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# SCHNYDER DECOMPOSITIONS FOR REGULAR PLANE GRAPHS AND APPLICATION TO DRAWING

OLIVIER BERNARDI AND ÉRIC FUSY

ABSTRACT. *Schnyder woods* are decompositions of simple triangulations into three edge-disjoint spanning trees crossing each other in a specific way. In this article, we define a generalization of Schnyder woods to  $d$ -angulations (plane graphs with faces of degree  $d$ ) for all  $d \geq 3$ . A *Schnyder decomposition* is a set of  $d$  spanning forests crossing each other in a specific way, and such that each internal edge is part of exactly  $d - 2$  of the spanning forests. We show that a Schnyder decomposition exists if and only if the girth of the  $d$ -angulation is  $d$ . As in the case of Schnyder woods ( $d = 3$ ), there are alternative formulations in terms of orientations (“fractional” orientations when  $d \geq 5$ ) and in terms of corner-labellings. Moreover, the set of Schnyder decompositions on a fixed  $d$ -angulation of girth  $d$  is a distributive lattice. We also show that the structures dual to Schnyder decompositions (on  $d$ -regular plane graphs of mincut  $d$  rooted at a vertex  $v^*$ ) are decompositions into  $d$  spanning trees rooted at  $v^*$  such that each edge not incident to  $v^*$  is used in opposite directions by two trees. Additionally, for even values of  $d$ , we show that a subclass of Schnyder decompositions, which are called even, enjoy additional properties that yield a reduced formulation; in the case  $d = 4$ , these correspond to well-studied structures on simple quadrangulations (2-orientations and partitions into 2 spanning trees).

In the case  $d = 4$ , the dual of even Schnyder decompositions yields (planar) orthogonal and straight-line drawing algorithms. For a 4-regular plane graph  $G$  of mincut 4 with  $n$  vertices plus a marked vertex  $v$ , the vertices of  $G \setminus v$  are placed on a  $(n - 1) \times (n - 1)$  grid according to a permutation pattern, and in the orthogonal drawing each of the  $2n - 2$  edges of  $G \setminus v$  has exactly one bend. Embedding also the marked vertex  $v$  is doable at the cost of two additional rows and columns and 8 additional bends for the 4 edges incident to  $v$ . We propose a further compaction step for the drawing algorithm and show that the obtained grid-size is strongly concentrated around  $25n/32 \times 25n/32$  for a uniformly random instance with  $n$  vertices.

## 1. INTRODUCTION

A *plane graph* is a connected planar graph drawn in the plane in such a way that the edges do not cross. A *triangulation* is a plane graph in which every face has degree 3. More generally, a  *$d$ -angulation* is a plane graph such that every face has degree  $d$ . In [22, 23], Schnyder defined a structure for triangulations which became known as *Schnyder woods*. Roughly speaking, a Schnyder wood of a triangulation is a partition of the edges of the triangulation into three spanning trees with specific incidence relations; see Figure 1 for an example, and Section 2 for a precise definition. Schnyder woods have been extensively studied and have numerous applications; they provide a simple planarity criterion [22], yield beautiful straight-line drawing algorithms [23], and are a key ingredient in bijections for counting triangulations [20, 17]. They also have connections with several famous lattices [2].

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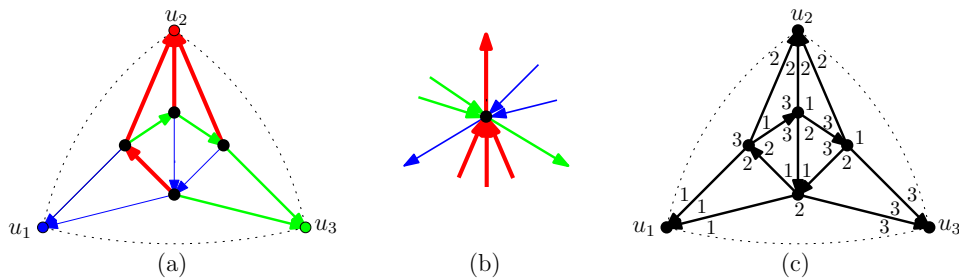


FIGURE 1. (a) A Schnyder wood of a triangulation. (b) Crossing rule for the three spanning trees at an internal vertex. (c) Corresponding 3-orientation and clockwise labelling.

In this article, we propose a generalized definition of Schnyder woods to  $d$ -angulations, for any  $d \geq 3$ . Roughly speaking, a *Schnyder decomposition* of a  $d$ -angulation  $G$  is a covering of the internal edges of  $G$  by  $d$  forests  $F_1, \dots, F_d$ , with specific crossing relations and such that each internal edge belongs to exactly  $d - 2$  of the forests; see Figure 2 for an example, and Section 3 for a precise definition. We show that a  $d$ -angulation admits a Schnyder decomposition if and only if it has girth  $d$  (i.e., no cycle of length less than  $d$ ).

**Other incarnations.** One of the nice features of Schnyder woods on triangulations is that they have two alternative formulations: one in terms of orientations with outdegree 3 at each inner vertex (so-called *3-orientations*) and one in terms of certain corner-labellings with labels in  $\{1, 2, 3\}$  which we call *clockwise labellings*, see Figure 1(c). We show in Section 3 that the same feature occurs for any  $d \geq 3$ . Precisely, on a fixed  $d$ -angulation of girth  $d$ , we give bijections between Schnyder decompositions, some generalized orientations called  $d/(d - 2)$ -orientations, and certain *clockwise labellings* of corners with colors in  $\{1, \dots, d\}$ . The  $d/(d - 2)$ -orientations recently appeared in [3], as a key ingredient in a bijection to count  $d$ -angulations of girth  $d$ . They are also a suitable formulation to show that the set of Schnyder decompositions on a fixed  $d$ -angulation of girth  $d$  is a distributive lattice (Corollary 15).

**Duality.** Schnyder decompositions can also be studied in a dual setting. Precisely, for  $G$  a  $d$ -angulation of girth  $d$  and  $G^*$  the dual graph ( $G^*$  is a  $d$ -regular plane graph of mincut  $d$  and rooted at a vertex  $v$ ), we show in Section 4 that the combinatorial structures of  $G^*$  dual to Schnyder decompositions of  $G$  are  $d$ -tuples of spanning trees  $T_1^*, \dots, T_d^*$  such that every edge incident to the root vertex belong to one spanning tree, while the other edges belong to two spanning trees in opposite directions; in addition around a non-root vertex  $v$  the edges leading  $v$  to its parent in  $T_1^*, \dots, T_d^*$  appear in clockwise order<sup>1</sup>. A duality property was well-known in the case  $d = 3$ , operating even more generally on 3-connected plane graphs [21].

**Bipartite case.** In Section 5, we show that when  $d$  is even,  $d = 2p$ , there is a subclass of Schnyder decompositions that enjoy additional properties. The so-called *even Schnyder decompositions* can be reduced to  $p$ -tuples of spanning trees (or rather, spanning forests if the external edges are not taken into account) with specific incidence relations and such that each edge belongs to  $p - 1$  trees; see Figure 6 for an example and Theorem 26 for precise properties. In the case of quadrangulations

<sup>1</sup>Actually, the existence of  $d$  spanning trees such that every edge non incident to the root is used twice is granted by the Nash-Williams Theorem (even for non-planar  $d$ -edge-connected  $d$ -regular graphs). Additionally, the decomposition can be taken in such a way that every edge is used once in each direction (for the orientation of the trees toward the root-vertex) by a “directed version” of the Nash-Williams theorem due to Edmonds [11]. However, these theorems do not grant any crossing rule for the spanning trees.

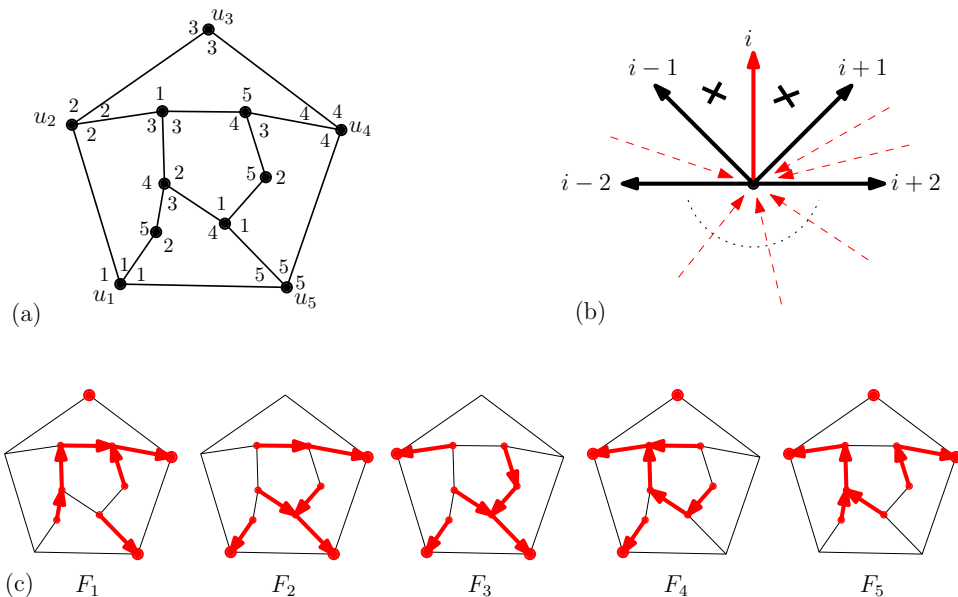


FIGURE 2. (a) clockwise labelling of a 5-angulation. (b) Crossing rule for the tree  $i$  at an internal vertex. (c) The five trees forming the Schnyder decomposition.

( $p = 2$ ) even Schnyder decompositions correspond to specific partitions into two non-crossing spanning trees; these partitions have been introduced in [9] under the name of *separating decompositions* and are in bijections with *2-orientations* (i.e., orientations of internal edges such that every internal vertex is has outdegree 2). The existence of 2-orientations on simple quadrangulations is also proved in [10, 9].

**Graph drawing.** Finally, as an application, we present in Section 6 a linear-time orthogonal drawing algorithm for 4-regular plane graphs of mincut 4, which relies on the dual of Schnyder decompositions. For a 4-regular plane graph  $G$  of mincut 4, with  $n$  vertices ( $2n$  edges,  $n + 2$  faces) one of which called *root-vertex*, the algorithm places each non-root vertex of  $G$  using face-counting operations, that is, one obtains the coordinate of a vertex  $v$  by counting the faces in areas delimited by the paths between  $v$  and the root-vertex in the spanning trees of the dual Schnyder decompositions. Such face-counting operations were already used in Schnyder's original procedure on triangulations [23], and in a recent straight-line drawing algorithm of [1] relying on Schnyder decompositions of quadrangulations. The placement of vertices given by our algorithm is such that each line and column of the grid  $\{1, \dots, n - 1\} \times \{1, \dots, n - 1\}$  contains exactly one vertex<sup>2</sup>, and each edge has exactly one bend (see Figure 7). Placing the root-vertex and its 4 incident edges requires two more columns and rows and 8 additional bends (1 bend for two of the edges and 3 bends for the other two). In addition we present a compaction step allowing one to reduce the grid size in certain cases. We show that for a uniformly random instance of size  $n$ , the grid size after compaction is strongly concentrated around  $25n/32 \times 25n/32$  (similar techniques for analyzing the average grid-size were used in [8, 16] for certain classes of triangulations).

For comparison with existing orthogonal drawing algorithms: our algorithm for 4-regular graphs of mincut 4 with  $n$  vertices has grid size (width  $\times$  height)  $n \times n$

<sup>2</sup>The placement of vertices thus follows a permutation pattern, and it is actually closely related to a recent bijection [7] between Baxter permutations and *plane bipolar orientations* (which are known to be in bijection with even Schnyder decompositions of quadrangulations [9, 13]).

the worst case and  $25n/32 \times 25n/32$  in average, with 1 bend per edge except two edges with 3 bends ( $2n + 4$  bends in total). In the less restrictive case of loopless biconnected 4-regular graphs Biedl and Kant's algorithm [6] has worst case grid size  $(n + 1) \times (n + 1)$ , with  $2n + 4$  bends in total and at most 2 bends per edge. The worst-case values are the same in Tamassia's algorithm [24] (analyzed by Biedl in [5]) but with the advantage that the total number of bends obtained is best possible for each fixed loopless biconnected 4-regular plane graphs. In the more restrictive case of 3-connected 4-regular graphs, Biedl's algorithm [5], which revisits an algorithm by Kant [19], has worst case grid size  $(\frac{2}{3}n + 1) \times (\frac{2}{3}n + 1)$ , with  $\lceil \frac{4}{3}n \rceil + 4$  bends in total, and at most 2 bends per edge except for the root-edge having at most 3 bends. We refer the reader to [4] for a survey on the worst-case grid size of orthogonal drawings.

## 2. PLANE GRAPHS AND FRACTIONAL ORIENTATIONS OF $d$ -ANGULATIONS

In this section we recall some classical definitions about plane graphs, and prove the existence of  $d/(d - 2)$ -orientations for  $d$ -angulations.

### 2.1. Graphs, plane graphs and $d$ -angulations.

Here *graphs* are finite; they have no loop but can have multiple edges. For an edge  $e$ , we use the notation  $e = \{u, v\}$  to indicate that  $u, v$  are the endpoints of  $e$ . The edge  $e = \{u, v\}$  has two *directions* or *arcs*; the arc with origin  $u$  is denoted  $(u, e)$ . A graph is  $d$ -regular if every vertex has degree  $d$ . The *girth* of a graph is the minimal length of its cycles. The *mincut* of a connect graph is the minimal number of edges that one needs to delete in order to disconnect the graph.

A *plane graph* is a connected planar graph drawn in the plane in such a way that the edges do not cross. A *face* is a connected component of the complement of the graph. The bounded faces are called *internal* and the unbounded one is called *external*. A vertex or an edge is said to be *external* if it is incident to the external face and *internal* otherwise. The numbers  $v(G), e(G), f(G)$  of vertices, edges and faces of a plane graph satisfy the *Euler relation*:  $v(G) - e(G) + f(G) = 2$ . A *corner* is a triple  $(v, e, e')$ , where  $e, e'$  are consecutive edges in clockwise order around  $v$ . A corner is said *incident* to the face which contains it. The *degree* of a face is the number of incident corners.

A plane graph is a  $d$ -angulation if every face has degree  $d$ . *Triangulations* and *quadrangulations* correspond to  $d = 3$  and  $d = 4$ , respectively. Clearly, a  $d$ -angulation has girth at most  $d$ , and if it has girth  $d$  then it has  $d$  distinct external vertices. The incidence relation between faces and edges in a  $d$ -angulation  $G$  gives  $df(G) = 2e(G)$ . Combining this equation with Euler relation gives

$$(1) \quad \frac{e(G) - d}{v(G) - d} = \frac{d}{d - 2}.$$

If the external face has  $d$  distinct vertices, the left-hand-side of (1) is the ratio between internal edges and internal vertices.

### 2.2. Existence of $d/(d - 2)$ -orientations.

Let  $G$  be a graph and let  $k$  be a positive integer. A  $k$ -fractional orientation of  $G$  is a function  $\Omega$  from the arcs of  $G$  to the set  $\{0, \frac{1}{k}, \frac{2}{k}, \dots, 1\}$  such that the values of the two arcs constituting an edge  $e = \{u, v\}$  add up to 1:  $\Omega(u, e) + \Omega(v, e) = 1$ . We identify 1-orientations with the classical notion of orientations by considering the arc with value  $\Omega = 1$  as the orientation of the edge. More generally, it is possible to think of  $k$ -fractional orientations as orientations of the graph  $kG$  obtained from  $G$  by replacing each edges by  $k$  *indistinguishable* parallel edges. The *outdegree* of a vertex  $v$  is the sum of the value of  $\Omega$  over the arcs having origin  $v$ .

**Definition 1.** A  $j/k$ -orientation of a plane graph is a  $k$ -fractional orientation of its internal edges such that each internal vertex has outdegree  $j/k$ . A  $j$ -orientation is a  $j/1$ -orientation (which can be seen as a non-fractional orientation).

By definition, a  $j/k$ -orientation is also a  $(nj)/(nk)$ -orientation for all positive integer  $n$ , but the converse is false. Given relation (1), it is natural to look for  $d/(d-2)$ -orientations of  $d$ -angulations. For  $d = 3$ ,  $d/(d-2)$ -orientations correspond to the notion of 3-orientations of triangulations considered in the literature. However, for  $d = 4$  the  $d/(d-2)$ -orientations are more general than the 2-orientations of quadrangulations considered for instance in [7, 13]. The existence of 3-orientations for simple triangulations and the existence of 2-orientations for simple quadrangulations were proved in [23] and [10] respectively. We now prove a more general existence result.

**Theorem 2.** A  $d$ -angulation admits a  $d/(d-2)$ -orientation if and only if it has girth  $d$ . Moreover, if  $d = 2p$  is even, then any  $d$ -angulation of girth  $d$  also admits a  $p/(p-1)$ -orientation.

In order to prove Theorem 2, we first give a general existence criteria for  $k$ -fractional orientations with prescribed outdegrees. Let  $H = (V, E)$  be a graph and let  $\alpha : V \rightarrow \mathbb{R}$ . A  $(\alpha, k)$ -orientation is a  $k$ -fractional orientation such that every vertex  $v$  has outdegree  $\alpha$ .

**Lemma 3** (Folklore). Let  $H = (V, E)$  be a graph and let  $\alpha : V \mapsto \mathbb{N}$ . There exists an orientation of  $H$  such that every vertex  $v$  has outdegree  $\alpha(v)$  (i.e., a  $(\alpha, 1)$ -orientation) if and only if

- (a)  $\sum_{v \in V} \alpha(v) = |E|$
- (b) for all  $S \subseteq V$ ,  $\sum_{v \in S} \alpha(v) \geq |E_S|$ , where  $E_S$  is the set of edges with both ends in  $S$ .

**Corollary 4.** Let  $H = (V, E)$  be a graph and let  $\alpha : V \mapsto \mathbb{R}$ . There exists a  $(\alpha, k)$ -orientation of  $H = (V, E)$  if and only if  $k\alpha$  takes integer values and Properties (a), (b) of Lemma 3 hold.

*Proof of Corollary 4.* The conditions are clearly necessary. We now suppose that the mapping  $\alpha$  satisfies the conditions and want to prove that a  $(\alpha, k)$ -orientation exists. For this, it suffices to prove the existence of an orientation of  $kH$  with outdegree  $k\alpha(v)$  for each vertex  $v$ . This is granted by Lemma 3.  $\square$

*Proof of Theorem 2.* Let  $G = (V, E)$  be a  $d$ -angulation with  $d$  distinct external vertices, and let  $G' = (V, E')$  be the graph obtained from  $G$  by erasing the  $d$  external edges. By (1), the  $d$ -angulation  $G$  has a  $d/(d-2)$ -orientation (resp.  $p/(p-1)$ -orientation) if and only if  $G'$  has a  $(\alpha, d-2)$ -orientation (resp.  $(\alpha, p-1)$ -orientation) with  $\alpha(v) = d$  on internal vertices and  $\alpha(v) = 0$  on external vertices. We next prove that a  $(\alpha, d-2)$ -orientation (resp.  $(\alpha, p-1)$ -orientation) exists for  $G'$  if and only if  $G$  has girth  $d$ .

We first show that the girth  $d$  is necessary. Suppose by contradiction that  $G$  has a simple cycle  $C$  of length  $k$  less than  $d$ . Let  $S$  be the set of vertices on  $C$  and outside. We can suppose that the cycle  $C$  is simple and that there is no edge inside  $C$  joining two vertices of  $C$ . In this case, the subgraph  $G_S = (S, E_S)$  (resp.  $G'_S = (S, E'_S)$ ) of  $G$  (resp.  $G'$ ) induced by  $S$  is obtained by deleting all the edges and vertices strictly inside  $C$ . The plane graph  $G_S$  has every face of degree  $d$  except one of degree  $k < d$ . Therefore, the incidence relation gives  $f(G_S) \leq e(G_S)$ , and Euler relation gives  $\frac{d}{d-2}(v(G_S) - d) < (e(G_S) - d)$ . Thus,  $\sum_{s \in S} \alpha(s) < |E'_S|$  which violates condition (b) of Lemma 3.

We now suppose that  $G$  has girth  $d$ , and apply Corollary 4 to the graph  $G'$ . The fact that  $k\alpha$  takes integer values is clear (both for  $k = d-2$  and for  $k = p-1$  in

the even case). Moreover, Condition (a) holds by (1). We now prove Condition (b). Since the value of  $\alpha$  is 0 for the external vertices, it suffices to show  $\sum_{v \in S} \alpha(v) \geq |E'_S|$  for subset  $S$  of vertices containing all the external vertices. Let  $S$  be such a set, and let  $G_S = (S, E_S)$ ,  $G'_S = (S, E'_S)$  be respectively the subgraphs of  $G$  and  $G'$  induced by  $S$ . We only need to show that  $\sum_{v \in R} \alpha(v) \geq |E'_R|$  for any  $R \subseteq S$  which is the vertex set of a connected component of  $G'_S$  (this will prove  $\sum_{v \in S} \alpha(v) \geq |E'_S|$ ). We consider a plane embedding of the connected component  $G_R$ . Since,  $G_R$  has no cycle of length less than  $d$ ,  $G_R$  is either a tree or have every faces of degree at least  $d$ . In both cases, one gets  $\frac{d}{d-2}(v(G_R) - d) \geq (e(G_R) - d)$  (by using the incidence relation and Euler relation). Thus,  $\sum_{v \in R} \alpha(v) \geq |E'_R|$  (one has to consider two cases depending on whether  $R$  contains all the external vertices or none).  $\square$

Note that our existence proof does not yield a computation algorithm. However, it is known that the computation of an  $\alpha$ -orientation can be reduced to a flow problem, and has polynomial complexity in the number of edges (see [12]). Hence, for fixed  $d$  a  $d/(d-2)$ -orientation can be computed in polynomial time. In the case  $d = 3, 4$  it is known that a  $d/(d-2)$ -orientation (and the associated Schnyder decomposition) can be computed in *linear time*, hence the following open question.

**Question 1.** *For a given  $d \geq 5$ , is it possible to compute a  $d/(d-2)$ -orientation of a  $d$ -angulation in time linear in the number of vertices?*

### 3. THREE INCARNATIONS OF SCHNYDER DECOMPOSITIONS

In this section we define *Schnyder decompositions* and define as well as certain labellings of the corners which we call *clockwise labellings*. We show that on a fixed  $d$ -angulation  $G$  of girth  $d$ , Schnyder decompositions, clockwise labellings and  $d/(d-2)$ -orientations are in bijective correspondence; hence the three definitions are actually three incarnations of the same structure.

We start with the definition of Schnyder decompositions, which is illustrated in Figure 2. In the following  $d$  is an integer greater or equal to 3 and  $G$  is a  $d$ -angulation with  $d$  distinct external vertices  $u_1, \dots, u_d$  in clockwise order around the external face. We will denote by  $[d]$  the set of integers  $\{1, \dots, d\}$  *considered modulo  $d$*  (the arithmetic operations are made modulo  $d$  so as to get a representant in  $\{1, \dots, d\}$ ).

**Definition 5.** *A Schnyder decomposition of  $G$  is a covering of the internal edges of  $G$  by  $d$  oriented forests  $F_1, \dots, F_d$  (one forest for each color  $i \in [d]$ ) such that*

- (i) *Each internal edge  $e$  appears in  $d-2$  of the forests.*
- (ii) *For each  $i$  in  $[d]$ , the forest  $F_i$  spans all vertices except  $v_i, v_{i+1}$ ; it is made of  $d-2$  trees each containing one of the external vertices  $u_j, j \neq i, i+1$ , and the tree containing  $u_j$  is oriented toward  $u_j$  which is considered as its root.*
- (iii) *Around any internal vertex  $v$ , the outgoing edges  $e_1, \dots, e_d$  leading  $v$  to its parent in  $F_1, \dots, F_d$  appear in clockwise order around  $v$  (some of these edges can be equal). Moreover, the ingoing edges of color  $i$  are between  $e_{i+1}$  and  $e_{i-1}$  in clockwise order around  $v$  and are distinct from  $e_{i+1}$  and  $e_{i-1}$ .*

We denote by  $(F_1, \dots, F_d)$  the decomposition.

The classical definition of *Schnyder woods* coincides with our definition of *Schnyder decomposition* for triangulations (case  $d = 3$ ). In the following, given a Schnyder decomposition, we will sometimes speak about the two *missing colors of an edge* (the colors of the two forests not containing this edge), the *colors of an arc* (the colors of the oriented forests using this arc) or of the *colors going out of a vertex*.

**Remark 6.** *Condition (iii) in Definition 5 immediately implies two properties:*

- (a) If an internal edge has  $i, j$  as missing colors, then the colors  $i+1, i+2, \dots, j-1$  are all in one direction of  $e$ , while the colors  $j+1, j+2, \dots, i-1$  are all in the other direction of  $e$ .
- (b) Let  $v$  be an internal vertex and  $i \in [d]$  be a color. Let  $e \neq e', f, f'$  be edges incident to  $v$  with  $e, e'$  of color  $i$  and  $f, f'$  of color  $i+1$ . If  $e, f$  are ingoing,  $e', f'$  are outgoing and  $e, e', f$  appear in clockwise order around  $v$  (with possibly  $e = f$ ), then the outgoing edge  $f'$  appears between  $e'$  and  $f$  in clockwise order around  $v$  (with possibly  $e' = f'$ ). As a consequence, a directed path of color  $i+1$  cannot cross a directed path of color  $i$  from right to left (however, a crossing from left to right is possible).

We now define *clockwise labellings*. An example is shown in Figure 2(a).

**Definition 7.** A clockwise labelling of  $G$  is the assignment to each corner of a color, or label, in  $[d]$  such that:

- (i) The colors  $1, 2, \dots, d$  appear in clockwise order around each face of  $G$ .<sup>3</sup>
- (ii) For all  $i$  in  $[d]$ , the corners incident to the external vertex  $u_i$  are of color  $i$ .
- (iii) In clockwise direction around each internal vertex, there is exactly one corner having a larger color than the next corner. Equivalently, if  $i_1, \dots, i_k$  are the colors of the corners in clockwise order around a vertex, then the sets  $\{i_j, i_j + 1, i_j + 2, \dots, i_{j+1} - 1\}$ ,  $j = 1 \dots k$  form a partition of  $[d]$ .

**Theorem 8.** The sets of Schnyder decompositions, clockwise labellings, and  $d/(d-2)$ -orientations of  $G$  are in bijection.

Given the existence result in Theorem 2, we obtain:

**Corollary 9.** A  $d$ -angulation  $G$  admits a Schnyder decomposition (respectively, a clockwise labelling) if and only if it has girth  $d$ .

In the next two subsections we prove Theorem 8 and explicit the bijections between Schnyder decompositions, clockwise labellings, and  $d/(d-2)$ -orientations. Then, we present a lattice structure on the set of Schnyder decompositions of a given  $d$ -angulations.

### 3.1. Bijection between clockwise labellings and $d/(d-2)$ -orientations.

Let  $L$  be a clockwise labelling of  $G$ . For each arc  $a = (u, e)$  of  $G$ , the *clockwise-jump* across  $a$  is the quantity  $J(a) = \ell_2 - \ell_1$  in  $\{0, 1, \dots, d-1\}$ , where  $\ell_1$  and  $\ell_2$  are respectively the labels of the corners preceding and following  $e$  in clockwise order around  $u$ .

**Lemma 10.** For each vertex  $v$  of  $L$ , the sum of the clockwise-jumps over the arcs incident to  $v$  is  $d$  if  $v$  is an internal vertex and is  $0$  if  $v$  is an external vertex. For each internal edge  $e = \{u, v\}$  of  $L$ , the sum of the clockwise-jumps on the two arcs constituting  $e$  is  $d-2$ .

*Proof.* The first assertion is a direct consequence of Properties (ii), (iii) of clockwise labellings. This assertion implies that the sum of clockwise-jumps over all arcs of  $G$  equals  $d \cdot n$ , where  $n$  is the number of internal vertices of  $G$ . By (1) the number of internal edges is  $m = \frac{d}{d-2}n$ , so the sum of clockwise-jumps over all arcs of  $G$  equals  $(d-2)m$ . Let us now consider the sum  $S(e) := J(v, e) + J(u, e)$  for an internal edge  $e = \{u, v\}$ . Let  $i, j$  denote the colors of the corners preceding and following the edge  $e$  in clockwise order around  $u$ . By Property (i) of clockwise labellings, the colors preceding and following  $e$  in clockwise order around  $v$  are  $j+1$  and  $i-1$  respectively. Hence if  $j = i + (d-1)$ , then  $S(e) = 2d-2$ , and otherwise  $S(e) = d-2$ .

<sup>3</sup>Considering the embedding on the sphere rather than in the plane has the effect of reversing the order of the corners around the root-face, i.e., the labels around the root-face are  $1, 2, \dots, d$  in counterclockwise order in a sphere embedding.



Given that the sum of the  $J(v, e)$  over all the internal arcs of  $G$  is equal to  $(d-2)m$ , we conclude that  $S(e) = d - 2$  for all internal edges.  $\square$

Lemma 10 implies that in a clockwise labelling a clockwise-jump can not equal  $d - 1$ , which ensures that our definition of clockwise labellings coincides with the Schnyder labellings [22] in the case  $d = 3$ . It also ensures that the orientation  $O$  with flow  $\Omega(a) := J(a)/(d-2)$  at each arc  $a$  is a  $d/(d-2)$ -orientation. The mapping associating  $O$  to  $L$  is called  $\Psi$ . Denote by  $\mathcal{L}$  the set of clockwise labellings of  $G$  and by  $\mathcal{O}$  the set of  $d/(d-2)$ -orientations of  $G$ .

**Proposition 11.** *The mapping  $\Psi$  is a bijection between  $\mathcal{L}$  and  $\mathcal{O}$ .*

*Proof.* First of all, it is clear that  $\Psi$  is injective, since the values of  $\Omega$  suffice to determine the color of every corner by starting from the corners incident to external vertices (whose color is known by definition) and propagating the colors according to the rule:

- (1) if the color of a corner is  $i$ , the color of the next corner in clockwise order around its face is  $i + 1$ ,
- (2) if the color of a corner is  $i$ , the color of the next corner in clockwise order around its vertex  $v$  is  $i + (d - 2)\Omega(v, e)$ , where  $e$  is the edge between these corners

We will now prove that it is possible to apply these rules starting with any  $d/(d-2)$ -orientation  $\Omega$  without encountering any conflict (thereby proving the surjectivity of  $\Psi$ ). Let  $\Omega$  be a  $d/(d-2)$ -orientation. For this purpose we consider the “cycles of corners” which occur when propagating these rules. We therefore introduce a few notations. Let  $C_G$  be the *graph of corners* whose vertices are the internal corners of  $G$  and whose edges are the pairs  $\{c, c'\}$  of corners which are consecutive around a vertex or a face of  $G$ . For an arc  $a = (c, c')$  in the graph of corners  $C_G$ , we define  $\omega(a) = 1$  (resp.  $\omega(a) = -1$ ) if the corner  $c'$  follows  $c$  in clockwise (resp. counterclockwise) order around a face of  $G$  and we define  $\omega(a) = (d-2)\Omega(v, e)$  (resp.  $\omega(a) = -(d-2)\Omega(v, e)$ ) if  $c'$  follows  $c$  in clockwise (resp. counterclockwise) order around a vertex of  $G$  and  $e$  is the separating edge. Lastly, if  $C$  is a directed cycle on  $C_G$ , we denote by  $\omega(C)$  the sum of  $\omega(a)$  over the arcs composing  $C$ . We will now prove that the value  $\omega(C)$  is a multiple of  $d$  for any (simple) directed cycle  $C$  of  $C_G$ . Observe first that the graph  $C_G$  can be drawn in a planar way (by superimposing it over the plane graph  $G$ ) in which the internal faces correspond either to an internal vertex, edge, or face of  $G$ . See Figure 3 for an example. Moreover, if  $C_f$  is the cycle delimiting a face  $f$  of  $C_G$  in clockwise direction, the value  $\omega(C_f)$  is  $d$  (resp.  $-d$ ,  $d$ ) if  $f$  corresponds to an internal vertex (resp. edge, face) of  $G$ . Furthermore, if  $C$  is a simple clockwise (counterclockwise) directed cycle of  $C_G$ ,  $\omega(C) = \sum_f \omega(C_f)$  (resp.  $\omega(C) = -\sum_f \omega(C_f)$ ) where the sum is over the faces of  $C_G$  enclosed in  $C$  (since the contributions of arcs strictly inside  $C$  cancel out in this sum). Thus, the value of  $\omega(C)$  is a multiple of  $d$  for any directed cycle  $C$ . This proves that applying rules (1), (2) for determining the colors of the corner can lead to no contradiction. Hence, these rules allow one to define a clockwise labelling from any  $d/(d-2)$ -orientation. This completes the proof that the mapping  $\Psi$  is surjective, hence bijective.  $\square$

### 3.2. Bijection between clockwise labellings and Schnyder decompositions.

We now define a mapping from clockwise labellings to Schnyder decompositions. Let  $L$  be a clockwise labelling of  $G$ . For each internal arc  $a = (u, e)$  of  $G$ , we give to  $a$  the colors  $i, i + 1, \dots, j - 1$  (no color if  $i = j$ ), where  $i$  and  $j$  are respectively the labels preceding and following  $e$  in clockwise order around  $u$ . We denote by  $F_i, i \in [d]$  the oriented graph of color  $i$ , and we call  $\Phi$  the mapping that associates  $(F_1, \dots, F_d)$  to  $L$ . The mapping  $\Phi$  is represented in Figure 4.

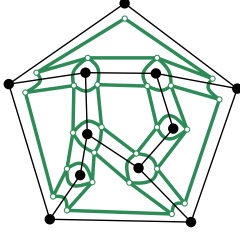


FIGURE 3. The corner graph.

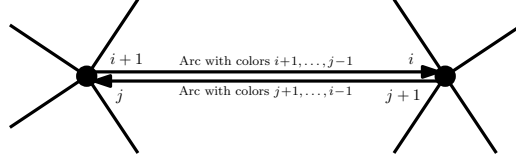


FIGURE 4. Mapping  $\Phi$  from clockwise labellings to Schnyder decompositions.

**Lemma 12.** *For each  $L \in \mathcal{L}$ ,  $(F_1, \dots, F_d) = \Phi(L)$  is a Schnyder decomposition.*

*Proof.* According to Lemma 10, each internal edge receives a total of  $d - 2$  colors (see Figure 4). Properties (iii) of clockwise labellings ensures that every internal vertex  $v$  has exactly one outgoing edge in each oriented subgraphs  $F_1, F_2, \dots, F_d$ , and that these edges appear in clockwise order around  $v$ . Moreover, if an edge  $e$  has color  $k \in [d]$  on the arc directed toward an internal vertex  $v$ , then in clockwise order around  $v$  the edge  $e$  is preceded and followed by corners with colors  $i + 1, j$  satisfying  $k \in \{j + 1, \dots, i - 1\}$ . Hence,  $i + 1 \notin \{k, k + 1\}$  and  $e$  is not between the outgoing edges of color  $k - 1$  and  $k + 1$  in clockwise order around  $v$ . This proves Property (iii) of Schnyder decompositions, provided we show that the oriented subgraphs  $F_1, \dots, F_d$  are forests oriented toward the external vertices.

Suppose that one of the subgraph  $F_i$  is not a forest oriented toward the external vertices. Since every internal vertex has exactly one outgoing edge in  $F_i$ , this implies the existence of a directed cycle in  $F_i$ . Consider a directed cycle in one of the subgraphs  $F_1, F_2, \dots, F_d$  enclosing a minimal number of faces. Clearly, the cycle  $C$  is simple and has only internal vertices. Suppose first that the cycle  $C$  is clockwise. Let  $i$  be a color in  $[d]$  such that the directed cycle  $C$  has all its arc of color  $i$  but not all its arcs of color  $i + 1$ . There exists a vertex  $v$  on  $[d]$  such that the edge  $e_{i+1}$  of color  $i + 1$  going out of  $v$  is not equal to the the edge  $e_i \in C$  of color  $i$  going out of  $v$ . In this case, the edge  $e_{i+1}$  is strictly inside  $C$ . Indeed, we have established above that the ingoing arcs of color  $i$  at  $v$  (one of which belongs to  $C$ ) are strictly between the outgoing edges of color  $i - 1$  and  $i + 1$  in clockwise direction around  $v$ . Let  $v_1$  be the other end of the edge  $e_{i+1}$ . Since  $v_1$  is an internal vertex, there is an edge of color  $i + 1$  going out of  $v_1$  and leading to a vertex  $v_2$ . Continuing in this way we get a directed path  $P$  of color  $i + 1$  starting at  $v$  and either ending at an external vertex or creating a directed cycle of color  $i + 1$ . Moreover, a local examination shows that the path  $P$  remains inside the cycle  $C$  of color  $i$  (the path  $P$  cannot cross the directed cycle  $C$  from right to left, even if it can use certain edges of  $C$ ). This means that the path  $P$  creates a directed cycle of color  $i + 1$  which is strictly contained inside  $C$ . This cycle encloses less faces than  $C$ , contradicting our minimality assumption on  $C$ . Similarly, if the cycle  $C$  is directed counterclockwise there is a cycle (of color  $i - 1$ ) which enclose less faces than  $C$ . We thereby reach a contradiction. This proves that there is no monochromatic directed cycle. Therefore  $F_1, \dots, F_d$  are forests directed toward the external vertices.

In order to completes our proof that  $(F_1, \dots, F_d)$  is a Schnyder decomposition, it only remains to show that for all  $i \in [d]$  the forest  $F_i$  is not incident to the external vertices  $u_i, u_{i+1}$ . For this, recall that the corners incident to the external vertex  $u_i$  have color  $i$ , so that for any edge  $e = \{u_i, v\}$  the two corners incident to  $v$  have colors  $i - 1$  and  $i + 1$ . Hence, the edge  $e$  belongs to  $F_{i+1}, F_{i+2}, \dots, F_{i-2}$  but not to  $F_{i-1}$  or  $F_i$ . Thus,  $(F_1, \dots, F_d)$  is a Schnyder decomposition.  $\square$

Call  $\mathcal{S}$  the set of Schnyder decompositions of  $G$ .

**Proposition 13.** *The mapping  $\Phi$  is a bijection between  $\mathcal{L}$  and  $S$ .*

*Proof.* It is clear from the definition of the mappings  $\Psi$  and  $\Phi$  that the orientation  $O = \Psi(L)$  associated with a clockwise labelling  $L$  is obtained from the Schnyder decomposition  $(F_1, \dots, F_d) = \Phi(L)$  by forgetting the colors. Denoting by  $\Gamma$  the “color-deletion” mapping, we thus have  $\Psi = \Gamma \circ \Phi$ . By Proposition 11, the mapping  $\Psi$  is injective, thus  $\Phi$  is also injective.

To prove the surjectivity of  $\Phi$ , we consider a Schnyder decomposition  $S = (F_1, \dots, F_d)$ . Clearly,  $\Gamma(S)$  is a  $d/(d-2)$ -orientation, hence  $L = \Psi^{-1} \circ \Gamma(S)$  is a clockwise labelling. We now show that  $\Phi(L) = S$  (thereby showing the surjectivity of  $\Phi$ ). Let  $S' = \Phi(L)$ . First observe that for any arc  $a$  of  $G$  the number of colors of  $a$  in  $S$  and in  $S'$  is the same: it is equal to the clockwise jump across  $a$  in  $L$ . We say that  $S$  and  $S'$  agree on an arc  $a = (u, e)$  if its colors are the same in  $S$  and in  $S'$ . We say that  $S$  and  $S'$  agree on an edge  $e$  if and if it agrees on both arcs of  $e$ . By Property (ii) of Schnyder decompositions, the internal edges incident to the external vertex  $u_i$  have missing colors  $i$  and  $i-1$  in both  $S$  and  $S'$  (and all their colors are oriented toward  $u_i$ ), hence  $S$  and  $S'$  agree on these edges. We now suppose that  $v$  is an internal vertex incident to an edge  $e$  on which  $S$  and  $S'$  agree and show that in this case  $S$  and  $S'$  agree on every edge incident to  $v$  (this shows that  $S$  and  $S'$  agree on every edge). Suppose first that the edge  $e$  has some colors going out of  $v$  (which by hypothesis are the same in  $S$  and in  $S'$ ). In this case,  $S$  and  $S'$  agree on each arc going out of  $v$ , since the colors going out of  $v$  are  $1, 2, \dots, d$  in clockwise order. Now, for an edge  $e' = \{v, v'\}$ , either the arc  $a = (v, e')$  has some colors  $i+1, \dots, j-1$  in which case the arc  $a' = (v', e')$  has the colors  $j+1, \dots, i-1$  (both in  $S$  and in  $S'$ ), or the arc  $a = (v, e')$  has no color in which case the  $d-2$  colors of  $a' = (v', e')$  are imposed (both in  $S$  and in  $S'$ ) by Property (iii) of clockwise labellings (the missing colors are  $i$  and  $i+1$  if  $e'$  is between the outgoing arcs of color  $i$  and  $i+1$  around  $v$ ). We now suppose that the edge  $e$  has no color going out of  $v$ . Let  $i, i+1$  be the missing colors of  $e$ . By Property (iii) of clockwise labellings, the edge  $e$  is between the outgoing arcs of color  $i$  and  $i+1$  around  $v$  (both for  $S$  and  $S'$ ), thus it shows that  $S$  and  $S'$  agree on these arcs. Hence  $S$  and  $S'$  also agree on all the edges incident to  $v$  by the same reasoning as above. This shows that  $S = S'$  hence that  $\Psi$  is surjective.  $\square$

### 3.3. Lattice property.

In this subsection we explain how to endow the set of  $d/(d-2)$ -orientations of a  $d$ -angulations (equivalently its set of Schnyder decompositions) with the structure of a distributive lattice. The construction applies in fact to any set of fractional orientations characterized by indegree conditions.

Let  $G$  be a plane graph. Given a simple cycle  $C$ , the *clockwise* (resp. *counterclockwise*) arcs of  $C$  are the arcs obtained by orienting  $C$  in clockwise (resp. counterclockwise) direction. Given a  $k$ -fractional orientation  $O$ , the cycle  $C$  is a *counterclockwise circuit* if the flow of every counterclockwise arc is positive. We then denote by  $O_C$  the orientation obtained from  $O$  by increasing the flow of clockwise arcs by  $1/k$  and decreasing the flow of counterclockwise arcs by  $1/k$ . The transition from  $O$  to  $O_C$  is called a *cycle-pushing operation*. We then prove the following lemma as an easy consequence of [15].

**Lemma 14.** *Let  $G$  be a plane graph and let  $\alpha$  be a function from the vertices to the real. If the set of  $(\alpha, k)$ -orientations is not empty, the transitive closure of cycle-pushing operations gives a partial order which is a distributive lattice.*

**Corollary 15.** *The set of Schnyder decompositions of a fixed  $d$ -angulation of girth  $d$  is a distributive lattice. The partial order relation is the transitive closure of cycle-pushing on the associated  $d/(d-2)$ -orientations.*

*Proof.* First we recall the result from [15]. Let  $H$  be a (loopless) directed plane graph with set of vertices  $V_H$  and set of arcs  $A_H$ . For each function  $\Delta : V_H \rightarrow \mathbb{Z}$ , a  $\Delta$ -bond is a function  $\Omega : A_H \rightarrow \mathbb{Z}$  such that for any vertex  $v$  the net flow  $\sum_{a \in \text{out}(v)} \Omega(a) - \sum_{a \in \text{in}(v)} \Omega(a)$  equals  $\Delta(v)$ . For two functions  $\ell : A_H \rightarrow \mathbb{Z}$  and  $u : A_H \rightarrow \mathbb{Z}$ , a  $(\Delta, \ell, u)$ -bond is a  $\Delta$ -bond  $\Omega$  such that  $\ell(a) \leq \Omega(a) \leq u(a)$  for each arc  $a$ . In that context a “clockwise-non-saturated circuit” is a simple cycle  $C$  on  $H$  such that each clockwise arc  $a$  of  $C$  satisfies  $\Omega(a) < u(a)$  and each counterclockwise arc  $a$  of  $C$  satisfies  $\Omega(a) > \ell(a)$ . “Pushing” the cycle  $C$  means increasing by 1 the flow of clockwise arcs and decreasing by 1 the flow of counterclockwise arcs of  $C$ . It is shown in [15] that, if the set of  $(\Delta, \ell, u)$ -bonds is not empty, the transitive closure of the pushing operation gives a distributive lattice.

We now use this result in our context. Let  $H$  be the directed plane graph obtained from  $G$  by inserting a vertex, called an *edge-vertex*, in the middle of each edge, and orienting the two half-edges toward the incident edge-vertex (hence, each edge of  $G$  gives two arcs of  $H$ ). Call  $\Delta$  the function defined as  $\Delta(v) = k \cdot \alpha(v)$  for each vertex  $v$  of  $G$  and  $\Delta(v) = -k$  for each edge-vertex. Clearly,  $(\alpha, k)$ -orientations of  $G$  correspond bijectively to the  $(\Delta, \ell, u)$ -bonds of  $H$ , where  $\ell(a) = 0$  and  $u(a) = k$  for all arc  $a$  of  $H$ . Hence, the above mentioned result guarantees that the set of  $(\alpha, k)$ -orientations of  $G$  is a distributive lattice. In our context, the pushing operations on  $\Delta$ -bonds comes down to cycle-pushing.  $\square$

#### 4. DUAL OF SCHNYDER DECOMPOSITIONS

In this section, we explore the structures obtained as “dual” of Schnyder decompositions. Recall that the *dual*  $G^*$  of a plane graph  $G$  is the map obtained by drawing a vertex  $v_f$  of  $G^*$  in each face  $f$  of  $G$  and drawing an edge  $e^* = \{v_f, v_g\}$  of  $G^*$  across each edge  $e$  of  $G$  separating the faces  $f$  and  $g$  (this process is well-defined and unique up to the choice of the infinite face of  $G^*$  which can correspond to any of the external vertices of  $G$ ). An example is given in Figure 5(a). The vertex dual to the external face of  $G$  is called the *root-vertex* of  $G^*$  and is denoted  $v^*$ . The edges  $e$  and  $e^*$  are called *primal* and *dual* respectively. The *dual of a corner* of  $G$  is the corner of  $G^*$  which faces it. Observe that the degree of the vertex  $v_f$  is equal to the degree of the face  $f$ , hence the dual of a  $d$ -angulation is a  $d$ -regular plane graph. Moreover, the  $d$ -angulation has girth  $d$  if and only if its dual has mincut  $d$ .

From now on  $G^*$  is a  $d$ -regular plane graph rooted at a vertex  $v^*$ . The edges  $e_1^*, \dots, e_d^*$  in counterclockwise order around  $v^*$  are called *root-edges*, and the faces  $f_1^*, \dots, f_d^*$  in counterclockwise order ( $e_i^*$  being between  $f_i^*$  and  $f_{i+1}^*$ ) are called *root-faces*.

**Definition 16.** A regular labelling of  $G^*$  is the assignment to each corner of a color, or label, in  $[d]$  such that

- (i) The colors  $1, 2, \dots, d$  appear in clockwise order around each non-root vertex of  $G$  and appear in counterclockwise order around the root-vertex.
- (ii) For all  $i$  in  $[d]$ , the corners incident to the root-face  $f_i^*$  have color  $i$ .
- (iii) In a clockwise order around each non-root face, there is exactly one corner having a larger color than the next corner.

Note that this is exactly the dual definition of clockwise labellings on  $d$ -angulations. Consequently we have:

**Lemma 17.** A vertex-rooted  $d$ -regular plane graph  $G^*$  admits a regular labelling if and only if  $G^*$  has mincut  $d$ . In that case, regular labellings of  $G^*$  are in bijection (by giving to a corner the same color as its dual) with clockwise labellings of  $G$ .

A regular labelling and the corresponding clockwise labelling are said to be *dual* to each other. We now define a structure which is dual of Schnyder decompositions.

**Definition 18.** A regular decomposition of  $G^*$  is a covering of the edges of  $G^*$  by  $d$  spanning trees  $T_1^*, \dots, T_d^*$  (one tree for each color  $i \in [d]$ ) oriented toward  $v^*$  such that

- (i) Each edge  $e^*$  not incident to  $v^*$  appears in two of the trees  $T_i^*$  and  $T_j^*$ , and the directions of  $e^*$  in  $T_i^*$  and  $T_j^*$  are opposite.
- (ii) Each edge  $e_i^*$  incident to  $v^*$  appears only in  $T_i^*$ .
- (iii) Around any internal vertex  $v$ , the outgoing edges  $e_1, \dots, e_d$  leading  $v$  to its parent in  $T_1^*, \dots, T_d^*$  appear in clockwise order around  $v$ .

We denote by  $(T_1^*, \dots, T_d^*)$  the decomposition.

Figure 5(d) represent a regular decomposition of a 5-regular graph  $G^*$ . As for Schnyder decomposition, we shall talk about the *colors* of edges and arcs of  $G^*$ .

**Remark 19.** Property (iii) implies that a directed path of color  $i+1$  cannot cross a directed path of color  $i$  from right to left.

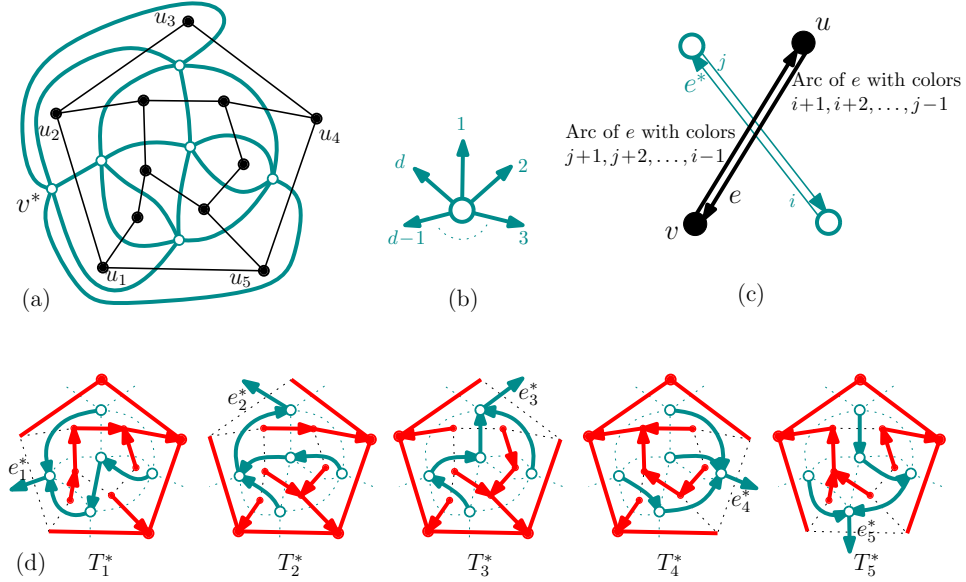


FIGURE 5. (a) Dual of a pentagulation. (b) Rule (iii) for the outgoing edges around a non-root vertex. (c) Correspondence between the colors and orientations of a edge  $e \in G$  and its dual  $e^* \in G^*$  through the mapping  $\chi$ . (d) Regular decomposition  $(T_1^*, \dots, T_d^*) = \chi(F_1, \dots, F_d)$  of  $G^*$ .

We now establish the bijective correspondence between regular labellings and regular decompositions of  $G^*$ . For a regular labelling  $L^*$  of  $G^*$ , we define  $\xi(L^*) = (T_1^*, \dots, T_d^*)$ , where  $T_i^*$  is the oriented subgraph of  $G^*$  made of every arc  $(u, e)$  such that  $u \neq v^*$  and the label of the corner preceding  $e$  in clockwise order around  $u$  is  $i$ . By duality,  $L^*$  corresponds to a clockwise labelling  $L$  of  $G$ , which itself corresponds via the bijection  $\Phi$  to a Schnyder decomposition  $(F_1, \dots, F_d)$  of  $G$ ; we then define  $\chi(F_1, \dots, F_d) = (T_1^*, \dots, T_d^*)$ .

**Lemma 20.** Let  $S = (F_1, \dots, F_d)$  be a Schnyder decomposition of  $G$  and let  $R = (T_1^*, \dots, T_d^*) = \chi(F_1, \dots, F_d)$ . For each internal edge  $e$  of  $G$ , the missing colors of  $e$  on  $S$  are the colors of the dual edge  $e^*$  on  $R$ , and the orientations obey the rule represented in Figure 5(c). Hence, if we denote by  $T_i$  the spanning tree of  $G$  obtained from the forest  $F_i$  by adding every external edge except  $\{v_i, v_{i+1}\}$ , then  $T_i^*$

is the spanning tree of  $G^*$  which is the dual of  $T_i$ : it is made of all the edges of  $G^*$  which are dual of the edges of  $G$  not in  $T_i$ . Lastly, the spanning tree  $T_i^*$  is oriented toward the root-vertex  $v_i^*$ .

Lemma 20 gives a more direct way to define the mapping  $\chi$ : the subgraphs  $(T_1^*, \dots, T_d^*) = \chi(F_1, \dots, F_d)$  can be defined as the duals of  $T_1, \dots, T_d$  oriented toward  $v^*$ ; see Figure 5(d).

*Proof.* The first assertion is clear from the definition of the mappings  $\Phi$  and  $\xi$ . From this, it follows that  $T_i^*$  is made of all the edges of  $G^*$  which are dual of edge not in  $T_i$ . It is well-known that the dual of a spanning tree is a spanning tree, hence  $T_i^*$  is a spanning tree. Lastly, since every vertex of  $G^*$  except  $v^*$  has one outgoing edge of color  $i$ , the tree  $T_i^*$  is oriented toward  $v^*$ .  $\square$

**Theorem 21.** *The mapping  $\xi$  is a bijection between regular labellings and regular decompositions of  $G^*$ . Consequently the mapping  $\chi$  is a bijection between the Schnyder decompositions of  $G$  and the regular decompositions of  $G^*$ .*

*Proof.* We first show that the image  $(T_1^*, \dots, T_d^*) = \xi(L^*)$  of any regular labelling is a regular decomposition. By Lemma 20, for all  $i$  in  $[d]$ ,  $T_i^*$  is a spanning tree of  $G^*$  oriented toward  $v^*$  and each edge non-incident to  $v^*$  have two colors in opposite directions. Hence, the Properties (i) and (ii) of regular labelling hold. Moreover, it is clear from the definition of  $\xi$  that (iii) holds. Thus,  $(T_1^*, \dots, T_d^*)$  is a regular decomposition.

We now prove that  $\xi$  is bijective. The injectivity holds since the regular labelling  $L^*$  is clearly recovered from  $(T_1^*, \dots, T_d^*)$  by giving the color  $i \in [d]$  to each corner of  $G^*$  preceding an outgoing arc of color  $i$  in clockwise order around a non-root vertex, and giving the color  $i$  also to the corner incident to  $v^*$  in the root-face  $f_i^*$ . To prove the surjectivity of  $\xi$  we must show that applying this rule to any regular decomposition  $(T_1^*, \dots, T_d^*)$  gives a regular labelling. For this purpose, we use an alternative characterization of regular labellings (which we state only as a sufficient condition):

**Claim.** *If  $L^*$  is a coloring in  $[d]$  of the corners of  $G^*$  satisfying Property (i) of regular labellings and additionally the Properties (ii') (iii') (iv') below, then  $L^*$  is a regular labelling.*

- (ii') For each non-root edge  $e = \{u, v\}$ , the colors of the corners preceding  $e$  in clockwise order around  $u$  and  $v$  respectively are different.
- (iii') For each root-edge  $e_i^* = \{v_i^*, v^*\}$ , the colors of the corners before and after  $v^*$  in clockwise order are  $i + 1, i$ , and the colors of the corners before and after  $v_i^*$  in clockwise order are  $i, i + 1$ .
- (iv') No face  $f \notin \{f_1^*, \dots, f_d^*\}$  has all its corners of the same color.

*Proof of the claim.* We suppose that  $L^*$  satisfies (ii'), (iii'), (iv') and have to prove that it satisfies Properties (ii) and (iii) of regular labellings. For this purpose it is convenient to define the *jump along an arc*  $a = (u, e)$  of  $G^*$ : if the end of the arc  $a$  is the vertex  $v$ , then the jump along  $a$  is the quantity  $j - i \in \{0, \dots, d - 1\}$ , where  $i$  (resp.  $j$ ) is the color of the corner following (resp. preceding)  $e$  in clockwise order around  $u$  (resp.  $v$ ). Property (ii') implies that a jump is never  $d - 1$ . Hence the sum of jumps along the two arcs forming a non-root edge always add up to  $d - 2$  (because this sum plus 2 is a multiple of  $d$ ). Property (iii') implies that the jump along both arcs of a root-edge is 0. Thus the sum of jumps over all arcs is  $(d - 2) \cdot m$ , where  $m$  is the number of non-root edges of  $G^*$ . Moreover, by Equation (1),  $(d - 2) \cdot m = d \cdot F$ , where  $F$  is the number of non-root faces of  $G^*$ . We now compute the sum of jumps as a sum over faces. Note that for any face  $f$ , the sum of the jumps along the arcs incident to  $f$  and having  $f$  to their right is a multiple of  $d$ . Moreover, Property (iv') guarantees that this is a positive multiple of  $d$  when  $f$  is a non-root face. This

implies that the sum of jumps is  $d$  for any non-root face (implying Property (iii)) and 0 for any root-face (implying Property (ii) since the color of one of the corner of  $f_i$  is  $i$  by (iii')).  $\triangle$

Given the Claim, it is now easy to check that the coloring  $L^*$  obtained from a regular decomposition  $S = (T_1^*, \dots, T_d^*)$  is a regular labelling. Since  $S$  satisfies Property (iii) of regular decompositions, the coloring  $L^*$  satisfies Property (i) of regular labellings. Since  $S$  satisfies Property (i) (resp. (ii)) of regular decompositions, the coloring  $L^*$  satisfies Property (ii') (resp. (iii')) of the Claim. Lastly,  $L^*$  satisfies Property (iv') of the Claim since if a face  $f \notin \{f_1^*, \dots, f_d^*\}$  had all its corner of the same color  $i$ , then the whole contour of  $f$  would belong to  $T_i^*$ , contradicting the fact that  $T_i^*$  is a tree. Hence,  $L^*$  is a regular labelling, and  $\xi$  is surjective. Thus,  $\xi$  and  $\chi$  are bijections.  $\square$

## 5. EVEN SCHNYDER DECOMPOSITIONS AND THEIR DUALS

In this section we focus on even values of  $d$  and study a special class of Schnyder decompositions and clockwise labellings (and their duals).

### 5.1. Even Schnyder decompositions and even clockwise labellings.

Let  $d = 2p$  be an even integer greater or equal to 4. Recall that Theorem 2 grants the existence of a  $p/(p-1)$ -orientation for any  $2p$ -angulation of girth  $2p$ . Recall that a  $p/(p-1)$ -orientation is a special case of  $d/(d-2)$ -orientation (where all flows are of the form  $2i/(d-2) = i/(p-1)$ ), hence there are Schnyder decompositions and clockwise labellings associated to these orientations. We call *even Schnyder decompositions* and *even clockwise labellings* the Schnyder decompositions and clockwise labellings corresponding to  $p/(p-1)$ -orientations of  $2p$ -angulations by the bijections  $\Phi$  and  $\Psi$  defined in Section 3.

In the following we consider a  $2p$ -angulation  $G$  of girth  $2p$  with external vertices denoted  $u_1, \dots, u_{2p}$  in clockwise order around the external face. The  $2p$ -angulation  $G$  is bipartite (since faces generate all cycles) hence its vertices can be properly colored in black and white. We fix the coloring by requiring the external vertex  $u_1$  to be black (so that  $u_i$  is black if and only if  $i$  is odd). We first characterize even clockwise labellings, an example of which is presented in Figure 6(a).

**Lemma 22.** *A clockwise labelling of  $G$  is even if and only if the corners incident to black vertices have odd colors, while corners incident to white vertices have even colors.*

*Proof.* By definition, a clockwise labelling is even if and only if the associated  $d/(d-2)$ -orientation  $O = \Psi(L)$  has only flows of the form  $2i/(d-2)$ . By definition of the bijection  $\Psi$ , this condition is equivalent to the fact that  $L$  has only even jumps across arcs. Equivalently, *the colors of the corners incident to a common vertex all have the same parity*. This, in turns, implies that the parity is odd around black vertices and even around white vertices because the parity change from a corner to the next around a face (by Property (i) of clockwise labellings).  $\square$

**Remark 23.** *The parity condition ensures that, in an even clockwise labelling, one can replace every label  $i$  by  $\lfloor (i-1)/2 \rfloor$  with no loss of information. For quadrangulations ( $p = 2$ ), this means that the labels are 0, 0, 1, 1 around each face. Such labellings (and extensions of them) for quadrangulations were recently studied by Felsner et al [14]. The even clockwise labellings were also considered in [1] in our form (labels 1, 2, 3, 4 around a face) to design a straight-line drawing algorithm for quadrangulations.*

We now come to the characterization of even Schnyder decompositions.

**Lemma 24.** *A Schnyder decomposition of  $G$  is even if and only if the two missing colors of each internal edge have different parity (equivalently, the edge has as many even as odd colors in  $\{1, \dots, 2p\}$ ). In this case, for all  $i \in [p]$  and for each black (resp. white) internal vertex  $v$ , the edges leading  $v$  to its parent in  $F_{2i}$  and in  $F_{2i-1}$  (resp.  $F_{2i+1}$ ) are the same.*

*Proof.* Recall that the bijection  $\Gamma = \Psi \circ \Phi^{-1}$  from Schnyder decompositions to  $d/(d-2)$ -orientations is simply the ‘‘color deletion’’ mapping. Thus a Schnyder decomposition is even if and only if the number of colors of every internal arc is even. Recall from Remark 6(a) that if  $i, j$  are the colors missing from an edge  $e$  then the colors  $i+1, \dots, j-1$  are all in one direction and the colors  $j+1, \dots, i-1$  are all in the other direction. Therefore the emphasized property above is equivalent to saying that in the Schnyder decomposition the two missing colors of any internal edge have different parity.

To prove the second statement, recall that, by Lemma 22, the colors of the corners around a black vertex are odd in an even clockwise labelling. Thus, in the corresponding even Schnyder decomposition, the colors of an arc going out of a black vertex are of the form  $2i-1, 2i, \dots, 2j-1, 2j$  for some  $i, j \in [p]$ . Similarly, the colors of an arc going out of a white vertex are of the form  $2i, 2i+1, \dots, 2j, 2j+1$  for some  $i, j \in [p]$ .  $\square$

Lemma 24 shows that there are redundancies in considering both the odd and even colors of an even Schnyder decomposition. Let  $\Lambda$  be the mapping which associates to an even Schnyder decomposition  $(F_1, \dots, F_{2p})$  the covering  $(F'_1, \dots, F'_p)$  of the internal edges of  $G$  by the forests of even color, i.e.,  $F'_i := F_{2i}$  for all color  $i$  in  $[p]$ . The forests  $F'_i = F_{2i}, i = 1, 2, 3$  are represented in Figure 6.

**Definition 25.** *Let  $G$  be a  $2p$ -angulation, with distinct external vertices denoted  $v_1, \dots, v_{2p}$ . A reduced Schnyder decomposition of  $G$  is a covering of the internal edges of  $G$  by oriented forests  $F'_1, \dots, F'_p$  such that*

- (i') *Each internal edge  $e$  appears in  $p-1$  of the forests.*
- (ii') *For each  $i \in [p]$ ,  $F'_i$  spans all vertices except  $v_{2i}, v_{2i+1}$ ; it is made of  $2p-2$  trees each containing one of the external vertices  $u_j, j \neq 2i, 2i+1$ , and the tree containing  $u_j$  is oriented toward  $u_j$  which is considered as its root.*
- (iii') *Around any internal vertex  $v$ , the edges  $e'_1, \dots, e'_p$  leading  $v$  to its parent in  $F'_1, \dots, F'_p$  appear in clockwise order around  $v$  (some of these edges can be equal). Moreover, if  $v$  is a black (resp. white) vertex, the ingoing edges of color  $i$  are between  $e'_{i+1}$  and  $e'_i$  ( $e'_i$  and  $e'_{i-1}$ ) clockwise around  $v$  and are distinct from these edges; see Figure 6(c).*

**Theorem 26.** *The mapping  $\Lambda$  is a bijection between even Schnyder decompositions of  $G$  and reduced Schnyder decompositions of  $G$ .*

As mentioned in the introduction, the case  $p = 2$  of even Schnyder decompositions already appeared in many places in the literature. Adding to  $F'_1$  the external edges  $\{v_4, v_1\}$  and  $\{v_1, v_2\}$ , and adding  $\{v_2, v_3\}$  and  $\{v_3, v_4\}$  to  $F'_2$ , one obtains a pair of non-crossing spanning trees. Such pair of trees on quadrangulations were first studied in [9] and a bijective survey on related structures appeared recently [13].

*Proof.* We first show that if  $(F'_1, \dots, F'_p) = \Lambda(F_1, \dots, F_{2p})$  (i.e.,  $F'_i = F_{2i}$ ), then the properties (i'), (ii') and (iii') are satisfied. Properties (i') and (ii') are obvious from Properties (i) and (ii) of Schnyder decompositions. For Property (iii') we first consider the situation around a black internal vertex  $v$ . By Lemma 24 the edges  $e_{2i-1}$  and  $e_{2i} = e'_i$  leading  $v$  to its parent in  $F_{2i-1}$  and  $F_{2i}$  respectively are equal, hence Property (iii) of Schnyder decompositions immediately implies Property (iii') for black vertices. The proof for white vertices is similar.



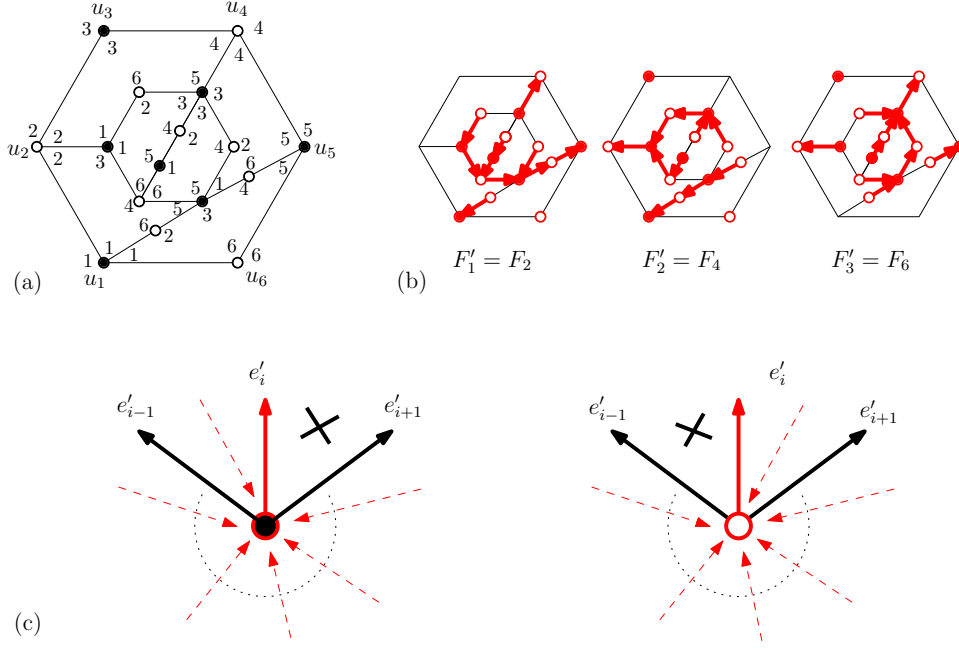


FIGURE 6. (a) An even clockwise labelling. (b) Reduced Schnyder decomposition. (c) Property (iii') of reduced Schnyder decompositions.

We now prove that  $\Lambda$  is bijective. Injectivity is clear since Lemma 24 ensures that the forests of odd colors  $(F_1, \dots, F_{2p})$  can be recovered from the even ones: starting from  $(F'_1, \dots, F'_p)$ , one defines  $F_{2i} = F'_i$  and then, for all black (resp. white) internal vertex  $v$ , one gives the color  $2i - 1$  (resp.  $2i + 1$ ) to the edge leading from  $v$  to its parent in the forest  $F_{2i}$ . To prove surjectivity we must show that applying the emphasized rule to  $(F'_1, \dots, F'_p)$  satisfying (i') (ii') (iii') always produces an even Schnyder decomposition. Properties (i) (ii) (iii) of Schnyder decompositions clearly hold, as well as the characterization of even Schnyder decomposition given in Lemma 24. The only non-trivial point is to prove that the subgraphs  $F_{2i-1}, i \in [p]$  are forests oriented toward the external vertices. If, for  $i \in [p]$  the subgraph  $F_{2i-1}$  is not a forest, then there is a simple directed cycle  $C$  of color  $2i - 1$  with only internal vertices (since each internal vertex has exactly one outgoing edge of color  $2i - 1$ ). If  $C$  is directed clockwise, we consider a vertex  $v_0$  on  $C$  and the edge of color  $2i$  going out of  $v_0$ . It is easy to see (by local constraint on the crossing of colors) that the other end  $v_1$  of this edge is either on  $C$  or inside  $C$ . Hence,  $v_1$  is an internal vertex and we can consider the edge of color  $2i$  going out of  $v_1$ . It is easy to see (by local constraint on the crossing of colors) that the other end  $v_2$  of this edge is either on  $C$  or inside  $C$ . Hence, continuing the process we find a directed cycle of color  $2i$ , which contradicts the fact that  $F_{2i} = F'_i$  is a forest. Similarly, if we assume that the cycle  $C$  is counterclockwise, we obtain a cycle of color  $2i - 2$  and reach a contradiction. This shows that there is no directed cycle in the subgraphs  $F_{2i-1}, i \in [p]$ , hence that they are forests oriented toward external vertices. Thus,  $(F_1, \dots, F_{2p})$  is an even Schnyder decomposition.  $\square$

## 5.2. Duality on even Schnyder decompositions.

Consider a  $2p$ -regular graph  $G^*$  of mincut  $2p$ . The faces of  $G^*$  are said to be *black* or *white* respectively if they are the dual of black or white vertices of the primal graph  $G$ . Call *even* the regular decompositions of  $G^*$  that are dual to even Schnyder

decompositions. We first characterize even regular decompositions. An example is shown in Figure 7.

**Lemma 27.** *A regular decomposition  $(T_1^*, \dots, T_{2p}^*)$  of  $G^*$  is even if and only if the two colors of every non-root edge have different parity. Equivalently, the spanning trees  $T_2^*, T_4^*, \dots, T_{2p}^*$  form a partition of the edges of  $G^*$  distinct from the root-edges  $e_1^*, \dots, e_{2p-1}^*$  (while  $T_1^*, T_3^*, \dots, T_{2p-1}^*$  form a partition of the edges of  $G^*$  distinct from  $e_2^*, \dots, e_{2p}^*$ ). Moreover, in this case, the arcs having an even (resp. odd) color have a black (resp. white) face on their right.*

*Proof.* The first part of Lemma 27 is obvious from Lemma 24. To prove that the arcs of even color have a black face on their right, we consider a black vertex  $v$  of the primal graph  $G$  and an incident arc  $a = (v, e)$ . By Lemma 24, the colors of  $a$  are of the form  $2i - 1, 2i, \dots, 2j - 1, 2j$  for certain integers  $i, j \in [p]$ . Therefore, by Lemma 20, the arc of the dual edge  $e^*$  having the even color (i.e., the color  $2i$ ) has the black face of  $G^*$  corresponding to  $v$  on its right.  $\square$

We denote by  $\Lambda^*$  the mapping which associates to an even regular decomposition  $(T_1^*, \dots, T_{2p}^*)$  the subsequence  $(T_1'^*, \dots, T_p'^*)$  of trees of even color,  $T_i'^* := T_{2i}^*$  for all  $i$  in  $[p]$ .

**Definition 28.** *A reduced regular decomposition of  $G^*$  is a partition of the edges of  $G^*$  distinct from the root-edges  $e_1^*, e_3^*, \dots, e_{2p-1}^*$  by  $p$  spanning trees  $T_1'^*, \dots, T_p'^*$  (one tree for each color  $i \in [p]$ ) oriented toward  $v^*$  such that*

- (i') *Every arc of the oriented tree  $T_1'^*, \dots, T_p'^*$  has a black face on its right.*
- (ii') *The root-edge  $e_{2i}^*$  is in  $T_i'^*$ .*
- (iii') *Around any internal vertex  $v$ , the edges  $e_1', \dots, e_p'$  leading  $v$  to its parent in  $T_1'^*, \dots, T_p'^*$  appear in clockwise order around  $v$ .*

We denote by  $(T_1'^*, \dots, T_p'^*)$  the decomposition.

**Theorem 29.** *The mapping  $\Lambda^*$  establishes a bijection between the even regular decompositions and the reduced regular decompositions of  $G^*$ .*

*Proof.* It is clear from Lemma 27 that the image  $(T_1'^*, \dots, T_p'^*)$  of any even regular decomposition by the mapping  $\Lambda^*$  satisfies (i'), (ii'), (iii'). Moreover the mapping  $\Lambda^*$  is injective since the odd colors can be recovered from the even ones: *starting from  $(T_1'^*, \dots, T_p'^*)$ , one defines  $T_{2i}^* = T_i'^*$  and then around each vertex  $v \neq v^*$  one gives the color  $2i - 1$  to the arc going out of  $v$  preceding the arc of color  $2i$  going out of  $v$ .* In order to show that  $\Lambda^*$  is surjective, we must show that applying the emphasized rule to a coloring  $(T_1'^*, \dots, T_p'^*)$  satisfying (i'), (ii'), (iii') always produces an even regular decomposition. Clearly, Property (i') implies that the ingoing and outgoing arcs of  $(T_1'^*, \dots, T_p'^*)$  alternate around any non-root vertex. Thus the coloring  $(T_1^*, \dots, T_{2p}^*)$  obtained is such that every non-root edges has two colors of different parity in opposite direction: Property (ii) of dual Schnyder decomposition holds as well as the condition of Lemma 27. Property (iii) of dual Schnyder decomposition also holds. It remains to show that each oriented subgraph  $T_{2i-1}^*, i \in [p]$  is a spanning tree oriented toward  $v^*$  and contains the root-edge  $e_{2i-1}^*$ . Suppose that, the subgraph  $T_{2i-1}^*$  is not a tree. In this case, there is a simple directed cycle  $C$  of color  $i$  with only non-root vertices (since each non-root vertex has exactly one outgoing edge of color  $2i - 1$ ). If  $C$  is directed clockwise, we consider a vertex  $v_0$  on  $C$  and the arc of color  $2i$  going out of  $C$ . It is easy to see (by local constraints on the colors) that the end  $v_1$  of this arc is either on  $C$  or inside  $C$ . Hence,  $v_1$  is a non-root vertex and we can consider arc of color  $2i$  going out of  $v_1$ . It is easy to see (by local constraint on the crossing of colors) that the end  $v_2$  of this arc is either on  $C$  or inside  $C$ . Hence, continuing the process we find a directed cycle of color  $2i$ , which contradicts the fact that  $T_{2i}^* = T_i'^*$  is a tree. Similarly, if the cycle

$C$  is counterclockwise, we obtain a cycle of color  $2i - 2$  and reach a contradiction. Thus the subgraphs  $T_{2i-1}^*$ ,  $i \in [p]$  are spanning trees oriented toward  $v^*$ . Lastly, we must show that the root-edge  $e_{2i-1}^*$  is in the tree  $T_{2i-1}^*$ . We denote  $v_i^*$ ,  $i \in [d]$  the non-root vertices incident to the root-edge  $e_i^*$ . The directed path  $P$  of color  $2i$  from  $v_{2i-1}^*$  to  $v^*$  goes through  $v_{2i}^*$  (since it uses  $e_{2i}^*$ ). We consider the cycle  $C$  made of  $P$  together with the root-edge  $e_{2i-1}^*$ . It is easily seen (by local constraints on the crossing of colors) that the path  $P'$  of color  $2i - 1$  from  $v_{2i}^*$  to  $v^*$  starts and stays inside  $C$  (its edges are either part of  $C$  or lies on the region enclosed by  $C$  on the left of  $C$ ). Thus, the path  $P'$  of color  $2i - 1$  must use the root-edge  $e_{2i-1}^*$  to reach  $v^*$ . Hence,  $e_{2i-1}^*$  is in  $T_{2i-1}^*$ . This completes the proof that  $(T_1^*, \dots, T_{2p}^*)$  is an even dual Schnyder decomposition and that  $\Lambda^*$  is a bijection.  $\square$

## 6. ORTHOGONAL AND STRAIGHT-LINE DRAWING OF 4-REGULAR PLANAR GRAPHS

A *straight-line drawing* of a (planar) graph is a planar drawing where each edge is drawn as a segment. An *orthogonal drawing* is a planar drawing where each edge is represented as a sequence of horizontal and vertical segments. We present and analyse an algorithm for obtaining (in linear time) straight-line and orthogonal drawings of 4-regular plane graphs of mincut 4.

In all this section,  $G$  denotes a 4-regular plane graph of mincut 4 rooted at a vertex  $v^*$ , and with  $n+1$  vertices (hence with  $2n+2$  edges and  $n+3$  faces by the Euler relation). As in previous sections, we denote by  $e_1^*, e_2^*, e_3^*, e_4^*$  and call *root-edges* the edges incident to  $v^*$  in counterclockwise order, and we denote by  $v_1^*, v_2^*, v_3^*, v_4^*$  the other end of these edges (these vertices are not necessarily distinct). We call *root faces* the 4 faces incident to  $v^*$  and non-root the  $n - 1$  others. As shown in the previous section, there exists an even regular decomposition  $(T_1^*, \dots, T_4^*)$  of  $G$  (with  $e_i^* \in T_i^*$ ) and we now work with this decomposition. The spanning trees  $(T_1^*, \dots, T_4^*)$  satisfy the properties of regular decompositions (Definition 18) and the additional property of even Schnyder decomposition given by Lemma 27. An even dual Schnyder decomposition  $(T_1^*, \dots, T_4^*)$  is shown in Figure 7(left).

### 6.1. Vertex placement and planar drawings using face-counting operations.

For a vertex  $v \neq v^*$  and a color  $i$  in  $\{1, 2, 3, 4\}$ , we denote by  $P_i(v)$  the directed path of color  $i$  from  $v$  to  $v^*$ . We first establish a technical lemma about these paths.

**Lemma 30.** *For all  $i \in \{1, 2, 3, 4\}$  the paths  $P_i(v)$  and  $P_{i+2}(v)$  only intersect at  $v$  and  $v^*$ . Equivalently,  $P_i(v) \cup P_{i+2}(v)$  is a simple cycle.*

*Proof.* Assume the contrary, and consider the first vertex  $v' \neq v$  on the directed path  $P_i(v)$  which belongs to  $P_{i+2}(v)$ . Let  $P'_i, P'_{i+2}$  be the part of the paths  $P_i(v), P_{i+2}(v)$  from  $v$  to  $v'$ . Clearly,  $C = P'_i \cup P'_{i+2}$  is a simple cycle not containing  $v^*$ . For  $j \in \{1, 2, 3, 4\}$ , let  $e_j$  be the edge of color  $j$  going out of  $v'$ . Since the two colors of any non-root edge of  $G$  have different parity,  $e_i$  and  $e_{i+2}$  are not part of the cycle  $C = P'_i \cup P'_{i+2}$ . Instead, one of these edges is strictly inside the cycle  $C$ . Let us first suppose that  $e_i$  is inside  $C$  and consider the directed path  $P_i(v')$  starting with edge  $e_i$ . This path cannot cross  $P'_i$  because it would create a cycle of color  $i$  and it cannot cross  $P'_{i+2}$  by the color incidence rules. Therefore, the directed path  $P_i(v')$  is trapped in the cycle  $C$  and cannot reach  $v^*$ . We reach a contradiction. Similarly, the assumption that  $e_{i+2}$  is inside  $C$  leads to a contradiction.  $\square$

For  $i$  in  $\{1, 2, 3, 4\}$ , the cycle  $P_i(v) \cup P_{i+2}(v)$  separates two regions of the plane. We denote by  $R_{i,i+2}(v)$  the region containing the root-edge  $e_{i+1}^*$ . We also denote by  $x(v)$  and  $y(v)$  respectively the number of faces *minus 1* in the regions  $R_{1,3}(v)$  and

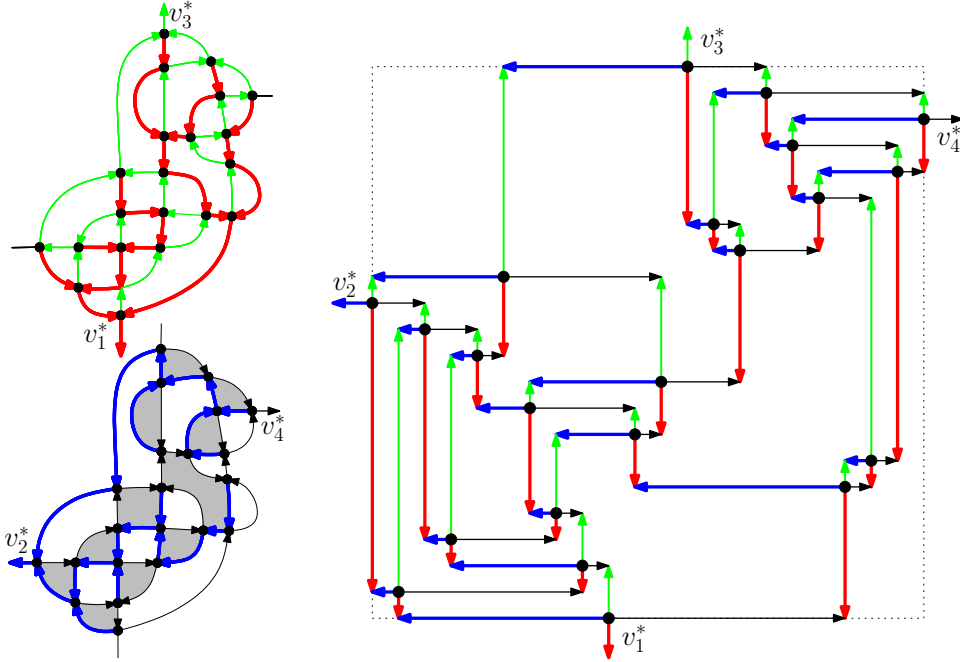


FIGURE 7. Left: an even dual Schnyder decomposition of a 4-regular plane graph  $G$  (the spanning trees  $T_1^*, T_3^*$  are represented in the upper part and the trees  $T_2^*, T_4^*$  are represented in the lower part). Right: the orthogonal drawing of  $G$ .

$R_{4,2}(v)$ . This associates a point  $p(v) = (x(v), y(v))$  in the grid  $[n] \times [n]$  to any non-root vertex  $v$  of  $G$ . Informally, if the vertices  $v_1^*, v_2^*, v_3^*, v_4^*$  are thought as down, left, up, right, then the coordinate  $x(v)$  corresponds to the number of faces on the left of the “vertical line”  $P_1(v) \cup P_3(v)$  and the coordinate  $y(v)$  corresponds to the number of faces below the “horizontal line”  $P_2(v) \cup P_4(v)$ . This placement of vertices is represented in Figure 7. As stated next, it yields both a planar orthogonal drawing and a planar straight-line drawing.

Before stating the straight-line drawing result, we make the following observation: if a 4-regular graph of mincut 4 has a double edge, then the double edge must delimit a face (of degree 2). Denote by  $\tilde{G}$  the simple graph obtained from  $G$  by emptying all faces of degree 2 (i.e., turning such a double edge into a single edge).

**Theorem 31** (straight-line drawing). *The placement of each non-root vertex  $v$  at the point  $p(v) = (x(v), y(v))$  of the  $[n] \times [n]$  grid gives a planar straight-line drawing of  $\tilde{G} \setminus v^*$ . Moreover the points  $p(v_1^*), p(v_2^*), p(v_3^*), p(v_4^*)$  are respectively on the down, left, up, and right boundaries of the grid.*

For a vertex  $v \neq v^*$  of  $G$  we call *ray in the direction 1* (resp. 2,3,4) from the point  $p(v)$  the half-line starting from  $p(v)$  and going in the negative  $y$  direction (resp. negative  $x$  direction, positive  $y$  direction, positive  $x$  direction). For an edge  $e = \{u, v\}$  of  $G$  not incident to  $v^*$ , we denote  $p(e)$  the intersection of the ray in direction  $i$  from  $p(u)$  and the ray in direction  $j$  from  $p(v)$ , where  $i$  is the color of arc  $(u, v)$  and  $j$  is the color of the arc  $(v, u)$ . Observe that one of the ray is horizontal while the other is vertical (because  $i$  and  $j$  have different parity), hence the intersection  $p(e)$  is *a priori* either empty or a point. If  $p(e)$  is a point (this is always the case, as we will prove shortly), then we call the union of segment  $[p(u), p(e)] \cup [p(v), p(e)]$  the *bent-edge* corresponding to  $e$ . We say that the bent-edge from  $u$  to  $v$  is *down-left* (resp. *down-right*, *up-left*, *up-right*) if the vector from

$p(u)$  to  $p(e)$  is down (resp. down, up, up) and the vector from  $p(e)$  to  $p(v)$  is left (resp. right, left, right). We now state the main result of this section.

**Theorem 32** (orthogonal drawing). *For each non-root edge  $e = \{u, v\}$  of  $G$ , the intersection  $p(e)$  is a point. Moreover, if one places each non-root vertex  $v$  of  $G$  at the points  $p(v)$  of the grid  $[n] \times [n]$  (which is of size  $(n-1) \times (n-1)$ ) and draws the bent-edge  $[p(u), p(e)] \cup [p(v), p(e)]$  for each non-root edge  $e = \{u, v\}$  of  $G$ , one obtains a planar orthogonal drawing of  $G \setminus v^*$  with one bend per edge. Moreover, the drawing has the following properties:*

- (1) Each line and column of the grid  $[n] \times [n]$  contains exactly one vertex.
- (2) The spanning tree  $T_1^*$  (resp.  $T_2^*, T_3^*, T_4^*$ ) is made of all the edges  $e = \{u, v\}$  such that the vector from  $p(u)$  to  $p(e)$  is going down (resp. left, up, right).
- (3) Every non-root face  $f$  has two distinct edges  $f_a = \{a, a'\}$   $f_b = \{b, b'\}$  called special. If the face  $f$  is black the bent-edges in clockwise direction around  $f$  are as follows: the special bent-edge  $\{a, a'\}$  is right-down, the edges from  $a'$  to  $b$  are left-down, the special bent-edge  $\{b, b'\}$  is left-up, the edges from  $b'$  to  $a$  are right-up; see Figure 8. The white faces satisfy the same property with right,down,left,up replaced by up,right,down,left. Moreover, for black (resp. white) faces, one has  $x(a) + 1 = y(b)$  and  $y(a') + 1 = y(b')$  (resp.  $x(a') - 1 = x(b')$  and  $y(a) + 1 = y(b)$ ).

Moreover, adding the root-vertex  $v^*$  and its 4 incident edges  $e_1^*, e_2^*, e_3^*, e_4^*$  to the embedding adds two more rows and columns and requires 8 additional bends (3 for each of  $e_1^*, e_2^*$  and 1 for each of  $e_3^*, e_4^*$ ), see Figure 10. Overall the planar orthogonal drawing a 4-regular plane graph of mincut 4 with  $n$  vertices is on the  $n \times n$  grid and has a total of  $2n + 4$  bends.

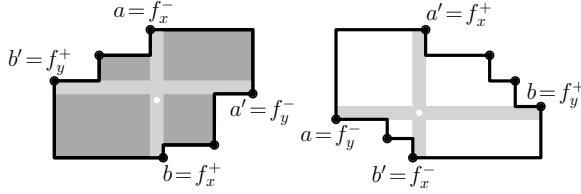


FIGURE 8. Property (3) for a white face (left) and a black face (right).

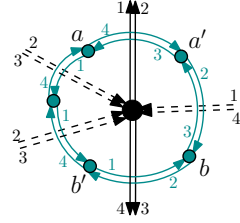


FIGURE 9. Around a black vertex of a quadrangulation.

Before starting the proof of Theorems 31 and 32 we make a few easy observations. Let  $i \in \{1, 2, 3, 4\}$ , let  $v$  be a non-root vertex of  $G$ , and let  $u \neq v$  be a non-root vertex on the cycle  $P_i(v) \cup P_{i+2}(v)$ . The edges of  $u$  belonging to the cycle  $P_i(v) \cup P_{i+2}(v)$  are consecutive around  $u$  (because the two colors of any edge have different parity). Therefore, the two edges incident to  $u$  and not in the cycle  $P_i(v) \cup P_{i+2}(v)$  are either both strictly in the region  $R_{i,i+2}(v)$  or both strictly outside this region. A non-root vertex  $w \neq v$  is said to be *weakly inside* the region  $R_{i,i+2}(v)$  if it is either strictly

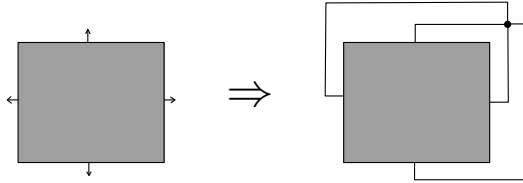


FIGURE 10. Adding the root-vertex to the orthogonal drawing.

inside this region or if it is on the cycle  $P_i(v) \cup P_{i+2}(v)$  with two edges strictly inside this region.

**Lemma 33.** *Let  $i$  be a color in  $\{1, 2, 3, 4\}$  and let  $u, v$  be distinct non-root vertices of  $G$ . Then either  $R_{i,i+2}(u) \subsetneq R_{i,i+2}(v)$  or  $R_{i,i+2}(v) \subsetneq R_{i,i+2}(u)$ . Moreover  $R_{i,i+2}(u) \subsetneq R_{i,i+2}(v)$  if and only if  $u$  is weakly inside  $R_{i,i+2}(v)$ .*

Observe that Lemma 33 implies that for all  $i \in \{1, 2, 3, 4\}$  the regions  $R_{i,i+2}(v)$  are (strictly) totally ordered by inclusion. In particular, the non-root vertices of  $G$  all have distinct  $x$  coordinates and distinct  $y$  coordinates. Since the  $x$  and  $y$  coordinates are constrained to be in  $[n]$ , this implies that the vertices are placed according to a permutation: each line and column of the grid contains exactly one vertex<sup>4</sup>. The rest of this section is devoted to the proof of Lemma 33 and then Theorems 32 and 31.

*Proof of Lemma 33.* • We first prove that  $R_{i,i+2}(u) \subsetneq R_{i,i+2}(v)$  if and only if  $u$  is weakly inside  $R_{i,i+2}(v)$ . First suppose that  $R_{i,i+2}(u) \subsetneq R_{i,i+2}(v)$ . In this case, either  $u$  is strictly inside  $R_{i,i+2}(v)$  (hence weakly inside) or on the cycle  $P_i(v) \cup P_{i+2}(v)$ . If  $u \in P_i(v)$ , the edge  $e_{i+2}$  of color  $i+2$  going out of  $u$  is not in  $P_i(v) \cup P_{i+2}(v)$  (because each edge of  $G$  has two colors of different parity in opposite direction). Therefore the edge  $e_{i+2}$  which belongs to  $R_{i,i+2}(u) \subsetneq R_{i,i+2}(v)$  is strictly inside  $R_{i,i+2}(v)$ . Thus  $u$  is weakly inside  $R_{i,i+2}(v)$ . A similar proof shows that if  $u \in P_{i+2}(v)$ , then  $u$  is weakly inside  $R_{i,i+2}(v)$ . Thus, in all cases  $u$  is weakly inside  $R_{i,i+2}(v)$ .

We now prove the other direction of the equivalence: we suppose that  $u$  is weakly inside  $R_{i,i+2}(v)$  and want to prove that  $R_{i,i+2}(u) \subsetneq R_{i,i+2}(v)$ . It suffices to show that the paths  $P_i(u)$  and  $P_{i+2}(u)$  have no edge strictly outside of the region  $R_{i,i+2}(v)$ . We first prove that  $P_i(u)$  has no edge strictly outside of  $R_{i,i+2}(v)$ . Let  $e_1$  be the edge of color  $i$  going out of  $u$  (i.e. the first edge of the directed path  $P_i(u)$ ). Since  $u$  is weakly inside  $R_{i,i+2}(v)$  either  $e_1$  belongs to  $P_i(v)$  (in which case  $P_i(u)$  has no edge strictly outside of  $R_{i,i+2}(v)$ ) or  $e_1$  is strictly inside  $R_{i,i+2}(v)$  (the endpoints of  $e_1$  might belong to the cycle  $P_i(v) \cup P_{i+2}(v)$ , but not the edge  $e_1$ ). In the second case, the end  $u_1$  of  $e_1$  is weakly inside  $R_{i,i+2}(v)$ . Thus, the edge  $e_2$  of color  $i$  going out of  $u_1$  is either in  $P_i(v)$  or strictly inside the region  $R_{i,i+2}(v)$ . Continuing in this way, it is clear that the path  $P_i(u) = e_1, e_2, \dots$  has no edge strictly outside of  $R_{i,i+2}(v)$ . Similarly, the path  $P_{i+2}(u)$  has no edge strictly outside of region  $R_{i,i+2}(v)$ .

• We now suppose that the region  $R_{i,i+2}(u)$  is not included in  $R_{i,i+2}(v)$  (and want to prove  $R_{i,i+2}(v) \subsetneq R_{i,i+2}(u)$ ). By the preceding point, this implies that  $u$  is not weakly inside  $R_{i,i+2}(v)$ . In this case,  $u$  is weakly inside the complementary region  $R_{i+2,i}(v)$ . By the preceding point (applied to color  $i+2$ ) this implies  $R_{i+2,i}(u) \subsetneq R_{i+2,i}(v)$ . Or equivalently  $R_{i,i+2}(v) \subsetneq R_{i,i+2}(u)$ .  $\square$

*Proof of Theorem 32.* • We first prove that for a non-root edge  $e = \{u, v\}$ , the intersection  $p(e)$  is a point. It is easy to see that if the arc of  $e$  from  $u$  to  $v$  is colored  $i$ , then the vertex  $v$  is weakly inside the region  $R_{i-1,i+1}(u)$ . By Lemma 33, this implies that for  $i = 1$  (resp.  $i = 2, 3, 4$ ), the point  $p(v)$  is below (resp. on the left of, above, on the right of) the point  $p(u)$ . This shows that the intersection  $p(e)$  is non-empty, hence, a point.

• We now show that the orthogonal drawing is planar. Consider an edge  $e = \{u, v\}$  and the segment  $[p(u), p(e)]$  (which is the embedding of the arc  $(u, e)$ ). This segment contains no point  $p(w)$  for  $w \neq u$  since every line and column of the grid contains exactly one point. Assume  $[p(u'), p(e')]$  crosses the segment  $[p(u), p(e)]$  for another

<sup>4</sup>As mentioned in the introduction, our placement of vertices is closely related to the bijection between plane bipolar orientations and Baxter permutations in [7], but we will not need to explicit the relation with this bijection here.

arc  $(u', e')$ , with  $v'$  the other extremity of  $e'$ . Clearly,  $u' \neq u, v$  and  $v \neq u'$  (but the case  $v = v'$  is possible). By symmetry between the colors, we can assume that the color of the direction of  $e$  from  $u$  to  $v$  is 1. Thus, the segment  $[p(u), p(e)]$  is vertical with  $x(e) = x(u)$  and  $y(e) = y(v) < y(u)$ . Moreover the segment  $[p(u'), p(e')]$  (which we assume to cross  $[p(u), p(e)]$ ) is horizontal with  $x(e') = x(v')$  and  $y(v) < y(e') = y(u') < y(u)$ . We now consider the case  $x(u) < x(v)$  (the case  $x(u) > x(v)$  being symmetric). This means that the color of the direction of  $e$  from  $v$  to  $u$  is 2 (in the other case  $x(u) > x(v)$ , the color would be 4). The situation is represented in Figure 11. Observe that path  $P_1(u)$  is equal to  $\{e\} \cup P_1(v)$ , while the path  $P_2(v)$  is equal to  $\{e\} \cup P_2(u)$ . Moreover, all the edges of the path  $P_4(v)$  (resp.  $P_4(u)$ ) lie strictly inside the region  $R_{3,1}(u)$  (resp.  $R_{3,1}(v)$ ). Indeed the local constraints on the crossing of colors prevents  $P_4(u)$  (resp.  $P_4(v)$ ) from crossing the cycle  $P_3(u) \cup P_1(u)$  (resp.  $P_3(v) \cup P_1(v)$ ). Furthermore,  $R_{3,1}(v) \subseteq R_{3,1}(u)$  (because  $x(u) < x(v)$  and Lemma 33), hence the paths  $P_4(u), P_4(v)$  have all their edges strictly inside  $R_{3,1}(u)$ .

Let  $C$  be the cycle made of  $e$  and the paths  $P_4' = P_4(u)$ ,  $P_4'' = P_4(v)$  between  $u$ ,  $v$  and their common ancestor in the tree  $T_4^*$  (the region enclosed by  $C$  is shaded in Figure 11). By the arguments above, we know that all the edges of  $C$  except  $e$  are strictly inside  $R_{3,1}(u)$ . By Lemma 33, the inequality  $y(v) < y(u') < y(u)$  implies that  $u'$  is weakly inside  $R_{2,4}(v)$  and weakly inside  $R_{4,2}(u)$ . Thus  $u'$  is either strictly inside  $C$  or on  $C$  with two edges strictly inside  $C$ . Thus,  $u'$  has its four edges strictly inside the region  $R_{3,1}(u)$ . In particular,  $x(u) < x(u')$  by Lemma 33. Hence, if the segment  $[p(u'), p(e')]$  is to cross  $[p(u), p(e)]$ , one must have  $x(v') < x(u)$ . By Lemma 33, this implies that  $v'$  is weakly in the region  $R_{1,3}(u)$ . Thus,  $v'$  has none of its four edges strictly inside the region  $R_{1,3}(u)$ . But this prevents  $u'$  and  $v'$  from being adjacent, and gives a contradiction.

- We shall now prove Properties (1), (2), (3). Property (1) has already been proved and Property (2) is immediate from the definitions. We now prove Property (3) for a black face  $f$  (the case of a white face being symmetric). We first study the colors of the arcs which appear in clockwise direction around  $f$ . Let  $Q$  be the quadrangulation which is the dual of  $G$  (that is,  $G = Q^*$ ) and let  $v$  be the vertex of  $Q$  corresponding to  $f$ . By Lemma 24 on even Schnyder decomposition, the edges in clockwise order around  $v$  are made of an edge with colors 1,2 oriented away from  $v$ , a (possibly empty) sequence of edges with colors 4,1 oriented toward  $v$ , an edge with colors 3,4 oriented away from  $v$ , and a (possibly empty) sequence of edge with colors 2,3 oriented toward  $v$ . Consequently, by Lemma 20 on the duality relations of edge colors, the edges of  $G$  in clockwise order around the black face  $f$  are made of an edge with clockwise color 4 and counterclockwise color 3 (right-down bent-edge), a sequence of edges with clockwise color 2 and counterclockwise color 3 (left-down bent-edges), an edge with clockwise color 2 and counterclockwise color 1 (left-up bent-edge), a sequence of edges with clockwise color 4 and counterclockwise color 1 (right-up bent-edges). Denoting by  $e_a = \{a, a'\}$  the edge with clockwise color 4 and counterclockwise color 3 and by  $e_b = \{b, b'\}$  the edge with color 2 and counterclockwise color 1, we have proved the first part of Property (3). It remains to prove  $x(a) + 1 = x(b)$  and  $y(a') + 1 = y(b')$ . Observe that the counterclockwise path from  $a$  to  $b$  around  $f$  has color 1 while the counterclockwise path from  $b$  to  $a$  around  $f$  has color 3. Therefore the regions  $R_{1,3}(a)$  and  $R_{1,3}(b)$  only differ by the face  $f$ :  $R_{1,3}(a) \cup \{f\} = R_{1,3}(b)$ . Hence,  $x(a) + 1 = x(b)$ . Similarly, the clockwise path from  $a'$  to  $b'$  has color 2 and the clockwise path from  $b'$  to  $a'$  has color 4. Thus,  $R_{4,2}(a') \cup \{f\} = R_{4,2}(b')$  and  $x(a') + 1 = x(b')$ .  $\square$

Before embarking on the proof of Theorem 31, let us first frame an easy consequence of Property (3) in Theorem 32 (see Figure 8).

**Lemma 34.** *The orthogonal drawing satisfies the following property.*

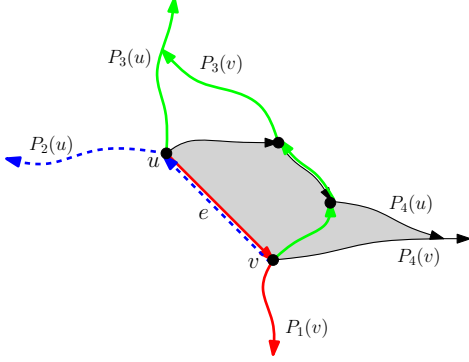


FIGURE 11. Edge  $e = \{u, v\}$  in the proof of Theorem 32.

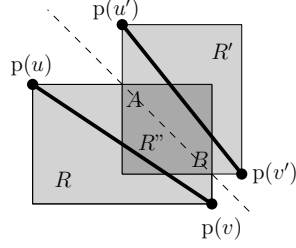


FIGURE 12. Rectangles  $R, R'$  in the proof of Theorem 31.

- (4) Each of the  $2n - 2$  non-root edges  $e = \{u, v\}$  is special for exactly one non-root face  $f_e$ . Let  $R$  be the rectangle with diagonal  $[p(u), p(v)]$  (and sides parallel to the axes). Then  $f_e$  is characterized as the unique face of the embedding that contains  $R$ . Moreover, there is no vertex other than  $u, v$  in  $R$  or on its boundary.

*Proof of Theorem 31.* We have to prove that the straight-line drawing is planar. Let  $e = \{u, v\}$  be a non-root edge of  $G$ . Lemma 34 ensures that no vertex lies on the segment  $[p(u), p(v)]$ . We consider another non-root edge  $e' = \{u', v'\}$  (with  $\{u, v\} \cap \{u', v'\} = \emptyset$ ) and want to prove that the segments  $[p(u), p(v)]$ ,  $[p(u'), p(v')]$  are not intersecting. We consider the rectangles  $R, R'$  with diagonal  $[p(u), p(v)]$  and  $[p(u'), p(v')]$  respectively (and sides parallel to the axes) and their intersection  $R''$ . If  $R'' = \emptyset$  the segments  $[p(u), p(v)]$ ,  $[p(u'), p(v')]$  do not intersect, hence we consider the case  $R'' \neq \emptyset$ . By Lemma 34, the rectangle  $R''$  contains none of the points  $p(u), p(v), p(u'), p(v')$ . Let  $A, B$  be the intersections of the boundaries of  $R$  and  $R''$  (see Figure 12). It is easy to see that the points  $p(u), p(v)$  are both on the same side of the line  $(A, B)$ , while the points  $p(u'), p(v')$  are both on the other side; see Figure 12. Thus, the segments  $[p(u), p(v)]$  and  $[p(u'), p(v')]$  are on different sides of  $(A, B)$  and do not intersect.  $\square$

## 6.2. Reduction of the grid size.

In this subsection, we present a way of reducing the size of the grid for the drawing of  $G$ . We consider the placement  $p(v) = (x(v), y(v))$  of the non-root vertices of  $G$  defined in the previous subsection. Recall from Property (3) in Theorem 32 that any non-root face  $f$  of  $G$  has two special edges  $f_a = \{a, a'\}$  and  $f_b = \{b, b'\}$ . If the face  $f$  is black (resp. white), we denote  $f_x^- = a$ ,  $f_x^+ = b$ ,  $f_y^- = a'$ ,  $f_y^+ = b'$  (resp.  $f_x^- = b'$ ,  $f_x^+ = a'$ ,  $f_y^- = a$ ,  $f_y^+ = b$ ); see Figure 8. We then define the  $x$  and  $y$  coordinates of the face  $f$  by:  $x(f) = x(f_x^+)$  and  $y(f) = y(f_y^+)$ .

**Lemma 35.** *The  $x$  (resp.  $y$ ) coordinates of the non-root faces of  $G$  are all distinct.*

*Proof.* We make the proof for the  $x$  coordinates. Suppose  $f, f'$  are non-root faces such that  $x(f) = x(f')$ . Because the  $x$  coordinates of the non-root vertices are all distinct, this implies  $f_x^+ = f'^+_x$  (since  $x(f_x^+) = x(f'^+_x) = f(x)$ ) and also  $f_x^- = f'^-_x$  (since  $x(f_x^-) = x(f'^-_x) = f(x) - 1$ ). By Property (3) of Theorem 32, the rectangle with diagonal  $[f_x^-, f_x^+]$  is included in the face  $f$  (see Figure 8), hence  $f = f'$ .  $\square$



Let  $f$  be a face such that the vertices  $f_x^-, f_x^+, f_y^-, f_y^+$  are all distinct. We say that  $f$  is *partly-reducible* if there are no other vertices and *fully-reducible* if there are some other vertices. The drawing in Figure 13 has 1 partly-reducible face and 4 fully-reducible faces. A *reduction choice* is a pair  $(I, J)$  with  $I, J \subseteq \{2, 3, \dots, n\}$  such that  $I$  contains the  $x$  coordinates of the fully-reducible faces together with the  $x$  coordinates of a subset of the partly-reducible faces, while  $J$  contains the  $y$  coordinates of the fully-reducible choices together with the  $x$  coordinates of the complementary subset of partly-reducible faces.

We now prove that deleting the columns and lines corresponding to any reduction choice gives *non-crossing* orthogonal and straightline drawings. More precisely, given a reduction choice  $(I, J)$ , we define new coordinates  $p'(v) = (x'(v), y'(v))$  for any non-root vertex  $v$ , where  $x'(v) = x(v) - |[x(v)] \cap I|$  and  $y'(v) = y(v) - |[y(v)] \cap J|$ . Clearly,  $x(u) < x(v)$  implies  $x'(u) \leq x'(v)$ , and  $y(u) < y(v)$  implies  $y'(u) \leq y'(v)$  for any vertices  $u, v$ . Thus the orthogonal drawing of  $G$  with placement  $p'$  is well defined (the rays from  $p'(u)$  and  $p'(v)$  intersect each others for any edge  $e = \{u, v\}$ ). We call *reduced* the orthogonal and straight-line drawing obtained with the the new placement  $p'$  of vertices. These drawings are represented in Figure 13.

**Proposition 36.** *For any reduction choice  $(I, J)$  for  $G$ , the reduced orthogonal drawing is non-crossing, and every edge has exactly one bent. Moreover, the reduced straight-line drawing is non-crossing provided that  $G$  has no multiple edges.*

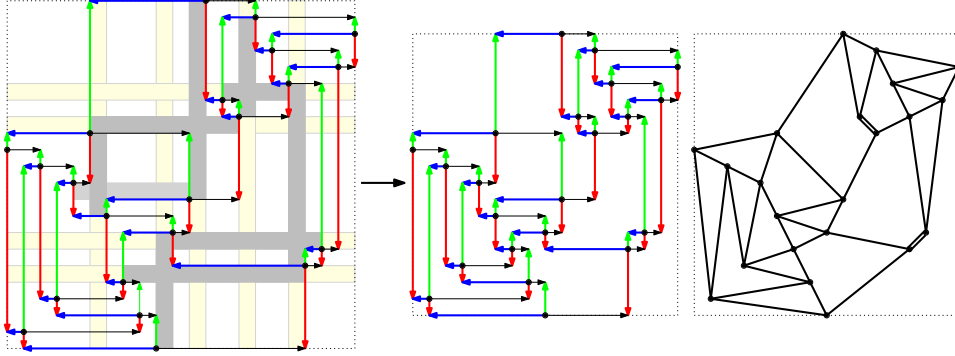


FIGURE 13. Reduced orthogonal and straight-line drawings.

The rest of this subsection is devoted to the proof of Proposition 36.

**Lemma 37.** *Let  $u, v$  be adjacent non-root vertices of  $G$ . If  $x(u) < x(v)$ , then  $x'(u) < x'(v)$ . Similarly, if  $y(u) < y(v)$ , then  $y'(u) < y'(v)$ . In particular the reduced orthogonal drawing has exactly one bend per edge.*

*Proof.* We show the property for the  $x$  coordinates. Suppose by contradiction that there exists an edge  $e = \{u, v\}$  with  $x(u) < x(v)$  and  $x'(u) = x'(v)$ . We can choose  $e$  such that the difference  $x(v) - x(u)$  is minimal. Clearly  $x'(u) = x'(v)$  means that all the columns between  $p(u)$  and  $p(v)$  have been erased:  $\{x(u)+1, x(u)+2, \dots, x(v)\} \subseteq I$ . By Lemma 34, we know that  $e$  is the special edge of a face  $f$ . Since  $x(u) < x(v)$ , one has either  $u = f_x^-$  or  $v = f_x^+$  (see Figure 8). We first assume that  $u = f_x^-$ . In this case,  $x(f) = x(u) + 1$  is in  $I$ , thus  $f$  is either partly- or fully-reducible. Hence,  $v \neq f_x^+$ . We consider the non-special edge  $e' = \{f_x^+, v'\}$  around  $f$ . Since  $x(u) + 1 = x(f_x^+) \leq x(v') \leq x(v)$ , one has  $x'(f_x^+) = x'(v')$  in contradiction with our minimality assumption on  $e$ . The alternative assumption  $v = f_x^+$  also leads to a contradiction by a similar argument.  $\square$

Lemma 37 gives the following Corollary illustrated in Figure 14.

**Corollary 38.** *The following property holds after deletion of any subset of columns in  $I$  and any subset of lines in  $J$ .*

- (3') *Let  $f$  be a non-root face of  $G$  and  $f_a = \{a, a'\}$ ,  $f_b = \{b, b'\}$  be its special edges. If  $f$  is black the bent-edges in clockwise direction are as follows: the special bent-edge  $\{a, a'\}$  is right-down, the edges from  $a'$  to  $b$  are left-down, the special bent-edge  $\{b, b'\}$  is left-up, the edges from  $b'$  to  $a$  are right-up. The white faces satisfy the same property with right,down,left,up replaced by up,right,down,left.*

*Moreover, for black (resp. white) faces, the coordinates  $x'', y''$  of the point after the partial reduction satisfy  $x''(a) + \epsilon_x = x''(b)$ , where  $\epsilon_x = 0$  if the column  $x(f)$  has been deleted and 1 otherwise. Similarly  $y''(a') + \epsilon_y = y''(b')$  (resp.  $y''(a) - \epsilon_y = y''(b)$ ), where  $\epsilon_y = 0$  if the line  $y(f)$  has been deleted and 1 otherwise.*

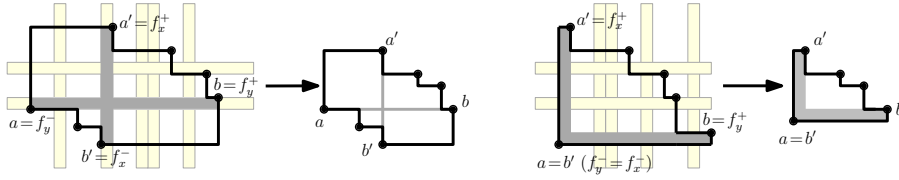


FIGURE 14. Effect of reduction on a fully-reducible face (left) and a non-reducible face (right).

*Proof of Property 36.* We first prove the planarity of the orthogonal drawing. We call *reduction process* the fact of deleting the columns in  $I$  and lines in  $J$  one at a time (in any order). We want to prove that there is no crossing at any time of the reduction process. Suppose the contrary and consider the first crossing. We suppose by symmetry that it occurs when deleting a column. In this case, it has to be that two vertical segments of the orthogonal drawing are made to superimpose by the column deletion. Just before the collision the two vertical segments had to be part of a common face  $f$ . By Corollary 38 the only possible collision is between the vertical rays out of the vertices  $f_x^-$  and  $f_x^+$  (indeed around  $f$  the vertical segments that were initially strictly to the left of  $f_x^-$  remain so, and the vertical segments that were initially strictly to the right of  $f_x^+$  remain so, see Figure 14). However, for the vertical rays out of  $f_x^-$  and  $f_x^+$  to collide it is necessary that  $x(f) \in I$  (so that the rays get to the same vertical line) and that  $y(f) \in J$  (since one of the ray is below  $f_y^-$  and the other is above  $f_y^+$ ). Thus, the face  $f$  is fully-reducible. But in this case it is clear that the vertical rays out of  $f_x^-$  and  $f_x^+$  cannot collide (because either one is strictly below  $f_y^-$  or the other is strictly above  $f_y^+$ , see Figure 14). We reach a contradiction, hence the reduced orthogonal drawing is non-crossing.

The proof that the reduced straight-line drawing is non-crossing is identical to the proof given in Section 6 for the non-reduced straight-line drawing (proof of Theorem 31). Indeed this proof is entirely based on the non-crossing property of the orthogonal drawing and Property (4) given in Lemma 34. Moreover, Property (4) holds for the reduced orthogonal drawing because of Corollary 38.  $\square$

### 6.3. Analysis of the grid-reduction for a random instance.

We analyze here the average grid-size of the drawing after the reduction step. We denote by  $\mathcal{R}_n$  the set of pairs  $(G, X)$  where  $G$  is a 4-regular plane graph of mincut 4 with  $n$  vertices and  $X$  is an even regular decomposition of  $G$ . In order to know the grid size obtained for a random instance in  $\mathcal{R}_n$  we must study the number of partly- and fully-reducible faces. We say that a sequence of real random variables

$(X_n)_{n \in \mathbb{N}}$  is *strongly concentrated* around  $\alpha n$  (for a real number  $\alpha \neq 0$ ) if for all  $\epsilon > 0$ , there exists  $c > 0$  such that  $\mathbb{P}(|X_n - \alpha n| > \epsilon n) = o(\exp(-cn))$ .

**Proposition 39.** *The number of partly-reducible and fully-reducible faces in a uniformly random instance of  $\mathcal{R}_n$  are strongly concentrated around  $n/16$  and  $3n/16$  respectively. Consequently, for a “balanced” reduction choice, the grid-size after the reduction step is highly concentrated around  $25n/32 \times 25n/32$ .*

The rest of the subsection is dedicated to a sketch of proof. First we come to the dual setting. The dual of a regular decomposition  $(G, X) \in \mathcal{R}_n$  is an even Schnyder decomposition of a quadrangulation  $Q = G^*$ , whose reduced form is a partition of  $Q$  into two non-crossing spanning trees: a *red* one and a *blue* one. Such partitions, called *separating decompositions* have been introduced in [9]. Let  $\mathcal{R}_n^*$  be the set of separating decompositions with  $n$  faces. Let  $v$  be a vertex of a separating decomposition  $(Q, X) \in \mathcal{R}_n^*$  and let  $f$  be the corresponding face of  $G = Q^*$ . It is easily seen by duality (see Figure 8 and 9) that the face  $f$  is partly-reducible (resp. fully-reducible) if and only if the number of children of  $v$  in the red and blue trees of the separating decomposition  $X$  are both equal to 1 (resp. both at least 1 and not both equal to 1).

We now use some results from [18], where a bijection for separating decompositions is presented. Let  $\mathcal{B}_n$  be the set of triples  $(w_1, w_2, w_3)$  of binary words of length  $n$  (that is,  $w_i \in \{0, 1\}^n$ ). Let  $\mathcal{B}'_n$  be the subset of triples such that  $w_1, w_2, w_3$  have the same numbers of letters 1 and such that for all  $k < n$ , the numbers  $x_i^k, i = 1, 2, 3$  of letters 1 in the prefix of length  $k$  of  $w_i$  satisfy  $x_1^k \leq x_2^k \leq x_3^k$ . We call *0-runs* of a binary word  $w$  the subsequences of 0 delimited by the letters 1 (if  $w$  has  $k$  letters 1, then it has  $k + 1$  0-runs). We will use the following result from [18]:

**Lemma 40.** *There is a bijection  $\varphi$  between  $\mathcal{R}_n^*$  and  $\mathcal{B}'_n$  such that for any separating decomposition  $(Q, X) \in \mathcal{R}_n^*$  the white vertices of  $Q$  are in bijective correspondence with the 0-runs of  $w_1$ , where  $(w_1, w_2, w_3) = \varphi(G, X)$ . Moreover, if a white vertex  $v$  of  $Q$  corresponds to the  $i$ th 0-run of  $w_1$ , then the number of children of  $v$  in the red and blue trees of the separating decomposition  $X$  are equal to the length of the  $i$ th 0-run of  $w_1$  and of  $w_2$  respectively. Consequently, the white face  $f$  of  $G = Q^*$  corresponding to  $v$  is partly-reducible (resp. fully-reducible) if and only if the  $i$ th 0-run of  $w_1$  and the  $i$ th 0-run of  $w_2$  have both length 1 (resp. both length at least 1 and not both equal to 1).*

It remains to study the length of 0-runs in a random instance of  $\mathcal{B}'_n$ . First of all, it is easily shown (by calculations on binomial coefficients) that the number  $X_n$  of 0-runs in a uniformly random binary word of length  $n$  is strongly concentrated around  $\frac{n}{2}$ . Moreover, it can be shown that the number  $Y_n$  (resp.  $Z_n$ ) of 0-runs of length at least 1 (resp. exactly 1) is strongly concentrated around  $\frac{1}{2} \cdot \frac{n}{2} = \frac{n}{4}$  (resp.  $\frac{1}{4} \cdot \frac{n}{2} = \frac{n}{8}$ ). Thus, in a uniformly random instance  $(w_1, w_2, w_3)$  of  $\mathcal{B}_n$ , the number  $H_n$  (resp.  $F_n$ ) of integers  $i$  such that  $w_1$  and  $w_2$  both have at least  $i$  0-runs and the  $i$ th 0-runs of  $w_1$  and of  $w_2$  have both length 1 (resp. length at least 1 and not both equal to 1) is strongly concentrated around  $\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{n}{2} = \frac{n}{32}$  (resp.  $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{n}{2} - \frac{n}{32} = \frac{3n}{32}$ ). An asymptotic counting result in [18] shows that the subset  $\mathcal{B}'_n$  occupies a polynomially large proportion of  $\mathcal{B}_n$ , that is, there exists  $\alpha, \beta > 0$  such that  $\frac{|\mathcal{B}'_n|}{|\mathcal{B}_n|} \geq \beta n^{-\alpha}$ . Therefore, the strong concentration results can be transferred from  $\mathcal{B}_n$  to  $\mathcal{B}'_n$ : for a uniformly random instance  $(w_1, w_2, w_3)$  of  $\mathcal{B}'_n$ , the number  $H'_n$  (resp.  $F'_n$ ) of integers  $i$  such that the  $i$ th 0-run of  $w_1$  and of  $w_2$  have both length 1 (resp. length at least 1 and not both equal to 1) is strongly concentrated around  $\frac{n}{32}$  (resp.  $\frac{3n}{32}$ ). By Lemma 40, this implies that the number partly-reducible (resp. fully-reducible) white faces in a uniformly random instance  $(G, X)$  of  $\mathcal{R}_n$  is strongly concentrated around  $\frac{n}{32}$  (resp.  $\frac{3n}{32}$ ). By symmetry the same result applies to black

faces. Thus, the number of faces which are partly-reducible (resp. fully-reducible) is strongly concentrated around  $\frac{n}{16}$  (resp.  $\frac{3n}{16}$ ). Since  $n - \frac{1}{2} \cdot \frac{n}{16} - \frac{3n}{16} = \frac{25n}{32}$ , a balanced reduction choice gives a grid-size strongly concentrated around  $\frac{25n}{32} \times \frac{25n}{32}$ .  $\square$

#### 6.4. Linear-time implementation.

We discuss the complexity of the drawing algorithms in terms of the number  $n$  of vertices of  $G$ . First of all, it is known that a reduced Schnyder decomposition of the quadrangulation  $Q = G^*$  can be computed in linear time (indeed, these are obtained from bipolar orientations of the planar map associated with  $Q$ , as explained in [9]). By duality, this implies that an even regular decomposition  $(T_1^*, \dots, T_4^*)$  can be computed in linear time.

We now explain how to obtain the coordinates  $x(v), y(v)$  of all internal vertices in linear time. Let us concentrate on the  $x$  coordinates (similar arguments apply to the  $y$  coordinates). Recall that  $v_1, v_2, v_3, v_4$  denote the end of the 4 root-edges. We know that  $x(v_2) = 1$  and  $x(v_4) = n$ . We will now show that we can compute in linear time the mapping  $\theta$  which associates to any vertex  $v$  distinct from  $v^*, v_4$  the vertex  $\theta(v)$  satisfying  $x(\theta(v)) = x(v) + 1$  (by *computing*  $\theta$  we mean creating a pointer from each vertex  $v$  distinct from  $v^*, v_4$  to the vertex  $\theta(v)$ ). This will prove that the  $x$  coordinates can be computed in linear time (by starting from the vertex  $v_2$  and following the pointers given by  $\theta$ ).

We use the notations of subsection 6.2: for a non-root face  $f$  we denote by  $f_x^-$  and  $f_x^+$  the vertices incident to  $f$  satisfying  $x(f_x^-) + 1 = x(f_x^+) = f(x)$ . By Lemma 35, the  $x$  coordinates of the  $n - 1$  root-faces are all distinct in  $\{2, 3, \dots, n\}$ . Thus, for each vertex  $v \neq v^*, v_4$ , there is a non-root face such that  $v = f_x^-$ . Hence, computing the mapping  $\theta$  amounts to setting  $\theta(f_x^-) = f_x^+$  for every non-root face. This can be done in linear time. Indeed, for a black (resp. white) face  $f$  it suffices to turn around  $f$  in order to identify the origin  $f_x^-$  and end  $f_x^+$  of the path of counterclockwise (resp. clockwise) color 3; and each of the  $2n - 2$  non-root edges appears around at most two non-root faces.

This concludes the proof that the coordinates of all the non-root vertices can be computed in linear time. Thus the complexity of the drawing algorithm is linear. Note that the additional grid-size reduction also clearly takes linear time (again it is done by inspecting the contour of each face).

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