A moving boundary problem describing the growth of a droplet in its vapour

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IN [1–5] AND [12] THE THEORY of shells is generalized: nonmaterial bidimensional continua are introduced in order to model capillarity phenomena. In this paper we solve some mathematical problems arising when the quoted models are used to describe the growth in its vapour of a sufficiently small drop in the neighbourhood of an equilibrium state. We start to consider the source terms appearing in the integro-differential parabolic evolution equation (IDE) deduced in [5] for the temperature field in the vapour phase. We prove that, due to coupling between the capillarity and thermomechanical phenomena occurring close to the interface, these terms have both space and time Hölder coefficients equal to that one relative to the second time-derivative of the radius of the droplet. To our knowledge only GEVREY [14] partially treated this case for PDE of parabolic type. We improve his results in order to prove the well-posedness of the moving boundary problem formulated in [5] for IDE.

1. Introduction: physical motivation and discussion of proof strategy

1.1. Physical motivation

THE IMPORTANT role played in technology and applied science by capillarity phenomena (see for instance the classical books [13]–[14]) has drawn a growing interest in the theoretical study of models suitable for their description. Indeed, the classical theoretical studies, of capillarity mainly due to Gibbs (for a more detailed discussion and more references see [12] and [13]), are confined to the consideration of equilibrium states, while non-equilibrium phenomena have a lot of relevance in applications. Our attention was drawn by the surface tension-elastograms like that on p. 92 in [13]: a periodically time-varying surface tension is induced by changing concentrations in a biphasic solution. Our idea was to look for a similar effect induced, in a biphasic mono-component system, by a periodically varying supersaturation vapour temperature and/or pressure. Therefore we want to study, on a theoretical ground, the time evolution of the radius $R(t)$ of a liquid drop in presence of a periodical time variation of supersaturation pressure or temperature, in the neighbourhood of an eqilibrium state for the system

$$S = (\text{liquid small drop} + \text{interface} + \text{surrounding vapour}).$$

1.2. Models for capillarity phenomena

The models we use to develop our theoretical treatment are those proposed in [1–4] and more recently improved in [12]. In those papers, in order to model the interfaces between different phases the concept of bidimensional directed nonmaterial continuous system is introduced, generalizing the one classically introduced in the theory of shells: indeed in this theory such systems are modelled which, during their evolution in time, always consist of the same set of material points, while the study of interfacial phenomena obviously requires the introduction of a continuum which at different time instants contain different sets of material points. To bidimensional nonmaterial continua surface densities of material properties are attached, for which the evolution equations have to be found (see [12] for a more detailed discussion).

Let us now quote some of the results derived in [5], the paper upon which the present one is mainly based.

i. The concept of *Soap-Bubble-like* (SB-) continuum is introduced: it is a nonmaterial bidimensional continuum for which some properties which hold for true soap bubbles are still valid. More precisely, it is assumed that the temperature field is continuous across the interface, the total amount of surface mass is constant, the surface stress tensor is pressure-like so that a surface tension is sufficient to describe the stress state in the interface, the interfacial inner energy is an affine function of specific area. It is clear that one can reasonably expect that, at least when S evolves in the neighbourhood of a given equilibrium state, SB-continua suitably describe the behaviour of considered interface.

ii. For SB – continua the

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is proved; if the equilibrium temperature v and vapour pressure P_e belong to $]\vartheta^t, \vartheta^c[$ and $]P_e^t, P_e^c[$, where the indices t and c denote triple and critical values, then there exists a unique (uniform) field of pressure in the liquid phase, and a unique radius R of the droplet for which the equilibrium conditions are satisfied.

We therefore get some equilibrium functions of the equilibrium parameters, $\mathscr{E} = (\vartheta, P_e)$:

$R(\mathscr{E})$, i.e. the equilibrium droplet radius;

$\rho_v(\mathscr{E})$ i. e. the mass density of vapour phase, which is determined by constitutive equations of the vapour;

$p_l(\mathscr{E})$ i. e the pressure of the liquid phase, which is determined when the surface tension $\gamma(\mathscr{E})$ is given;

$p_v(\mathscr{E}) \equiv P_e$ for consistency of notations;

$\rho_\sigma(\mathscr{E})$ i.e. the equilibrium surface mass density.

iii. It is proved that, if the interface is assumed to be SB-like, the liquid phase is incompressible, the vapour is a perfect gas and all fields are spherically symmetric, then the linearized (1) balance equation for mass, velocity and energy, valid for liquid,

(1) In the neighbourhood of one of the previously characterized equilibrium states.

vapour and interfacial phases yield some evolution equations for mass, velocity and temperature fields which are decoupled.

More precisely, this means that the evolution of temperature field can be determined once an Integro-Differential Equation (IDE) is solved in which only the temperature appears, while the mass and velocity fields are derived from a hyperbolic problem, in which the (known) temperature field appears as source term.

1.3. Physical meaning of IDE and related FMB: their dimensionless forms

The IDE is quoted in Sec. 2, together with the Free Moving Boundary problem for its arising in the treatment exposed in [5].

In IDE the coupling between thermal phenomena, mechanical and capillarity is modelled by:

a) its source terms (cf. Eq. (2.2)), whose space and time Hölder continuity exponents are both equal to that of the function $R(t)$;

b) the initial and boundary conditions for the temperature;

c) the new (with respect the classical Stefan condition) contributions appearing in the free moving boundary condition (2.3); moreover, in the most relevant of them the second derivative of $R(t)$ appears.

We are thus facing the following problem: are the Hölder continuity conditions, satisfied by the source terms in Eq. (2.2), able to assure the Hölder continuity of the heat flux jump? We will see that, in order to get a positive answer to this question, we are obliged to improve the results found by GEVREY [6]. Indeed he manages to find solutions of heat flow equations, starting even from the Hölder continuity properties for the heat source which are weaker than those usually applied in the literature (see for instance CILIBERTO [7]).

We explicitly remark here that, in order to recognize that the IDE and its initial and boundary conditions, as formulated in Sec. 3 can be easily regarded as adimensional equations, we only need to change slightly the meaning of the symbols in Sec. 2 by using the set of physical quantities listed in ii) together with the characteristic time ϑ (also appearing in the following Eq. (2.5)) defined as follows

$$\vartheta = \alpha_3\left(\frac{M}{4\pi R(E)}\right),$$

where M is the total mass of the interface and α_3 appears in the generalized non-equilibrium Laplace equation for pressure at the interface. More precisely, it represents the proportionality coefficient which relates the pressure lag

$$p_v - p_l + 2\gamma/R,$$

which is not vanishing far from equilibrium, with the speed lag

$$v - \dot{R},$$

where v is the normal barycentric speed of the material particles lying in the interface.

For the sake of self-consistence we conclude this subsection recalling that

1. IDE stems from the balance of energy when it is considered in the vapour phase in the case of spherically symmetric fields; moreover (cf. [5]), in the quoted balance of energy the time derivative of mass density field appears; this derivative is determined (also in terms of temperature field) once the system of linearized hyperbolic equations (to which the balance of mass and linear momentum reduce) is solved. This is the reason for which an integral operator acting along some characteristic lines, appears in Eq. (2.2).

2. BE (i.e. the following condition (2.3)) is determined when balance of energy is postulated for the bidimensional continuum modelling of the interface: it generalizes, in the considered instance, the classical Stefan condition. We underline here that, while in the latter only the first order time-derivative of the drop radius $R(t)$ appears, in BE an extra term, containing the second order derivative of $R(t)$ times the capillarity coefficient, is found.

3. The boundary conditions for IDE on the moving boundary (2.5), (2.6) and $(2.9)_{1,2}$ in the following Sec. 2, are obtained from the balance of interfacial mass and linear momentum and the dynamic version, found in [3], of Gibbs' conditions at the interface. In [3] these dynamic conditions are determined assuming that:

3.1. the jump of Gibbs' dynamic potential at the interface is proportional to the average mass flux through the interface;

3.2. the previously introduced speed and pressure lags are mutually proportional;

3.3. the increase of interfacial mass is proportional to the interfacial average Gibbs' potential lag.

1.4. Discussion of the proof strategy

In this paper we prove the existence and uniqueness theorem for the solution of the moving boundary problem formulated in Sec.2, when $R(t)$ is known, in a space of functions which show some regularity properties which are compatible with the free boundary condition (2.3). The proof is based upon a fixed point method which uses the results on the solutions of parabolic equations available in the classical works of GEVREY [6], CILIBERTO [7] and FRIEDMAN [8].

Indeed we always consider a space of functions whose strong derivative are Hölder continuous; however, we cannot use the results found neither in Ciliberto's nor Friedman's papers, but we need to implement them with Gevrey's results (or better to say, with Gevrey's techniques), which are more general. This impossibility is due to the quoted coupling properties between thermodynamic and capillarity phenomena: a thermal source arises in IDE because of surface phenomena, and, as we prove in the following sections, this source is space and time Hölder continuous, but the space Hölder exponent is equal to the Hölder time exponent, and both are equal to time Hölder exponent of the function $\ddot{R}(t)$. Those are difficulties we solve in this paper: indeed, in the literature only space Hölder exponents which are twice the time

exponents, are considered. If one tries to apply the classical results in the Boundary Equation (BE) (2.3), one is lead to regard the boundary value of space derivative of the solution of IDE (i.e. the heat flux), *which is part of the source appering in BE*, as a function of time whose Hölder continuity exponent is one half of that of $\ddot{R}$, which is clearly a contradiction.

We shall prove that this contradition can be solved considering the space of source terms in heat equation whose space and time Hölder exponents are equal to $\alpha < 1/2$. The solution of heat equation still exists in this case (this result is mainly due to Gevrey, who really needs only time Hölder continuity), and is sufficiently regular to supply a heat flux that is space-time Hölder continuous with exponent α.

Therefore we will use the following proof strategy:

a) we study the regularity properties of source terms in IDE as determined by the regularity of the function $R(t)$. We assume that the rate of growth of the drop is smaller than the speed of sound in the vapour (cf. [9] for physical meaning of such an assumption);

b) we define a class $\mathscr{K}$ of functions and discuss the existence and uniqueness in $\mathscr{K}$ of the solution of IDE when $R(t)$ is chosen in the class $C^{2+\alpha}$ and boundary data are given as in Sec.2. We are considering the Moving Boundary Problem MBP which is obtained from the corresponding Free Moving Boundary Problem found in [5];

c) we prove that in BE the heat flux appearing as source term has the same Hölder continuity of the second time derivative of $R(t)$;

d) we prove the continuous dependence of the solution of MBP, as an element of $\mathscr{K}$, on the initial and boundary data.

2. Statement of the problem. Regularity properties of the source terms

In this section we shall summarize the main features of the problem outlined in [5].

Let now $\mathscr{D}$ be a spherical region of R^3 and $\partial\mathscr{D}$ its boundary. We shall consider a spherical liquid droplet $\mathscr{D}^-$, its center coinciding with that of $\mathscr{D}$, with time-dependent radius $R(t) \in [R(0), b]$, where $R(0)$ is a positive number and b the radius of $\mathscr{D}$. We suppose moreover that the droplet is surrounded by its vapour which occupies the domain $\mathscr{D}^+ = \mathscr{D} - \mathscr{D}^-$. The temperature on $\partial\mathscr{D}$ is a given function of time $\Phi_E(t)$. According to the theory exposed in [9], we assume that

$$|\dot{R}(t)| < a, \tag{2.0}$$

where a means the speed of sound in the vapour.

Introducing a spherical system of coordinates and denoting by r the distance of the generic point in $\mathscr{D}$ from its center, we define

$$D^+ \equiv \{(r, t) \in R^+ \times R^+ : \ R(t) \leqslant r \leqslant b\},$$

$$D^- \equiv \{(r, t) \in R^+ \times R^+ : \ 0 \leqslant r \leqslant R(t)\}.$$

In [5] it is proved that the problem of determining the evolution of the radius of the liquid droplet near the equilibrium condition, when surface and convective phenomena are not negligible, is solved if the following problem is solved:

Find a triple $(\vartheta_r^{\pm}, R(t))$ which satisfies the conditions

$$\delta\vartheta_r^- \equiv \vartheta_{r,t}^- - \vartheta_{r,rr}^- = 0 \quad \text{in } D^-, \tag{2.1}$$

$$\vartheta_{r,t}^+ = \vartheta_{r,rr}^+ + \lambda_0\Phi_{,t} + \tilde{k}\int_0^t \vartheta_{r,rr}^+(\rho,\tau)\Big|_{\rho=I_\tau}^{\rho=F_\tau} d\tau \quad \text{in } D^+, \tag{2.2}$$

$$\lambda_1\ddot{R} + \lambda_2\dot{R} + \lambda_3 v' + R^{-1}(C^+\vartheta_{r,r}^+ - C^-\vartheta_{r,r}^-) + \vartheta' R^{-1}\Delta C = 0, \tag{2.3}$$

$$\vartheta_r(0,t) = 0, \quad \vartheta_r(R(t),t) = \Phi_R(t), \quad \vartheta_r(r,0) = u_0(r),$$
$$\vartheta_r(b,t) = \Phi_E(t), \quad R(0) = R_0, \quad \dot{R}(0) = \dot{R}_0. \tag{2.4}$$

where:

1) in $(R(0), 0)$ and $(b, 0)$, respectively, the following equalities hold:

$$\dot{\Phi} - u_{0,r}\dot{R} = u_{0,rr} + \lambda_0\Phi_{,t}; \quad \dot{\Phi}_E = u_{0,rr} + \lambda_0\Phi_{,t}, \tag{2.4}'$$

2) the physical case which is considered in [5] leads to the identifications

$$\Phi_R(t) = R(t)\vartheta'(t), \quad u_0 = r\vartheta_0(r),$$
$$v'(t) = \left(\int_0^t \dot{R}(\tau)\, e^{\tau/\theta}\, d\tau\right) e^{-t/\theta}/\theta + v_0' \tag{2.5}$$

where v_0' is an initial value and θ is a given constant,

3) $\tilde{k}$, $C^{\pm}$ and λ_j are suitable constants and $\Delta C = C^+ - C^-$, ϑ' is a linear function of the variables R, $\dot{R}$, v',

$$\vartheta' = b_{19}(R - R(0)) + b_{29}\dot{R} + b_{39}v', \tag{2.6}$$

4) F_τ, I_τ, Φ are functionals depending on (r, t), the function $R(t)$ and, in the case of Φ, on suitable initial and boundary conditions, [5]. They are defined as follows: once the curve $R(t)$ ([2]) is fixed, it is well known (see [10]) that there exists a unique couple of functions $u_1(r, t)$ and $u_2(r, t)$ defined in $\{(r, t) : R(t) \leqslant r\}$ which is the solution of the system:

$$u_{1,t} + cu_{2,r} + cu_2/r = 0, \tag{2.7}$$

$$cu_{2,t} + a^2(u_{1,r} - u_1/r) + d(\vartheta_{r,r} - \vartheta_r/r) = 0, \tag{2.8}$$

satisfying the initial and boundary conditions:

([2]) The curve $R(t)$, because of Eq. (2.0), is not characteristic so that mixed data problem for the hyperbolic system (2.2)–(2.9) in D^+ is well-posed.

$$
\begin{aligned}
&u_1(r, 0) = u_{10}(r), \qquad u_2(r, t) = u_{20}(r), \quad \forall r \in [R(t), \infty]; \\
&u_1(R(t), t) = R(t)\rho_{v'}(t), \\
&u_2(R(t), t) = m\dot{R}(t)\, R(t), \quad \forall t \in [0, T],
\end{aligned} \tag{2.9}
$$

where m is a given constant and $\rho_{v'}$ a suitable linear function of v', R, $\dot{R}$:

$$\rho_{v'} = c_1(R(t) - R(0)) + c_2\dot{R}(t) + c_3 v'(t).$$

Moreover, we assume that

$$
\begin{aligned}
&u_{10}(0) = R(0)\rho_{v'}(0), \qquad u_{20} = m\dot{R}(0)\, R(0), \\
&u_{10}R(0)\, R^{-1}(0) = u_{10,\xi}R(0) + a u_{20}R(0).
\end{aligned} \tag{2.9}'
$$

Using the results found in [10] we can see that the characteristic curves of Eqs. (2.8)–(2.9) are the lines whose angular coefficient is a.

We set

$$\Phi(R, r, t, IBC) \equiv u_1(r, t).$$

Let

$$D_C(r, t) \equiv \{(\xi, \tau) \in R^+ \times R^+ : \ 0 \leqslant \tau \leqslant t, \quad r + a(\tau - t) \leqslant \xi \leqslant r - a(\tau - t)\}.$$

Once the curve $R(t)$ is fixed, we define the application

$$
\begin{aligned}
&P_R : R^+ \times R^+ \to R^+ \times R^+, \\
&P_R(r, t) \equiv (R(\tau_1), \tau_1)
\end{aligned} \tag{2.10}
$$

as follows:

- either it is the unique intersection, if it exists, of the characteristic line stemming from the point (r, t), whose angular coefficient is a, with the curve $R(t)$ ([3]);
- or it is the intersection of the quoted line with the line $t = 0$.

We remark here that $\tau_1 = \tau_1(r, t)$.

Let us moreover assume

$$D^*(r, t) \equiv (D^+ \cap D_C(r, t)) - D_C(P_R(r, t)).$$

The functionals $I_\tau(R, r, t)$ and $F_\tau(R, r, t)$ are defined as follows:

$$D^*(r, t) \equiv \bigcup_{\tau \in [0, t]} \{\tau\} \times [I_\tau, F_\tau]. \tag{2.11}$$

In order to find the regularity of the time derivative of the function Φ we represent the solution of the system (2.7)–(2.8) in the whole plane R^2 using the method of characteristic curves, as it was done in [10], and prove some lemmas.

([3]) Uniqueness of this intersection can be easily proved starting from assumption (2.0).

This solution will be given by a couple (ψ_1, ψ_2) which will satisfy in the interval $[R(0), \infty]$ the initial data (2.9) and (2.9'), and in the interval $[-\infty, R(0)]$, some suitable initial data of the type

$$\psi_1(r, 0) = \chi_1(r), \quad \psi_2 = \chi_2(r).$$

Hence we have

$$\psi_1(R(t), t) = R(t)\rho_{v'}(t), \quad \psi_2(R(t), t) = m\dot{R}(t)R(t).$$

The existence of such initial data is assured by arguments completely analogous to those found in [10].

Due to the uniqueness theorems quoted in [10], together with the D'Alambert representation formula, using some simple algebra we conclude that:

When (r, t) is such that $\tau_1(r, t) > 0$, then

$$\Phi(R, r, t, IBC) = R(\tau_1)\rho_{v'}(\tau_1) - 1/2\, u_{10}(R(\tau_1) + a\tau_1) + 1/2\, u_{10}(r+at) + 1/2\, a \int_{R(\tau_1)+a\tau_1}^{r+at} u_{20}(\xi)\, d\xi, \tag{2.12}$$

while if (r, t) is such that $\tau_1(r, t) = 0$, then

$$\Phi(R, r, t, IBC) = 1/2\,(u_{10}(r+at) + u_{10}(r-at)) + 1/2\, a \int_{r-at}^{r+at} u_{20}(\xi)\, d\xi. \tag{2.12'}$$

LEMMA 1. If $\tau_1(r, t)$ is defined according to Eqs. (2.10), then

$$\frac{\partial \tau_1}{\partial t} = a(a - \dot{R}(\tau_1(r, t)))^{-1},$$

$$\frac{\partial \tau_1}{\partial r} = (\dot{R}(\tau_1(r, t)) - a)^{-1}. \tag{2.13}$$

Proof. It is a trivial application of the Dini Theorem. ■

LEMMA 2. If $\Phi(R, r, t, IBC)$ is given by Eq. (2.12) and $\ddot{R}(t)$ is α-Hölder continuous, then $\Phi_{,t}$ is a continuous function in D^+ which is α-Hölder continuous in both the space and time variables.

Proof. In fact, using Eqs. (2.12) and (2.12') we have: when (r, t) is such that $\tau_1(r, t) > 0$, then

$$\Phi_{,t} = \tau_{1,t}\{\dot{R}(\tau_1\rho_{v'}(\tau_1) + (R(\tau_1) - R(0))[c_1\dot{R}(\tau_1) + c_2\ddot{R}(\tau_1) + (c_3/v)(\dot{R}(\tau_1) - v'(\tau_1))] - 1/2\, u_{10,\xi}(R(\tau_1) + a\tau_1)[\dot{R}(\tau_1) + a] - 1/2a\, u_{20}(R(\tau_1) + a\tau_1)[\dot{R}(\tau_1) + a]\} + a/2\, u_{10,\xi}(r+at) + 1/2\, u_{20}(r+at), \tag{2.13}$$

while, if (r, t) is such that $\tau_1(r, t) = 0$, then

(2.13') $\Phi_{,t} = a/2\,(u_{10,\xi}(r+at) - u_{10,\xi}(r-at)) + 1/2\,(u_{20}(r+at) + u_{20}(r-at))$,

so that the regularity of $\Phi_{,t}$ follows from Eqs. (2.9), (2.9') and (2.13) and from the assumed regularity of $\ddot{R}$. ∎

LEMMA 3. Let $f(r, t)$ be a given function whose domain is D^+, and let $G(r, t)$ be defined as follows:

$$G(r, t) = \int_0^t f(\rho, \tau)\Big|_{\rho=I_\tau}^{\rho=F_\tau} d\tau.$$

If $f(r, t)$ is

i) either α-Hölder continuous with respect to r with Hölder constant $H_r(f)$,

ii) or α-Hölder continuous with respect to t with Hölder constant $H_t(f)$, then $G(r, t)$ is α-Hölder continuous with respect to both r and t.

Proof. We prove this statement only in the case i) which will be used later. The proof when ii) holds is completely analogous.

We start defining

$$\Lambda(\pm a, B, t) = \int_0^t f(\pm a\tau + B, \tau)\, d\tau, \tag{2.14}$$

Λ is α-Hölder continuous with respect to B.

Indeed, from (2.14) we get

$$|\Lambda(\pm a, B_1, t) - \Lambda(\pm a, B_2, t)| \leqslant H_r(f)t\,|B_1 - B_2|, \tag{2.15}$$

where $H_r(f)$ is the space Hölder coefficient of f.

We now split $G(r_1, t) - G(r_2, t)$ into four parts and prove that $G(r, t)$ is space Hölder continuous.

In fact, it is easily seen that

$$G(r_1, t) - G(r_2, t) = \int_0^t (f(F(r_1, t, \tau), \tau) - f(F(r_2, t, \tau), \tau)\, d\tau$$

$$+ \int_{\tau_1(r_1,t)}^{t} (f(I(r_1, t, \tau) - f(I(r_2, t, \tau), \tau)\, d\tau + \int_0^{\tau_1(r_2,t)} (f(I(r_1, t, \tau) - f(I(r_2, t, \tau), \tau)\, d\tau$$

$$+ \int_{\tau_1(r_2,t)}^{\tau_1(r_1,t)} (f(I(r_1, t, \tau) - f(I(r_2, t, \tau), \tau)\, d\tau.$$

Now the first three terms are of the form (2.14), where b_i have suitable values.

In the first two integrals we have

$$B_1 - B_2 = x_1 - x_2,$$

while in the third one we have

$$| B_1 - B_2 | \leqslant | \dot{R} | \ | \partial\tau_1/\partial r | \ | r_1 - r_2 |.$$

Finally, the absolute value of the last integral is easily bounded when we remark that

$$\forall\tau \in [\tau_1(r_1, t), \tau_1(r_2, t)],$$

$$| I(r_1, t, \tau) - I(r_2, t, \tau) | \leqslant | \partial\tau_1/\partial r | \ 2a | r_1 - r_2 |.$$

We can conclude, using Eqs. (2.15) and (2.13'), that

$$(2.16) \quad H_r(G) \leqslant 2H_r(f)t + H_r(f)t \ | \ a(\dot{R} - a)^{-1} |^{\alpha} + 2H_r(f) \ | \ (\dot{R} - a)^{-1} | \ abt.$$

In order to prove the time Hölder continuity of G, we split

$$G(r, t_1) - G(r, t_2)$$

in a way similar to that used in space Hölder continuity to obtain finally

$$(2.17) \qquad H_t(G) \leqslant 2H_r(f)\, ta^{\alpha} + 2H_r(f)\, ta^{\alpha} \quad \blacksquare$$

3. Existence and uniqueness of the solution of MBP. Class K

Let us recall some definitions:

A real function f is said to be Hölder continuous with exponent α if there exists $\alpha \in (0, 1)$ such that

$$| f(x+h) - f(x) | \leqslant H_x(f)\, h^{\alpha},$$

where $H_x(f)$ is a constant Hölder coefficient of f.

For every $m \in N$ we shall say that f belongs to the class $C^{m+\alpha}$ if its m-th derivative exists and it is α-Hölder continuous.

A function of n real variables $f(x_1, ..., x_n)$, whose domain is a compact subset X of R^n, belongs to the class $C^{m_1+\alpha_1 \ldots m_n+\alpha_n}$ if the m_i-th partial derivative with respect to $x_i (i = 1, ..., n)$ is α_i-Hölder continuous with respect to x_i itself.

Let

$$D_T^+ \equiv D^+ \cap \{(r, t): \ 0 \leqslant t \leqslant T\}.$$

We now define the class $\mathscr{K}(D_T^+)$ as follows:

$$\mathscr{K}(D_T^+) \equiv \{v: D^+ \to \mathbb{R} \text{ with } v \in C^{2+\alpha, 1};\ v_r \in C^{1+\alpha, 0+\alpha};\ v_t \in C^{0+\alpha, 0}\}.$$

We define in $\mathscr{K}$ the following norm:

$$\| v \|_{\mathscr{K}} = \sup | v | + \sup | v_r | + \sup | v_{rr} | + H_t(v_r) + H_r(v_{rr}).$$

Let us introduce the following notation:

$$\lhd D_T^+ = \partial D_T^+ - \{(r, t): \ R(t) \leqslant r \leqslant b; \ \ t = T\},$$

so that we can define

$$\mathscr{K}_0(D_T^+) \equiv \left\{ v \in \mathscr{K},\ v \Big|_{\lhd D_\tau^+} = 0 \right\}.$$

Consider now the problem (2.1)–(2.2) when $R(t)$ is a given function belonging to $C^{2+\alpha}$ and the initial and boundary data, listed in Eq. (2.4) and imposed on the function ϑ_r, are vanishing.

The existence of the solution of Eq. (2.1) is assured by standard theorems on the parabolic equations which can be found, for instance, in [8].

Let us introduce the function

$$\mathscr{S}_R : \mathscr{K}_0 \to \mathscr{K}_0 \tag{3.0}$$

such that $\mathscr{S}_R(v)$ is the solution, satisfying the conditions $\mathscr{S}_R(v)\Big|_{\lhd D_\tau^+} = 0$, of the following equation:

$$\mathscr{S}_R(v)_{,t} - \mathfrak{s}_R(v)_{,rr} = \lambda_0 \Phi_{,t} + \tilde{k} \int_0^t v_{,rr}(\rho, \tau) \Big|_{\rho = I_\tau}^{\rho = F_\tau} d\tau + \mathfrak{s}(r, t) \equiv f \quad \text{in } D^+, \tag{3.1}$$

where $\mathscr{S}_R(r, t)$ is a source term belonging to $C^{0+\alpha, 0+\alpha}$.

Moreover, we introduce the following quantity:

$$\mu = \inf_{t \in [0,\ T]} \{ | R(t) |\ ,\ | R(t) - b | \}\ ,$$

which will be assumed in the following to be strictly positive.

PROPOSITION 1. As the solution of Eq. (3.1) belongs to $\mathscr{K}_0$, definition (3.0) makes sense and there exists T' such that $\mathscr{S}_R(v)$ is a contraction in $\mathscr{K}_0$. Moreover, there exists a unique solution of Eq. (3.2) with vanishing initial and boundary data.

Proof. As $v \in \mathscr{K}_0$, the Lemmas 2 and 3 assure us that the right-hand member of Eq. (3.1) belongs to $C^{0+\alpha, 0+\alpha}$.

Moreover, in Appendix A, theorem A.1, we shall use the last result together with some theorems stated in [6-8] in order to prove that, if $R(t)$ belongs to $C^{2+\alpha}$, then $\mathscr{S}_R(v) \in \mathscr{K}_0$ and there exists a constant $L(\mu^{-1}, T, b, a)$, which is bounded when T tends to zero, such that:

$$\forall (w_1, w_2) \in \mathscr{K}_0^2 \qquad \| \mathscr{S}_R(w_1) - \mathscr{S}_R(w_2) \|_{\mathscr{K}_0} \leqslant L(\mu^{-1}, T, b, a)\ \| w_1 - w_2 \|_{\mathscr{K}_0} t\ . \tag{3.2}$$

Obviously, for suitable value T' of t, we have $LT' < 1$ so that $\mathscr{S}_R(v)$ is a contraction in $\mathscr{K}_0(D_T^+)$.

Using the Banach–Caccioppoli fixed point theorem we conclude that

$$\exists!\ v \in \mathscr{K}_0 : \mathscr{S}_R(v) = v\ ,$$

i.e., when $R(t)$ is fixed, there exists a unique solution of Eq. (2.1) with vanishing initial and boundary data (4). ■

Let us consider a couple

$$(\Phi_R(t), \Phi_E(t)) \in C^{1+\alpha} \times C^{1+\alpha}$$

and use it to define the function

$$v_2(r, t) = A(r, t) + B(r, t), \tag{3.3}$$

where

$$A(r, t) \equiv [\Phi_E(t) - \Phi_R(t)](r - R(t))(b - R(t))^{-1} + \Phi_R(t),$$
$$B(r, t) \equiv v_0(\xi(r, t)) \equiv u_0(\xi) - A(\xi, 0) \equiv u_0(\xi) - p\xi + q,$$
$$\xi \equiv [(b - R(0))(b - R(t)^{-1}](r - R(t)) + R(0),$$
$$p \equiv (\Phi_E(0) - \Phi_R(0))(b - R(0))^{-1},$$
$$q \equiv -R(0)(\Phi_E(0) - \Phi_R(0))(b - R(0))^{-1} - \Phi_R(0).$$

It is easily seen that

$$v_2(R(t), t) = \Phi_R(t), \quad v_2(b, t) = \Phi_E(t), \quad v_2(r, 0) = u_0(r).$$

If $R(t) \in C^{2+\alpha}$ then, because of Proposition 1, we can assume that $\Phi_R(t) = \vartheta_r(R(t), t)$ belongs to $C^{1+\alpha}$.

LEMMA 4. If the functions

$$(\Phi_E(t), \Phi_R(t)) \in C^{1+\alpha} \times C^{1+\alpha}$$

and $u_0(r)$ belong to $C^{2+\alpha}$, then the functions $v_{2,rr}$ and $v_{2,t}$ are α-Hölder continuous both in the space and time.

P r o o f. Since evidently $v_{2,t}$ and $v_{2,rr}$ exist and are continuous, in order to prove this statement it is sufficient to calculate the space and time Hölder constants of both $v_{2,t}$ and $v_{2,rr}$. Now it is easily seen that

$$A_{,rr} = 0,$$
$$H_r(B_{,rr}) \leqslant b^3\mu^{-3}H_r(v_{0,rr}),$$
$$H_t(B_{,rr}) \leqslant b^4\mu^{-4}H_r(v_{0,rr}) + \sup \mid v_{0,rr} \mid 2ab\mu^{-3},$$
$$H_r(A_{,t}) \leqslant \sup \mid \Phi_{E,t} - \Phi_{R,t} \mid a^2b\mu^{-1} + \sup \mid \Phi_E - \Phi_R \mid a\mu^{-2},$$
$$H_t(A_{,t}) \leqslant (H(\Phi_{R,t} - \Phi_{E,t})b\mu^{-1} + H(\Phi_{R,t}) + \sup \mid \Phi_E - \Phi_R \mid 2ba^2\mu^{-3},$$
$$H_r(B_{,t}) \leqslant H(v_{0,r})ab\mu^{-2},$$
$$H_t(B_{,t}) \leqslant \sup \mid v_{0,r} \mid a^2b^3.$$

so that the lemma is proved ■.

(4) Note that Eq. (2.1) coincides with Eq. (2.2) when $s(r, t) = 0$.

We put now

$$v = v_1 + v_2, \tag{3.4}$$

where v_1 is the solution of Eq. (3.1) with vanishing initial and boundary data.

Lemma 4 assures us that the function

$$-\tilde{k}\int_0^t v_{2,rr}(\rho,\tau)\Big|_{\rho=I_\tau}^{\rho=F_\tau} d\tau + c_1 v_{2,t} - c_2 v_{2,rr} = g(r,t) \tag{3.5}$$

belongs to $C^{0+\alpha,\,0+\alpha}$.

Setting $g(r, t) = \mathfrak{s}(r, t)$ we conclude that v satisfies Eq.(2.2) with arbitrary initial and boundary conditions.

In this way we have proved the following:

THEOREM 1. *If* $R(t)\in C^{2+\alpha}$ is such that $\mu>0$, *then in the interval* $[0, T']$ *there exists a unique solution of* Eqs.(2.1), (2.2) *with initial and boundary conditions given by* Eq. (2.4) *and satisfying* Eq. (2.4').

4. Continuous dependence of the solution of MBP on initial and boundary data

We now introduce the space $\mathscr{D}$, i.e. the space of initial and boundary data:

(4.1)

$$\mathscr{D} = C^{1+\alpha}[0, T]\times C^{1+\alpha}[0, T] \times C^{2+\alpha}[R(0), b] \times C^{1+\alpha}[R(0), b] \times C^{0+\alpha}[R(0), b],$$

which satisfy all compatibility conditions, in $(R(0), 0)$ and $(b, 0)$, listed in Sec.2.

An element of $\mathscr{D}$ is the set $(\Phi_E, \Phi_R, u_0, u_{10}, u_{20})$ which will be denoted in what follows by IBD.

In order to define a norm in $\mathscr{D}$, we recall that:

if $f(x)\in C^{n+\alpha}$,

$$\| f(x) \|_{C^{n+\alpha}} \equiv \sup|f| + \sup|f_{,x}| + \ldots + \sup|f_{,x^n}| + |f_{,x^n}|_x^{\alpha}, \tag{4.2}$$

so that we can define

$$\| \mathrm{IBD} \|_{\mathscr{D}} \equiv \| \Phi_R \|_{C^{1+\alpha}} + \| \Phi_E \|_{C^{1+\alpha}} + \| u_0 \|_{C^{1+\alpha}} + \| u_{10} \|_{C^{1+\alpha}} + \| u_{20} \|_{C^{0+\alpha}}. \tag{4.3}$$

■ We begin with the observation

$$v_2(r, t) = A(r, t) + B(r, t). \tag{4.4}$$

Simple calculations yield the result

$$\| A \|_{\mathscr{X}} \leqslant \| \mathrm{IBD} \|_{\mathscr{D}} [a_1 + a_2 \| R_{,t} \|_{C^{1+\alpha}} + a_3 \| R_{,t} \|_{C^{1+\alpha}}^2], \tag{4.5}$$

$$\| B \|_{\mathscr{X}} \leqslant \| \mathrm{IBD} \|_{\mathscr{D}} [a_1 + a_2 \| R_{,t} \|_{C^{1+\alpha}} + a_3 \| R_{,t} \|_{C^{1+\alpha}}^2], \tag{4.6}$$

where a_1, a_2, a_3, are three suitable constans such that the following limits are finite:

$$\lim_{\mu\to 0}\mu^2 a_1, \quad \lim_{\mu\to 0}\mu^3 a_2, \quad \lim_{\mu\to 0}\mu^4 a_3 . \tag{4.7}$$

Consider now function v_1. We remark that v_1 is the solution of the following equation:

$$v_{1,t} - v_{1,rr} = \lambda_0 \Phi_{,t} + \tilde{k}\int_0^t v_{1,rr}(\rho,\tau)\Big|_{\rho=I_\tau}^{\rho=F_\tau} d\tau + \mathfrak{s}(r,t) \qquad \text{in } D^+, \tag{4.8}$$

where $\mathfrak{s}(r, t) = -\delta v_2$ is a source term belonging to $C^{0+\alpha,\,0+\alpha}$ which vanishes on $\lhd D_T^+$.

We split the function v_1 in the following way:

$$v_1 = V_1 + U_1 . \tag{4.9}$$

V_1 is such that

$$V_{1,t} - V_{1,rr} = \lambda_0 \Phi_{,t} + \mathfrak{s}(r,t) \tag{4.10}$$

and U_1 satisfies the equation

$$U_{1,t} - U_{1,rr} = \tilde{k}\int_0^t U_{1,rr}(\rho,\tau)\Big|_{\rho=I_\tau}^{\rho=F_\tau} d\tau + \tilde{k}\int_0^t V_{1,rr}(\rho,\tau)\Big|_{\rho=I_\tau}^{\rho=F_\tau} d\tau . \tag{4.11}$$

We can apply to the solution of Eq. (4.10) the Theorem A.2 of the Appendix and Lemma 4 in order to prove that $V_1 \in \mathscr{K}_0$ and, moreover,

$$\| V_1 \|_{\mathscr{K}_0} \leqslant L(R(t), t) K_0 , \tag{4.12}$$

where K_0 is $\| \lambda_0 \Phi_{,t} + \mathfrak{s}(r, t) \|_{C^{0+\alpha,\,0+\alpha}}$.

Let now U_1^1 be such that

$$U_{1,t}^1 - U_{1,rr}^1 = \tilde{k}\int_0^t V_{1,rr}(\rho,\tau)\Big|_{\rho=I_\tau}^{\rho=F_\tau} d\tau . \tag{4.13}$$

Lemma 3 assures us that the right-hand side of Eq. (3.13) $\in C^{0+\alpha,\,0+\alpha}$, so that we have

$$\| U_1^1 \|_{\mathscr{K}_0} \leqslant L_1(R(t), t) K_1 , \tag{4.14}$$

where K_1 is the $C^{0+\alpha,\,0+\alpha}$ norm of $\tilde{k}\int_0^t V_{1,rr}(\rho,\tau)\Big|_{\rho=I_\tau}^{\rho=F_\tau} d\tau$..

On the other hand, owing to Lemma 3, we can write

$$K_1 \leqslant J(R(t), t) K_0 t . \tag{4.15}$$

Using Eqs. (4.12), (4.13), (4.15) we conclude that

$$\| v_1^1 \|_{\mathcal{K}_0} \leqslant L(R(t), t) K_0 + J(R(t), t) K_0 t, \tag{4.16}$$

where

$$v_1^1 = V_1 + U_1^1.$$

Consider now the solution of the equation

$$U_{1,t}^n - U_{1,rr}^n = \tilde{k} \int_0^t U_{1,rr}^{n-1}(\rho, \tau)\Big|_{\rho=I_\tau}^{\rho=F_\tau} d\tau + \tilde{k} \int_0^t V_{1,rr}(\rho, \tau)\Big|_{\rho=I_\tau}^{\rho=F_\tau} d\tau.$$

As proved in the Appendix, the application

$$\mathcal{S} : U_1^n \longrightarrow U_1^{n+1}$$

is contracting in $\mathcal{K}_0$, so we conclude that

$$\lim_{n\to\infty} U_1^n = U_1 \tag{4.17}$$

and, if $v_1^n \equiv V_1 + U_1^n$,

$$\lim_{n\to\infty} v_1^n = v_1. \tag{4.18}$$

On the other hand, for every n we have

$$\| v_1^n \|_{\mathcal{K}_0} \leqslant L(R(t), t) K_0 + J(R(t), t) K_{n-1} t, \tag{4.19}$$

where K_{n-1} is the $C^{0+\alpha, 0+\alpha}$ norm of

$$\tilde{k} \int_0^t U_{1,rr}^{n-1}(\rho, \tau)\Big|_{\rho=I_\tau}^{\rho=F_\tau} d\tau + \tilde{k} \int_0^t V_{1,rr}(\rho, \tau)\Big|_{\rho=I_\tau}^{\rho=F_\tau} d\tau.$$

As n tends to infinity, the constant K_{n-1} is bounded by $\| v_1 \|_{\mathcal{K}_0}$ so that we can write

$$\| v_1 \|_{\mathcal{K}_0} \leqslant (L(R(t), t) K_0)(1 - J(R(t), t) t)^{-1}. \tag{4.20}$$

On the other hand, we have

$$K_0 \leqslant (L_R) \| \mathrm{IBD} \|_{\mathcal{D}}, \tag{4.21}$$

so that (4.20) reads

$$\| v_1 \|_{\mathcal{K}_0} \leqslant (L_R)'(\| \mathrm{IBD} \|_{\mathcal{D}})(1 - J(R(t), t) t)^{-1}, \tag{4.22}$$

where $J(R(t), t)$ is found in the proof of Lemma 3, and is easily seen to be bounded when

$$\inf_{t\in[0,T]} | \dot{R}(t) - a | \equiv \mu' > 0.$$

■

The last formula, together with Eqs. (4.5) and (4.6), proves the

THEOREM 2. *If $R(t) \in C^{2+\alpha}$ and is such that μ and μ' are not vanishing and the initial and boundary data belong to $\mathscr{D}$, then the operator which maps* IBD *onto the unique solution v of* MBP, *$v \in \mathscr{K}$, in a suitable interval* $[0, T'']$ *is continous and*

$$\| v \|_{\mathscr{K}^0} \leqslant (L_R)'' \| \text{IBD} \|_{\mathscr{D}}. \tag{4.23}$$

Appendix

In this section we prove

THEOREM A.1. *If $v \in \mathscr{K}_0$, $IBD \in \mathscr{D}$ ([5]) and* Eqs. $(2.4)_1$ *holds, then $\mathscr{S}_R(v) \in \mathscr{K}_0$ and* Eqs. (2.2) *holds.*

P r o o f. We note that $\mathscr{S}_R(v)$ is the solution of the heat equation with a source term ([6]) $f \in C^{0+\alpha, 0+\alpha}$, satisfying

$$\text{IBD} = 0. \tag{A.0}$$

As it is well known (see for instance [6]) we can write $\mathscr{S}_R(v)$ as follows:

$$\mathscr{S}_R(v) = Z_f(r, t) - z_0(r, t), \tag{A.1}$$

where

i) $Z_f(r, t)$ represents the following integral:

$$Z_f(r, t) = -(4\pi)^{-1/2} \int_{D^+} U_P(r, t, \xi, \eta) f(\xi, \eta)\, d\xi\, d\eta, \tag{A.2}$$

$$U_P(r, t, \xi, \eta) = e^{(r-\xi)^2/4(t-\eta)} (t-\eta)^{-1/2}. \tag{A.3}$$

ii) $z_0(r, t)$ is the solution of $\delta z_0 = 0$ corresponding to the IBD given by $Z_f\big|_{\lhd D^+}$.

In GEVREY [6] it is proved that:

a) $| Z_{f,r} | < (L)$ ([7]) $H(f)\, t^{\alpha+1/2}$ (p.344),

b) $H_t(Z_{f,r}) < (L)\ | f |\ t^{1/2-\alpha}$ (p.360),

c) $| Z_f | < (L)\ | f |\ t$ (p.358),

([5]) The space $\mathscr{D}$ of initial and boundary data *IBD* is defined by Eq. (3.1).

([6]) The source f was defined in (2.1) where $s = -g$ and g is given by Eq. (2.5).

([7]) (L) is a constant which is bounded, when t tends to zero, and its value is varying in different formulas.

d) $Z_{f,t}$ exists and is time-Hölder continous with exponent $\gamma \leqslant \alpha$; its Hölder constant will be denoted by $H_t^{\gamma}(Z_{f,t})$ (p.361 – 362).

Note moreover that, as it is easily seen from Eq. (A.2),

$$Z_f(r,0) = 0, \quad Z_{f,r}(r,0) = 0. \tag{A.4}$$

In the hypotheses d) and (A.4) the following results hold (Gevrey, p.342 and footnote 3 p.362):

e) $| z_0 | < (L) H_t(f) t$;

f) $| z_{0,r} | < (L) H_t(f) t^{1/2-\alpha}$;

g) $H_t(z_{0,r}) < (L) H_t(f) t^{1/2-\alpha}$.

h) $z_{0,t}$ exists and is time-Hölder continous with exponent $\gamma \leqslant \alpha$; its Hölder constant will be denoted by $H_t^{\gamma}(z_{0,t})$. The last result is an obvious consequence of d), the footnote at p.361 and the theorem before Eq. (22), p.342 in GEVREY [6].

i) Statements d) and h) imply that $\mathscr{S}_R(v)_{,t}$ exists and is time Hölder continous with Hölder constant $H_t^{\gamma}(\mathscr{S}_R(v)_{,t})$.

Moreover,

$$H_t^{\gamma}(\mathscr{S}_R(v)_{,t}) \leqslant (L) \mathrm{H}(f)$$

because of Eq. (34)" p.363, the footnote p.361, and footnote 3 p.362 in Gevrey.

The inequalities a)...c) and e)...g) prove that

$$| \mathscr{S}_R(v) | + | \mathscr{S}_R(v)_{,r} | + H_t \mathscr{S}_R(v)_{,r} < (L) \| f \|_{C^{0+\alpha, 0+\alpha}}. \tag{A.5}$$

On the other hand, using d), h), i) and Eq. (A.0) we can see that

$$| \mathscr{S}_R(v)_{,rr} \leqslant t^{\gamma} H_t^{\gamma}(\mathscr{S}_R(v)_{,rr}) \leqslant t^{\gamma}(H_t^{\gamma}(\mathscr{S}_R(v)_{,t}) + H_t(f)) \leqslant (L) t^{\gamma} H(f). \tag{A.6}$$

In order to prove that $H_r(\mathscr{S}_R(v)_{,rr})$ exists and is bounded by $(L)H(f)$, we perform the following transformation:

$$\begin{aligned} \xi &\equiv [(b - R(0))(b - R(t))^{-1}](r - R(t)) + R(0), \\ \tau &\equiv \int_0^t (b - R(0))^2 (b - R(\sigma))^{-2} \, d\sigma, \end{aligned} \tag{A.7}$$

which maps the domain D^+ on the rectangle $R^+ \equiv [R(0), b] \times [0, T]$.

If we introduce the notation ($\mathscr{S}_R = \mathscr{S}_R(v)$)

$$\mathscr{S}_R(r(\xi, \tau), t(\tau)) \equiv \mathscr{I}_R(\xi, \tau), \tag{A.9}$$

we have

$$\mathscr{S}_R(r(\xi, \tau)_{,r} = \mathscr{I}_R(\xi, \tau)_{,\xi} \xi_{,r}, \tag{A.10}$$

$$\mathscr{S}_R(r(\xi, \tau)_{,rr} = \mathscr{I}_R(\xi, \tau)_{,\xi\xi} (\xi_{,r})^2, \tag{A.11}$$

$$\mathscr{S}_R(r(\xi, \tau)_{,t} = \mathscr{I}_R(\xi, \tau)_{,\tau} \tau_{,t} + \mathscr{I}_R(\xi, \tau)_{,\xi} \xi_{,t}. \tag{A.12}$$

Under such a transformation the equation $\delta \mathscr{S}_R = f$ becomes

(A.13) $$\mathscr{I}_R(\xi, \tau)_{,\xi\xi} - \mathscr{I}_R(\xi, \tau)_{,\tau} = \mathfrak{f},$$

where

$$\mathfrak{f} = f(r(\xi, \tau), t(\tau))t_{,\tau} + \mathscr{I}_R(\xi, \tau)_{,\xi}\, \xi_{,t}\, t_{,\tau}.$$

We note that as $\mathscr{S}_{R,rr} = \mathscr{S}_{R,t} + f$ we have

(A.14) $$H_r(\mathscr{S}_{R,rr}) \leqslant H_r(\mathscr{S}_{R,t}) + H_r(f).$$

Moreover,

(A.15) $$H_r(\mathscr{S}_{R,t}) = H_r((\mathscr{I}_{R,\tau}\tau_{,t} + \mathscr{I}_{R,\xi}\xi_{,t})(r, t) \leqslant H_r((\mathscr{I}_{R,\tau}\tau_{,t})(r, t))$$

$$+ H_r((\mathscr{I}_{R,\xi}\xi_{,t})(r, t)) \leqslant |\ \tau_{,t}\ |\ H_r((\mathscr{I}_{R,\tau})(r, t)) + H_r((\mathscr{I}_{R,\xi}\xi_{,t})(r, t).$$

(A.16) $$H_r((\mathscr{I}_{R,\xi}\xi_{,t})(r, t)) = H_r((\mathscr{S}_{R,r}r_{,\xi}\xi_{,t})(r, t)$$

$$\leqslant |\ \mathscr{S}_{R,rr}\ |\ ab^3\mu^{-3} + a\mu^{-1}\ |\ \mathscr{S}_{R,r}\ | \leqslant (L)H_t(f)\, a\,(b^3\mu^{-3}t^{\gamma} + \mu^{-1}t^{1/2}),$$

where we used formula (A.6).

(A.17) $$H_r((\mathscr{I}_{R,\tau})(r, t) \leqslant H_\xi((\mathscr{I}_{R,\tau})(r, t))\ |\ \xi_{,x}\ |^{\alpha} \leqslant b^{\alpha}\mu^{-\alpha}H_\xi(\mathscr{I}_{R,\tau}).$$

We split now $\mathscr{I}_R(\xi, \tau)$ as in Eq. (A.3) and write

(A.18) $$\mathscr{I}_R(\xi, \tau) = \mathscr{Z}_{\mathfrak{f}}(\xi, \tau) + \mathfrak{z}_0(\xi, \tau).$$

Because of Eq. (A.18) we have

(A.19) $$H_\xi(\mathscr{I}_{R,\tau}) \leqslant H_\xi(\mathscr{Z}_{\mathfrak{f}}(\xi, \tau)_{,\tau}) + H_\xi(\mathfrak{z}_0(\xi, \tau)_{,\tau}).$$

In order to apply to $\mathscr{Z}_{\mathfrak{f}}$ the results found by FRIEDMAN in [11] we remark that, because of the hypothesis (2.4') $\mathscr{Z}$ is vanishing in $(R(0), 0)$ and $(b, 0)$ so that (see [11] Lemma 1 and its proof):

(A.20) $$H_\xi(\mathscr{Z}_{\mathfrak{f}}(\xi, \tau)_{,\tau}) \leqslant (L)H_\xi(\mathfrak{f}).$$

On the other hand it is possible to find an upper bound for $H_\xi(\mathfrak{z}_0(\xi, \tau)_{,\tau})$ by using the following arguments:

I. Owing to footnote p.361 Eq. (21') p.338, and to the results quoted at the beginning of p.362 in GEVREY [6]

$$\mathscr{Z}_{f,\tau}(R(0), t) \in C^{0+\alpha/2}, \quad \mathscr{Z}_{f,\tau}(b, t) \in C^{0+\alpha/2}. \tag{A.21}$$

II. Owing to Eq. (3) p.469, which is obtained only for rectangular domains, we conclude, taking again into account footnote 3 p.362 in Gevrey:

$$H_\xi(\mathfrak{z}_0(\xi, \tau)_{,\tau}) \leqslant (L)(H_\xi(\mathfrak{f}) + H_\tau(\mathfrak{f})). \tag{A.22}_1$$

Finally, we have to estimate $H_\xi(\mathfrak{f})$ and $H_\tau(\mathfrak{f})$ in terms of $\| f \|_{C^{0+\alpha, 0+\alpha}}$. It is easily seen that

$$H_\xi(\mathfrak{f}) \leqslant 1(L)\mathscr{Q}_1(\mu^{-1}) \| f \|_{C^{0+\alpha, 0+\alpha}}, \tag{A.22}_2$$

$$H_\tau(\mathfrak{f}) \leqslant (L)\mathscr{Q}_2(\mu^{-1}) \| f \|_{C^{0+\alpha, 0+\alpha}}, \tag{A.22}_3$$

where $\mathscr{Q}_i$ are bounded functions of the variable μ^{-1}.

Using now all results (A.15) ... (A.22)$_{1,2,3}$ we observe that (A.14) becomes

$$H_\xi(\mathscr{S}_R(r, t)_{,rr}) \leqslant \mathscr{Q}(\mu^{-1}) \| f \|_{C^{0+\alpha, 0+\alpha}}, \tag{A.23}$$

where $\mathscr{Q}$ is a bounded function of the variable μ^{-1}.

Equations (A.5), (A.6) and (A.23) prove that $\mathscr{S}_R(v) \in \mathscr{K}_0$.

Consider now v_1 and $v_2 \in \mathscr{K}_0$. The function $\mathscr{S}_R(v_1) - \mathscr{S}_R(v_2)$ belongs to $\mathscr{K}_0$ and is a solution of the following equation:

$$\delta[\mathscr{S}_R(v_1) - \mathscr{S}_R(v_2)] = \tilde{k} \int_0^t (v_1 - v_2)_{1,rr}(\rho, \tau) \Big|_{\rho = I_\tau}^{\rho = F_\tau} d\tau. \tag{A.24}$$

The right-hand side of Eq. (A.24) belongs to $C^{0+\alpha, 0+\alpha}$, owing to Lemma 3, and vanishes when $t \to 0$ so that we can apply to $\mathscr{S}_R(v_1) - \mathscr{S}_R(v_2)$ formulas (34") p.363 in Gevrey and (A.23).

In our notation they read

$$\| \mathscr{S}_R(v)_1 - \mathscr{S}_R(v_2) \|_{\mathscr{K}_0} \leqslant S(R(t), T, b) \| \Xi \|_{C^{0+\alpha, 0+\alpha}}, \tag{A.25}$$

where

$$\Xi = \tilde{k} \int_0^t (v_1 - v_2)_{,rr}(\rho, \tau) \Big|_{\rho = I_\tau}^{\rho = F_\tau} d\tau.$$

On the other hand, using the estimates found in proving the Lemma 3, it is easily seen that there exists a constant K, bounded when t tends to zero, such that

$$\| \Xi \|_{C^{0+\alpha, 0+\alpha}} \leqslant K(R(t), t, b)\, t \| v_1 - v_2 \|_{\mathscr{K}_0}. \tag{A.26}$$

Equation (3.2) is proved when we choose $L = SK$ ■

We remark that in proving Theorem A.1, we also proved the following

THEOREM A.2 *If* $l \in C^{0+\alpha, 0+\alpha}$ *and* V_1 *is the solution of*

$$V_{1,t} - V_{1,rr} = l(r, t)$$

which vanishes on $\lhd D^+$, *then* $V_1 \in \mathscr{K}_0$ *and, moreover,*

$$\| V_1 \|_{\mathcal{X}_0} \leq L(R(t), t) \| l \|_{C^{0+\alpha, 0+\alpha}},$$

where $L(R(t), t)$ is bounded when μ and μ' are not vanishing.

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