On a general balance law for continua with an interface

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1. Introduction

In the literature two different types of reasoning are developed in order to obtain local evolution equations and jump conditions for a continuous system with an interface (C, S).

The first one (we used in [1]) starts with an integral balance equation valid only for the *whole* system. (see also [2], [3]).

In this case it is seen in a simple way that the general balance equation is equivalent to a local volume equation and a surface jump condition where the so-called *localization residual* appears.

This residual is usually interpreted as describing a *non local* interaction and therefore is neglected when this interaction does not need to be taken into account.

The second one, on the other hand, starts assuming a general integral balance equation valid for all material volumes included in (C, S).

The local form of this integral equation is then obtained owing to the arbitrariness of material volumes (see [4], [5]).

Notwithstanding its more clear physical justification this second approach makes use of the concept of material volume, which is ambiguous when this volume includes a portion of a nonmaterial (i.e. not made up with always the same particles) and adsorbing interface (for instance in the case of phase transition).

In this paper we formulate (sec. 3) a general integral balance law (GIBL) valid for a continuosly moving family of *Eulerian volumes* at any instant t included in the region $C(t) \subset R^3$ occupied by the continuous system (C, S).

(*) Work supported by G.N.F.M. of the Italian C.N.R.

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Moreover, in:

– subsection 3 A) we define (when the projection of velocity on S is *continuous across S*) a family $\{c_m(t)\}$ of Eulerian regions which can be regarded as [1] a «first order» material volume and assume that for this $\{c_m(t)\}$ the GIBL holds. It results that it reads as the usually postulated integral balance law for material volumes;

– in subsection 3 B) we deduce even when the projection of the velocity on S in *discontinuous across S* the local balance law by applying GIBL to a *fixed* arbitrary Eulerian volume included in (C, S).

– in subsection 3 C) we finally observe that when applied to the region occupied at the instant t by the *whole* system (C, S), GIBL reduces to the global integral balance law postulated for instance by SUHUBI [3].

In the following sec. 2 some kinematical formulas are shown which will play a relevant role in sec. 3.

2. Some kinematical formulas

Let us assume that the family of Euleriam volumes $C(t)$ and of Eulerian surfaces $S(t)$ (representing the regions of space occupied by respectively the continuous system C and the interface S) satisfy the following properties:

i) A regular vector function $\mathbf{u}$ defined in $C(0) \times R^+$ exists such that $\mathbf{u}(C(0), t) = C(t)$ with jacobian determinant $J \neq 0$.

ii) $\forall\, t \in R^+$, $\partial C(t)$ is a regular surface whose outward pointing (unitary) normal $\mathbf{N}$ always exists.

iii) $\forall\, t \in R^+$, $S(t) \subset C(t)$ is a regular surface having (unitary) normal $\mathbf{n}$ almost everywhere and such that:

either $\qquad \partial S(t) = \emptyset$

or $\qquad \partial S(t) \subset \partial C(t)$.

We underline that because of iii) the region $C(t)$ is shared by the surface $S(t)$ into two disjoint sets $C_i(t)$ $(i = 1, 2)$ whose boundary $\partial C_i(t)$ is in general only almost everywhere regular. The normal $\mathbf{n}$ is always assumed to point towards C_1.

iv) $\qquad \mathbf{u}(C_i(0), t) = C_i(t). \qquad \forall\, t \in R^+$.

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[1] In an intuitive sense to make clearer later.

Physical volume properties of (C, S) will be represented by means of scalar fields $f_i(x, t)$ which are regular in $\overline{C}_i$, $\forall\, t \in R^+$.

We will call f the couple:

$$\{f_1, f_2\}$$

and $f^{\pm}$ the restriction of f_i to S respectively, with $i = 1, 2$.

In a completely analogous way physical surface properties will be represented by means of regular surface fields f_σ.

In literature, and in the following sections, integral balance laws for the fields describing (C, S) are postulated for a variety of families of Eulerian volumes $k(t)$ included in $C(t)$.

This circumstance induces us to prove the following theorems:

THEOREM 1.

Let us consider a family $k(t)$ of subsets of $C(t)$. Let $\sigma(t) \equiv k(t) \cap S(t)$. $k(t)$ results to be shared as $C(t)$ in two disjoint sets $k_i(t)$.

If $k(t)$ satisfies properties analogous to i) $\div$ iv), *then the following derivation formula holds:*

$$(1) \qquad \frac{d}{dt}\int_{k(t)} f\,dc = \int_{k(t)} \frac{\partial f}{\partial t}\,dc - \int_{\sigma(t)} [\![f c_n]\!]\,d\sigma + \int_{\partial k(t)} f\,\mathbf{V}_k \cdot \mathbf{N}_k\,d\sigma$$

where c_n is the geometrical normal speed of the surface σ,

$\mathbf{V}_k = \dfrac{\partial \mathbf{u}_k}{\partial t}$ *(the function $\mathbf{u}_k$ is that appearing in properties* i), iv)*);*

$\mathbf{N}_k$ *is the unitary outward pointing normal to the surface $\partial k(t)$.*

Proof: Formula (1) is easily proved when we note that:

$$\frac{d}{dt}\int_{k(t)} f\,dc = \frac{d}{dt}\int_{k_1(t)} f_1\,dc + \frac{d}{dt}\int_{k_2(t)} f_2\,dc\,.$$

where $k_i \equiv k \cap C_i$.

In fact we can apply to both derivatives on right hand side the following formula (see also SERRIN [6]) valid for an arbitrary moving volume veri-

fying property i):

$$\frac{d}{d\,t}\int_{k_i(t)} f_i\,dc = \int_{k_i(t)} \frac{\partial f_i}{\partial\,t}\,dc + \int_{\partial k_i(t)} f_i\,\mathbf{V}_k\cdot\mathbf{N}_{k_i}\,d\sigma \tag{2}$$

where $\mathbf{V}_k\cdot\mathbf{N}_{k_i}$ is the normal speed of the boundary of k_i and is equal on $\sigma(t)$ to $c_n\,.\,\mathbf{N}_{k_i}$ obviously is the outward pointing unitary normal to $k_i(t)$).

Adding this last two formulas each other we obtain:

$$\frac{d}{d\,t}\int_{k(t)} f\,dc = \int_{k(t)} \frac{\partial f}{\partial\,t}\,dc + \int_{\partial k(t)} f\mathbf{V}_k\cdot\mathbf{N}_k\,d\sigma - \int_{\sigma(t)} [\![f c_n]\!]\,d\sigma$$

We report here for completeness that formula (2) can be deduced, for instance, making use of vector function $\mathbf{u}_k$. In fact if J_k is the Jacobian determinant of this function we have:

$$\frac{d}{d\,t}\int_{k_i(t)} f_i\,dc = \frac{d}{d\,t}\int_{k_i(0)} f_i\,\mathrm{J}_k\,dc_0 = \int_{k_i(0)}\Big[f_i\,\mathrm{J}_k\,\mathrm{div}\,\mathbf{V}_k +$$

$$+\left(\frac{\partial f_i}{\partial\,t} + \mathbf{V}_k\cdot\nabla f_i\right)\mathrm{J}_k\Big]\,dc_0 = \int_{k_i(t)}\frac{\partial f_i}{\partial\,t}\,dc + \int_{\partial k_i(t)} f_i\,\mathbf{V}\cdot\mathbf{N}_{k_i}\,d\sigma$$

THEOREM 2.

Let $k(t)$ and $\sigma(t)$ defined as in Theorem 1) satisfy together with the hypothesis i) ÷ iv) *the following one:*

v) $$\partial\,k(t)\cap\partial\,\mathrm{C}(t) = \emptyset\,.$$

Then the following derivation formula holds:

$$\frac{d}{dt}\int_{\sigma(t)} f_\sigma\,d\sigma\Big|_{t=t_0} = \int_{\sigma(t_0)}\frac{\delta_n}{\delta\,t}\,(f_\sigma\,\mathrm{J}_{t_0})\,d\sigma$$

(3)

$$-\int_{\partial\sigma(t_0)} f_\sigma\,\mathrm{D}_n\,\frac{\sqrt{1-(\mathbf{v}\cdot\mathbf{N}_k)^2}}{\mathbf{v}\cdot\mathbf{N}_k}\,\frac{\mathbf{n}\cdot\mathbf{N}_k}{|\mathbf{n}\cdot\mathbf{N}_k|}\,dl + \int_{\partial\sigma(t_0)} f_\sigma\,\mathbf{V}_k\cdot\nu\,dl$$

with:

$$J_{t_0}(\mathbf{r}, t) \equiv \frac{d\sigma(\mathbf{r}, t)}{d\sigma(\mathbf{r}_0(\mathbf{r}, t), t_0)}$$

where $\mathbf{r} \in S(t)$ *and* $\mathbf{r}_0(\mathbf{r}, t)$ *is the intersection with* $S(t_0)$ *of the curve passing through* $\mathbf{r}$ *and perpendicular to the family of surfaces* $S(t)$,

$$D_n \equiv (\mathbf{V}_k \cdot \mathbf{n} - c_n),$$

ν *the outward pointing unitary normal vector to the curve* $\partial\sigma(t_0)$ *tangent to the surface* $\sigma(t_0)$,

We remark that:

a) with simple geometrical considerations (see for instance [3]) it is possible to show that the following equality holds:

$$\frac{\delta_n}{\delta t} J_{t_0} = -2 H c_n$$

so that it results:

$$\frac{\delta_n}{\delta t}(f_\sigma J_{t_0})\Big|_{t=t_0} = \left(\frac{\delta_n}{\delta t} f_\sigma\right)\Big|_{t=t_0} - 2 H c_n f_\sigma .$$

b) when $\nu \cdot \mathbf{N}_k = 0$, a portion of $\sigma(t)$ of surface-measure not vanishing can be included in $\partial k(t)$. This means that the left hand side in formula (3) can be infinite. This is the reason for which $(\nu \cdot \mathbf{N}_k)^{-1}$ appears in its right hand side.

Proof: We begin observing that, if we choose a suitably small ε, $\sigma(t_0)$ can be regarded as the image through the application $\mathbf{r}$ of suitable surface $\sigma_n(t) \subset S(t)$ for every instant $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$, so that the following equality is meaningful:

(Remark that $\sigma_n(t_0) = \sigma(t_0)$)

$$\sigma_n(t) \cup (\sigma(t) - \sigma_n(t)) = \sigma(t) \cup (\sigma_n(t) - \sigma(t)).$$

Additivity of integration implies that:

$$(4) \quad \frac{d}{dt}\int_{\sigma(t)} f_\sigma \, d\sigma = \frac{d}{dt}\int_{\sigma_n(t)} f_\sigma \, d\sigma + \frac{d}{dt}\int_{\sigma(t)-\sigma_n(t)} f_\sigma \, d\sigma - \frac{d}{dt}\int_{\sigma_n(t)-\sigma(t)} f_\sigma \, d\sigma$$

A simple application of change-of-variable theorems leads to:

$$\frac{d}{dt}\int_{\sigma_n(t)} f_\sigma \, d\sigma \Big|_{t=t_0} = \int_{\sigma(t_0)} \frac{\delta_n}{\delta t}\,(f_\sigma \, J_{t_0})\, d\sigma \tag{5}$$

In order to evaluate the remaining derivatives we note that when at a point of $\partial\sigma(t_0)$ the quantity:

$$s \equiv \frac{\mathbf{n}\cdot\mathbf{N}_k}{|\mathbf{n}\cdot\mathbf{N}_k|}\,\frac{D_n}{|D_n|}$$

is equal to + 1, part of the quantity f_σ contained in $\sigma_n(t)$ is leaving $\sigma(t)$ while if it is equal to − 1 some f_σ not included in σ_n is added in $\sigma(t)$. (see fig. 1).

Moreover (see fig. 1) the measure of $d(\sigma(t)-\sigma_n(t))$ or $d(\sigma_n(t)-\sigma(t))$ is given by:

$$d(\partial\sigma(t_0))\,(\mathbf{V}_{k_i}\cdot\nu + s\,|D_n|\,\mathrm{tg}\,\alpha)\,dt$$

where α is the angle appearing in fig. 1.

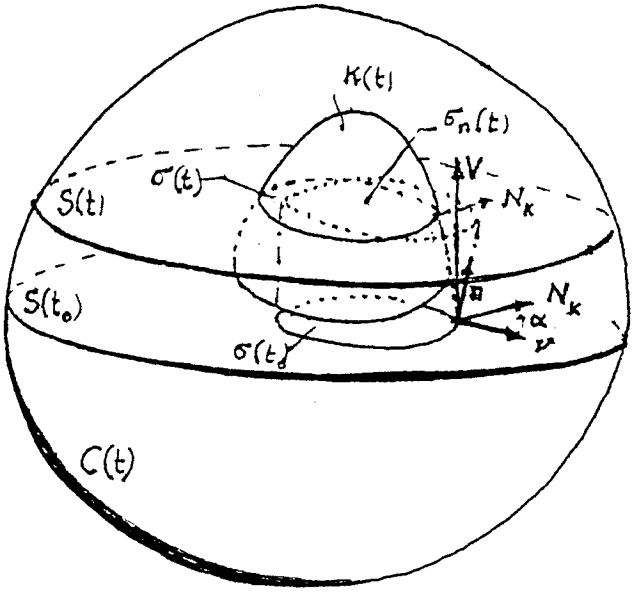

FIG. 1.

We obviously have that:

$$\operatorname{tg} \alpha = \frac{\sqrt{1 - (\nu \cdot \mathbf{N}_k)^2}}{\nu \cdot \mathbf{N}_k}$$

These considerations imply that the sum of the considered derivatives is the line integral appearing in formula (3), which is thus shown.

THEOREM 3.

Hypothesis v) *in Theorem 2 actually can be skipped, formula* (3) *being valid more generally even if* $\partial k(t) \cap \partial C(t)$ *is a nonempty set.*

Proof: We begin remarking that $\forall \varepsilon > 0$ it is possible to find a family $k_\varepsilon(t)$ of Eulerian volumes included in $k(t)$ for which the hypotheses of Theorem 2 are verified and such that: (if $\forall \varepsilon > 0$ $\sigma_\varepsilon \equiv k_\varepsilon \cap S(t)$)

$$\mu_2(\sigma(t) - \sigma_\varepsilon(t)) \leq \varepsilon \quad \text{and} \quad |\mu_1(\partial \sigma(t)) - \mu_1(\partial \sigma_\varepsilon(t))| \leq \varepsilon$$

where μ_2 indicates the Lebesgue measure on the surface $S(t)$ and μ_1 the Lebesgue measure on the curves $\partial \sigma$ and $\partial \sigma_\varepsilon$.

Let us calculate the limit for $\varepsilon \to 0$ in both the members of equations (3) written for the family of volumes k_ε.

The limit of both the terms in the second member of this equation are easily calculated using absolute continuity of Lebesgue integral and the circumstance that Thomas' derivative is not defined in a subset of $\partial \sigma(t)$ which has two dimensional Lebesgue measure zero. The sum of these limits gives second member of eq. (3) written for the volumes $k(t)$.

In order to calculate the limit of the first member we observe that if the surface $S(t)$ can be covered with just and the same chart Φ at every instant t belonging to a suitably small neighbourhood of t_0 we have: (if U^Δ are the coordinates in this chart)

$$\frac{d}{dt} \int_{\sigma_\varepsilon(t)} f_\sigma \, d\sigma \Big|_{t=t_0} \equiv F(\varepsilon, t) = \frac{d}{dt} \int_{\Sigma\varepsilon(t)} f_\sigma J_\sigma \, d\Sigma \Big|_{t=t_0} =$$

$$= \int_{\Sigma\varepsilon(t_0)} \frac{\partial f_\sigma J_\sigma}{\partial t} \, d\Sigma + \int_{\partial\Sigma\varepsilon(t_0)} f_\sigma J_\sigma \nu \cdot \nu \, dL$$

where:

$$\Sigma_\varepsilon(t) \equiv \Phi(\sigma_\varepsilon(t), t); \; J_\sigma \equiv \frac{d\sigma}{d\Sigma};$$

ν is the normal vector to $\partial \Sigma_\varepsilon(t_0)$; and

$$\mathbf{v}^\Delta = \frac{d\, U'^\Delta}{\partial\, t}$$

with $U'^\Delta(U^\Delta, t)$ that function mapping $\Sigma_\varepsilon(t_0)$ in $\Sigma_\varepsilon(t)$ (it exists because for the volume k property i), iv) hold).

(This formula can be obtained repeating in R^2 the reasoning used for showing formula (2)).

In a completely analogous way we obtain: (with obvious meaning of symbols)

$$\frac{d}{d\, t}\int_{\sigma(t)} f_\sigma\, d\sigma = \int_{\Sigma(t)} \frac{\partial f_\sigma\, J_\sigma}{\partial\, t}\, d\Sigma + \int_{\partial\Sigma(t)} f_\sigma\, J_\sigma\, \mathbf{v}\cdot\nu\, d\,L \equiv F(t)$$

Absolute continuity of Lebesgue integral and the uniform continuity of the functions $f_\sigma J_\sigma$ and $\mathbf{v}$ in $\Sigma(t_0)$ assure that when $\varepsilon \to 0$ the function $F(\varepsilon, t)$ converges uniformly to $F(t)$.

Formula (3) in thus proved even if property v) is skipped when $S(t)$ is covered by just one chart.

When $S(t)$ can be covered by a finite number of charts repeatedly applying the obtained result it is easily seen that formula (3) still holds.

3. A general integral balance equation for continuously moving Eulerian volumes

Let us consider a family of Eulerian volumes $k(t) \subset C(t)$ for which derivation (1) and (3) hold. We postulate the following general integral balance law (GIBL): (we use the symbols defined in sec. 2)

$$\begin{aligned}
&\frac{d}{d\, t}\left(\int_{k(t)} f\, dc + \int_{\sigma(t)} f_\sigma\, d\sigma\right) = \int_{\partial k(t)} (f(\mathbf{V}_k - \dot{\mathbf{x}}) - \underset{\sim}{\Phi})\cdot \mathbf{N}_k\, ds + \\
&+ \int_{k(t)} R\, dc + \int_{\sigma(t)} R_\sigma\, d\sigma + \int_{\partial\sigma(t)} \left\{\left(f_\sigma(\mathbf{v}_\tau - \mathbf{V}_k) + \underset{\sim}{\Phi}_\sigma\right)\cdot \underset{\sim}{\nu} + \right. \\
&\left. + \frac{\sqrt{1-(\nu\cdot\mathbf{N}_k)}}{\nu\cdot\mathbf{N}_k}\, D_n f_\sigma\, \frac{\mathbf{n}\cdot\mathbf{N}_k}{|\mathbf{n}\cdot\mathbf{N}_k|}\right\} dl
\end{aligned} \tag{6}$$

where f and f_σ represent the volume and surface density of a same physical quantity moving with the velocities $\dot{\mathbf{x}}$ and $\mathbf{v}_\tau$ in $k(t)$ and on $\sigma(t)$, respectively.

Moreover R and R_σ represent volume and surface production density of the quantities f and f_σ while the vector fields Φ and Φ_σ represent the non-convective fluxes of the same quantities in the material volume and surface instantaneously coinciding with $k(t)$ and $\sigma(t)$.

In (6) the terms containing:

$$\dot{\mathbf{x}} - \mathbf{V}_k \quad \text{and} \quad \mathbf{v}_\tau - \mathbf{V}_k$$

can be regarded as representing the unit fluxes through ∂k related to the quantities f and f_σ, and due to the motion relative to $k(t)$ of the particles constituting the continua, while the last term in the line integral takes into account the relative motion of the surface with respect to the boundary of the volume $k(t)$.

In fact, if the surface moves with normal speed c_n, a part of it, carrying the property represented by f_σ, goes out (or enters) the volume k because the angle between ν and $\mathbf{N}_k$ is different from zero and the normal speed $\mathbf{V}_k \cdot \mathbf{n}$ of ∂k is different from c_n.

The area of this part is proportional to the tangent of this same angle and to the relative normal speed $|D_n|$ of the surface with respect to the volume $k(t)$ (when $\nu \cdot \mathbf{N}_k$ is different from zero), while the quantity $D_n \mathbf{n} \cdot \mathbf{N}_k$ is positive when the surface carries out f_σ, otherwise negative. If $\nu \cdot \mathbf{N}_k = 0$, the previous formula has to fail, because the derivative at first member could be not finite. However, to our end it is enough to know the balance laws for volumes whose boundaries have normals different from $\mathbf{n}$.

In the following subsections we particularize GIBL to three important cases. More precisely we will take as family $k(t)$ the families:

A) $c_m(t)$ representing a «first order» material volume;

B) c representing a fixed Eulerian volume;

C) $C(t)$ representing the whole region occupied by (C, S).

It is obvious that because of our definitions all these three families satisfy conditions i) ÷ iv).

3 A. *«Greatly» viscous interface and «first order» material volumes*

When, as in phase-transition or in the case of two immiscibile fluids, the particles on the interface, sharing the fluids or the phases, continuously change, it is impossible both to find a material parametrization for the interface itself and coherently to identify a material volume. However, it is possible to justify the reasoning, frequently made in literature, in which the previous concepts appear.

In fact, if the assumption:

$$\dot{\mathbf{x}}_\tau^+ = \dot{\mathbf{x}}_\tau^- = \mathbf{v}_\tau \tag{7}$$

is made, it is possible to define in a suitably small time interval, a kind of «first order» material volume $c_m(t)$, as the image through the motion of the volume whose lateral surface is generated by the normal directions to S starting from the points of any closed regular curve on S. By virtue of its definition, the intersection $\sigma_m(t)$ of S with $c_m(t)$ is made up, at first order with respect to time (see how GIBL reads in this case), by material points belonging to the aforesaid volume at the initial instant.

We finally remark that because of (7) the just defined family of volumes $c_m(t)$ satisfies i) ÷ iv). GIBL thus reads for the volumes $c_m(t)$:

$$\begin{aligned} &\frac{d}{dt}\left(\int_{c_m(t)} f\,dc + \int_{\sigma_m(t)} f_\sigma\,d\sigma\right) = \\ &= \int_{\partial c_m(t)} \Phi \cdot \mathbf{N}\,ds + \int_{c_m(t)} \mathrm{R}\,dc + \int_{\sigma_m(t)} \mathrm{R}_\sigma\,d\sigma + \int_{\partial\sigma(t)} \Phi_\sigma \cdot \nu\,dl \end{aligned} \tag{8}$$

We remark that $\mathbf{v}_\tau - \mathbf{V}_k \cdot \nu = \mathbf{0}$ in this case together with geometrical flux terms, containing the normal surface speed c_n, which are eliminated because of our choice of the volumes ($\mathbf{N}$ is the outward pointing unitary normal to $c_m(t)$, so that $\mathbf{v} \cdot \mathbf{N} = 1$).

We conclude giving a more intuitive justification of the reason which led us to call $c_m(t)$ first order material volume. The intersection of $S(t+dt)$ with $c_m(t+dt)$ is not completely made by particles belonging at the instant t to $c_m(t)$ so that this volumes are not strictly a material volume. On the other hand it is easily seen that the measure of the portion of $\sigma_m(t+dt)$ made up of particles not belonging to $c_m(t)$ is:

$$\int_{\partial\sigma_m(t)} (\dot{\mathbf{x}}_\tau(\mathbf{r} + c_n\,\mathbf{n}\,dt) - \mathbf{v}_\tau(\mathbf{r})) \cdot \nu\,dt\,dl = \int_{\partial\sigma_m(t)} c_n \frac{\partial\,\dot{\mathbf{x}}_\tau}{\partial\,\mathbf{x}} : \mathbf{n}\,dt^2\,dl$$

where $\mathbf{u}$ is the generic point belonging to $\partial\,\sigma_m(t)$;

which is a second order quantity not appearing in (6). The previous equality is true because of (7).

Now we need to calculate the derivatives appearing in the first member of (6). It is easy to show that formulas (1) and (2) become:

$$\frac{d}{dt}\int_{c_n(t)} f\,dc\Big|_{t=t_0} = \int_{c_n(t_0)}\left(\frac{\partial f}{\partial t} + \operatorname{div} f\dot{\mathbf{x}}\right)dc + \int_{\sigma_m(t_0)} [\![f(\dot{\mathbf{x}} - \mathbf{c})\cdot\mathbf{n}]\!]\,d\sigma$$

$$\frac{d}{dt}\int_{\sigma_m(t)} f_\sigma\,d\sigma\Big|_{t=t_0} = \int_{\sigma_m(t_0)}\left(\frac{\delta_n}{\delta t}(f_\sigma J_{t_0}) + \operatorname{div}_S f_\sigma \mathbf{v}_\tau\right)d\sigma$$

The integrand in the second member of the last relation is equal to the derivative of the function $f_\sigma J_{t_0}$ (with $J_{t_0}(\mathbf{r}, \mathbf{r}_0, t)\,d\sigma(\mathbf{r}_0, t_0) = d\sigma(\mathbf{r}, t)$) along the integral curves of the field of components $c_n\,\mathbf{n}$ and $\mathbf{v}_\tau$, when $\mathbf{v}_\tau$ is different from zero, and to Thomas' derivative of the same function when $\mathbf{v}_\tau = \mathbf{0}$.

This circumstance gives some content to the fictitious material system of surface coordinates frequently introduced in literature (see for instance [4]).

Substituting these relations in the balance equation (6), and applying once more Gauss' theorem, we obtain local balance equation already found in literature:

$$\text{(9)}\qquad \begin{aligned} &\frac{\partial f}{\partial t} + \operatorname{div}(f\dot{\mathbf{x}} - \Phi) - R = 0 && \text{in } \mathring{C}_1(t) \cup \mathring{C}_2(t)\\ &\frac{\delta_n f_\sigma}{\delta t} - 2H c_n f_\sigma + \operatorname{div}_S(f_\sigma \mathbf{v}_\tau - \Phi_\sigma) + [\![f(\dot{\mathbf{x}} - \mathbf{c}) + \Phi]\!]\cdot\mathbf{n} = 0 && \text{on } S(t) - \partial S(t) \end{aligned}$$

the second of which, in the u^α system of coordinates on S, becomes:

$$\frac{\partial \tilde{f}_\sigma}{\partial t} + \tilde{f}_\sigma(c^\alpha_{|\alpha} - 2H c_n) - (\tilde{f}_\sigma c^\alpha)_{|\alpha} + (\tilde{f}_\sigma \mathbf{v}^\alpha - \Phi^\alpha_\sigma)_{|\alpha} + [\![f(\mathbf{x}-\mathbf{c}) + \Phi]\!]\cdot\mathbf{n} = 0$$

3 B. *Case of discontinuous motions across the interface. GIBL applied to an Eulerian fixed volume*

As already remarked, the necessity of taking into account the possibility for the interface of adsorbing or emitting particles (for a more careful discussion of this point see [7]) obliges ut to renounce to the Lagrangian description of the motion of the system under consideration. In fact, we can follow the material particle only until it reaches the interface, since, when it merges into this interface, it completely loses its identity.

We can conclude that we are forced to use the Eulerian formalism in order to describe with the same mathematical model phenomena including both the cases of a material interface and a nonmaterial one (but eventually carrying physical properties). So, as we have at our disposal Eulerian quantities only, it seems quite natural to try to find a general balance law valid for every Eulerian fixed volume.

Moreover, this kind of equation allows us to treat the more general case in which tangential velocity is discontinuous across the interface. The interest paid to this last case could be also justified in view of the possibility of describing by means of this model a system composed of two perfect fluids, without being forced to introduce a boundary layer with a viscous behaviour contiguous to the interface. Hence, GIBL is assumed to be valid for every fixed Eulerian volume c contained in the interior part of the region $C_1(t) \cup C_2(t) \cup S(t)$ occupied at the time t by the system C_1, C_2, S when $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$. It is obvious that this family of Eulerian volumes satisfies all hypotheses assumed in sec. 2. In this case GIBL reads:

$$\frac{d}{dt}\left(\int_c f\,dc + \int_{\sigma(t)} f_\sigma\,d\sigma\right)\Big|_{t=t_0} = -\int_{\partial c}(f\dot{\mathbf{x}} + \Phi)\cdot\mathbf{N}\,ds + \int_c \mathrm{R}\,dc + \int_{\sigma(t_0)} \mathrm{R}_\sigma\,d\sigma +$$

(10)

$$+\int_{\partial\sigma(t)}\left\{(f_\sigma\,\mathbf{v}_\tau + \Phi_\sigma)\cdot\nu - \frac{\sqrt{1-(\nu\cdot\mathbf{N})^2}}{\nu\cdot\mathbf{N}}\,c_n f_\sigma\,\frac{\mathbf{n}\cdot\mathbf{N}}{|\mathbf{n}\cdot\mathbf{N}|}\right\}dl \qquad \sigma(t) \equiv c \cap S(t)$$

(where $\mathbf{N}$ is the outward pounting normal to ∂c).

Another application of formulas (1) and (2) to such a volume leads to:

$$\int_c \left(\frac{\partial f}{\partial t} + \operatorname{div} f\dot{\mathbf{x}} - \operatorname{div}\Phi + \mathrm{R}\right)dc +$$

(11)

$$+\int_{\sigma(t_0)}\left(\frac{\delta_n}{\delta t}(f_\sigma\,\mathrm{J}_{t_0}) + \operatorname{div}_S(f_\sigma\,\mathbf{v}_\tau - \Phi_\sigma) + [\![f(\dot{\mathbf{x}} - \mathbf{c}) + \Phi]\!]\cdot\mathbf{n}\right)d\sigma = 0$$

Arbitrariness of c and t_0 implies again eq. (9).

3 C. *Balance equation for the whole continuous system as a particular case of GIBL*

Let us particularize GIBL to the case in which $k(t)$ actually is the whole volume $C(t)$. It is easily seen that in this case eq. (6) becomes that postulated by SUHUBI [3], or KOSINSKI [2].

In fact because of assumption iii) about the motion of the considered system we have that either the linear measure of $\partial S(t)$ or the line integral term of eq. (6) in which $(\mathbf{v}_\tau - \mathbf{V}_k)$ and D_n appear, vanish. In both the cases we can conclude that formula (6) thus reads:

$$\frac{d}{dt}\left(\int_{C(t)} f\,dc + \int_{S(t)} f_\sigma\,d\sigma\right) = \tag{11}$$
$$= \int_{\partial C(t)} (f\dot{\mathbf{x}} - \Phi)\cdot \mathbf{N}\,ds + \int_{C(t)} R\,dc + \int_{S(t)} R_\sigma\,d\sigma + \int_{\partial S(t)} \Phi_\sigma \cdot \nu\,dl$$

This integral equation is equivalent to local and jump condition (9) where localization residuals have to be added.

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Pervenuto alla Redazione l'11 dicembre 1986