Outlooks in Saint Venant theory
I. Formal expansions for torsion of Bredt-like sections

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STARTING FROM a regular curve (middle line), we define a section of a Saint Venant cylinder transporting the curve along its normal; this we call a Bredt-like section. A formal series expansion for Prandtl stress flow function valid for these sections is given. Starting from it, and using standard relationships, formal expansions for warping, shear stress vector field and torsional stiffness are obtained. Although the expansions are affected by degeneration in regularity, at least where curvature of the middle line is not smooth or at its ends, we obtain all traditional results of the theory of strength of materials and are able to give an estimate of the error associated with them. Moreover, some interesting open problems are formulated, concerning both the problems of regularizing the expansions and of generalizing them for sections with variable thickness.

1. Introduction

IT IS WELL KNOWN [1–7] that in the Saint Venant linearized theory of torsion there are two different ways to find the tangential stress vector field, each leading to a boundary value problem. Both ways are equivalent; one is based on the search of a potential function for the stress, sometimes called the Saint Venant function, which is also proportional to the warping; the other leads to the so-called Prandtl stress flow function. If we look at the general expression of the problem, one has to find that vector field $\mathbf{t}$, defined over the section $\mathcal{D}$ of a Saint Venant cylinder, which obeys the Cauchy balance equations

$$\operatorname{div}\mathbf{t} = 0 \quad \text{in } \mathcal{D}^o, \tag{1.1}$$

$$\mathbf{t}\cdot\mathbf{n}_{\partial\mathcal{D}} = 0 \quad \text{along } \partial\mathcal{D}, \tag{1.2}$$

and the integrability condition

$$\oint_{\mathcal{L}} \mathbf{t}\cdot\mathbf{l}_{\mathcal{L}} = 2G\tau A_{\mathcal{L}}, \qquad \forall\mathcal{L}. \tag{1.3}$$

Here $\mathbf{n}_{\partial\mathcal{D}}$ denotes the outer normal unit vector to the boundary of the domain $\mathcal{D}^o$, where $\mathcal{D}^o$ denotes the interior of $\mathcal{D}$, τ is the angle of twist, G is the modulus of elasticity in shear and $\mathcal{L}$ is an arbitrary circuit (closed curve) in $\mathcal{D}$. By $\mathbf{l}_{\mathcal{L}}$ we denote the unit vector tangent to $\mathcal{L}$ while $A_{\mathcal{L}}$ is the area confined by $\mathcal{L}$, with sign depending on the orientation of the cycle with respect to the orientation of the plane $\mathcal{P}$ containing $\mathcal{D}$.

The first method to solve system (1.1)–(1.3) starts from a particular function which satisfies (1.3):

$$\tilde{\mathbf{t}} = G\tau * \mathbf{r}, \tag{1.4}$$

where $*$ is the Hodge operator, which rotates a vector contained in $\mathcal{P}$ by $\pi/2$ in the positive orientation of $\mathcal{P}$. By $\mathbf{r}$ we denoted the position vector of a typical point q in $\mathcal{P}$. Even if the domain is not simply linearly connected (we will suppose, that it is monoconnected), from (1.4) and (1.3) it follows that

$$\oint_{\mathcal{L}} (\mathbf{t} - \tilde{\mathbf{t}}) \cdot \mathbf{l}_{\mathcal{L}} = 0 \tag{1.5}$$

for any circuit $\mathcal{L}$ included in $\mathcal{D}$. Equation (1.5) is equivalent to

$$\mathbf{t} - \tilde{\mathbf{t}} = G\tau \operatorname{grad} \phi, \tag{1.6}$$

where $\phi : \mathcal{D} \to \mathbb{R}$ is a potential function for the field $\mathbf{t} - \tilde{\mathbf{t}}$, sometimes referred to as the Saint Venant warping function. By substitution of (1.6) in (1.1)–(1.2), one obtains

$$\Delta\phi = 0 \qquad \text{in } \mathcal{D}^o, \tag{1.7}$$

$$\operatorname{grad}\phi \cdot \mathbf{n}_{\partial\mathcal{D}} = - * \mathbf{r} \cdot \mathbf{n}_{\partial\mathcal{D}} \qquad \text{along } \partial\mathcal{D}, \tag{1.8}$$

where Δ is the Laplace operator. The problem of finding tangential stress vector field is now reduced to a boundary value problem which is in the standard Dini–Neumann form.

The second method to find the field $\mathbf{t}$ starts with observing that (1.2) implies

$$\oint_{\partial\lambda} \mathbf{t} \cdot \mathbf{n}_{\partial\lambda} = \oint_{\partial\lambda} * \mathbf{t} \cdot \mathbf{l}_{\partial\lambda} = 0. \tag{1.9}$$

With this result, we may now deduce from (1.1) that, for any cycle $\mathcal{L}$,

$$\oint_{\mathcal{L}} * \mathbf{t} \cdot \mathbf{l}_{\mathcal{L}} = \oint_{\partial\mathcal{S}} * \mathbf{t} \cdot * \mathbf{n}_{\partial\mathcal{S}} = \oint_{\partial\mathcal{S}} \mathbf{t} \cdot \mathbf{n}_{\partial\mathcal{S}} = \int_{\mathcal{S}} \operatorname{div} \mathbf{t} = 0, \tag{1.10}$$

$\mathcal{S} \subset \mathcal{D}$ being a surface whose boundary is $\mathcal{L} \cup \partial\lambda$. Equation (1.10) assures that

$$* \mathbf{t} = G\tau \operatorname{grad} \psi \quad \Rightarrow \quad \mathbf{t} = -G\tau * \operatorname{grad} \psi, \tag{1.11}$$

where $\psi : \mathcal{D} \to \mathbb{R}$ is known as Prandtl stress flow function [8]. If we substitute this result in (1.1)–(1.3), we obtain another boundary value problem, this time

in a Dirichlet form:

$$\Delta\psi + 2 = 0 \qquad \text{in } \mathcal{D}^o, \tag{1.12}$$

$$\psi = \overline{\psi}_i \qquad \text{along } \partial\mathcal{D}_i, \tag{1.13}$$

$$\oint_{\mathcal{L}} \operatorname{grad}\psi \times \mathbf{l}_{\mathcal{L}} = -2A_{\mathcal{L}} \qquad \forall \text{ closed line } \mathcal{L}. \tag{1.14}$$

In (1.14) the $\partial\mathcal{D}_i$ are all the connected components of $\partial\mathcal{D}$ and the ψ_i' in (1.13) are uniquely determined by (1.14) up to an additive constant. The constant can be chosen in such a way that $\psi = 0$ along the outher connected part of the boundary.

Both systems (1.7)–(1.8) and (1.12)–(1.14) in general do not have a solution in analytical form: such a solution is known only for a few domains, that is, for sections whose boundary is a circle, an ellipse, a square or a triangle with equal sides; in the last two cases the solution is given in terms of a series and is due to Saint Venant himself.

There always is, of course, the possibility of searching solutions of (1.7)–(1.8) and (1.12)–(1.14) by a numerical approach, but this would lead to list of numbers giving no qualitative understanding of the problems. Qualitative results are a good tool for the designer: they put into evidence which parameters are of importance in the design of a structure undergoing torsion.

Among these there are also "technical" results derived to searching solutions sufficiently close to the exact one by approximating the functions of interest by means of linear functions; such an approximation is valid only when the section is made up by thin stripes. These results are usually attributed to VLASOV [9] for open sections and to BREDT [10] for closed, hollow (tubular) sections: Vlasov starts from a constrained material (the section is rigid in its own plane and the middle line of the domain is free of shear), while Bredt uses a particular geometrical construction of the domain which lets him attain his result quite easily.

Our aim is, starting from Bredt's way to imagine the geometry of sections, to develop a different approach to the problem: we will use a "rational" series expansion for the functions of interest (warping, stress, torsional stiffness) and a perturbation technique [11]. This is a well known standard procedure that will provide a set of equations, valid for thicker domains. We will prove that this method will let us recover "technical" results and give us further qualitative results for sections which cannot be considered "thin", both for open and closed sections.

We choose to study a section whose geometry can be described starting from its "middle line", trying to follow Bredt in his analysis of torsion of "thin" closed hollow sections. Let us choose in $\mathcal{P}$ a line Γ of length $\bar{l}$, which can be either open or closed, and parametrize it through an abscissa s:

$$\Gamma(s) := \left\{ q \in \mathcal{P} \,:\, q - o = \mathbf{r}_0(s), \;\; s \in [0, \bar{l}] \right\}, \tag{1.15}$$

where $\mathcal{P}$ is the plane to which the section belongs and o is one of its points, chosen as the origin. It is possible to define a Frenet basis over Γ:

$$\mathbf{l}(s) := \frac{d\mathbf{r}_0(s)}{ds} = \mathbf{r}_0'(s), \tag{1.16}$$

$$\mathbf{m}(s) := \frac{d\mathbf{r}_0'(s)}{ds} = \kappa(s) * \mathbf{l}(s), \tag{1.17}$$

where $\kappa(s)$ is the curvature of Γ at s. After this, let us build the section $\mathcal{D}$ by "thickening" the middle line along $\mathbf{m}$; more precisely, we say that the domain we consider is

$$\mathcal{D}_{\Gamma,\delta} := \left\{ p \in \mathcal{P} : \; p - o = \mathbf{r}_0(s) + z * \mathbf{r}_0'(s), \;\; z \in \left[-\frac{\delta(s)}{2}, \frac{\delta(s)}{2} \right] \right\}, \tag{1.18}$$

$\delta(s)$ being section's thickness at s; the section will be said to be open or closed, depending on Γ being open or closed. We can also write:

$$\mathcal{D} := \left\{ p \in \mathcal{P} : \; p - o = \mathbf{r}_0(s) + z\frac{\delta(s)}{2} * \mathbf{r}_0'(s), \;\; s \in [0, \bar{l}], \;\; z \in [-1, 1] \right\}; \tag{1.19}$$

we choose the orientation of the boundary of $\mathcal{D}$, in order to have always outer normals expressed by the equation

$$\mathbf{n}_{\partial\mathcal{D}} = -\mathbf{m}. \tag{1.20}$$

In this paper we will confine ourselves to the case of sections with constant thickness. This will imply that the curvilinear coordinate system $\mathcal{C} = \{(s, z) \backslash s \in [0, \bar{l}], \; z \in [-1, 1]\}$ (see (1.19)) is orthogonal and therefore the Laplace operator can be represented without use of covariant derivatives, i.e. Christoffel symbols. The choice of such a coordinate system does not lead to a well behaving asymptotic expansion (for a discussion of this problem see [11] Sec. 2.5): indeed, although the first terms of our expansions perfectly match the results already known and supply their generalization, some regularity problems are left open in the case of middle lines whose curvature is not analytic. Moreover, in the case of open sections, there occurs a problem of matching inner and outer solutions of an edge boundary layer. These phenomena most likely arise from the fact that the particular expansion chosen is singular: the starting elliptic operator is turned into a hierarchy of ordinary differential equations in which either a *shift in singularity* or a *coordinate singularity* appears (see [11] Sec. 2.4 and following).

2. Representation of Laplace operator in system $\mathcal{C}$

One of the most interesting features of torsion is that, in general, sections cannot remain plane and undergo a deformation traditionally known as *warping*.

Such a strain is strictly linked with section's shape and resistance and so we will begin our study analyzing it. If we wish to know the warping of a section undergoing torsion, we should remember the following relation between warping and the potential function ϕ:

$$w(p) = \tau\phi(p), \quad p \in \mathcal{D}. \tag{2.1}$$

To find warping we have now a boundary value problem similar to (1.7)–(1.8):

$$\Delta w = 0 \qquad \text{in } \mathcal{D}^o, \tag{2.2}$$

$$\operatorname{grad} w \cdot \mathbf{n}_{\partial\mathcal{D}} = -\tau * \mathbf{r} \cdot \mathbf{n}_{\partial\mathcal{D}} \qquad \text{along } \partial\mathcal{D}. \tag{2.3}$$

Problem (2.2)–(2.3) is well-posed, that is to say, as

$$\oint_{\partial\mathcal{D}} \operatorname{grad} w \cdot \mathbf{n}_{\partial\mathcal{D}} = 0. \tag{2.4}$$

Condition (2.4) assures that there exists a unique solution of the system (2.2)–(2.3) and that this solution is the only one. However, it is well known that in order to determine this solution for sections of interest in technical applications, there are two possible approaches:

a) numerical methods of integration;

b) the so-called "technical" theories for cylinders with "thin" sections, i.e. "thin-walled beams" (for example [9, 10]). These theories give solutions under the hypotheses usually based on the fact that if the section is "thin", one may approximate functions involved in the problem by first order polynomials.

Let us now use curvilinear coordinates system $\mathcal{C}$, that is to say, abscissa s along the middle line Γ and ordinate z along the direction of Frenet normal $\mathbf{m}$; let us also suppose $\delta(s)$ to be constant with respect to s. A basis for $\mathcal{P}$ can then be obtained considering vectors tangent to the coordinate lines:

$$\mathbf{i}_1(s,z) := \frac{\partial p(s,z)}{\partial s} = \mathbf{r}_0'(s)\left(1 - z\kappa(s)\frac{\delta}{2}\right), \tag{2.5}$$

$$\mathbf{i}_2(s,z) := \frac{\partial p(s,z)}{\partial z} = \frac{\delta}{2} * \mathbf{r}_0'(s); \tag{2.6}$$

such a basis is orthogonal. To avoid tedious notation let us omit the dependence of the functions of interest on the coordinates, where this does not lead to confusion. If we wish to have an orthonormal basis, it is sufficient to divide (2.5)–(2.6) by their norm, and to obtain (cf. Fig. 1)

$$\mathbf{e}_1 = \frac{\mathbf{i}_1}{\|\mathbf{i}_1\|} = \mathbf{r}_0', \tag{2.7}$$

$$\mathbf{e}_2 = \frac{\mathbf{i}_2}{\|\mathbf{i}_2\|} = *\mathbf{r}_0'. \tag{2.8}$$

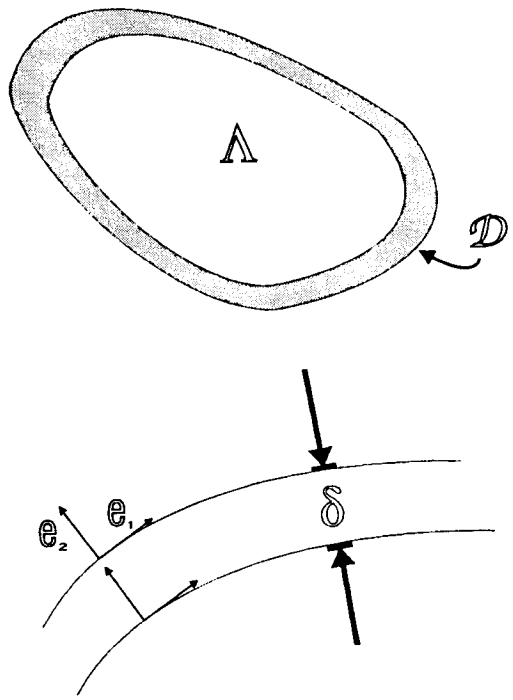

FIG. 1.

We may therefore formulate the problem (2.2)–(2.3) in the following way [12, 13, 14]:

$$\frac{1}{\|\mathbf{i}_1\|\,\|\mathbf{i}_2\|}\left[\frac{\partial}{\partial s}\left(\frac{\|\mathbf{i}_2\|}{\|\mathbf{i}_1\|}\frac{\partial w}{\partial s}\right)+\frac{\partial}{\partial z}\left(\frac{\|\mathbf{i}_1\|}{\|\mathbf{i}_2\|}\frac{\partial w}{\partial z}\right)\right]=0 \qquad \text{in } \mathcal{D}^{\circ}, \tag{2.9}$$

$$\left(\frac{1}{\|\mathbf{i}_1\|}\frac{\partial w}{\partial s}\mathbf{r}_0'+\frac{1}{\|\mathbf{i}_2\|}\frac{\partial w}{\partial z}*\mathbf{r}_0'\right)\cdot *\mathbf{r}_0'=-\tau\mathbf{r}_0\cdot\mathbf{r}_0' \qquad \text{along } \partial\mathcal{D}. \tag{2.10}$$

In our case it is easy to see that Eq. (2.9) assumes the form

$$\frac{1}{\delta\left(1-z\kappa\frac{\delta}{2}\right)}\left[\frac{\partial}{\partial s}\left(\frac{\frac{\delta}{2}}{1-z\kappa\frac{\delta}{2}}\frac{\partial w}{\partial s}\right)+\frac{\partial}{\partial z}\left(\frac{1-z\kappa\frac{\delta}{2}}{\frac{\delta}{2}}\frac{\partial w}{\partial z}\right)\right]=0; \tag{2.11}$$

we have to assume that

$$\delta<\sup_{s\in[0,\bar{l}]}\left|\frac{2}{\kappa}\right|, \tag{2.12}$$

to be sure that the Jacobian of $\mathcal{C}$ is not singular.

To solve problem (2.9)–(2.10) by means of an "asymptotic" analysis, let us suppose to have a set of domains built exactly as $\mathcal{D}$, but with different thickness, constant for each domain. It is then obvious that warping depends on δ as well as on s, z: we suppose that it is possible to write a formal series expansion for w in terms of δ:

$$w(s,z,\delta) = \sum_{n=0}^{\infty} w_n(s,z)\delta^n; \tag{2.13}$$

with (2.13), system (2.9)–(2.10) becomes now

$$\begin{aligned}\sum_{n=0}^{\infty}\Bigg\{&\delta^{n+3}\left[-\frac{z\kappa}{8}w_{n,ss} - \frac{z^3\kappa^3}{8}w_{n,zz} + \frac{z\kappa'}{8}w_{n,s} - \frac{z^2\kappa^3}{8}w_{n,z}\right]\\ &+\delta^{n+2}\left[\frac{1}{4}w_{n,ss} + \frac{3}{4}z^2\kappa^2 w_{n,zz} + z\frac{\kappa^2}{2}w_{n,z}\right]\\ &+\delta^{n+1}\left[-\frac{3}{2}z\kappa w_{n,zz} - \frac{\kappa}{2}w_{n,z}\right] + \delta^n[w_{n,zz}]\Bigg\} = 0 \qquad \text{in } \mathcal{D}^o,\end{aligned} \tag{2.14}$$

$$\sum_{n=0}^{\infty} w_{n,z}\delta^n = -\tau\frac{\delta}{2}\mathbf{r}_0 \cdot \mathbf{r}_0' \qquad z = \pm 1, \forall s; \tag{2.15}$$

here $g_{,x}$ stands for the derivative of g with respect to x. In particular, (2.15) implies

$$w_{n,z} = 0, \qquad n \neq 1, \tag{2.15}'$$

$$w_{n,z} = -\frac{\tau}{2}\mathbf{r}_0 \cdot \mathbf{r}_0', \qquad n = 1. \tag{2.15}''$$

To (2.14)–(2.15) we shall also add, for closed sections, the condition of continuity

$$w(\bar{l}) = w(0), \qquad \forall z. \tag{2.16}$$

Thus we have transformed (2.2)–(2.3) into a hierarchy of systems, one for each power of δ. One could expect that, since the originary problem was well-posed, so should also be each problem of the hierarchy; unfortunately, it is not so.

This may be proved by induction: one can at once see that the zeroth step of the hierarchy gives us an equation for $w_{0,zz}$ so that we know the dependence of w_0 on z; however, to have an equation for $w_{0,ss}$, one must wait for the third step of the hierarchy. It is clear though that, in order to solve the same third step, one is supposed to know also the exact expressions for the other terms that appear in that equation; in particular that implies one should know $w_{1,z}$. This is possible because of the very simple relations one has at the first two steps; but proceeding further this implies overwhelming difficulties.

In general, at the k-th step one has a relation between $w_{k,zz}$ and the derivatives of w at previous steps. Unfortunately, boundary conditions (2.15) do not completely determine w_k, leaving in it an undetermined function of s. This implies that fundamental information for that step are at our disposal only by solving the following steps, whereas for this solution one has to know all the preceding solutions. It seems to be impossible to solve the system (2.14)–(2.15) using the proposed procedure: even though (2.2)–(2.3) was a well-posed problem, its analysis through a perturbation technique leads to a ill-posed set of successive problems.

We may give a justification of this phenomenon by observing that the expression of Laplace operator as given by (2.11) is affected by the possible singularities occurring when (2.12) is not verified: moreover, the regularity of coordinates C is C^1, while for a theorem due to KELLOGG [19] the solution belongs to the class $C^2(\overline{\mathcal{D}}) \cap C^\omega(\mathcal{D}^o)$. As a matter of fact, a singularity (see [11]) occurs in expansion (2.13) whose terms have a regularity lower than that of the exact solution. Such an effect is in our case enhanced by the circumstance that warping is given as the solution of a Dini–Neumann problem, i.e. a boundary value problem with data on normal derivatives, whose regularity is even weaker.

Our idea is then to look for a solution of the warping problem by an analogous technique, starting from basic torsion problem expressed in terms of Prandtl stress function.

3. Asymptotic expansion for Prandtl function

We choose an expansion for Prandtl function of the form

$$\psi(s,z,\delta) = \sum_{n=0}^{\infty} \psi_n(s,z)\delta^n; \tag{3.1}$$

in this way, system (1.12)–(1.14) is turned into

$$\begin{aligned}\sum_{n=0}^{\infty} \Bigg\{ & \delta^{n+3}\left[-\frac{z\kappa}{8}\psi_{n,ss} - \frac{z^3\kappa^3}{8}\psi_{n,zz} + \frac{z\kappa'}{8}\psi_{n,s} - \frac{z^2\kappa^3}{8}\psi_{n,z}\right] \\ & + \delta^{n+2}\left[\frac{1}{4}\psi_{n,ss} + \frac{3}{4}z^2\kappa^2\psi_{n,zz} + z\frac{\kappa^2}{2}\psi_{n,z}\right] \\ & + \delta^{n+1}\left[-\frac{3}{2}z\kappa\psi_{n,zz} - \frac{\kappa}{2}\psi_{n,z}\right] + \delta^n[\psi_{n,zz}]\Bigg\} \\ & = -\frac{\delta^2}{2}\left(1 - \frac{3}{2}z\kappa\delta + \frac{3}{4}z^2\kappa^2\delta^2 - \frac{1}{8}z^3\kappa^3\delta^3\right) \qquad \text{in } \mathcal{D}^o,\end{aligned} \tag{3.2}$$

$$\sum_{n=0}^{\infty} \psi_n\delta^n = \sum_{n=0}^{\infty} \overline{\psi}_n\delta^n \qquad \text{along } \partial\mathcal{D}_i, \tag{3.3}$$

$$\oint_{z=1} 2\sum_{n=0}^{\infty} \psi_{n,z}\delta^n - \oint_{z=1} \kappa \sum_{n=0}^{\infty} \psi_{n,z}\delta^{n+1} = 2\overline{A}\delta - \overline{l}\delta^2 + \frac{\pi}{4}\delta^3 \quad \text{along } \partial\lambda. \tag{3.4}$$

If the section is simply connected (in our case, if it is *open*), its boundary is only one closed curve, and Eqs. (3.2) and (3.3) supply a hierarchy of well-posed problems, as will be proved below. If the section is not simply connected (in our case, if it is *closed*, with a hollow), we must consider, in addition to (3.2) and (3.3), Eq. (3.4) as a condition of integrability; $\overline{A}$ is the area enclosed by the middle line and $\partial\lambda$ is the boundary of the hollow, which has opposite orientation to that of the same curve regarded as part of the domain boundary.

We must observe, as it was remarked at the end of the previous section, that also asymptotic expansion (3.2) is affected by a degeneration in singularity phenomenon; however, as the boundary value problem is in a Dirichlet form, the sequence of ψ_n allows an evaluation of all quantities of interest in torsion, at least when Γ is analytic. However, although we do not prove here that this is really the case for expansion (3.2), we believe that at least its first steps reasonably approximate the *true* solutions; we limit ourselves to refer to [11] for a more detailed discussion on this subject.

We now look for the solution of system (3.2)–(3.4), step by step, both for closed and open sections; we remember that one may always choose ψ so that at the outer boundary ($z = -1$) it equals zero.

3.1. Closed sections

Solution at step δ^0

Considering in system (3.2)–(3.4) the coefficient of the zeroth power of δ, we have

$$\psi_{0,zz} = 0 \;\Rightarrow\; \psi_0(s,z) = z\psi_{0:1}(s) + \psi_{0:0}(s); \tag{3.5}$$

substituting (3.5) into (3.3) we obtain

$$\psi_0(s,-1) = 0 \;\Rightarrow\; \psi_0(s,z) = (z+1)\psi_{0:0}(s), \tag{3.6}$$

$$\psi_0(s,1) = \text{const} \;\Rightarrow\; \psi_0(s,z) = \psi(z) = (z+1)\psi_{0:0}, \tag{3.7}$$

where $\psi_{0:0}$ is a constant independent both of s and z. It is now possible to use (3.4) and (3.6):

$$2\oint_{z=1} \psi_{0,z} = 0 \;\Rightarrow\; \psi_{0:0} = 0 \;\Rightarrow\; \psi_0(z) \equiv 0. \tag{3.8}$$

Solution at step δ^1

Let us go on with our solution; keeping in mind (3.8), the second step of Eq. (3.2) gives us

$$\psi_{1,zz} = 0 \;\Rightarrow\; \psi_1(s,z) = z\psi_{1:1}(s) + \psi_{1:0}(s); \tag{3.9}$$

again applying (3.3) it results that

$$\psi_1(s,-1) = 0 \;\Rightarrow\; \psi_1(s,z) = (z+1)\psi_{1:0}(s), \tag{3.10}$$

$$\psi_1(s,1) = \text{const} \;\Rightarrow\; \psi_1(s,z) = \psi_1(z) = (z+1)\psi_{1:0}, \tag{3.11}$$

$\psi_{1:0}$ being a constant; another relation must be used, that is, Eq. (3.4); substituting in it (3.11), we have

$$2\oint_{z=1} \psi_{1,z} = 2\overline{A} \;\Rightarrow\; \psi_{1:0} = \frac{\overline{A}}{\overline{l}} \;\Rightarrow\; \psi_1(z) = \frac{\overline{r}}{2}(z+1), \tag{3.12}$$

where $\overline{r} := 2\dfrac{\overline{A}}{\overline{l}}$ is some sort of "equivalent radius" for the middle line Γ. It is worth noting that expression (3.12) is the same as that which can be found in "technical" books (for example [15, 16, 17]) under the assumption that the cylinder is "thin", that is to suppose the Prandtl function t be linear along z.

Solution at step δ^2

If we take (3.2) at its third step, Eqs. (3.8) and (3.12) lead to

$$\psi_{2,zz} - \frac{\kappa}{2}\frac{\overline{r}}{2} = -\frac{1}{2} \;\Rightarrow\; \psi_2(s,z) = \frac{z^2}{4}\left(\kappa\frac{\overline{r}}{2} - 1\right) + z\psi_{2:1}(s) + \psi_{2:0}(s). \tag{3.13}$$

Equation (3.13) and the boundary conditions (1.13) in the form of (3.3) give

$$\psi_2(s,-1) = 0 \;\Rightarrow\; \psi_2(s,z) = \frac{1}{4}(z^2-1)\left(\kappa\frac{\overline{r}}{2} - 1\right) + (z+1)\psi_{2:1}(s), \tag{3.14}$$

$$\psi_2(s,1) = \text{const} \;\Rightarrow\; \psi_2(s,z) = \frac{1}{4}(z^2-1)\left(\kappa\frac{\overline{r}}{2} - 1\right) + (z+1)\psi_{2:1}, \tag{3.15}$$

where $\psi_{2:1}$ is a constant. As usual, we now turn to apply (3.15) to (3.4):

$$2\oint_{z=1} \psi_{2,z} - \oint_{z=1} \kappa\psi_{1,z} = -\overline{l} \;\Rightarrow\; \psi_{2:1} = 0 \tag{3.16}$$

$$\Rightarrow\; \psi_2(s,z) = \frac{1}{4}(z^2-1)\left(\kappa\frac{\overline{r}}{2} - 1\right).$$

Solution at step δ^3

Equations (3.8), (3.12) and (3.16) give us

$$(3.17)\quad \psi_{3,zz} = z\frac{\kappa}{4}(\kappa\bar{r}-1) \;\Rightarrow\; \psi_3(s,z) = z^3\frac{\kappa}{24}(\kappa\bar{r}-1) + z\psi_{3:1}(s) + \psi_{3:0}(s).$$

Equation (3.17) and boundary conditions (3.3) lead us to the results

$$(3.18)\quad \psi_3(s,-1) = 0 \;\Rightarrow\; \psi_3(s,z) = \frac{\kappa}{24}(z^3+1)(\kappa\bar{r}-1) + (z+1)\psi_{3:1}(s),$$

$$(3.19)\quad \psi_3(s,1) = c \;\Rightarrow\; \psi_3(s,z) = \frac{\kappa}{24}(z^3-z)(\kappa\bar{r}-1) + c(z+1),$$

where c is a constant. If we apply (3.19) to (3.4), we obtain

$$(3.20)\quad 2\oint_{z=1}\psi_{3,z} - \oint_{z=1}\kappa\psi_{2,z} = \oint_{z=1}\frac{\kappa}{4} \;\Rightarrow\; c = \frac{1}{\bar{l}}\oint_{z=1}\frac{\kappa}{24}(\kappa\bar{r}-1) = \frac{1}{\bar{l}}(a\bar{r}-2\pi)$$

$$\Rightarrow\; \psi_3(s,z) = \frac{\kappa}{24}(z^3-z)(\kappa\bar{r}-1) + \frac{1}{24\bar{l}}(a\bar{r}-2\pi)(z+1),$$

where

$$(3.21)\qquad a := \oint_{z=1}\kappa(\kappa\bar{r}-1).$$

Solution at step δ^4

It is a straightforward calculation which gives us (κ'' is the second-order derivative of κ)

$$(3.22)\qquad \psi_{4,zz} = (1-9z^2)\frac{\kappa^2(\kappa\bar{r}-1)}{48} + (1-3z^2)\frac{\kappa''\bar{r}}{32} + \frac{\kappa(a\bar{r}-2\pi)}{48\bar{l}};$$

and

$$(3.23)\qquad \psi_4(s,z) = (1+2z^2-3z^4)\frac{\kappa^2(\kappa\bar{r}-1)}{192} - (5-6z^2+z^4)\frac{\kappa''\bar{r}}{384}$$

$$+(z^2-1)\frac{\kappa(a\bar{r}-2\pi)}{48\bar{l}}.$$

Without going on to further steps, we observe that every ψ_i is uniquely determined. We may prove this by induction: having proved that we are able to find a complete solution at the first four steps, it is easy to see that at each step i one has:

a. One equation, derived from (3.2), which gives us a relationship between $\psi_{i,zz}$ and an algebraic sum of derivatives of ψ_j with $j < i$; as these constitute a

homogeneous polynomial, it turns out that $\psi_i(s,z)$ is a polynomial of degree i in z and it depends on two functions of s alone;

b. Two equations, implied by (3.3), which define the form of the polynomial $\psi_i(s,z)$ and give us some information on the functions of s above mentioned; more precisely, we obtain one of these functions in terms of the other. Therefore we are left with only one function to determine, about which we have some information, of the kind: it is a constant, or equals a known expression plus a constant;

c. Such information is employed by the equation derived from (3.4) which finally lets us solve the problem and find the expression of $\psi_i(s,z)$. One interesting property of $\psi_i(s,z)$ is that it presents only odd or even powers of z, depending on whether i is odd or even.

We may say that this way of seeking a solution of the torsion problem for hollow cylinders is rather succesful (see Conclusions); let us then turn to examine the case of open sections.

3.2. Open sections

As already said, an open section is simply connected, which implies that the problem is now ruled by Eqs. (3.2) and (3.3) alone; in particular, if $\partial\mathcal{D}$ is connected, we can say that Eq. (3.3) reduces to

$$\psi = 0 \qquad \text{along } \partial\mathcal{D}. \tag{3.24}$$

Let us now look for the solution of system (3.2)–(3.24). It should be reminded that the solution which will be given here is an *outer* solution (see [11]), that is, it coincides with the exact solution only outside a thin region near the shorter edges of the section. Indeed, in closed sections an edge effect layer arises, since solutions given in this section are not capable to satisfy *all* boundary conditions. They need to be matched with an *inner* solution valid in the edge layer which satisfies boundary condition at $s = 0$, $s = \bar{l}$. It will be a further step of our research to investigate what happens in the edge layer and to provide complete matched asymptotic expansions. We remark explicitly here that the outer solution we supply coincides with those found in the literature [4, 17].

Solution at step δ^0

It is quite easy to see in this case that (3.2) implies

$$\psi_{0,zz} = 0, \tag{3.25}$$

which means that ψ_0 is affine in z; but, as (3.24) tells us that ψ_0 vanishes both at $z = -1$ and $z = 1$, we may at once conclude that

$$\psi_0 = 0. \tag{3.26}$$

Solution at step δ^1

At this step results (3.25) and (3.26) show us that we have exactly the same situation as in the previous section concerning step δ^0; we may thus conclude that

$$\psi_1 = 0. \tag{3.27}$$

Solution at step δ^2

Equations (3.2), (3.26) and (3.27) tell us that now we have

$$\psi_{2,zz} = -\frac{1}{2} \Rightarrow \psi_2(s,z) = -\frac{z^2}{4} + z\psi_{2:1}(s) + \psi_{2:0}(s), \tag{3.28}$$

while the restriction of Eq. (3.24) to the lines $z = \pm 1$ gives us

$$\psi_2(s,z) = \psi_2(z) = \frac{1-z^2}{4}. \tag{3.29}$$

The remaining part of the boundary condition corresponding to the lines $s = 0$ and $s = \bar{l}$ cannot be fulfilled. This is a well-known phenomenon in singular perturbation problems. We will return to this problem in the final section. We can observe that this is the solution one usually finds in ordinary handbooks on the strength of materials.

Solution at step δ^3

This time we have

$$\psi_{3,zz} = -\frac{z\kappa}{4} \Rightarrow \psi_3(s,z) = -\frac{z^3\kappa}{24} + z\psi_{3:1}(s) + \psi_{3:0}(s), \tag{3.30}$$

which, with the help of (3.24), becomes

$$\psi_3(s,z) = (1-z^2)\frac{z\kappa}{24}. \tag{3.31}$$

Solution at step δ^4

This step is also easy to solve; we obtain from (3.2), (3.27), (3.29), (3.31),

$$\psi_{4,zz} = \frac{\kappa^2}{48}(1-9z^2) \Rightarrow \psi_3(s,z) = \frac{\kappa^2}{192}(2z^2 - 3z^4) + \psi_{4:1}(s)z + \psi_{4:0}(s) \tag{3.32}$$

and this, with the help of (3.24), becomes

$$\psi_4(s,z) = \frac{\kappa^2}{192}(2z^2 - 3z^4 + 1). \tag{3.33}$$

We can stop now and repeat what we said about the solution of system (3.2)–(3.4) for closed sections: we have, as a matter of fact, the same structure at each step as that observed there; we can conclude that also for open sections the torsion problem in terms of Prandtl function may be solved in this way. Let us now look for the relations between warping and ψ.

4. Asymptotic expansion for warping

An equation which lets us link the warping and Prandtl functions is given directly by (1.5), (1.7), (1.13), (2.1):

$$\operatorname{grad} w = -\tau * (\operatorname{grad} \psi + \mathbf{r}) \tag{4.1}$$

where each function is a field defined over the domain. We may now explicitely express the gradients in (4.1) in terms of the coordinates s and z, use expansions (2.13) and (3.1) for w and ψ and write as well the complete expression for $\mathbf{r}$ given by (1.19). We get

$$\frac{1}{2}\sum_{n=0}^{\infty} w_{n,s}\delta^{n+1} = \tau\left[\sum_{n=0}^{\infty}\psi_{n,z}\delta^{n} - z\frac{\kappa}{2}\sum_{n=0}^{\infty}\psi_{n,z}\delta^{n+1} + \left(\frac{\delta}{2} - z\kappa\frac{\delta^2}{4}\right)\left(z\frac{\delta}{2} - \mathbf{r}_0 \times \mathbf{r}_0'\right)\right], \tag{4.2}$$

$$\sum_{n=0}^{\infty} w_{n,z}\delta^{n} - z\frac{\kappa}{2}\sum_{n=0}^{\infty} w_{n,z}\delta^{n+1} = -\tau\left[\frac{1}{2}\sum_{n=0}^{\infty}\psi_{n,s}\delta^{n+1} + \left(\frac{\delta}{2} - z\kappa\frac{\delta^2}{4}\right)\mathbf{r}_0 \cdot \mathbf{r}_0'\right]; \tag{4.3}$$

having proved in the previous section that we are able to find the complete expression for ψ_k for each k, we can as well find complete expressions for w_k at each step, what we were not able to do following the method described in Sec. 2, both for closed and open sections.

4.1. Closed sections

Solution at step δ^0

It follows directly from (4.2) and (4.3) that

$$\tau\psi_{0,z} = 0, \tag{4.4}$$

$$w_{0,z} = 0. \tag{4.5}$$

It is worth noting that (4.4) corresponds to (3.8) and (4.5) could be derived also from an attempt to solve (2.2)–(2.3) at the first step; in this case we may note that at the first step the shift in singularity phenomenon did not occur, so that also system (2.2)–(2.3) provides a correct solution.

Solution at step δ^1

Again starting from (4.2)–(4.3) and keeping in mind (3.8), (3.12) we have

$$w_{0,s} = 2\tau\frac{\overline{r}}{2} - \tau\mathbf{r}_0 \times \mathbf{r}_0', \tag{4.6}$$

$$w_{1,z} = -\frac{\tau}{2}\mathbf{r}_0 \cdot \mathbf{r}_0'. \tag{4.7}$$

Simple integration of (4.5) and (4.6) gives

$$w_0(s,z) = w_0(s) = 2\tau\left(\frac{\bar{r}}{2}s - \Omega(s)\right) + w_0(0), \tag{4.8}$$

where $\Omega(s)$ is the area enclosed by the arc comprised by the values 0 and s of the abscissa on the middle line and the two radii $\mathbf{r}(0)$ and $\mathbf{r}(s)$. Equation (4.8) is the same which is possible to find in "technical" books under assumptions of "thin" section (see the already quoted books [9] and [18]).

Solution at step δ^2

If we now substitute (3.8), (3.12), (3.16) and (4.8) into (4.2), (4.3), we have

$$w_{1,s} = -z\frac{\tau}{2}(1 + \kappa\mathbf{r}_0' \times \mathbf{r}_0), \tag{4.9}$$

$$w_{2,z} = 0. \tag{4.10}$$

By integration we find

$$w_1(s,z) = -z\frac{\tau}{2}\mathbf{r}_0 \cdot \mathbf{r}_0', \tag{4.11}$$

plus an inessential constant; this is a new result which enables us to predict the warping of a hollow section of the kind described in Sec. 2 when the wall is not so "thin"; we may see that warping is not constant and varies with z, such a variation might be called *tilting*. Equation (4.10) corresponds to the boundary conditions (2.3).

Solution at step δ^3

At this step, with the same procedure as that used in previous sections, we find

$$w_{2,s} = \frac{\tau}{12}\left[\kappa(1 - \kappa\bar{r}) + \frac{a\bar{r} - 2\pi}{\bar{l}}\right], \tag{4.12}$$

$$w_{3,z} = -\frac{\tau}{16}(z^2 - 1)\kappa'\bar{r}, \tag{4.13}$$

and integrating we obtain

$$w_2(s,z) = w_2(s) = \frac{\tau}{12}\left[\frac{a\bar{r} - 2\pi}{\bar{l}} - \int_0^s \kappa(\kappa\bar{r} - 1)\right]. \tag{4.14}$$

Solution at step δ^4

We have now

$$w_{3,s} = \frac{\tau}{48}(3z^3 - z^3)\kappa''\bar{r} \tag{4.15}$$

which implies

$$w_3(s,z) = \frac{\tau}{48}(3z^3 - z^3)\kappa'\bar{r}. \tag{4.16}$$

4.2. Open sections

In this section we will apply the same procedure as that used in Sec. 4.1, starting from Eqs. (4.2) – (4.3); we shall make use, of course, of expression for the Prandtl (outer) function for open domains, that is to say, Eqs. (3.26), (3.27), (3.29), (3.31), (4.33).

Solution at step δ^0

We have immediately

$$\tau\psi_{0,z} = 0, \tag{4.17}$$

$$w_{0,z} = 0, \tag{4.18}$$

about which we may make the same remarks as those concerning Eqs. (4.4), (4.5).

Solution at step δ^1

Going on to the next step we have

$$w_{0,s} = -\tau\mathbf{r}_0 \times \mathbf{r}_0', \tag{4.19}$$

$$w_{1,z} = -\frac{\tau}{2}\mathbf{r}_0 \cdot \mathbf{r}_0'. \tag{4.20}$$

From (4.18) and (4.19), one obtains

$$w_0(s,z) = w_0(s) = -\tau\mathbf{r}_0 \times \mathbf{r}_0', \tag{4.21}$$

which is the well known Vlasov equation of warping for "thin-walled beams" of open section [9].

Solution at step δ^2

This time we have

$$w_{1,s} = -z\frac{\tau}{2}\left(1 - \kappa\mathbf{r}_0 \times \mathbf{r}_0'\right), \tag{4.22}$$

$$w_{2,z} = 0, \tag{4.23}$$

integration of (4.20) and (4.22) gives

$$w_1(s,z) = -z\frac{\tau}{2}\mathbf{r}_0 \cdot \mathbf{r}_0', \tag{4.24}$$

plus a constant; this is a new result concerning sections which are not so "thin"; note the *tilting* already mentioned concerning (4.11).

Solution at step δ^3

If we want to follow the procedure for further steps, we have

$$w_{2,s} = \tau\frac{\kappa}{12}, \tag{4.25}$$

$$w_{3,z} = 0, \tag{4.26}$$

we may integrate (4.23) and (4.25) to obtain

$$w_2(s,z) = w_2(s) = \frac{\tau}{12}\int_0^s \kappa, \tag{4.27}$$

which is another new result.

Solution at step δ^4

Let us examine the last step:

$$w_{3,s} = 0, \tag{4.28}$$

$$w_{4,z} = -\tau\frac{\kappa'}{48}(z - z^3); \tag{4.29}$$

one integration of (4.26), (4.28) gives us

$$w_3(s,z) = \text{const}, \tag{4.30}$$

which is a new and interesting result that shows us that there is a gap between the third and fifth step in this calculation.

We remark that at all steps the cross-derivatives of w_k are equal, which is an important test of the validity of our derivations.

So we may say to have found a kinematical solution of the torsion problem for these sections. Let us now examine some interesting statical properties.

5. Tangential stress field

Let us suppose to have a formal series expansion of the tangential stress field $\mathbf{t}$:

$$\mathbf{t}(s,z,\delta) = \sum_{n=0}^{\infty} \mathbf{t}_n(s,z)\delta^n; \tag{5.1}$$

as we know each coefficient of the formal series expansion (3.1), we may use (1.11) and (1.6) to find an equation for $\mathbf{t}$. In particular, we have

$$\sum_{n=0}^{\infty} \mathbf{t}_n\delta^{n+1} - \frac{z\kappa}{2}\sum_{n=0}^{\infty} \mathbf{t}_n\delta^{n+2} = 2G\tau\left[\left(\sum_{n=0}^{\infty}\psi_{n,z}\delta^n - \frac{z\kappa}{2}\sum_{n=0}^{\infty}\psi_{n,z}\delta^{n+1}\right)\mathbf{r}_0' - \frac{1}{2}\sum_{n=0}^{\infty}\psi_{n,s}\delta^{n+1}(*\mathbf{r}_0')\right]; \tag{5.2}$$

so, operating as in the previous sections, we can solve system (6.2)–(6.3) step by step both for the closed and open sections.

5.1. Closed sections

Solution at step δ^0

We immediately have, on the basis of (5.2),

$$\psi_{0,z} = 0, \tag{5.3}$$

conciding with a result already obtained, cf. Eq. (3.8).

Solution at step δ^1

We have now

$$\mathbf{t}_0 = G\tau\bar{r}\mathbf{r}_0' = \frac{2G\tau\overline{A}}{\bar{l}}. \tag{5.4}$$

Equation (5.4) can be found in technical literature: due to the assumption of "thin" section, the state of stress is plane, directed along the middle line, and besides one may neglect terms of higher than the second order in a power series expansion of Prandtl function, and obtain a constant distribution of tangential stress.

Solution at step δ^2

Simply applying (5.2) and (5.4), we get

$$\mathbf{t}_1 = G\tau z\left(\kappa\frac{\bar{r}}{2} - 1\right)\mathbf{r}_0'; \tag{5.5}$$

equation (5.5) is a new result that, as already said in connection with (4.11), takes into account the fact that the section is not so "thin".

Solution at step δ^3

If we go on with our simple procedure, it is quite easy to prove that

$$\mathbf{t}_2 = G\tau\left\{\left[\frac{1}{4}(\kappa\bar{r} - 1)\left(z^2\kappa - \frac{1}{3}\right) + \frac{1}{12}(a\bar{r} - 2\pi)\right]\mathbf{r}_0' + \left[\frac{\kappa'\bar{r}}{16}(1 - z^2)\right](*\mathbf{r}_0')\right\}, \tag{5.6}$$

which shows that this is the first step in which a component of shear stress along $\mathbf{m}$ appears.

Solution at step δ^4

Our last result for this case is

$$\mathbf{t}_3 = z\kappa\mathbf{t}_2 + G\tau + \frac{z\kappa'}{24}(2\kappa\bar{r} - 1)(1 - z^2)(*\mathbf{r}_0'), \tag{5.7}$$

which is another new result. Let us examine what happens in the case of an open section, keeping in mind that we have at our disposal only a partial knowledge of the complete solution.

5.2. Open sections

Solution at step δ^0

We immediately have from (5.2)

$$\psi_{0,z} = 0 \tag{5.8}$$

about which we may repeat the remark made before, concerning (5.3).

Solution at step δ^1

We have now the possibility to write tangential stress vector field at the first step:

$$\mathbf{t}_0(s, z) = 0, \tag{5.9}$$

this is a result already known: stress distribution in an open section cannot be determined before arriving at the third step of Prandtl function.

Solution at step δ^2

We get now from (5.2), (5.8), (5.9),

$$\mathbf{t}_1(s, z) = -G\tau z\mathbf{r}_0'. \tag{5.10}$$

Solution at step δ^3

We have at this step, remembering the results of previous steps,

$$\mathbf{t}_2(s, z) = G\tau\frac{\kappa}{12}(1 - 3z^2)\mathbf{r}_0', \tag{5.11}$$

which is the first of our new results for open sections.

Solution at step δ^4

Equations (5.2), (5.8), (5.9), (5.10) and (5.11) yield

$$\mathbf{t}_3(s, z) = G\tau\left[z\frac{\kappa^2}{8}(\frac{1}{3} - z^2)\mathbf{r}_0' + z\frac{\kappa'}{24}(z^2 - 1) * \mathbf{r}_0'\right], \tag{5.12}$$

which is another new result. We see that, starting from this step, the effect of tangential stress along the normal to middle line becomes appreciable.

6. Higher order torsional stiffnesses

Usually the torsional stiffness is defined as the ratio of torque divided by unit torsion angle, that is to say,

$$K = \frac{T}{\tau}; \tag{6.1}$$

since it is possible to express tangential stresses in two ways, either by means of a potential function or through Prandtl function; it is also possible to derive two different equations for torque, each starting from a different point of view. That implies we have also two different formulae for torsional stiffness:

$$K = G(J - D), \tag{6.2}$$

$$K = 2G\left(\int_{\mathcal{D}} \psi + A_\lambda \overline{\psi}_\lambda\right); \tag{6.3}$$

in (6.2) J is the section's polar moment of inertia and D is the so-called Dirichlet integral, which is given by

$$D = \int_{\mathcal{D}} \operatorname{grad}\phi \times \mathbf{r}. \tag{6.4}$$

In (6.3) A_λ is the area of the hole. To simplify the calculation, we use (6.3). Let us suppose for K a formal power series expansion similar to those already used:

$$K(\delta) = \sum_{n=0}^{\infty} K_n \delta^n; \tag{6.5}$$

we may then substitute (6.5) into (6.3), as well as the formal series expansion (3.1) for Prandtl function. We then have the following results.

6.1. Closed sections

We may see at once that

$$K_0 = 0 \tag{6.6}$$

which means that at the zero order, that is to say, when thickness tends to zero, the considered section exhibits no resistance to torque, which is physically reasonable. Going on with our calculation we have

$$K_1 = 2G\left(\int_{\mathcal{D}} \psi_0 + \overline{A}\frac{2\overline{A}}{\bar{l}}\right) = \frac{4G\overline{A}^2}{\bar{l}}, \tag{6.7}$$

which coincides with the well-known Bredt formula that one may found in every book on the strength of materials. At next step we find

$$K_2 = 2G\left(\frac{1}{2}\int_{\mathcal{D}} \psi_1 - \overline{A}\right) = 0 \tag{6.8}$$

which is a new result and gives us a mathematical reason for the validity of Bredt formula: since there are no immediate corrections to the torsional stiffness of a Bredt-like section, its approximate value has proved to be good enough for some applications. If we go on, we find

$$K_3 = G\left[\frac{\bar{l} - \pi\bar{r}}{3} + \frac{a\bar{r}}{12}\right], \tag{6.9}$$

which gives us a third-order correction for Bredt formula. At the next step we find

$$K_4 = 0. \tag{6.10}$$

6.2. Open sections

It is absolutely trivial to show that

$$K_0 = K_1 = K_2 = 0; \tag{6.11}$$

it is also easy to see that

$$K_3 = \frac{G\bar{l}}{3} \tag{6.12}$$

which is the usually known result for "thin" sections; going on we have

$$K_4 = 0 \tag{6.13}$$

which is a new result, interesting because it enables us to see that for open sections there are corrections for K only at higher orders of δ. However, we have to remark that we have not developed a matching of Prandtl function close to the ends of the section to account for the edge effects. This seems not to affect (6.12), but it could change the result given by (6.13). Further investigations will clear up this point.

7. Conclusion and open problems

The only exact solution available in literature with which we can compare our results is that of a circular ring. Our asymptotic expansions supply exact solutions

after three steps. All sections belonging to the class $\mathcal{B}$ we have studied share the property of having a vanishing second order torsional stiffness. We remark that the only other solution in analytical form for our problem which may be found in literature is that of an elliptic shaft [4] and that, unfortunately, this section does not belong to $\mathcal{B}$: indeed, its second order torsional stiffness does not vanish. Further investigations will be developed in the following directions:

a. To find an expansion valid for Bredt-like sections with variable thickness;

b. To develop a matched asymptotic procedure to handle the edge layer effect appearing in open sections;

c. To cure the loss of regularity arising in present expansions using the methods proposed in [11];

d. To determine mathematically the range of validity of the proposed expansions using the potential theory, as first done by KELLOG [19];

e. To estimate rigorously the errors associated with the Bredt results in some cases interesting from the point of view of applications.

In our opinion, the expansions proposed in this paper show a new example of a well-known phenomenon occurring in asymptotic analysis: first terms supply an accurate approximation of true solutions, while starting from a given step, a loss of regularity may be observed.

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