3 - 6\nu + 6\nu^2) \\ & + \mu(2\gamma - \nu - 1)](1 - \delta)\dot{\mathbf{c}} \} \quad \text{in } \mathcal{A} \times [0, \ell], \end{aligned} \quad (34)$$

$$\ddot{\mathbf{t}} \cdot \mathbf{N} = \mu\zeta \varepsilon^2 [\nu\varphi(*\dot{\mathbf{c}}) + (1 + \nu)(1 - \delta)\dot{\mathbf{c}}] \cdot \mathbf{N} \quad \text{on } \partial\mathcal{A} \times [0, \ell], \quad (35)$$

$$\ddot{\mathbf{S}}\mathbf{N} = -\varepsilon^2 [\mu\zeta^2 \dot{\mathbf{c}} \otimes \dot{\mathbf{c}} + k\hat{\mathbf{I}}]\mathbf{N} \quad \text{on } \partial\mathcal{A} \times [0, \ell]. \quad (36)$$

We will find a solution for (33)–(36) in Section 4.

3. Resultant Forces and Moments on a Cross-section

Integrating (26) over $\mathcal{A} \times [0, \zeta]$, using the divergence theorem and boundary conditions (27), we obtain

$$\int_{\mathcal{A}_\zeta} \ddot{\mathbf{S}}\mathbf{e} \, dA = \text{const.} \quad (37)$$

for every ζ . Thus from (28) and (20),

$$\begin{aligned} \int_{\mathcal{A}_\zeta} \ddot{\sigma} \, dA + (\lambda + \mu)\zeta^2 \varepsilon^2 \|\dot{\mathbf{c}}\|^2 \mathcal{A}_\zeta + \lambda\nu^2 \varepsilon^2 (\mathbf{J}\dot{\mathbf{c}}) \cdot \dot{\mathbf{c}} \\ + \varepsilon^2 \left(\frac{1}{2}\lambda(1 - 4\nu + 10\nu^2) + \mu\beta \right) (\mathcal{A}_\zeta - (\mathbf{J}(*\dot{\mathbf{c}}) \cdot (*\dot{\mathbf{c}}))) = \text{const.}, \end{aligned} \quad (38)$$

$$\int_{\mathcal{A}_\zeta} \ddot{\mathbf{t}} \, dA + \mu\varepsilon^2 (1 + \nu)\zeta \dot{\mathbf{c}} \mathcal{A}_\zeta = \text{const.}, \quad (39)$$

where

$$\mathbf{J} := \int_{\mathcal{A}_\zeta} (*\mathbf{r}) \otimes (*\mathbf{r}) \, dA \quad (40)$$

is an inertia tensor. The moment \mathbf{M}_ζ of tractions on a cross-section \mathcal{A}_ζ is given by

$$\mathbf{M}_\zeta = \int_{\mathcal{A}_\zeta} (\mathbf{p}_0 + \varepsilon \dot{\mathbf{u}} + \varepsilon^2 \ddot{\mathbf{u}} + \dots) \times [(\varepsilon \dot{\mathbf{S}} + \varepsilon^2 \ddot{\mathbf{S}} + \dots) \mathbf{N}] dA, \quad (41)$$

where \mathbf{p}_0 is the position vector of a point in the reference configuration. Thus

$$\dot{\mathbf{M}}_\zeta = \int_{\mathcal{A}_\zeta} (\mathbf{p}_0 \times \ddot{\mathbf{S}} \mathbf{N} + \dot{\mathbf{u}} \times \dot{\mathbf{S}} \mathbf{N}) dA. \quad (42)$$

Taking moments of tractions on the boundary of the domain $\mathcal{A} \times [0, \zeta]$, recalling (27) and since the resultant moment of second-order tractions at the end faces vanishes, we obtain

$$\int_{\mathcal{A}_\zeta} [(\mathbf{r}_0 + \zeta_0 \mathbf{e}) \times \ddot{\mathbf{S}} \mathbf{e} + \dot{\mathbf{u}} \times \dot{\mathbf{S}} \mathbf{e}] dA = \text{const.} \quad (43)$$

Substitution for $\ddot{\mathbf{S}}$ from (20) and for $\dot{\mathbf{S}}$ into (43) and some simplification gives

$$\int_{\mathcal{A}_\zeta} [(*\mathbf{r}) \cdot \ddot{\mathbf{t}} - \mu \zeta \varepsilon^2 \dot{\mathbf{c}} \cdot (\mathbf{J}(*\dot{\mathbf{c}}))] dA = \text{const.} \quad (44)$$

$$\begin{aligned} & \zeta \int_{\mathcal{A}_\zeta} (*\ddot{\mathbf{t}}) dA - \int_{\mathcal{A}_\zeta} (\ddot{\sigma}(*\mathbf{r})) dA \\ & + \varepsilon^2 \left\{ \left(\frac{1}{2} \lambda (1 - 4\nu + 10\nu^2) + \mu \beta \right) \right. \\ & \quad \times \left(2\mathbf{J}(*\dot{\mathbf{c}}) - \int_{\mathcal{A}_\zeta} [(*\mathbf{r}) \otimes (*\mathbf{r}) \cdot (*\dot{\mathbf{c}} \otimes *\dot{\mathbf{c}})(* \mathbf{r})] dA \right) \\ & \quad - \lambda \nu^2 \int_{\mathcal{A}} [(*\mathbf{r}) \otimes (*\mathbf{r}) \cdot (\dot{\mathbf{c}} \otimes \dot{\mathbf{c}})](*\mathbf{r}) dA - 2\mu(1 + \nu) \\ & \quad \left. \times \left[\frac{\nu}{2} \int_{\mathcal{A}_\zeta} (\mathbf{r} \vee (*\mathbf{r})) (1 - \delta) \dot{\mathbf{c}} dA + \nu J(*\dot{\mathbf{c}}) \right] \right\} = \text{const.} \end{aligned} \quad (45)$$

4. Saint-Venant Solutions

Assuming that $\ddot{\mathbf{u}}(\mathbf{r}, \zeta)$ is a polynomial in ζ , we write it as

$$\ddot{\mathbf{u}}(\mathbf{r}, \zeta) = \mathbf{u}_0(\mathbf{r}) + \zeta \mathbf{u}_1(\mathbf{r}) + \frac{\zeta^2}{2!} \mathbf{u}_2(\mathbf{r}) + \frac{\zeta^3}{3!} \mathbf{u}_3(\mathbf{r}) + \dots, \quad (46)$$

where we have dropped the superimposed double dots on the right-hand side of (46). We note that such a solution will not capture well the boundary layer effect, if any, near the end faces $\zeta = 0$ and $\zeta = \ell$. Saint-Venant solutions represent well the deformation fields away from the end faces. Recognizing that the right-hand sides of (33)–(36) are polynomials of order at most one in ζ , we obtain for $\mathbf{u}_4, \mathbf{u}_5, \dots$ etc., equations analogous to those in linear elasticity with zero body forces and zero traction boundary conditions. Because of the conditions (7) at the centroid of one end face, and the uniqueness of solutions in linear elasticity, $\mathbf{u}_4 = \mathbf{u}_5 = \dots = \mathbf{0}$. A similar reasoning for \mathbf{u}_3 gives

$$\mathbf{u}_3 = w_3^0 \mathbf{e} + \mathbf{v}_3^0 + \omega_3^0(*\mathbf{r}). \quad (47)$$

Here and below, quantities with superscript zero denote constants. From (46), (47) and $\mathbf{u}_4 = \mathbf{u}_5 = \dots = \mathbf{0}$, we obtain

$$\begin{aligned} \ddot{\mathbf{v}}(\mathbf{r}, \zeta) &= \mathbf{v}_0(\mathbf{r}) + \zeta \mathbf{v}_1(\mathbf{r}) + \frac{\zeta^2}{2!} \mathbf{v}_2(\mathbf{r}) + \frac{\zeta^3}{3!} (\mathbf{v}_3^0 + \omega_3^0(*\mathbf{r})), \\ \ddot{w}(\mathbf{r}, \zeta) &= w_0(\mathbf{r}) + \zeta w_1(\mathbf{r}) + \frac{\zeta^2}{2!} w_2(\mathbf{r}) + \frac{\zeta^3}{3!} w_3^0. \end{aligned} \quad (48)$$

Substitution from (48) into (29)–(32) yields

$$\begin{aligned} 2\ddot{\mathbf{E}} &= \frac{\zeta^2}{2} (\text{grad } \mathbf{v}_2(\mathbf{r}))_s + \zeta (\text{grad } \mathbf{v}_1(\mathbf{r}))_s + (\text{grad } \mathbf{v}_0(\mathbf{r}))_s, \\ \ddot{\sigma} &= (\lambda + 2\mu) \left[\frac{\zeta^2}{2} w_3^0 + \zeta w_2(\mathbf{r}) + w_1(\mathbf{r}) \right] \\ &\quad + \lambda \text{div} \left[\frac{\zeta^2}{2} \mathbf{v}_2(\mathbf{r}) + \zeta \mathbf{v}_1(\mathbf{r}) + \mathbf{v}_0(\mathbf{r}) \right], \\ \ddot{\mathbf{t}} &= \mu \left[\frac{\zeta^2}{2} (\mathbf{v}_3^0 + \omega_3^0(*\mathbf{r}) + \text{grad } w_2(\mathbf{r}) + \zeta (\mathbf{v}_2(\mathbf{r}) \right. \\ &\quad \left. + \text{grad } w_1(\mathbf{r}) + \mathbf{v}_1(\mathbf{r}) + \text{grad } w_0(\mathbf{r})) \right] \\ \ddot{\mathbf{S}} &= \mu \left[\frac{\zeta^2}{2} (\text{grad } \mathbf{v}_2(\mathbf{r}))_s + \zeta (\text{grad } \mathbf{v}_1(\mathbf{r}))_s + (\text{grad } \mathbf{v}_0(\mathbf{r}))_s \right] \\ &\quad + \lambda \left[\frac{\zeta^2}{2} (w_3^0 + \text{div } \mathbf{v}_0(\mathbf{r})) + \zeta (w_2(\mathbf{r}) \right. \\ &\quad \left. + \text{div } \mathbf{v}_1(\mathbf{r}) + w_1(\mathbf{r}) + \text{div } \mathbf{v}_0(\mathbf{r})) \right] \hat{\mathbf{I}}, \end{aligned} \quad (49)$$

where

$$(\text{grad } \mathbf{v}(\mathbf{r}))_s = \text{grad } \mathbf{v}(\mathbf{r}) + (\text{grad } \mathbf{v}(\mathbf{r}))^T. \quad (50)$$

Substitution from (49) into (33)–(36) and equating like powers of ζ on both sides yields equations for determining w_2 , \mathbf{v}_2 , w_1 , \mathbf{v}_1 and w_0 , \mathbf{v}_0 . Equations for the determination of w_2 and \mathbf{v}_2 are

$$\begin{aligned} \Delta_R w_2 &= 0, \quad \text{in } \mathcal{A}, \\ F \mathbf{v}_2 &= \mathbf{0}, \quad \text{in } \mathcal{A}, \\ (\text{grad } w_2) \cdot \mathbf{N} &= -(\mathbf{v}_3^0 + \omega_3^0(*\mathbf{r})) \cdot \mathbf{N}, \quad \text{on } \partial \mathcal{A}, \\ (\mathbf{G} \mathbf{v}_2) \mathbf{N} &= -[\lambda w_3^0 \hat{\mathbf{I}} + 2\varepsilon^2(\mu \dot{\mathbf{c}} \otimes \dot{\mathbf{c}} + \lambda \|\dot{\mathbf{c}}\|^2 \hat{\mathbf{I}})] \mathbf{N}, \quad \text{on } \partial \mathcal{A}, \end{aligned} \quad (51)$$

where

$$\begin{aligned} \mathbf{G}(\cdot) &= \mu(\text{grad } (\cdot))_s + \hat{\mathbf{I}} \lambda \text{div } (\cdot), \\ F &= \text{div } \mathbf{G} = \mu \Delta_R + (\lambda + \mu) \text{grad div}, \end{aligned} \quad (52)$$

Δ_R is the two-dimensional (in-plane) Laplacian operator, and F is the Navier operator of linear elasticity.

The solution of (51) is

$$\begin{aligned} w_2(\mathbf{r}) &= w_2^0 - \mathbf{v}_3^0 \cdot \mathbf{r} + \omega_3^0 \phi(\mathbf{r}), \\ \mathbf{v}_2(\mathbf{r}) &= \mathbf{v}_2^0 + \omega_2^0(*\mathbf{r}) - \nu w_3^0 \mathbf{r} \\ &\quad - \varepsilon^2[(1 + \nu)(\dot{\mathbf{c}} \otimes \dot{\mathbf{c}}) + \nu(*\dot{\mathbf{c}} \otimes *\dot{\mathbf{c}})] \mathbf{r}, \end{aligned} \quad (53)$$

where $\phi(\mathbf{r})$ is the standard Saint-Venant warping function. The Poisson effect proportional to ε^2 varies quadratically with ζ , the distance from the ‘fixed’ end, and is not uniform throughout the cross-section. The displacement of a point in the bending direction $\dot{\mathbf{c}}$ is more than that in the direction perpendicular to $\dot{\mathbf{c}}$. Substitution from (53) into (49), the result into (38) and equating coefficients of like powers of ζ on both sides yields

$$w_3^0 = -\varepsilon^2 \|\dot{\mathbf{c}}\|^2, \quad \omega_3^0 = 0. \quad (54)$$

Thus

$$\begin{aligned} w_2(\mathbf{r}) &= w_2^0 - \mathbf{v}_3^0 \cdot \mathbf{r}, \\ \mathbf{v}_2(\mathbf{r}) &= \mathbf{v}_2^0 + \omega_2^0(*\mathbf{r}) - \varepsilon^2(\dot{\mathbf{c}} \otimes \dot{\mathbf{c}}) \mathbf{r}. \end{aligned} \quad (55)$$

Hence for an unloaded mantle, the second-order Poisson effect proportional to ε^2 is only in the $\dot{\mathbf{c}}$ direction.

Equations governing displacement fields \mathbf{v}_1 and w_1 are

$$\begin{aligned} \Delta_R w_1 &= 0, & \text{in } \mathcal{A}, \\ F\mathbf{v}_1 &= \lambda \mathbf{v}_3^0, & \text{in } \mathcal{A}, \\ (\text{grad } w_1) \cdot \mathbf{N} &= -[\mathbf{v}_2^0 + \omega_2^0(*\mathbf{r}) + \varepsilon^2(\nu(\dot{\mathbf{c}} \otimes \dot{\mathbf{c}}) \\ &\quad - (*\dot{\mathbf{c}} \otimes *\dot{\mathbf{c}})\mathbf{r} - (1 + \nu)\dot{\mathbf{c}}] \cdot \mathbf{N}, & \text{on } \partial\mathcal{A}, \\ (\mathbf{G}\mathbf{v}_1)\mathbf{N} &= [\lambda(\mathbf{v}_3^0 \cdot \mathbf{r} - \omega_2^0)\hat{\mathbf{I}}]\mathbf{N}, & \text{on } \partial\mathcal{A}. \end{aligned} \quad (56)$$

The solution of (56) is

$$\begin{aligned} w_1(\mathbf{r}) &= w_1^0 - \mathbf{v}_2^0 \cdot \mathbf{r} + \omega_2^0 \phi(\mathbf{r}) - \varepsilon^2(\mathbf{C}\mathbf{r}) \cdot \mathbf{r} + \varepsilon^2(1 + \nu)\dot{\mathbf{c}} \cdot \mathbf{r}, \\ \mathbf{v}_1(\mathbf{r}) &= \mathbf{v}_1^0 + \omega_1^0(*\mathbf{r}) - \nu w_2^0 \mathbf{r} + \mathbf{K}\mathbf{v}_3^0, \end{aligned} \quad (57)$$

where

$$\mathbf{C} = \nu(\dot{\mathbf{c}} \otimes \dot{\mathbf{c}} - (*\dot{\mathbf{c}}) \otimes (*\dot{\mathbf{c}}))/2. \quad (58)$$

Equations (44) and (45) for the resultant force and the resultant moment do not restrict the constants appearing in (57). However, (38) and (44) imply the following

$$w_2^0 = 0, \quad \omega_2^0 = \dot{\mathbf{f}} \cdot \dot{\mathbf{c}}/K_t, \quad (59)$$

where

$$\begin{aligned} \dot{\mathbf{f}} &= E\mathbf{J}(*\dot{\mathbf{c}}), & E &= 2\mu(1 + \nu), \\ K_t &= \mu(J_0 - D), & J_0 &= \int_{\mathcal{A}} r^2 \, dA, \\ D &= \int_{\mathcal{A}} (*\text{grad } \phi) \cdot \mathbf{r} \, dA, \end{aligned} \quad (60)$$

$\dot{\mathbf{f}}$ equals the resultant bending moment, and K_t the torsional rigidity of the Saint-Venant first-order bending and torsion problems respectively. In the absence of torques on the mantle, the angular twist is proportional to ζ^2 . The constant ω_2^0 vanishes when $\dot{\mathbf{c}}$ is parallel to a principal axis of the cross-section and is extremum when $\dot{\mathbf{c}}$ is parallel to the direction bisecting the principal axes of the cross-section. This second-order angular twist proportional to ζ^2 does not depend upon the second-order elasticities of the material but is determined by Poisson's ratio, the bending vector, and the properties associated with the cross-section. The last term

on the right-hand side of (57)₁ corresponds to the extra out-of-plane rigid rotation of a cross-section around $(*\dot{\mathbf{c}})$, which is proportional to its distance from the fixed end, and arises only when the bar is simultaneously prebent and prestretched.

Equations governing displacements \mathbf{v}_0 , w_0 are

$$\begin{aligned} \Delta_R w_0 &= 2\mathbf{v}_3^0 \cdot \mathbf{r}, \quad \text{in } \mathcal{A}, \\ F\mathbf{v}_0 &= \lambda\mathbf{v}_2^0 - \omega_2^0[(\lambda + \mu)\text{grad } \phi + \mu(*\mathbf{r})] - 2\lambda\nu(*\dot{\mathbf{c}} \otimes *\dot{\mathbf{c}})\mathbf{r} \\ &\quad - \lambda(\dot{\mathbf{c}} \otimes \dot{\mathbf{c}})\mathbf{r} + \varepsilon^2(1 - \delta)\dot{\mathbf{c}}\{\lambda(6\nu^2 - 7\nu - 4) \\ &\quad + \mu[6\nu^2 - 6\nu - 2 + 2\alpha_3(1 + 2\nu)^2 + 2\alpha_4(\nu^2 - 2\nu) \\ &\quad + 2\alpha_5(2\nu^2 - \nu) + 2\alpha_6\nu^2]\}, \quad \text{in } \mathcal{A}, \\ (\text{grad } w_0) \cdot \mathbf{N} &= -(\mathbf{v}_1^0 + \omega_1^0(*\mathbf{r}) + \mathbf{K}\mathbf{v}_3^0) \cdot \mathbf{N}, \quad \text{on } \partial\mathcal{A}, \\ (\mathbf{G}\mathbf{v}_0)\mathbf{N} &= -\{\lambda[w_1^0 + (\varepsilon^2(1 + \nu)\dot{\mathbf{c}} - \mathbf{v}_2^0) \cdot \mathbf{r} \\ &\quad - \varepsilon^2(\mathbf{C}\mathbf{r}) \cdot \mathbf{r} + \omega_2^0\phi] + \varepsilon^2(\lambda + \mu)\nu^2\phi^2 \\ &\quad + \varepsilon^2[\lambda(3 - 6\nu + 6\nu^2)/2 + \mu(3\nu^2 - 2\nu) + \alpha_3(1 - 2\nu)^2 \\ &\quad + \alpha_4(\nu^2 - 2\nu) + \alpha_5(2\nu^2 - \nu) + \alpha_6\nu^2]\} \\ &\quad \times (1 - \delta)^2 \hat{\mathbf{I}}\mathbf{N}, \quad \text{on } \partial\mathcal{A}. \end{aligned} \tag{61}$$

The solution of (61) is

$$\begin{aligned} w_0(\mathbf{r}) &= w_0^0 - \mathbf{v}_1^0 \cdot \mathbf{r} + \omega_1^0\phi(\mathbf{r}) + \tilde{w}_0(\mathbf{r}), \\ \mathbf{v}_0(\mathbf{r}) &= \mathbf{v}_0^0 + \omega_0^0(*\mathbf{r}) - \nu w_1^0\mathbf{r} + \mathbf{K}\mathbf{v}_2^0 + \tilde{\mathbf{v}}_0(\mathbf{r}). \end{aligned} \tag{62}$$

Functions $\tilde{w}_0(\mathbf{r})$ and $\tilde{\mathbf{v}}_0(\mathbf{r})$ are solutions of (61) after setting $\omega_1^0 = 0$, $w_1^0 = 0$, $\mathbf{v}_1^0 = 0$ and $\mathbf{v}_2^0 = \mathbf{0}$, and constant w_0^0 , ω_0^0 , \mathbf{v}_0^0 and \mathbf{v}_1^0 are to be determined from the six conditions (7) at the centroid of one end-face. Constants w_1^0 , ω_1^0 , \mathbf{v}_2^0 and \mathbf{v}_3^0 determine the Saint-Venant solution for extension, torsion, bending, and flexure problem respectively. The terms linear in $\dot{\mathbf{c}}$ on the right-hand side of (61)₂ and (61)₄ are due to the simultaneous prebending and prestretching of the bar.

5. Generalized Poynting Effects in a Prebent, Prestretched and Pretwisted Bar

A second-order effect in a bar bent by loads applied only at the end faces is that the angular twist varies quadratically with the distance from the 'fixed' end. This twist vanishes only when the bending vector $\dot{\mathbf{c}}$ is parallel to one of the principal axes of the cross-section, and hence is always zero for a circular bar.

The resultant second-order normal and shear tractions on an end face are given by

$$\begin{aligned}
& \int_{\mathcal{A}_\ell} (\ddot{\mathbf{S}}\mathbf{e}) \cdot \mathbf{e} \, dA \\
&= E \mathcal{A} w_1^0 - \varepsilon^2 \left\{ (\lambda + 2\mu) \int_{\mathcal{A}_\ell} [(\mathbf{C}\mathbf{r}) \cdot \mathbf{r} - \omega_2^0 \phi] \, dA \right. \\
&\quad \left. - \lambda v^2 (\mathbf{J}\dot{\mathbf{c}}) \cdot \dot{\mathbf{c}} + ((\lambda(1 - 4\nu + 10\nu^2)/2) + \mu\beta) \right. \\
&\quad \left. \times (\mathcal{A} - (\mathbf{J}(*\dot{\mathbf{c}})) \cdot (*\dot{\mathbf{c}})) \right\} + \lambda \int_{\mathcal{A}_\ell} (\operatorname{div} \tilde{\mathbf{v}}_0) \, dA, \tag{63}
\end{aligned}$$

$$\int_{\mathcal{A}_\ell} (\ddot{\mathbf{S}}\mathbf{e})_\perp \, dA = \mu \int_{\mathcal{A}_\ell} (\operatorname{grad} \tilde{w}_0) \, dA - \frac{\mu\nu}{2} (J + *J*) \mathbf{v}_3^0,$$

and the resultant second-order torque T , and the bending moment \mathbf{M} , have the expressions

$$T = K_t \omega_1^0 + \mu \left(\int_{\mathcal{A}_\ell} (*\mathbf{r}) \cdot (\operatorname{grad} \tilde{w}_0) \, dA - \frac{\nu}{2} \int_{\mathcal{A}_\ell} r^2 (*\mathbf{r}) \cdot \mathbf{v}_3^0 \, dA \right), \tag{64}$$

$$\begin{aligned}
\mathbf{M} &= EJ(*\mathbf{v}_2^0) - (\lambda + 2\mu)[\varepsilon^2(1 + \nu)J(*\dot{\mathbf{c}})] \\
&\quad - \varepsilon^2 \int_{\mathcal{A}_0} ((\mathbf{C}\mathbf{r}) \cdot \mathbf{r})(*\mathbf{r}) \, dA + \omega^0 [(*\mathbf{r})\phi \, dA] - \lambda [(\operatorname{div} \tilde{\mathbf{v}}_0)(*\mathbf{r}) \, dA] \\
&\quad + \varepsilon^2 \left\{ \left(\frac{1}{2} \lambda (1 - 4\nu + 10\nu^2) + \mu\beta \right) \left(2J(*\dot{\mathbf{c}}) - \int_{\mathcal{A}_0} (*\mathbf{r}) \otimes (*\mathbf{r}) \right. \right. \\
&\quad \left. \left. \times ((*\dot{\mathbf{c}}) \otimes (*\dot{\mathbf{c}}))(*\mathbf{r}) \, dA \right. \right. \\
&\quad \left. \left. - \lambda v^2 \int_{\mathcal{A}_0} ((*\mathbf{r}) \otimes (*\mathbf{r})) \cdot (\dot{\mathbf{c}} \otimes \dot{\mathbf{c}})(*\mathbf{r}) \, dA \right) \right. \\
&\quad \left. - E \left(\frac{1}{2} \nu \int_{\mathcal{A}_0} (\mathbf{r} \vee (*\mathbf{r}))(1 - \delta)\dot{\mathbf{c}} \, dA + \nu J(*\dot{\mathbf{c}}) \right) \right\}. \tag{65}
\end{aligned}$$

Here $E = \mu(3\lambda + 2\mu)/(\lambda + \mu)$ is Young's modulus for the material of the bar. Equations (61)₁ and (63)₂ imply that the resultant second-order shear traction is linear in \mathbf{v}_3^0 . Therefore, there is no second-order flexure in the absence of second-order

resultant shear force. However, (63)₁ implies that in the absence of the resultant second-order axial traction, there can be a second-order extension of the bar. This second-order extension depends upon the first and second-order elasticities of the material, area of cross-section, first-order torsional rigidity, bending vector $\dot{\mathbf{c}}$, and the inertia tensor \mathbf{J} . Equation (63)₁ generalizes Rivlin's result for the resultant axial force in a pretwisted bar to that in a prebent bar. For a given cross-section and zero resultant normal traction, it is a nontrivial task to find the bending direction $\dot{\mathbf{c}}$, which will give extreme values of the second-order elongation w_1^0 since the function $\tilde{\mathbf{v}}_0$ is a solution of a plane linear elastic problem with non-zero body forces.

For a zero resultant shear force, the second and third terms on the right-hand side of (64)₁ vanish. Thus when the torque also vanishes, $\omega_1^0 = 0$ and there is no angular twist proportional to ζ .

Equation (64)₂ implies that even when $\mathbf{M} = \mathbf{0}$, \mathbf{v}_2^0 need not vanish. Thus there is in general a second-order bending effect in the absence of the external resultant second-order moment at the end faces. It is rather difficult to find the bending direction $\dot{\mathbf{c}}$ which will extremize \mathbf{v}_2^0 .

In [8] we studied Saint-Venant type deformations of a pretwisted bar, computed the resultant forces and moments on an end face, and discussed the usual Poynting effect. The expressions for the resultant second-order normal and shear tractions on an end face are as follows

$$\int_{\mathcal{A}_\ell} (\ddot{\mathbf{S}}\mathbf{e}) \cdot \mathbf{e} \, dA = E w_1^0 \mathcal{A} + \lambda \int_{\mathcal{A}_\ell} \operatorname{div} \tilde{\mathbf{v}}_0 \, dA + \tau^2 \left(\frac{1}{2} \lambda - \frac{\mu(\alpha_4 - \alpha_6)}{4} \right) \times (J_0 - D) + \tau^2 (\lambda + \mu) D, \quad (66)$$

$$\int_{\mathcal{A}_\ell} (\ddot{\mathbf{S}}\mathbf{e})_\perp \, dA = \mu \left[\mathbf{v}_1^0 \mathcal{A} - (J + *J*) \mathbf{v}_3^0 + \int_{\mathcal{A}_\ell} (\operatorname{grad} \tilde{w}_0) \, dA \right]. \quad (67)$$

Functions $\tilde{\mathbf{v}}_0$ and \tilde{w}_0 in (66) and (67) are different from those appearing in earlier equations, and are defined in [8]. As in this paper, they are also solutions of plane elliptic problems but with different source terms and boundary conditions. τ is the pretwist in the bar. The second-order resultant torque, T , and the moment, \mathbf{M} , are given by

$$T = \mu \left[\omega_1^0 (J_0 - D) - \frac{1}{2} \nu \mathbf{v}_3^0 \cdot \int_{\mathcal{A}_0} \|\mathbf{r}\|^2 (*\mathbf{r}) \, dA + \int_{\mathcal{A}_0} ((*\mathbf{r}) \cdot \operatorname{grad} \tilde{w}_0) \, dA \right], \quad (68)$$

$$\begin{aligned}
\mathbf{M} = & EJ * \mathbf{v}_2^0 - \lambda \int_{\mathcal{A}_0} (*\mathbf{r}) \operatorname{div} \tilde{\mathbf{v}}_0 \, dA \\
& + \tau^2 (\lambda + \mu) \int_{\mathcal{A}_0} (*\mathbf{r} \otimes *\mathbf{r}) \operatorname{grad} \phi \, dA \\
& - \tau^2 \left(\frac{1}{2} \lambda - \frac{\mu(\alpha_4 - \alpha_6)}{4} \right) \int_{\mathcal{A}_0} \|\mathbf{g}\|^2 (*\mathbf{r}) \, dA \\
& + \tau^2 \mu \int \phi (*\mathbf{g}) \, dA, \tag{69}
\end{aligned}$$

where

$$\mathbf{g} = (*\mathbf{r} + \operatorname{grad} \varphi). \tag{70}$$

Expressions (67) and (68) for the second-order resultant shear force and the twisting moment respectively imply that the second-order shear deformation and the second-order twist vanish if and only if these resultants equal zero. However, expression (69) implies that, in general, there is a second-order bending deformation even when the resultant bending moment on an end face is zero. This bending deformation depends upon the first and second-order elasticities, the inertia tensor, and

$$\int (*\mathbf{r}) \operatorname{div} \tilde{\mathbf{v}}_0 \, dA,$$

where $\tilde{\mathbf{v}}_0$ is the solution of the elliptic problem defined by equations (51) of [8].

Conclusions

We have found a second-order solution of the Saint-Venant problem for a straight, prismatic, homogeneous body made of a second-order elastic material. The bar is initially stretched and bent by an infinitesimal amount from the unstressed reference configuration, and then deformed by tractions applied at the end faces. The displacements and infinitesimal rotations at the centroid of one end face are taken to be zero. The problem is analysed by the Signorini expansion method, and its solution reduced to that of solving two linear plane elliptic problems. The resultant second-order forces and moments at an end face are given by equations (63), (64), and (65).

Equation (53)₂ reveals a new Poisson's effect proportional to the square of the initial infinitesimal deformation. The Poisson effect in a cross-section varies as the

square of its distance from the ‘fixed’ end, and the in-plane displacement of a point in the bending direction is more than that in the orthogonal direction.

Unless the direction of prebending is along a principal axis of a cross-section, the cross-section twists in the absence of external torques at the end faces. This twist is proportional to the square of the distance of the cross-section from the ‘fixed’ end, and is determined by Poisson’s ratio, the bending vector and properties of the cross-section, but is independent of the second-order elasticities.

Equation (63)₁ indicates that in the absence of a second-order resultant normal force, there can be a second-order extension of the bar which depends upon the first and second order elasticities of the material, the bending vector and geometric properties of the cross-section.

From (64)₂ we conclude that a second-order bending effect may occur in the absence of corresponding bending moments at the end faces.

For a pretwisted bar, it follows from (69) that there is a second-order bending deformation for vanishing second-order bending moments at the end faces.

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