Instability of a pre-stressed solid-fluid mixture

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INSTABILITY OF A PRE-STRESSED SOLID–FLUID MIXTURE

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A solid-fluid mixture is generally modelled assuming that the state of stress in the reference configuration is identically equal to zero. However, such an assumption is not always appropriate to take into account some instability phenomena occurring in Nature. In this contribution, the continuum mechanics point of view is used and the reference configuration of the solid-fluid mixture has a state of stress, i.e. the pre-stress is different from zero. The instability of the mixture with respect to the perturbation fields given by a general plane wave is then studied.

1. Introduction

In the present macroscopic continuum theory, the balance equations of mass and linear momentum are derived in the form given by Truesdell[7,8]. Moreover, a set of constitutive equations is given in order to characterize the material and to close the problem, see also[1,2]. To do this, we assign the energy per unit mass of the mixture and then we derive the constitutive quantities. The solid part is considered as an elastic material and the fluid part is non viscous. The interactions between the two species is also considered through a non dissipative and a dissipative (Darcy) terms. The Cauchy stress tensor of a given species depends on the constitutive variables of the

other species. We note that this interaction is not in contraddiction with the second law of thermodynamics: the interaction force, in this formulation, depends upon the gradient of mass densities of both species, see e.g.[4,9]. The propagation of harmonic waves is studied and normal modes for the transverse and longitudinal velocities are considered. Since the perturbed fields blow up, we have a set of instability conditions derived by the dispersion relations. Such conditions define the constraints of the constitutive parameters of the model derived from the present analysis.

2. Preliminary and Balance Equations

Let B_s and B_f be the reference configurations of the solid and of the fluid constituents, respectively. The actual configuration is denoted by B. Let $\mathbf{x} \in B$ be the Eulerian coordinates and X_s, X_f be Lagrangian coordinates of the solid and the fluid, respectively. From the mathematical point of view, this means that there exist two functions χ_s and χ_f, called the current placements of the solid and of the fluid, such that the position $\mathbf{x} \in B$ in the actual configuration is given by,

$$\mathbf{x} = \chi_s(\mathbf{X}_s, t) = \chi_f(\mathbf{X}_f, t), \quad \forall \mathbf{X}_s \in B_s, \quad \forall \mathbf{X}_f \in B_f, \quad \forall t \in \mathbb{R}. \tag{1}$$

The derivative with respect to time of (1) gives the velocity fields $\mathbf{v}_s$ and $\mathbf{v}_f$ of the solid and of the fluid, respectively, i.e.,

$$\mathbf{v}_s = \mathbf{v}_s(\mathbf{x}, t) = \frac{d_s \mathbf{x}}{dt} = \frac{\partial \chi_s}{\partial t}, \quad \mathbf{v}_f = \mathbf{v}_f(\mathbf{x}, t) = \frac{d_f \mathbf{x}}{dt} = \frac{\partial \chi_f}{\partial t}, \tag{2}$$

where $\frac{d_s}{dt}$ and $\frac{d_f}{dt}$ denote the time derivatives keeping $\mathbf{X}_s$ and $\mathbf{X}_f$ constant. The derivative with respect to the particles $\mathbf{X}_s$ and $\mathbf{X}_f$ give the deformation gradients $\mathbf{F}_s$ and $\mathbf{F}_f$ of the solid and of the fluid, i.e.,

$$\mathbf{F}_s = \mathbf{F}_s(\mathbf{x}, t) = \frac{\partial \chi_s}{\partial \mathbf{X}_s} = \mathbf{I} + \mathbf{H}_s, \quad \mathbf{F}_f = \mathbf{F}_f(\mathbf{x}, t) = \frac{\partial \chi_f}{\partial \mathbf{X}_f} = \mathbf{I} + \mathbf{H}_f, \tag{3}$$

where $\mathbf{H}_s$ and $\mathbf{H}_f$ are the displacement gradients of the solid and of the fluid, respectively. In the following, the symmetric part $\mathbf{E}_s$

$$\mathbf{E}_s = \tfrac{1}{2}\left(\mathbf{H}_s + \mathbf{H}_s^T\right) \tag{4}$$

of the displacement gradient $\mathbf{H}_s$ of the solid constituent, also called the infinitesimal deformation, will be extensively used.

The balance equations of masses and of linear momenta of the solid and of the fluid are assumed to be the following,

$$\left.\begin{array}{l} \dfrac{\partial \varrho_s}{\partial t} + \nabla \cdot [\varrho_s \mathbf{v}_s] = \varrho_s \Gamma_{sf}, \quad \dfrac{\partial \varrho_f}{\partial t} + \nabla \cdot [\varrho_f \mathbf{v}_f] = \varrho_f \Gamma_{fs}, \\ \dfrac{\partial \varrho_s \mathbf{v}_s}{\partial t} + \nabla \cdot [\varrho_s \mathbf{v}_s \otimes \mathbf{v}_s - \mathbf{T}_s] = \mathbf{m}_{sf} + \varrho_s \mathbf{g}_s, \\ \dfrac{\partial \varrho_f \mathbf{v}_f}{\partial t} + \nabla \cdot [\varrho_f \mathbf{v}_f \otimes \mathbf{v}_f - \mathbf{T}_f] = \mathbf{m}_{fs} + \varrho_f \mathbf{g}_f, \end{array}\right\} \tag{5}$$

where Γ_{sf}, Γ_{fs}, $\mathbf{m}_{sf}$ and $\mathbf{m}_{fs}$ are interaction terms; $\mathbf{T}_s$ and $\mathbf{T}_f$ are the partial Cauchy stress tensors of the solid and of the fluid and $\mathbf{g}_s$ and $\mathbf{g}_f$ are the specific external supplies of linear momenta (e.g., the gravity acceleration $\mathbf{g}$) of the solid and of the fluid.

3. Constitutive Equations

The solid and the fluid are underposed to a certain state of initial stress, called pre-stress, that is assumed to be constant. In both components it is assumed to be spherical, i.e.,

$$\mathbf{T}_0^s = -p_0^s \mathbf{I}, \quad \mathbf{T}_0^f = -p_0^f \mathbf{I}. \tag{6}$$

We underline that $\mathbf{T}_0^s$ and $\mathbf{T}_0^f$ are intended to be Cauchy stress tensors. This means that the forces producing this kind of pre-stress are prescribed in the actual configuration and are per unit area in the actual configuration, too. Note that not all the forces fulfil this characteristic. The forces due to an hydrostatic pressure acting on the mixture is one of this kind; the forces due to the weight or to the traction in one fixed direction are not !

The characterization of the initial (reference) configuration is now complete once one assigns the partial mass densities ϱ_0^s and ϱ_0^f of the solid and of the fluid constituents. We will denote the fields evaluated in this (reference) configuration with the subscript 0 and we can write,

$$\varrho_0^s = \text{const.}\,, \quad \varrho_0^f = \text{const.}\,, \quad \mathbf{v}_0^s = \mathbf{v}_0^f = \mathbf{0}.\,, \quad p_0^s = \text{const.}\,, \quad p_0^f = \text{const.} \tag{7}$$

Superimposed to this equilibrated and pre-stressed state, we consider an arbitrary perturbation, that is defined by the two displacement fields $\mathbf{u}_s$ and $\mathbf{u}_f$ and by $\tilde{\varrho}_s$ and $\tilde{\varrho}_f$, the change of partial mass densities, i.e.,

$$\mathbf{x} = \mathbf{X}_s + \mathbf{u}_s(\mathbf{X}_s, t) = \mathbf{X}_f + \mathbf{u}_f(\mathbf{X}_f, t); \quad \varrho_s = \varrho_0^s + \tilde{\varrho}_s, \quad \varrho_f = \varrho_0^f + \tilde{\varrho}_f. \tag{8}$$

The constitutive characterization of the non-simple mixture considered in this paper is derived by Sciarra[5,6] in 2001 in the limit of small perturbation. Thus, it is reasonable to consider the solid in the regime of linear

elasticity; the fluid is modelled as a perfect and compressible medium. The objective form of the conservative part Ψ of the energy per unit mass (so that Ψ is the conservative part of the specific energy) is defined as follows,

$$\left.\begin{aligned}\varrho_0\Psi = \mathbf{T}_0^s\cdot\mathbf{H}^s + \frac{p_0^f}{\varrho_0^f}\tilde{\varrho}_f + \frac{1}{2}\mathbb{C}^s\mathbf{E}^s\cdot\mathbf{E}^s - \frac{1}{4}\mathbf{H}^s\cdot\left(\mathbf{T}_0^s\mathbf{H}^{sT} - \mathbf{H}^s\mathbf{T}_0^s\right)\\ +\frac{1}{8}\mathbf{H}^s\cdot\left(\mathbf{H}^s\mathbf{T}_0^s - \mathbf{T}_0^s\mathbf{H}^s + \mathbf{T}_0^s\mathbf{H}^{sT} - \mathbf{H}^{sT}\mathbf{T}_0^s\right) + \tilde{\varrho}_f\mathbf{K}^{sf}\cdot\mathbf{H}^s + \frac{1}{2}\varrho_0\gamma_{ff}\tilde{\varrho}_f^2,\end{aligned}\right\} \tag{9}$$

where the superscript $(\mathbf{A})^T$ denotes the transpose of a general tensor $\mathbf{A}$ and the constitutive quantities have the following general form, i.e.,

$$\left.\begin{aligned}\mathbf{m}_{fs} = -\mathbf{m}_{sf} = \mathbf{m} = -\varrho_f\left(\nabla\mathbf{F}^s\right)^T\frac{\partial\Psi}{\partial\mathbf{F}^s} + \varrho_s\frac{\partial\Psi}{\partial\varrho_f}\nabla\tilde{\varrho}_f + D\left(\mathbf{v}^s - \mathbf{v}^f\right)\\ \mathbf{T}_s = \varrho\frac{\partial\Psi}{\partial\mathbf{F}^s}\mathbf{F}^{sT}, \quad \mathbf{T}_f = -p_f\mathbf{I}, \;\; p_f = \varrho\varrho_f\frac{\partial\Psi}{\partial\varrho_f},\end{aligned}\right\} \tag{10}$$

where the classical dissipative interaction Darcy term for the interaction force $\mathbf{m}$ in the equation $(10)_1$, i.e. $D\left(\mathbf{v}^s - \mathbf{v}^f\right)$, is added to the formulation of Sciarra[5]. Therefore, the constitutive equations of the mixture are completely characterized once one assigns the constitutive parameters D, $\mathbf{K}^{sf}$, γ_{ff} and $\mathbb{C}^s$. We remark that such parameters are referred to the given reference configuration and therefore are related to the prescribed state of pre-stress.

The scalar D and the tensor $\mathbf{K}^{sf}$ represent the dissipative and the non dissipative interaction between the species of the mixture; γ_{ff} is related to the compressibility of the fluid species and $\mathbb{C}^s$ is the fourth order elastic tensor related to the linear elastic solid species. Furthermore, we assume (see e.g.[3,6]),

$$\mathbf{K}^{sf} = -\varrho_0\varrho_0^s\gamma_{sf}\mathbf{I}, \qquad \mathbb{C}^s\mathbf{E}^s = 2\mu\mathbf{E}^s + \lambda\mathrm{tr}\left(\mathbf{E}^s\right)\mathbf{I}, \tag{11}$$

where γ_{sf} is a single interaction parameter and λ and μ are the Lamé coefficients evaluated for the given solid component and for the given state of pre-stress. If we insert the constitutive equations (10) and (9) into the balance equations (5), then the field equations take the form

$$\frac{\partial\tilde{\varrho}_s}{\partial t} = -\varrho_0^s\nabla\cdot\left[\mathbf{v}_s\right], \; \frac{\partial\tilde{\varrho}_f}{\partial t} = -\varrho_0^f\nabla\cdot\left[\mathbf{v}_f\right], \tag{12}$$

$$\varrho_0^s\frac{\partial\mathbf{v}_s}{\partial t} = \left(2\mu - p_0^s\right)\nabla\cdot\mathbf{E}_s + \tilde{\lambda}_{ss}\nabla\left(\mathrm{tr}\mathbf{E}_s\right) - \frac{\tilde{\lambda}_{sf}}{\varrho_0^f}\nabla\tilde{\varrho}_f - D\left(\mathbf{v}_s - \mathbf{v}_f\right), \tag{13}$$

$$\varrho_0^f\frac{\partial\mathbf{v}_f}{\partial t} = -\frac{\tilde{\lambda}_{ff}}{\varrho_0^f}\nabla\tilde{\varrho}_f + \tilde{\lambda}_{sf}\nabla\left(\mathrm{tr}\mathbf{E}_s\right) + D\left(\mathbf{v}_s - \mathbf{v}_f\right). \tag{14}$$

where the coefficients $\tilde{\lambda}_{\alpha\beta}$ depend on $\gamma_{\alpha\beta}$ and on the pre-stresses p_0^α with α and β be s and/or f (see also[3]). Moreover, the relation between the infinitesimal deformation $\mathbf{E}^s$ and the partial mass density of the solid constituent is derived from the mass balance, i.e., $\tilde{\varrho}_s = -\varrho_0^s \mathrm{tr}\mathbf{E}^s$, in the limit of small perturbation. Since we are interested in a solution of the form

$$\tilde{\varrho}_\alpha = \hat{\varrho}_\alpha e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}, \ \tilde{\mathbf{v}}_\alpha = \hat{\mathbf{v}}_\alpha e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}, \ \alpha = s, f, \tag{15}$$

where ω is the frequency and $\mathbf{k}$ the wave vector, both possibly complex, substituting the above expressions into the balance equations yields,

$$\begin{pmatrix} \mathbf{A_{ss}} & \mathbf{A_{sf}} \\ \mathbf{A_{fs}} & \mathbf{A_{ff}} \end{pmatrix} \begin{pmatrix} \hat{\mathbf{v}}_s \\ \hat{\mathbf{v}}_s \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \implies \mathrm{Det} \begin{pmatrix} \mathbf{A_{ss}} & \mathbf{A_{sf}} \\ \mathbf{A_{sf}} & \mathbf{A_{ff}} \end{pmatrix} = 0, \tag{16}$$

where $\mathbf{A}_{\alpha\beta}$ are the suitable matrix coefficients,

$$\begin{aligned} \mathbf{A_{ss}} &= \left(\left(\mu - \tfrac{1}{2}p_0^s\right)k^2 + i\omega D - \varrho_0^s\omega^2\right)\mathbf{I} + \left(\lambda + \mu + \frac{\varrho_0^s - 3\varrho_0^f}{2\varrho_0}p_0^s\right)(\mathbf{k}\otimes\mathbf{k}) \\ \mathbf{A_{sf}} &= -i\omega D\mathbf{I} + \tilde{\lambda}_{sf}\,(\mathbf{k}\otimes\mathbf{k}) \\ \mathbf{A_{ff}} &= \left(i\omega D - \varrho_0^f\omega^2\right)\mathbf{I} + \varrho_0^f\left(\gamma_{ff}\varrho_0\varrho_0^f + 2\frac{p_0^f}{\varrho_0}\right)(\mathbf{k}\otimes\mathbf{k}) \end{aligned} \tag{17}$$

The solution of the dispersion relation given in $(16)_2$ is not an easy task. It can be simplified decomposing the amplitudes of the velocity fields $\hat{\mathbf{v}}_\alpha$ into a solenoidal $\hat{\mathbf{v}}_\alpha^\perp$ and an irrotational $\hat{v}_\alpha^\parallel\mathbf{k}$ part, i.e.,

$$\hat{\mathbf{v}}_\alpha = \hat{\mathbf{v}}_\alpha^\perp + \hat{v}_\alpha^\parallel\mathbf{k}, \quad \mathbf{k}\cdot\hat{\mathbf{v}}_\alpha^\perp = 0, \quad \nabla\cdot\hat{\mathbf{v}}_\alpha^\perp = 0, \quad \nabla\times\hat{v}_\alpha^\parallel\mathbf{k} = 0, \quad \alpha = s, f \tag{18}$$

corresponding to transverse and longitudinal part of the velocity fields, respectively. Assuming that $\mathbf{k}\in\mathbb{R}^3$ is a real vector, then if the immaginary part ω_i of the frequency ω is positive, then the perturbations in the form of plane waves expressed in (15) are attenuated. Note that this does not mean that the system is stable under all kind of perturbations. On the other hand it is possible to express the prediction of our assumptions claiming that while ω_i is negative, the system is unstable.

For transverse waves the dispertion relation is reduced to a third degree polynomial in ω, the coefficient of which are real. Thus, the negativeness of the immaginary part ω_i of the frequency ω can be assured if

$$D > 0, \qquad p_0^s < 2\mu. \tag{19}$$

If such conditions are not fulfilled, then the system is unstable. The first condition is a direct consequence of the Second Law of Thermodynamics and simply tells us that the dissipation due to the Darcy term in the constitutive equation $(10)_1$ is not able to produce energy. The second restriction expressed in (19) gives an explicit restriction on the value of the pre-stress of the solid.

For longitudinal waves the dispersion relation is reduced to a fourth-degree polynomial in ω, the coefficient of which are real. We show that the immaginary part ω_i of the frequency ω is negative if,

$$\begin{aligned} & p_0^s < \frac{1}{2}(\lambda + 2\mu) + \frac{1}{2}\varrho_0 (\varrho_0^s)^2 (\gamma_{ff} - 2\gamma_{sf}), \\ & \left(\lambda + 2\mu - 2\frac{\varrho_0^f}{\varrho_0} p_0^s\right)\left(\varrho_0 \left(\varrho_0^f\right)^2 \gamma_{ff} + 2\frac{\varrho_0^f}{\varrho_0} p_0^f\right) > \tilde{\lambda}_{sf}^2, \end{aligned} \tag{20}$$

We remark that conditions $(19)_2$ and (20) are explicit or implicit restrictions on the values of the prestresses p_0^s and p_0^f. Whether $p_0^s = p_0^f = 0$, then such non-instability conditions must be satisfied. We can represent $(19)_2$ and (20) in terms of a graphic in which the coordinate axes are the prestresses p_0^s and p_0^f. It is easy to show that they characterize regions of the values of prestress that the mixture can admit.

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