Deduction of a generalized Stefan-problem and its solution by means of an iterative method

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A GENERALIZED Stefan-problem arises when the thermomechanical theory for continua with interface, proposed by A. ROMANO and others [10, 11], is used for modelling a „one-dimensional" system where a phase transition is taking place and the difference in density between the phases cannot be neglected. This problem is solved by means of an iterative method which allows for the construction of a solution in a suitable time interval. An integral representation of the moving boundary s is then shown, which immediately proves the uniqueness of the solution in its existence time interval. The repeated application of the iterative method allows us to construct the solution up to the instant T^* only, that is up to that instant when (eventually) of the phases disappears. Proof techniques we have used sum to be convenient for numerical applications and applicable to more general problems. Next we show a global continuous-dependence-on--coefficients- and data theorem. The existence andcontinuous dependence theorem thus proved confirms the applicability of the proposed thermomechanical model. We finally discuss the physical meaning of the failure of quoted proof-techniques when $t > T^*$.

Uogólniony problem Stefana powstaje przy zastosowaniu teorii termomechanicznej dla ośrodków z powierzchnią rozdziału (A. ROMANO [10, 11]) do modelowania układu „jednowymiarowego", w którym zachodzi przejście fazowe, przy którym nie można zaniedbać różnic gęstości poszczególnych faz. Problem ten rozwiązuje się sposobem iteracyjnym, który pozwala na konstrukcję rozwiązania w odpowiednim przedziale czasu. Ruchomy brzeg przedstawia się następnie w postaci całkowej, co pozwala zarazem stwierdzić jednoznaczność rozwiązania. Wielokrotne zastosowanie metody iteracyjnej pozwala nam konstruować rozwiązanie ważne jedynie do chwili T^*, to jest do chwili gdy jedna z faz znika. Metoda dowodu okazuje się szczególnie przydatna do obliczeń numerycznych i daje się zastosować do bardziej ogólnych zagadnień. Następnie przedstawia się twierdzenie o globalnej ciągłej zależności rozwiązania do współczynników i danych początkowych. Dowiedzione w ten sposób twierdzenie o istnieniu i ciągłej zależności potwierdza zasadność zaproponowanego modelu termomechanicznego. Na koniec omawia się sens fizyczny nieprzydatności stosowanych metod dowodu w przypadku, gdy $t > T^*$.

Обобщенная задача Стефана возникает при применении термомеханической теории для сред с поверхностью раздела (А. Романо, [10, 11]) для моделирования „одномерной" системы, в которой происходит фазовый переход, при котором нельзя пренебречь разницей плотности отдельных фаз. Эта задача решается итерационным способом, который позволяет построить решения в соответствующем интервале времени. Подвижная граница представляется затем в интегральном виде, что позволяет одновременно констатировать единственность решения. Многократное применение итерационного метода позволяет нам построить решения справедливые только к моменту T^*, т.е. к моменту, когда одна из фаз исчезает. Методика доказательства оказывается особенно пригодной для численных расчетов и дается применить к более общим проблемам. Затем представляется теорема о глобальной, непрерывной зависимости решения от коэффициентов и начальных данных. Доказанная таким образом теорема существования и о непрерывной зависимости подтверждает обоснованность предложенной термомеханической модели. Наконец обсуждается физический смысл непригодности применяемых методов доказательства в случае, когда $t > T^*$.

Conventions

IN THIS PAPER we will use the following notations:

1) the symbol $L^i u[L^{*i}u]$ will mean $u_{,xx}-k^i u_{,t}[u_{,xx}+k^i u_{,t}]$, where u is a real-valued function of the real variables x, t, $u_{,x}$, $u_{,t}$ represents the partial derivatives with respect to x and t variables, and k^i are real positive constants $[i = 1, 2]$;

2) $D_{s,r,T}$ is the set of couples $\{x, t: 0 \leqslant t \leqslant T, s(t) < x < r(t)\}$, where $s(t)$, $r(t) \in C^0[[0, T]] \cap C^1[(0, T)]$. If $s \leqslant r$, we will use D^1 instead of $D_{x=0,s,T}$ and D^2 instead of $D_{s,r,T}$;

3) if $f(\bar{x})$ is a real-valued function defined in a compact region K, then $||f||$ indicates $\underset{\bar{x}\in K}{\text{l.u.b.}} |f(\bar{x})|$;

4) if χ is a real function of the two variables x, t, χ^* is defined as $L^*\chi$;

5) Ω is the quarter plane $x > 0$, $t > 0$;

6) let us consider the real-valued function $s(t)$ defined in $[0, T]$ whose range $S \equiv s([0, T]) \subset |R^+$ (set of positive real numbers) together with the function $\bar{r}: S \times [0, T] \to |R$. We define:

i) $$a(T, s, \bar{r}) \equiv \underset{t\in[0,T]}{g.l.b.}\{\min\{s(t), \bar{r}(s(t), t)-s(t)\}\};$$

ii) $$A(T, s, \bar{r}) \equiv \underset{t\in[0,T]}{l.u.b.}\{\max\{|\dot{s}(t)|, |\dot{\bar{r}}(s(t), t)|\}\};$$

when it does not cause misunderstanding, we will use the notations a_S and A_S;

7) the symbol $\bar{p}$ will denote the nt-ple $[Q, H, \Phi, b, X, \bar{R}, M, k^i]$;

8) the symbol d will denote the mt-ple of functions $[k^i, b^i, h^i, \phi^i, \chi^i, q^i, \mu, \bar{r}]$;

9) the symbol $\Delta(d_1, d_2)$ denotes the sum of Δ listed in [1] III p. 9, where the integrals are calculated in the region $R \times [0, T]$ instead of in $[0, 1] \times [0, T]$, when double, and in R instead of $[0, 1]$, when simple, plus

$$\Delta\bar{r} \equiv \underset{(x,t)\in R\times[0,T]}{\text{l.u.b.}} \{\max\{|\bar{r}_1-\bar{r}_2|, |\bar{r}_{1,x}-\bar{r}_{2,x}|, |\bar{r}_{1,t}-\bar{r}_{2,t}|\}\};$$

10) $I_1 \equiv [0, s(0)]$, $I_2 \equiv [s(0), r(0)]$;

11) We mean by the symbol □ that Proofs are completed.

1. Introduction. Physical motivations of the proposed mathematical model for solid-liquid phase transition; an outline of the following sections

1.1. Remarks on references

In the present paper a Stefan-problem (called G-problem in the following) is formulated and analysed. It is a generalization of the one dealt with by Fasano–Primicerio [1,III] since it takes into account the density difference between the two phases. The proposed form of the problem is here deduced (as done in [7]) from a general thermomechanical

theory (see [2, 3, 8, 9, 10, 11]) (named in the whole paper TMT) when it is possible to neglect:

i) the influence which mechanical phenomena have on thermal ones;

ii) the thermomechanical action exerted by the interface between the phases.

We will show the theorem existence, uniqueness and continuous dependence on the data and coefficients for the solution of the G-problem, by means of an iterative process, taking into account the mobility of the second boundary, analogous to the one used in [1]. When the function $\bar{r}(s)/s$([1]) is one to one, the posed problem can be reduced to the one solved in [12], when a suitable variable change is performed, which makes the second boundary fixed. Nevertheless, the techniques used in [12], do not seem immediately applicable for those more general problems in which the mobility of the second boundary plays an essential role.

Such types of problems actually can arise, for instance in the framework of the quoted TMT. In conclusion, the main features of the demonstrative techniques here developed are the following:

a) they seem to be easily extensible to further generalization of the Stefan-problem,

b) by using them it is possible to show the continuous dependence on the coefficients in the Stefan condition of the solution of the problem to be considered,

c) they allow, in numerical applications, to use that qualitative information about the solution, which can be made available by considering the problem from a physical point of view.

In fact, as the first term in the iterative sequence whose limit is the moving boundary can be chosen in a large range of functions, it is possible to make this sequence converge more quickly.

d) they are applicable under very general hypotheses for the function $r(s, t)$. With regard to such techniques, we underline that they resemble those used by CANNON–HILL [4] and RUBISTEIN [6] in other situations. Finally, continuous dependence is explicitly shown in the whole existence interval of the solution.

1.2. Physical deduction of the mathematical problem posed in Sect.2

Let us now deduce the G-problem by means of TMT when the hypotheses i) and ii) are verified. The system of differential equations and jump conditions, whose solution gives the evolution of a thermomechanical system with an interface not carrying thermomechanical properties, becomes, in the unidimensional case, the following([2]):

$$\frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial x}(\varrho v) = 0, \tag{1.1}$$

$$\varrho \frac{\partial v}{\partial t} + \varrho v \frac{\partial v}{\partial x} = -\frac{\partial p}{\partial x}, \tag{1.2}$$

([1]) The previously mentioned function $\bar{r}$ mathematically describes the physical circumstance that the position of the second moving boundary is determined by that of the first.

([2]) The more general system of equations governing the evolution of an arbitrary thermomechanical system is deduced in ROMANO and others [3, 10, 11] and in KOSIŃSKI [8, 9].

$$\varrho\frac{\partial\varepsilon}{\partial t}+\varrho v\frac{\partial\varepsilon}{\partial x}=-p\frac{\partial v}{\partial x}+\frac{\partial h}{\partial x}, \tag{1.3}$$

$$[\![\varrho(v-c)]\!]=0 \tag{1.4}$$

the interface does not absorb mass,

$$[\![\theta]\!]=0 \tag{1.5}$$

thermal resistivity of the interface is negligible,

$$[\![\varrho v(v-c)+p]\!]=0 \tag{1.6}$$

the interface does not exert stress,

$$\left[\!\!\left[\varrho\left(\varepsilon+\frac{1}{2}v^2\right)(v-c)+pv-h\right]\!\!\right]=0 \tag{1.7}$$

the interface does not absorb or produce energy,

$$[\![\varrho\eta(v-c)-h/\theta]\!]=0 \tag{1.8}$$

there is no dissipation at the interface.

In the previous equations we used the following notations: ϱ is the mass density of bulk materials, v their velocity, p their pressure (in one-dimensional problems the stress tensor is always determined by just one component), ε and η, respectively, their internal mass density energy and entropy, h heat flux, c the speed of the interface between the phases, θ the temperature. Finally the symbol $[\![f]\!]$ means f^+-f^-, where $f^\pm$ is the limit from the $\pm$ side of the arbitrary function f.

We observe that in order to assure the consistence of the assumption that entropy inequality is valid as an equality for every process and the interface does not carry thermo-mechanical properties, with the approach developed in [11], the coefficients appearing in the constitutive equations there postulated have to assume some particular values. Subtracting Eq. (1.7) from Eq. (1.6), multiplied by c, we obtain

$$\left[\!\!\left[\varrho\left(\frac{1}{2}(v-c)^2+\varepsilon\right)(v-c)+p(v-c)-h\right]\!\!\right]=0. \tag{1.9}$$

Treating in the same way (1.8), it results

$$\left[\!\!\left[\varrho\left(\psi+\frac{1}{2}(v-c)^2\right)(v-c)+p(v-c)\right]\!\!\right]=0, \tag{1.10}$$

where $\psi\equiv\varepsilon-\theta\eta$.

Using Eq. (1.4), the formula (1.10) becomes

$$[\![g]\!]+\left[\!\!\left[\frac{2}{1}(v-c)^2\right]\!\!\right]=0, \tag{1.11}$$

where the function $g=\psi+p/\varrho$ coincides, in the case of fluid phases with the limit at the interface of Gibbs' potential. In what follows we will suppose that the densities of both phases are constant during the evolution of the system (incompressible materials). With this hypothesis the previous system of bulk equations becomes

$$\frac{\partial v}{\partial x}=0, \tag{1.12}$$

$$\varrho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial x}, \tag{1.13}$$

$$\varrho \frac{\partial \varepsilon}{\partial t} + \varrho v \frac{\partial \varepsilon}{\partial x} = \frac{\partial h}{\partial x}. \tag{1.14}$$

Equation (1.12) obviously implies that in both phases $v = v(t)$. We remark here that the system of equations (1.12), (1.13), (1.14) suitably describes, for example, solid-liquid phase transition. In what follows we will refer only to this physical case.

If we impose as a boundary condition for the solid phase that its free boundary is fixed, we obtain $v_s = 0$. Formula (1.4) then becomes

$$v_l = c \frac{\Delta\varrho}{\varrho_l}, \quad \text{where} \quad \Delta\varrho \equiv \varrho_l - \varrho_s. \tag{1.15}$$

Moreover, $p_s = p_s(t)$.

Formulae (1.13), (1.15) allow us to express p_l in terms of c and the externally applied pressure p_e. We obtain

$$p_l = \dot{c}\Delta\varrho\big(r(t)-x\big)p_e, \tag{1.16}$$

where $r(t)$ is the position of the free boundary of the liquid phase and is a given function of the position of the solid-liquid interface $s(t)$. We finally have:

in the solid phase

$$p_s = p_s(t), \quad v_s = 0,$$

$$\varrho_s C_s \frac{\partial \theta}{\partial t} = K_s \frac{\partial^2 \theta}{\partial x^2};$$

in the liquid phase

$$v_l = \frac{\Delta\varrho}{\varrho_l} c,$$

$$p_l = \dot{c}\Delta\varrho(r(t)-x)+p_e,$$

$$\varrho_l C_l \frac{\partial \theta}{\partial t} + C_l \Delta\varrho c \frac{\partial \theta}{\partial x} = K_l \frac{\partial^2 \theta}{\partial x^2},$$

at the interface

$$[\![p]\!] = -\Delta\varrho \frac{\varrho_s}{\varrho_l} c^2,$$

$$[\![g]\!] = \frac{1}{2} \frac{\Delta\varrho}{\varrho_l} \left(\frac{\varrho_s + \varrho_l}{\varrho_l} \right) c^2,$$

$$[\![\theta]\!] = 0,$$

$$[\![\eta]\!] \varrho_s \theta c = \left[\!\!\left[K \frac{\partial \theta}{\partial x} \right]\!\!\right].$$

In the previous equations we have assumed $\varepsilon = C\theta$ and $h = K \frac{\partial \theta}{\partial x}$. Moreover, as to the previous evolution equations we remark, in the first place, that it is experimentally known that the quantity $[\eta]$ is largely independent of temperature. In the second place, at least for metals, the ratio $\Delta\varrho/\varrho \ll 1$. When it can be neglected, the melting temperature

is constant and the evolution of the system is determined if the solution of the classical Stefan-problem with external boundary fixed ($v_1 = 0$) is known. On the other hand, if we want to take into account this difference, what is sometimes appropriate in dealing with metallurgy (see, for instance, [7]), we still can suppose that melting temperature is given. In fact, in order to obtain both in the liquid and at the interface pressure differences comparable with respect to the external one, the melting velocity should be much larger than is usually found ($c \simeq 10$ m/s in order to have $\Delta p \simeq 1$ atm). So the jump conditions found in the previous approximation can still be assumed to be valid, although the velocity in the liquid phase is not neglected. These physical considerations justify the position of the G-problem (Sect. 3), where the further approximation is made by neglecting, in the energy equation for the liquid phase, the term

$$c\Delta\varrho C_1 \frac{\partial\theta}{\partial x}.$$

Dealing with the more general case in which this last term is retained leads only to formal complications. In fact, the iterative method used in the following still applies with the only difference that the source function, occurring in the integral representations, concerns a more general parabolic equation, having the same kind of singularity than that actually treated.

1.3. An outline of the following sections

In Sect. 3 the problem is formulated from the mathematical point of view. Moreover, a list of the assumptions necessary for the proof is shown. Some of them have an obvious physical interpretation (for example, in the case of phase transition, hypothesis G5 expresses the finiteness of total mass involved in the process), while others have just a mathematical character. The last ones could be weakened as done in Fasano–Primicerio [1] II. In Sect. 4 some properties of solutions of heat equation in moving boundary regions are shown and the equivalence between the Stefan condition 5) and an integral relation is proved.

In Sect. 5 the iterative process which generates a solution of the G-Problem is defined. The convergence of the sequence thus obtained is then proved in a suitably small time interval. The integral relation obtained in Sect. 4 is used for showing that the uniform limit of the approximating sequence actually is a solution of the G-Problem.

In Sect. 6 an inequality is exhibited, which immediately implies the uniqueness of the solution in its definition interval. Next, it is shown that the repeated application of the iterative process defined in Sect. 5 actually allows the construction of the solution to the G-Problem up to the eventual time when one of the phases completely disappears or the upper limit of the time-derivative of the function giving the interface position is not finite.

In Sect. 7 a continuous dependence theorem for the solution of the G-Problem is shown, using the inequality in Sect. 6 and the properties of the sequence defined in Sect. 5.

In the section "Conclusions" physically relevant observations are made about the obtained mathematical results.

2. Formulation of the problem

Let us consider the following generalization of the Stefan-problem (G-problem):

a triple of real-valued functions $\{s(t), u^1(x, t), u^2(x, t)\}$ will be called a solution of he G-Problem in the $[0, T]$ interval if:

1) $s(t) \in \{C^1((0, T)) \cap C^0([0, T])\}$,
$s(0) = b > 0$,
$s(t) > 0 \quad \forall t \in [0, T)$;

2) $u^i(x, t)$ is defined and continuous in D_i
$u^i_{,x}(x, t) \in C^0(\bar{D}_i)$,
$u^i_{,xx}$ and $u^i_{,t} \in C^0(D_i)$, where $i = 1, 2$ and
$r(t) \equiv \bar{r}(s(t), t) > s(t)$

with $\bar{r}(x, t)$ a given real-valued function defined in $\bar{\Omega}$;

3) $L^i u^i = q^i(x, t)$ in D_i,
$u^1(0, t) = \phi^1(t), \quad u^2(r(t), t) = \phi^2(t) \quad \forall t \in [0, T]$,
$u^i(x, 0) = h^i(x) \quad \forall x \in I_i$,

where q^i, ϕ^i, h^i are given real-valued functions.

4) $u^1(s(t), t) = u^2(s(t), t) = 0 \quad \forall t \in (0, T]$,

5) $(\chi^1 u^1_{,x} - \chi^2 u^2_{,x})(s(t), t) = \dot{s}(t) + \mu(s(t), t) \quad \forall t \in (0, T]$,

where $\chi^i(x, t)$, $\mu(x, t)$ are given real-valued functions. In order to show the existence, uniqueness and continuous dependence theorem for the solutions of the G-problem, we will need the assumptions listed in the following section.

3. List of assumptions

G1) $q^i(x, t)$ are locally Hölder-continuous in $\bar{\Omega}$ with respect to x and t and $|q^i(x, t)| \leqslant \leqslant Q \cdot k_i$ in $\bar{\Omega}$.

G2) $\phi^i(t)$ are continuous in $|R^+$ and $|\phi^i| < \Phi \, \forall t \in |R^+$.

G3) $h^i(x)$ are continuous in I_i and $|h^1| \leqslant H(b-x)$, $|h^2| \leqslant H(x-b)$ where $Hb \geqslant \Phi$ and $H(\bar{r}(b, 0)-b) \geqslant \Phi$.

G4) $\chi^i, \chi^i_{,x}, \chi^i_{,xx}, \chi^i_{,t} \in C^0(\Omega)$ and
$$|\chi^i(x, t)| \leqslant X, \quad |\chi^i_{,x}(x, t)| \leqslant X \quad \forall (x, t) \in \Omega.$$

G5) $\bar{r}(x, t) \in C^2[\bar{\Omega}]$. Moreover we suppose that: $\bar{r}(x, t) \in \text{GTIF} \subset C^2[\bar{\Omega}]$, where the set GTIF is defined in what follows. To this aim, let us consider $\Gamma_t \equiv \{x: \bar{r}(x, t) \geqslant x\}$ and Γ^1_t the connected subset of Γ_t containing b (eventually empty). The function $\bar{r}$ belongs to GTIF if and only if the following conditions (3) are satisfied:

(3) The physical meaning of these assumptions is clear when we regard the function $s(t)$ as giving the position of the interface between solid and liquid phases, the point 0 the fixed extreme of the solid phase and $r(t)$ the freely moving extreme of the liquid phase (We refer to the introduction and p. 265 [13]).

G5)$_1$ $\bar{r}(b, 0) > b$ so that it results $\Gamma_0^1 \neq \phi$,

G5)$_2$ $0 \in \Gamma_t^1 \subset R$ if $t \in [0, T]$, where R is a given finite interval independent of t.

G5)$_3$ $\bar{r}$, and its derivatives, is bounded in $R \times [0, T]$ by the constant $\bar{R}$.

G6) μ is a Lipschitz-continuous, bounded function in every bounded subset of $\bar{\Omega}$. Its Lipschitz constant in R (def. in G5)$_2$) is M. M can be chosen so that it results l.u.b. $|\mu| < M$.
in R

4. Some useful results from heat-equation theory. An integral relation equivalent to 5), Sect. 3

LEMMA 1. An integral relation equivalent to 5) Sect. 3.

Under the assumption G1) ... G6), the triple $[s(t), u^i(x, t)]$ satisfying the conditions 1) ... 4) of the G-problem and $a(s, T) > 0$, verifies the condition 5) if and only if the following integral relation is verified:

$$s(t) - b = L_a(s(t), \bar{r}, h^i, \chi^i, q^i, \phi^i, u^i, \mu, b, t) \quad \forall\, t \in [0, T], \tag{4.1}$$

where the L_a-operator is given by the following integral expression:

$$\begin{aligned}
L_a(\ldots\ldots) \equiv & \iint_{D_{x=0,\, r(t),\, t}} d\xi\, d\tau (\chi q - u\chi^*) - \int_0 \mu(s(\tau), \tau) d\tau \\
& - \int_0^{r(t)} (ku\chi)(\xi, t) d\xi + 1/a \int_0^a (a-\xi)(k^1\chi^1 u^1)(\xi, 0) d\chi \\
& - 1/a \iint_{D_{x=0,\, x=a,\, t}} ((a-\xi)\chi^1 q^1 - u^1[(a-\xi)\chi^1_{\cdot xx} - 2\chi^1_{\cdot x} + (a-\xi)k^1\chi^1_{\cdot t}])(\xi, t) d\xi\, d\tau \\
& - 1/a \int_0^a (k^1(a-\xi)\chi^1 u^1)(\xi, t) d\xi - 1/a \int_0^t (\chi^1 u^1)(0, \tau) - (\chi^1 u^1)(a, \tau) d\tau \\
& - 1/a \int_0^t (\chi^2 u^2)(r(\tau), \tau) - (\chi^2 u^2)(r(\tau) - a, \tau) d\tau + 1/a \\
\times & \iint_{D_{r(\tau)-a,\, r(\tau),\, t}} (a - r(\tau) + \xi)\chi^2 q^2 - u^2[(a - r(\tau) + \xi)(k^2\chi^2_{\cdot t} + \chi^2_{\cdot xx}) + 2\chi^2_{\cdot x} - k^2\chi^2 \dot{r}(\tau)] d\xi\, d\tau \\
& + 1/a \int_{r(0)-a}^{r(0)} (a - r(0) + \xi)(k^2\chi^2 u^2)(\xi, 0) d\xi - 1/a \int_{r(t)-a}^{r(t)} (a - r(t) + \xi)(k^2\chi^2 u^2)(\xi, t) d\xi.
\end{aligned}$$

The relation (4.1) coincides with Eq. (3.3) in [1] III when $\bar{r} = 1$. We omit the proof of Lemma 1 because it is conceptually analogous to that used in order to prove Eq. (3.3) in [1] III. Moreover, its formally involved details can be found in [7].

LEMMA 2. *A priori* bounds for functions u^i verifying the conditions 1) — 4) Sect. 2.

Let us consider the functions $s(t)$, $u^i(x, t)$ verifying the conditions 1)–4) Sect. 2. Let the functions $v^i(x, t)$ be defined as the (unique) solution of the following problem

$$\begin{aligned}
&(4.2)_1 \quad L^i v^i = 0 \quad \text{in } D_i, \\
&\qquad v^i(x, 0) = H|x - b| \quad \forall\, x \in I_i,
\end{aligned}$$

$$v^1(0, t) = Hb \quad \forall t \in (0, T],$$
$$v^2(r(t), t) = H(r(0)-b) \quad \forall t \in (0, T],$$
$$v^1(s(t), t) = v^2(s(t), t) = -Qt \quad \forall t \in (0, T].$$

Under the hypotheses G1) ... G5), the following relations hold:

$$(4.2)_2 \quad |u^i(x, t)| \leqslant v^i(x, t)+Qt \quad \forall (x, t) \in \bar{D}_i,$$

$$(4.2)_3 \quad |u^i_{,x}(s(t), t)| \leqslant |v^i_{,x}(s(t), t)| \quad \forall t \in [0, T].$$

Indeed, the two couples of functions defined as

$$W^{i\pm} \equiv v^i+Qt \pm u^i \quad \text{in } D_i$$

are solutions of the following problems

a) $$L^iW^{i\pm} = L^iv^i+L^iu^i+L^i(Qt) = -k_iQ \pm q^i \leqslant 0 \quad \text{in } D_i,$$
$$W^{i\pm}(x, 0) = H|b-x| \pm h^i(x) \geqslant 0 \quad \forall x \in I_i,$$
$$W^{1\pm}(0, t) = Hb+Qt \pm \phi^1(t) \geqslant 0 \quad \forall t \in [0, T],$$
$$W^{2\pm}(r(t), t) = H(r(0)-b)+Qt \pm \phi^2(t) \geqslant 0 \quad \forall t \in [0, T],$$
$$W^{1\pm}(s(t), t) = W^{2\pm}(s(t), t) = 0 \quad \forall t \in [0, T].$$

The maximum principle for the heat equation implies

b) $$W^{i\pm}(x, t) \geqslant 0 \quad \text{in } D_1 \cup D_2,$$

from which $(4.2)_2$ is deduced.

By virtue of a), b) and the continuity of $v^i_{,x}$ and $u^i_{,x}$ up to the curve $(s(t), t)$ (see for example [6]), we obtain

$$W^{1\pm}_{,x}(s(t), t) \leqslant 0, \quad W^{2\pm}_{,x}(s(t), t) \geqslant 0$$

from which the problem $(4.2)_3$ follows. □

LEMMA 3. *A priori* bounds for solutions of the problem $(4.2)_1$.

Let v^i be the solution of the problem $(4.2)_1$. The following inequalities hold:

$$(4.3)_1 \quad |v^i_{,x}(s(t), t)| \leqslant h'(A_s, a_s, t, k^i, b, r(0), H, M, Q, \Phi, X),$$
$$|v^2_{,x}(r(t), t)| \leqslant h'(\cdots\cdots\cdots\cdots\cdots\cdots\cdots) \quad \forall t \in \Theta,$$

where $\Theta = \{t: t < a_s(t)/3A_s(t)\, t\}$ and h' is a regular function of the quoted arguments which is defined during the proof. We limit ourselves to the treatment of $v^2_{,x}$, as all the considerations concerning the function $v^1_{,x}$ are developed in [1].

Using an integral representation of $v^2(x, t)$ by means of suitable source functions, the following inequalities are obtained:

$$(4.3)_2 \quad |v^2_{,x}(s(t), t)| \leqslant D+C\int_0^t (t-\tau)^{-\frac{1}{2}}|v^2_{,x}(s(\tau), \tau)|\,d\tau$$
$$+E\int_0^t (t-\tau)^{-\frac{1}{2}}|v^2_{,x}(r(\tau), \tau)|d\tau \quad \forall t \in \Theta,$$

where

$$D \equiv 2/3\left(Hk^2+\frac{Qk^2t^{1/2}}{\sqrt{k^2\pi}}\right) \equiv \alpha+\beta t^{1/2},$$

$$C \equiv 2/3\,\frac{A}{4\sqrt{\pi}\,(k^2)^{3/2}},$$

$$E \equiv 2/3\left(\frac{16k^2}{ea^2}+\frac{3A}{a}\right)\frac{\bar{r}(b,0)-b+2At}{4\sqrt{\pi}\,(k^2)^{3/2}};$$

$$(4.3)_3 \qquad |v^2_{,x}(r(t),t)| \leqslant D+C\int_0^t (t-\tau)^{-\frac{1}{2}}|v^2_{,x}(r(\tau),\tau)|\,d\tau$$

$$+E\int_0^t (t-\tau)^{-\frac{1}{2}}|v^2_{,x}(s(\tau),\tau)|\,d\tau \qquad \forall\, t\in\Theta.$$

Summing the inequalities $(4.3)_2$ to $(4.3)_3$, after simple algebra we obtain

$$(4.3)_4 \qquad |v^2_{,x}(s(t),t)|+|v^2_{,x}(r(t),t)| \leqslant 2D+2(C+E)$$

$$\times\int_0^t (t-\tau)^{-\frac{1}{2}}\left(|v^2_{,x}\left(s(\tau),\tau\right)|+|v^2_{,x}(r(\tau),\tau)|\right)d\tau \qquad \forall\, t\in\Theta.$$

To the function $g(t) = |v^2_{,x}(s(t),t)|+|v^2_{,x}(r(t),t)|$ the following lemma, (due to T. H. Gronwall) quoted both in [5] and in [14], can now be applied:

LEMMA. If

$$g(t)\in C^0([0,T]) \quad \text{and} \quad g(t) \leqslant \gamma+\frac{\lambda}{\sqrt{\pi}}\int_0^t (t-\tau)^{-\frac{1}{2}}g(\tau)\,d\tau,$$

then

$$g(t) \leqslant \gamma\exp(\lambda^2 t)\left(1+2t^{\frac{1}{2}}\pi^{-\frac{1}{2}}\lambda\right) \qquad \forall\, t\in[0,T].$$

In our case we have $\gamma = 2D$ and $\lambda = 2\sqrt{\pi}[C+E]$. It is now necessary to introduce the following function G (the reasons which led us to define it in exactly this way will become clearer in the following section):

$$G(t,A,a,\bar{p}) \equiv M+(\alpha+\beta t^{1/2})\exp\left(\frac{\zeta^2 t(t+2)^2(1+A)^4}{a^4}\right)\left(1+\frac{2t^{1/2}\zeta(2+t)(1+A)^2}{a^2}\right)$$

where

$$\zeta \equiv \varepsilon\bar{R}C/A\,\Big(\varepsilon \text{ being defined as } \sup\Big\{3{,}32k^2/e,\ 3\left(\bar{r}(b,0)-b\right)$$

$$\equiv 3\delta(b),\ \left(\bar{r}(b,0)-b\right)32k^2/e\Big\}\Big)$$

and a is assumed to range in the interval $(0,\delta(b))$ with $\delta(b) < 1$. The last assumption is physically not relevant as it can always be fulfilled with a scale change.

With simple algebra it is possible to show that

$$\gamma\exp(\lambda^2 t)\left(1+2t^{\frac{1}{2}}\pi^{-\frac{1}{2}}\lambda\right) \leqslant h' \equiv (G-M)/2X,$$

which completes the proof of the inequalities $(4.3)_1$. □

It is possible to find an upper bound for $v^i_{,x}(s(t), t)$ appearing in Lemma 3 which depends only on the dimensions of D_i and $\bar{p}$.

In order to enunciate the following Lemma 4, we investigate the properties of the function G(...) (for more details see [7]). It can easily be seen that it verifies the following:

PROPERTY G

$\forall\ a$ and $\bar{p}$, a unique $T_0 > 0$ exists such that the equation $G(t, A, a, \bar{p}) = A$ admits a unique solution $A_0 > 0$. The so-defined functions $A_0(a, \bar{p})$ and $T_0(a, \bar{p})$ are continuous and, moreover, T_0 is increasing with respect to a.

LEMMA 4

If $s(t)$ is such that $a_s(t) > \bar{a} > 0$, $|\dot{s}(t)| < A_0(\bar{a}, \bar{p})$ and $|\dot{r}(t)| < A_s(A_0(\bar{a}, \bar{p}))$, then the solutions of the problem $(4.2)_1$ v^i satisfy

$$|v^i_{,x}(x, t)| \leqslant \frac{A_0(\bar{a}, \bar{p}) - M}{2X} + H \equiv B_0 > 0. \tag{4.4}$$

Indeed, it is well known from the heat equation theory that the solutions of the problem $(4.2)_1$ $v^i \in C^3\,[D_i] \cap C^1[\bar{D}_i]$. This implies that

$$L^i v^i_{,x} = 0 \quad \text{in } D_i \quad \text{and} \quad v^i_{,x} \in C^0(\bar{D}_i).$$

Under the hypotheses quoted in the next section, it is shown that

$$h'(\cdots\cdots\cdots) \leqslant \frac{A_0(\bar{a}, \bar{p}) - M}{2X}.$$

Using the inequalities $(4.3)_1$ and the maximum principle, we obtain the inequality $(4.4)_1$. □

LEMMA 5. *A priori* bounds for functions verifying the conditions 1) —4), Sect. 2.

Let us consider the triple $[s, u^i]$ verifying the conditions 1) ... 4), Sect. 2. Under the hypotheses G1)–G5) for data, and those established in Lemma 4 for the function $s(t)$, the following inequalities hold:

$$\begin{aligned} &|u^1(x, t)| \leqslant B_0|x - s(t)| \quad \forall\,(x, t) \in \bar{D}_1, \\ &|u^2(x, t) - u^2(r(t), t)| \leqslant B_0'|x - r(t)| \quad \forall\,(x, t) \in \bar{D}_2, \end{aligned} \tag{4.5}$$

where B_0 and B_0' are given functions of $\bar{a}$ and $\bar{p}$.

Proof

$v^i(x, t) = v^i(s(t), t) + v^i_{,x}(\theta_{xt})(x - s(t))\ \forall\,(x, t) \in D_i$ and $\theta_{xt} \in [x, s(t)] \cup [s(t), x]$.

Using $(4.2)_2$ we obtain:

$$|u^i(x, t)| \leqslant v^i(s(t), t) + Qt + |v^i_{,x}(\theta_{xt})(x - s(t))| \leqslant B_0|x - s(t)|$$

which is $(4.5)_1$, if $i = 1$.

In order to show the inequality $(4.5)_2$, we recall that, if $u(x, t)$ is the unique solution of the following problem,

$$\begin{aligned} L^2 u &= q^2(x, t) && \forall\,(x, t) \in D_{s, r, T}, \\ u(x, 0) &= h^2(x) && \forall\, x \in [b, r(0)], \\ u(s(t), t) &= 0 \quad u(r(t), t) = \phi^2(t) && \forall\, t \in (0, \bar{T}], \\ &\text{(or, respectively,} \quad u(r(t), t) = 0, \quad u(s(t), t) = \phi^2(t)) \end{aligned}$$

then (if the hypotheses of Lemma 4 are satisfied by $s(t)$) the relation

$$|u(x,t)| \leqslant B_0(\bar{a},\bar{p})|x-s(t)| \quad \text{(respectively } |u(x,t)| \leqslant B_0(\bar{a},\bar{p})|x-r(t)|\text{)}$$

holds. Let us define, in $D_{s,r,\bar{T}}$, the function $\bar{u} \equiv u-\phi^2(t)$. It results

$$L^2\bar{u} = q^2-\dot{\phi}^2, \quad \bar{u}(x,0) = h^2(x)-\phi^2(0),$$
$$\bar{u}(s(t),t) = -\phi^2(t), \quad \bar{u}(r(t),t) = 0.$$

Applying to $\bar{u}$ the previous result, we have

$$|\bar{u}(x,t)| \leqslant B_0(\bar{a},\bar{p}')|x-r(t)|$$

where $\bar{p}'$ is known in terms of $\bar{p}$. The preceding expression is just $(4.5)_1$. □

5. A local existence theorem for the solution of the G-problem

This theorem is proved by defining an approximating sequence of triples $[s_k(t), u_k^i(x,t)]$, with domains $[0, T_k]$, $D_{x=0,s_k,T_k}$, D_{s_k,r_k,T_k}, respectively, in a completely similar way as in [1], and by showing the convergence of this sequence to a solution. A number of problems, of course, arise from the eventual variability of the $\bar{r}$ function.

DEFINITION 1. *Let us consider a sequence b_k of positive real numbers converging to b. Let us choose a positive T_1 (eventually infinite) and $s_1(t) \equiv \lambda(t)$ where $\lambda(t)$ is an arbitratry real function defined in $[0, T_1]$ whose range is included in $\mathbb{R}^+$ and is such that $\lambda(0) = b_1$.*

Once defined, by induction, $s_k(t)$ and T_k, u_k^i are the unique functions satisfying the conditions 2)–4) Sect. 2 in the regions $D_{x=0,s_k,T_k}$ and D_{s_k,r_k,T_k}, while T_{k+1} and s_{k+1} are determined as follows:

$$s_{k+1}(0) = b_{k+1}$$
$$\dot{s}_{k+1}(t) = (\chi^1 u^1_{k,x})(s_k(t),t)-(\chi^2 u^2_{k,x})(s_k(t),t)-\mu(s_k(t),t)$$
$$\text{if} \quad T(k) \equiv \{t \in R^+ : s_{k+1}(t) = 0 \quad \text{or} \quad s_{k+1}(t) = r_{k+1}(t)\}$$
$$T_{k+1} = \min\{\min T(k), T_k\}.$$

We emphasize that it will result that both the function $\lambda(t)$ and the sequence b_k have to satysfy suitable conditions if the uniform convergence of the sequence s_k must be assured. These conditions are specified in the hypotheses to Theorem 1, Sect. 5, and in the Remark preceding it. We anticipate that they are not very stringent. This means that the information we eventually have about the motion of the free boundary can be employed in shaping suitably the first term of the built sequence. If our choice of this first term is well-grounded, this sequence will efficaciously approximate the searched function $s(t)$, in the sense that it will quickly converge. We conclude by noting that this "qualified" freedom of choice we happen to have is not at all something new. It is very strictly linked to the convergence of the sequence defined in the Banach–Caccioppoli's Theorem to the unique fixed point of a contraction in a complete metric space independently of the choice of its first term. In fact it seems to us that a more complete treatment of the problem under consideration will make us recognize that the results we are going to obtain are in fact available by applying

the quoted Theorem. In order to prove the uniform convergence of the sequence s_k in a suitable interval $[0, \bar{T}]$, we need the following:

LEMMA 6. Let us define

$$A_k(t) \equiv \max_{\tau \in [0,t]} \{|\dot{s}_k(\tau)|, |\dot{r}_k(\tau)|\}, \qquad a_k(t) \equiv \min_{\tau \in [0,t]} \{s_k(\tau), r_k(\tau) - s_k(\tau)\}.$$

Then

$$|\dot{s}_k(t)| \leqslant G(t, A_k, a_k, \bar{p}) \quad \text{in } \Theta \quad \forall k \in N,$$

where both G and Θ were defined in Sect. 4. In the course of the proof (not given here) that it is analogous to that given in [1] III concerning Lemma 6, Sect. 6, the function G naturally arises when we take into account the definition of the sequence s_k. For more detail about the calculation see Appendix G1, G2, G3 in [7].

Let us now reconsider the sequence $[s_k, u_k^i]$ and define

$$(5.1) \qquad \mathbf{a} \equiv \min\{b/2, (\bar{r}(0)-b)/2, 1\}, \qquad \bar{T} \equiv \min\{T_0(\mathbf{a}, \bar{p}), T_1, \mathbf{a}/3A(\mathbf{a}, \bar{p})\},$$

$$A(\mathbf{a}, \bar{p}) \equiv \max\{A_0(\mathbf{a}, \bar{p}), A_{\bar{r}}(A_0(\mathbf{a}, \bar{p}))\}.$$

Before proving by induction the following Theorem we remark that in what follows we assume the terms of the sequence $\{b_k\}$ to be equal to b (defined in 1) Sect. 3. This is done in order to make the presentation less formally involved. In fact it may be shown by simple reasonings (4) that all the following statements equally hold if we assume that:

The series whose general term is

$$(5.2) \qquad |b_{k+1} - b_k| \quad \text{converges} \quad \text{and} \quad \forall k \; b_k \in [\varepsilon, \bar{r}(b, 0) - \varepsilon], \quad \text{where } \varepsilon > 0.$$

THEOREM 1. *Let us assume that the following condition is verified*:

$$(5.3) \qquad |\dot{s}_1(t)| \leqslant A_0(\mathbf{a}, \bar{p}) \quad \forall t \in [0, \bar{T}], \; (\textit{and, as a consequence, } a_1(\bar{T}) \geqslant \mathbf{a}),$$

then

$$|\dot{s}_{k+1}(t)| \leqslant A_0(\mathbf{a}, \bar{p}) \quad \forall k \in N \quad \textit{and} \quad \forall t \in [0, \bar{T}]$$

$$(\textit{which implies} \quad a_{k+1}(\bar{T}) \geqslant \mathbf{a} \quad \textit{and} \quad T_{k+1} \geqslant \bar{T}).$$

Indeed, if $|\dot{s}_k(t)| \leqslant A_0(a) \; \forall t \in [0, \bar{T}]$, then the following relations hold:

$$\mathbf{a} \leqslant (\bar{r}(0) - b)/2 \leqslant \bar{r}(0) - b - 2A(\mathbf{a})t \leqslant a_k(t)$$

rom which, remembering the preceding Lemma 6,

$$|\dot{s}_{k+1}(t)| \leqslant G(A_k, a_k, t, \bar{p}) \overset{1}{\leqslant} G(A_0(\mathbf{a}, \bar{p}), a_k, \bar{T}, \bar{p}) \overset{2}{\leqslant} G(A_0(\mathbf{a}, \bar{p}), \mathbf{a}, T_0(\mathbf{a}, \bar{p}), \bar{p}) \overset{3}{=}$$

$$\overset{3}{=} A_0(\mathbf{a}, \bar{p})$$

what completes the proof.

The inequalities *1*, *2* hold because G is an increasing function of A, t, and a decreasing one of a. The equality *3* is true by virtue of the definition of A_0 and T_0. □

(4) It will obviously be necessary to change slightly the definition (5.1).

We are now ready to show:

THEOREM 2. *If both conditions* (5.2), (5.3) *are fulfilled, we have that the sequences* s_k *and* $\dot{s}_k$ *are uniformly convergent in the* $[0, \bar{T}^*(a, \bar{p})]$ *interval, where* $\bar{T}^*(a, \bar{p})$ *is a continuous function, defined in the course of the proof.* (This theorem is a generalization of Theorem 1, [1], I, p. 704).

After simple algebra, using the recursive definition of sequence s_k, we obtain

$$(5.4)_1 \qquad \dot{\delta}_{k+1} \leqslant \Sigma_{i=1,2}\{|u^i_{k,x}(s_k(\tau), \tau)\Delta_k\chi^i| + |\chi^i(s_{k-1}(\tau), \tau)\Delta_k u^i_{,x}|\} + \Delta_k\mu \qquad \forall\, \tau \in [0, T_{k+1}],$$

where

$$\delta_{k+1} \equiv |s_{k+1} - s_k|, \qquad \dot{\delta}_{k+1} \equiv |\dot{s}_{k+1} - \dot{s}_k|,$$
$$\Delta_k\mu \equiv \mu(s_k(\tau), \tau) - \mu(s_{k-1}(\tau), \tau),$$
$$\Delta_k\chi^i \equiv \chi^i(s_k(\tau), \tau) - \chi^i(s_{k-1}(\tau), \tau),$$
$$\Delta_k u^i_{,x} \equiv u^i_{k,x}(s_k(\tau), \tau) - u^i_{k-1,x}(s_{k-1}(\tau), \tau).$$

If we define $\forall\, k \in N$, v^i_k as the solution of the problem $(4.2)_1$ stated for the curve s_k with the same initial and boundary data $\forall\, k \in N$, then we obtain the following inequalities:

$$(5.4)_2 \qquad \begin{aligned} &|u^i_{k,x}(s_k(t), t)| \leqslant |v^i_{k,x}(s_k(t), t)| \leqslant B_0, \\ &|u^i_k(x, t)| \leqslant B_0|s_k(t) - x| \\ &|u^2_k(x, t)| \leqslant B_0'|r_k(t) - x| \qquad\qquad \forall\,(x, t) \in \begin{cases} \text{if } i = 1\ D_{x=0,\, s_k,\, \bar{T}}, \\ \text{if } i = 2\ D_{s_k,\, r_k,\, \bar{T}}, \end{cases} \\ &|v^2_{k,x}(r_k(t), t)| \leqslant B_0' \qquad\qquad \forall\, k \in N,\ t \in [0, \bar{T}]. \end{aligned}$$

It is useful to remember that B_0 and B_0' are just functions of the variables $\bar{p}$, a and are, therefore, independent of k. Using the inequalities $(5.4)_2$ the inequality $(5.4)_1$ becomes

$$(5.4)_3 \qquad \dot{\delta}_{k+1}(\tau) \leqslant 2XB_0\,\delta_k(\tau) + M\delta_k(\tau) + X|\Delta_k u^1_{,x}| + X|\Delta_k u^2_{,x}| \quad \forall\, \tau \in [0, \bar{T}].$$

Representing u^i by means of the Green formula, it is possible to show that the following inequalities hold (see [7]):

$$(5.4)_4 \qquad |\Delta_k u^i_{,x}| \leqslant L\left(t^{-\frac{1}{2}}, t^{\frac{1}{2}}, t^2, t\right)K(\mathbf{a}, \bar{p}, T^*)\|\delta_k\| \qquad t \in [0, T^*].$$

Here T^* is chosen during the necessary calculations, depends just on $\mathbf{a}$ and $\bar{p}$ and is bounded by $\bar{T}$; L is a linear function whose coefficients are positive; K is a given function. (The assertion is obtained using results in the potential theory for the heat equation).

Substituting for $|\Delta_k u^i_{,x}|$ in the inequality $(5.4)_3$, we obtain

$$(5.4)_5 \qquad \dot{\delta}_{k+1}(\tau) \leqslant (2XB_0 + M)\,\delta_k + 2LK\|\delta_k\| \qquad \forall\, \tau \in [0, T^*].$$

After integrating the inequality $(5.4)_5$ with respect to τ in the $[0, t]$ interval with $t \leqslant T^*$ and calculating the l.u.b. of both members of the formula thus obtained, it results

$$(5.4)_6 \qquad \|\delta_{k+1}\| \leqslant \|\delta_k\| F\left(T^{*\frac{1}{2}}, T^*, T^{*2}, T^{*3}\right) \quad \text{in} \quad [0, T^*],$$

where F is a linear function whose positive coefficients can easily be defined in terms of $L, K, \bar{p}, \mathbf{a}$.

Let us define

$$\bar{T}^* \equiv \min\{T^*, T_F\}$$

where T_F is such that $F(T_F) < 1$. We emphasize that $\bar{T}^*$ $(\mathbf{a}, \bar{p})$ is a continuous function of its arguments and is greater than zero if $\mathbf{a}$ is positive. We can conclude that

$$(5.4)_7 \qquad \|\delta_{k+1}\| < F(T_F)\|\delta_k\| \quad \text{in} \quad [0, T^*].$$

This obviously implies that, in $[0, \bar{T}^*]$, the series $\Sigma_k \delta_k$ is convergent. We remark that, by virtue of the inequality $(5.4)_5$ and the uniform convergence of the sequence s_k to a function s, it results that the sequence $\dot{s}_k$ also is uniformly convergent to $\dot{s}$, giving, moreover, that $a_s > \mathbf{a}$ and $|\dot{s}| \leqslant A_0$ in $[0, \bar{T}^*]$. This completes the proof. □

THEOREM 3. *Let s be the uniform limit of the sequence s_k and u^i the unique functions satisfying the conditions* 2), 3) 4) *in D^i. Let us define in $R\bar{\times}[0, T^*] = R_{\bar{T}^*}$ the following functions*:

$$\bar{u}_k \equiv \begin{cases} u_k^i & \text{in } D^i, \\ \phi^2 & \text{in } R_{\bar{T}^*} - UD^i, \end{cases}$$

$$\bar{u} \equiv \begin{cases} u^i & \text{in } D^i, \\ \phi^2 & \text{in } R_{\bar{T}^*} - UD^i. \end{cases}$$

The sequence $\bar{u}_k$ is uniformly convergent to $\bar{u}$ in the region $R_{\bar{T}^*}$. Moreover, the triple $[s(t), u^i(x, t)]$ is a solution of the G-problem in $[0, \bar{T}^*]$.

Let us set

$$w_k \equiv \bar{u} - \bar{u}_k \quad \text{in } R_{\bar{T}^*},$$

$$\sigma_k \equiv \min\{s, s_k\}, \qquad \varrho_k \equiv \min\{r, r_k\},$$

$$\Sigma_k \equiv \max\{s, s_k\}, \qquad R_k \equiv \max\{r, r_k\}.$$

Lemma 5 of Sect. 4 can be applied to the function $\bar{u}$ so that

$$|\bar{u}(x, t)| \leqslant B_0|x - s(t)| \quad \text{in } R_{\bar{T}^*},$$

$$|u^2 - \phi^2| \leqslant B_0'|x - r(t)| \quad \text{in } D^2.$$

We first prove that w_k is uniformly convergent to zero in $R_{\bar{T}^*}$. Indeed

$$w_k = 0 \quad \text{in} \quad \{R_{\bar{T}^*} - D_{x=0,\, R_k,\, \bar{T}^*}\} \cup \{x = 0\} \cup \{t = 0\},$$

$$|w_k(x, t)| \leqslant |\bar{u}| + |\bar{u}_k| \leqslant 2B_0|s(t) - s_k(t)| \quad \text{in } D_{\sigma_k,\, \Sigma_k,\, \bar{T}^*},$$

$$|w_k| \leqslant |\bar{u} - \phi^2| + |\bar{u}_k - \phi^2| \leqslant 2B_0|r - r_k| \leqslant 2B_0\bar{R}|s - s_k| \quad \text{in } D_{\varrho_k,\, R_k,\, \bar{T}^*}.$$

In the last two relations we used the formulae $(5.4)_2$. The uniform convergence of the sequence s_k and the maximum principle for the heat equation, when applied to the regions $D_{x=0,\, \sigma_k,\, \bar{T}^*}$ and $D_{\Sigma_k,\, \varrho_k,\, \bar{T}^*}$ imply the claimed uniform convergence of w_k. In order to show the second statement of the theorem, remember, by virtue of its definition, that the triple $[s, u^i]$ satisfies the conditions 1) ... 4). Condition 5) is then equivalent to the integral relation $(4.1)_1$, which we write here for convenience

$$(5.5)_1 \qquad s - b = L_a(s, u^i \ldots\ldots).$$

Using the recursive definition of s_k and the same reasonings leading to $(5.5)_1$, we can easily verify that the following relation is satisfied:

$$(5.5)_2 \qquad s_{k+1} - b = L_{\mathbf{a}}(s_k, u_k^i \ldots\ldots).$$

As it has been shown that s_k and u_k^i are uniformly convergent to s and u^i, the proof is completed by calculating the limit for $k \to \infty$ of the formula $(5.5)_2$ and remarking that it becomes the formula $(5.5)_1$.

6. Existence and uniqueness theorem in the large for the solution of the G-problem

THEOREM 4. *If* $[s_1, u_1^i]$ *and* $[s_2, u_2^i]$ *are solutions of the G-problem with the same initial, boundary data and coefficients in the* $[0, T]$ *interval, then they coincide.*

In fact, more generally, it is possible to obtain the following relation for any couple of solutions of the G-problem with different data and coefficients:

$$(6.1)_1 \qquad |s_1(t) - s_2(t)| \leqslant N(a, A, \bar{p}^1, \bar{p}^2)\{\Delta\phi + \Delta h + \Delta k + \Delta q + \Delta\chi + \Delta\mu + \Delta\bar{r}\} \quad \forall t \in [0, T]$$

with

$$0 < a < a_{s_i}, \qquad A_{s_i} < A < \infty.$$

The formula $(6.1)_1$ can be derived by: a) representing both s_i by means of the formula $(4.1)_1$; b) subtracting from each other the relations thus obtained. The result will be

$$(6.1)_2 \qquad s_1 - s_2 = L_a(1) - L_a(2);$$

c) equating the absolute value of both members of Eq. $(6.1)_2$ and bounding the second with an expression where suitable differences of data and coefficients appear, in addition to $|s_1 - s_2|$; to this aim we need to represent the u functions by means of suitable thermal potentials whose properties are very well-known: d) applying T. H. Gronwall's inequality to the obtained expression. [$g(t)$ being in this case $|s_1 - s_2|$]. Formula (6.1) obviously proves Theorem 4. □

DEFINITION 2. *Let* T^* *be defined as* l.u.b Λ, *where* $\Lambda \equiv \{t$: *in the* $[0, t]$ *interval there exists a solution to the G-problem*$\}$.

DEFINITION 3. *Let us define recursively the sequence* T_n, ζ_n

$$T_1 = \bar{T}^*(\mathbf{a}, \bar{p}) - \zeta_1, \qquad \zeta_1 < \bar{T}^*(\mathbf{a}, \bar{p})/2.$$

Let us suppose that in the $[0, T_n]$ *interval there exists the solution of the G-problem. We have chosen* $\zeta_n = (T_n - T_{n-1})/2^n$. *Let us consider the n-th G-problem with initial time* T_n, *boundary data and coefficients equal to those of the original G-problem and initial data equal to the value of* u^i *at the time* T_n.

Obviously, these new data satisfy the conditions G1) ... G5) *with appropriate coefficients* p_n.

We define

$$T_{n+1} \equiv \bar{T}_n - \zeta_n + T^*(a(T_n), \bar{p}_n) \quad \text{and} \quad T^{**} \equiv \text{l.u.b.}_{n \in N}\, T_n.$$

We note that this recursive definition is unambiguous because, for the $n+1$ th G-problem, the hypotheses G1) ... G5) are satisfied.

THEOREM 5. $T^{**} = T^*$.

By virtue of Definition 3, $T^{**} \in \Delta$ and therefore $T^{**} \leqslant T^*$. The proof is obvious if $T^{**} = \infty$. Let us show, under the hypothesis $T^{**} < \infty$, that an "ad absurdo" argumentation implies $T^* \leqslant T^{**}$.

If T^{**} were less than T^*, we would have $a(T^{**}) > 0$ [because of the statement of the G-problem and definition of T^*] and hence

$$\bar{T}^*\left(a(T^{**}), \bar{p}^{**}\right) = K > 0.$$

Moreover, because of the continuity of the $\bar{T}^*$ function, we can find an N such that $\forall\, n > N\, T^{**} < T_n + K/3$ and

$$\bar{T}^*\left(a(T_n), \bar{p}_n\right) + K/3 > \bar{T}^*\left(a(T^{**}), \bar{p}^{**}\right) = K.$$

We should have finally

$$T_{n+1} = T_n - \zeta_n + \bar{T}^*\left(a(T_n), \bar{p}_n\right) > T^{**} + 1/3K - \zeta_n \geqslant T^{**}(1 - 1/2^n) + 1/3K$$

which is absurd. This completes the proof. □

PROPOSITION 1. If $T^* < \infty$, then either

1) $\lim_{t \to T^*} s(t) = 0$, or

2) $\lim_{t \to T^*} r(t) - s(t) = 0$, or

3) $\lim_{t \to T^*} \{\text{l.u.b.}_{\tau \in [0,\, t]} |\dot{s}(\tau)|\} = \infty$.

When both 1) and 2) are not true, then either $\lim_{t \to T^*} s(t)$ exists or not. In the first case, if 3) were false, we could find a solution of the G-problem in an interval strictly including $[0, T^*]$, which is absurd. In the second case the following property holds:

$$\forall\, t < T^*, \forall\, k > 0\ \exists\, t_1, t_2 : t < t_i < T^* \quad \text{and} \quad |s(t_1) - s(t_2)| > k|t_1 - t_2|.$$

In our hypotheses this property obviously implies 3). □

7. Continuity theorem for the solution of the G-problem

We note that the formula $(6.1)_1$ does not mean any sort of continuity for the solution of the G-problem because of the dependence of the coefficient N on the variables a and A. Notwithstanding this circumstance, it is possible to use it in order to show the following

THEOREM 6. *Let us consider the unique solution of the G-problem with data*

$$d_1 \equiv (\phi^1, h^1, \ldots\ldots).$$

$\forall\, T < T_1^*, \forall\, \varepsilon > 0$ *it exists a* $\delta(T, \varepsilon, d_1)$ *such that if* $\Delta(d_1, d_2) < \delta$, *then the solution of the G-problem relative to* d_2 *exists up to* T *and* $\|s_1 - s_2\| < \varepsilon$ *in* $[0, T]$.

We emphasize that for classical results of the heat equation theory this last inequality implies that $\|\bar{u}_1 - \bar{u}_2\| < C\varepsilon$ in R_T (for definition of $\bar{u}_i$, see Sect. 5).

Indeed, it is possible to partition the $[0, T]$ interval into a finite number of subintervals $[T_i, T_{i+1}]$, defined in the same way as in Definition 3 Sect. 6. Let us define $\mathbf{a}_i = Ca_1(T_i)$, $0 < C < 1/2$.

LEMMA 7. For every sequence of ε_i satisfying the first inequality in iv), it is possible to determine a sequence δ_i such that if $T_i < T_2^*$ and $\Delta(d_1(T_i), d_2(T_i)) < \delta_i$, then:

i) s_2 is defined up to T_{i+1};

ii) $|s_1 - s_2| < \varepsilon_i$ in $[T_i, T_{i+1}]$;

iii) $\Delta(d_1(T_{i+1}), d_2(T_{i+1})) < \varepsilon_i$;

iv) $\mathbf{a}_{i+1} < a_1(T_{i+1}) - \varepsilon_i(1+\bar{R}) < a_2(T_{i+1})$.

i) is true by virtue of continuity of the $\bar{T}^*$ function and definition of the T_i sequence. In fact $T_{i+1} \equiv T_i + \bar{T}^*(a_1(T_i)/2, p_1^{-i}) - \zeta_i$ and a δ_i can be chosen such that

$$\bar{T}^*(a_2(T_i)/2, \bar{p}_2^i) > \bar{T}^*(a_1(T_i)/2, \bar{p}_1^i) - \zeta_i \quad \text{if} \quad \Delta(d_1(T_i), d_2(T_i)) < \delta_i.$$

In order to show ii), we note that if δ_i is such that $a_2(T_i) > 2\mathbf{a}_i$, the formula $(6.1)_1$ is valid even when in the function N, which appears in it, the a-variables assume the value $\mathbf{a}_i$. The previous statement is true because N is decreasing with respect to the same a-variables and the properties of $\bar{T}^*$ function assure that both $a_1(t_1)$ and $a_2(t)$ remain greater than $\mathbf{a}_i$ in $[T_i, T_{i+1}]$. The continuity of N and A_0 functions trivially implies the claimed statement. iii) can be deduced from ii) when recalling the well-known results of the heat equation theory. iv) is an obvious consequence of ii).

Let us fix $\varepsilon > 0$. We choose $0 < \varepsilon_i < \min\{\varepsilon, \delta_{i+1}(\varepsilon)\}$. Let us consider the sequence δ_i obtained in Lemma 2 using these ε_i and $\delta(\varepsilon) \equiv \min \delta_i$. It is easily verified that with this δ the statement of Theorem 1 is satisfied, which completes the proof.

8. Conclusions

In this section we want to observe what follows: i) Both the classical Stefan-problem and our G-problem can be easily and rigorously deduced in the framework of the Thermomechanical Theory proposed in [8–11] (TMT). This is done by applying the general theory to the case of incompressible phases shared by an interface not carrying thermomechanical properties when some physically reasonable hypotheses about the evolution of the considered system are satisfied. No such clear deduction (free from infinitesimal limits and similar reasonings) seems to us to have been available before. (See, for example, [13] p. 265 and [15] Chapt. 1, which is more updated and complete). ii) The circumstance that the existence, uniqueness and continuous dependence theorem can be shown for both ii)$_a$ the classical Stefan-problem, ii)$_b$ our G-problem (most likely even when the term considered in Remark 1, Subsect. 1.2 of the introduction, which we have neglected, is actually taken into account), is an important check of the well-posedness even of the more general theory. iii) Even if we proved a well-posedness theorem for a G-problem ([5]) only, it seems to us reasonable that slight changes in the demonstrated proof techniques will lead us to prove similar theorems for the following class of problems ([6]) iii)$_a$ those which take

([5]) We recall that the G-problem takes into account just the "minimum" motion not necessarily convective but necessary to produce phase transition when density difference between phases is appreciable (recall the set of evolution equations in Subsect. 1.2 of the Introduction).

([6]) which also can be deduced by means of TMT.

into account those changes of melting temperature and/or pressure at the interface which occur because of its motion (Subsect. 1.2 Introduction). We underline here that one of the successes of TMT is that it forecasts (with a pretty good numerical agreement) the experimentally verified circumstance that melting temperature changes when there is a quick solidification (more details in [7]). iii)$_b$ arising even when convective motions are modelled (when, for instance, at least one of the phases cannot be regarded as incompressible). As concerns the "well-posedness" theorem for Free-Moving Boundary (FMB-) Problem arising in the maximum generality from the application of TMT, we remark that the techniques developed in [14] seem suitably applicable to its proof. It has to be pointed out that they generalize those we have used. iv) Proposition 1 Sect. 6 singles out an important feature (see infra) shown not only by the treated particular G-problem but also by the quoted general FMB-problem, as it is determined by the nature of the theory which both of them stem from.

In fact an existence and uniqueness theorem is provable only in the $[0, T^*]$ interval where T^* is that time instant when one of the following physical circumstances occurs: iv)$_b$ one of the phases disappears; iv)$_a$ phase transition tends to take place with an "infinite" speed. Moreover, we face the following apparently contradictory evidence: It is mathematically possible that the function θ, solution of an arbitrary generalization of the Stefan-problem deduced by means of TMT ([7]), could cross the "melting point" (determined by the actual pressure) in a point belonging to one of the domains shared by the moving boundary ([8]) because of, for example, suitable heat production terms. Both of these circumstances are obvious consequences of the logical structure of TMT: in fact in this theory there are no specifications of the modalities with which an interface between two considered continua can be formed. (More details about this aspect of the question can be found in the Introduction of [16]). This simple physical consideration leads us to conclude that, when $T > T^*$ or the quoted mathematical circumstance occurs, TMT (together with all the models deduced by means of it) is bound to fail as it cannot completely describe the behaviour of the specified systems (continua shared by an interface).

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([7]) This solution is here intended to model the temperature, for instance, in one of the considered phases when the proposed model actually describes phase transition.

([8]) These domains obviously model the volumes occupied by both phases.

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