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# Tools for parsimonious edge-colouring of graphs with maximum degree three

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## Abstract

In a graph  $G$  of maximum degree  $\Delta$  let  $\gamma$  denote the largest fraction of edges that can be  $\Delta$ -edge-coloured. Albertson and Haas showed that  $\gamma \geq \frac{13}{15}$  when  $G$  is cubic [1]. The notion of  $\delta$ -minimum edge colouring was introduced in [3] in order to extend the so called *parsimonious edge-colouring* to graphs with maximum degree 3. We propose here an english translation of some structural properties already present in [2, 3] (in French) for  $\delta$ -minimum edge colourings of graphs with maximum degree 3.

**Keywords :** Cubic graph; Edge-colouring;

## 1 Introduction

Throughout this note, we shall be concerned with connected graphs with maximum degree 3. We know by Vizing's theorem [7] that these graphs can be edge-coloured with 4 colours. Let  $\phi : E(G) \rightarrow \{\alpha, \beta, \gamma, \delta\}$  be a proper edge-colouring of  $G$ . It is often of interest to try to use one colour (say  $\delta$ ) as few as possible. When an edge colouring is optimal, following this constraint, we shall say that  $\phi$  is  $\delta$ -*minimum*. In [2] we gave without proof (in French) results on  $\delta$ -*minimum* edge-colourings of cubic graphs. Some of them have been obtained later and independently by Steffen [5] and [6]. The purpose of Section 2 is to give with their proofs those results as structural properties of  $\delta$ -minimum edge-colourings.

An edge colouring of  $G$  using colours  $\alpha, \beta, \gamma, \delta$  is said to be  $\delta$ -*improper* provided that adjacent edges having the same colours (if any) are coloured with  $\delta$ . It is clear that a proper edge colouring (and hence a  $\delta$ -minimum edge-colouring) of  $G$  is a particular  $\delta$ -improper edge colouring. For a proper or  $\delta$ -improper edge colouring  $\phi$  of  $G$ , it will be convenient to denote  $E_\phi(x)$  ( $x \in \{\alpha, \beta, \gamma, \delta\}$ ) the set of edges coloured with  $x$  by  $\phi$ . For  $x, y \in \{\alpha, \beta, \gamma, \delta\}, x \neq y$ ,  $\phi(x, y)$  is the partial subgraph of  $G$  spanned by these two colours, that is  $E_\phi(x) \cup E_\phi(y)$  (this subgraph being a union of paths and even cycles where the colours  $x$  and  $y$  alternate). Since any two  $\delta$ -minimum edge-colourings of  $G$  have the same number of edges coloured  $\delta$  we shall denote by  $s(G)$  this number (the *colour number* as defined in [5]).

As usual, for any undirected graph  $G$ , we denote by  $V(G)$  the set of its vertices and by  $E(G)$  the set of its edges and we suppose that  $|V(G)| = n$  and

$|E(G)| = m$ . A *strong matching*  $C$  in a graph  $G$  is a matching  $C$  such that there is no edge of  $E(G)$  connecting any two edges of  $C$ , or, equivalently, such that  $C$  is the edge-set of the subgraph of  $G$  induced on the vertex-set  $V(C)$ .

## 2 Structural properties of $\delta$ -minimum edge-colourings

The graph  $G$  considered in the following series of Lemmas will have maximum degree 3.

**Lemma 1** [2, 3] *Any 2-factor of  $G$  contains at least  $s(G)$  disjoint odd cycles.*

**Proof** Assume that we can find a 2-factor of  $G$  with  $k < s(G)$  odd cycles. Then let us colour the edges of this 2-factor with  $\alpha$  and  $\beta$ , except one edge (coloured  $\delta$ ) on each odd cycle of our 2-factor and let us colour the remaining edges by  $\gamma$ . We get hence a new edge colouring  $\phi$  with  $E_\phi(\delta) < s(G)$ , impossible.  $\square$

**Lemma 2** [2, 3] *Let  $\phi$  be a  $\delta$ -minimum edge-colouring of  $G$ . Any edge in  $E_\phi(\delta)$  is incident to  $\alpha$ ,  $\beta$  and  $\gamma$ . Moreover each such edge has one end of degree 2 and the other of degree 3 or the two ends of degree 3.*

**Proof** Any edge of  $E_\phi(\delta)$  is certainly adjacent to  $\alpha, \beta$  and  $\gamma$ . Otherwise this edge could be coloured with the missing colour and we should obtain an edge colouring  $\phi'$  with  $|E_{\phi'}(\delta)| < |E_\phi(\delta)|$ .  $\square$

Lemma 3 below was proven in [4], we give its proof for sake of completeness.

**Lemma 3** [4] *Let  $\phi$  be a  $\delta$ -improper colouring of  $G$  then there exists a proper colouring of  $G$   $\phi'$  such that  $E_{\phi'}(\delta) \subseteq E_\phi(\delta)$*

**Proof** Let  $\phi$  be a  $\delta$ -improper edge colouring of  $G$ . If  $\phi$  is a proper colouring, we are done. Hence, assume that  $uv$  and  $uw$  are coloured  $\delta$ . If  $d(u) = 2$  we can change the colour of  $uv$  to  $\alpha, \beta$  or  $\gamma$  since  $v$  is incident to at most two colours in this set.

If  $d(u) = 3$  assume that the third edge  $uz$  incident to  $u$  is also coloured  $\delta$ , then we can change the colour of  $uv$  for the same reason as above.

If  $uz$  is coloured with  $\alpha, \beta$  or  $\gamma$ , then  $v$  and  $w$  are incident to the two remaining colours of the set  $\{\alpha, \beta, \gamma\}$  otherwise one of the edges  $uv, uw$  can be recoloured with the missing colour. W.l.o.g., consider that  $uz$  is coloured  $\alpha$  then  $v$  and  $w$  are incident to  $\beta$  and  $\gamma$ . Since  $u$  has degree 1 in  $\phi(\alpha, \beta)$  let  $P$  be the path of  $\phi(\alpha, \beta)$  which ends on  $u$ . We can assume that  $v$  or  $w$  (say  $v$ ) is not the other end vertex of  $P$ . Exchanging  $\alpha$  and  $\beta$  along  $P$  does not change the colours incident to  $v$ . But now  $uz$  is coloured  $\alpha$  and we can change the colour of  $uv$  with  $\beta$ .

In each case, we get hence a new  $\delta$ -improper edge colouring  $\phi_1$  with  $E_{\phi_1}(\delta) \subsetneq E_\phi(\delta)$ . Repeating this process leads us to construct a proper edge colouring of  $G$  with  $E_{\phi'}(\delta) \subseteq E_\phi(\delta)$  as claimed.  $\square$

**Lemma 4** [2, 3] *Let  $\phi$  be a  $\delta$ -minimum edge-colouring of  $G$ . For any edge  $e = uv \in E_\phi(\delta)$  there are two colours  $x$  and  $y$  in  $\{\alpha, \beta, \gamma\}$  such that the connected component of  $\phi(x, y)$  containing the two ends of  $e$  is an even path joining these two ends.*

**Proof** Without loss of generality assume that  $u$  is incident to  $\alpha$  and  $\beta$  and  $v$  is incident to  $\gamma$  (see Lemma 2). In any case ( $v$  has degree 3 or degree 2)  $u$  and  $v$  are contained in paths of  $\phi(\alpha, \gamma)$  or  $\phi(\beta, \gamma)$ . Assume that they are contained in paths of  $\phi(\alpha, \gamma)$ . If these paths are disjoint then we can exchange the two colours on the path containing  $u$ ,  $e$  will be incident hence to only two colours  $\beta$  and  $\gamma$  in this new edge-colouring and  $e$  could be recoloured with  $\alpha$ , a contradiction since we consider a  $\delta$ -minimum edge-colouring.  $\square$

**Lemma 5** [2, 3] *If  $G$  is a cubic graph then  $|A_\phi| \equiv |B_\phi| \equiv |C_\phi| \equiv s(G) \pmod{2}$ .*

**Proof**  $\phi(\alpha, \beta)$  contains  $2|A_\phi| + |B_\phi| + |C_\phi|$  vertices of degree 1 and must be even. Hence we get  $|B_\phi| \equiv |C_\phi| \pmod{2}$ . In the same way we get  $|A_\phi| \equiv |B_\phi| \pmod{2}$  leading to  $|A_\phi| \equiv |B_\phi| \equiv |C_\phi| \equiv s(G) \pmod{2}$ .  $\square$

**Remark 6** An edge of  $E_\phi(\delta)$  is in  $A_\phi$  when its ends can be connected by a path of  $\phi(\alpha, \beta)$ ,  $B_\phi$  by a path of  $\phi(\beta, \gamma)$  and  $C_\phi$  by a path of  $\phi(\alpha, \gamma)$ . It is clear that  $A_\phi$ ,  $B_\phi$  and  $C_\phi$  are not necessarily pairwise disjoint since an edge of  $E_\phi(\delta)$  with one end of degree 2 is contained in 2 such sets. Assume indeed that  $e = uv \in E_\phi(\delta)$  with  $d(u) = 3$  and  $d(v) = 2$  then, if  $u$  is incident to  $\alpha$  and  $\beta$  and  $v$  is incident to  $\gamma$  we have an alternating path whose ends are  $u$  and  $v$  in  $\phi(\alpha, \gamma)$  as well as in  $\phi(\beta, \gamma)$ . Hence  $e$  is in  $A_\phi \cap B_\phi$ . When  $e \in A_\phi$  we can associate to  $e$  the odd cycle  $C_{A_\phi}(e)$  obtained by considering the path of  $\phi(\alpha, \beta)$  together with  $e$ . We define in the same way  $C_{B_\phi}(e)$  and  $C_{C_\phi}(e)$  when  $e$  is in  $B_\phi$  or  $C_\phi$ . In the following lemma we consider an edge in  $A_\phi$ , an analogous result holds true whenever we consider edges in  $B_\phi$  or  $C_\phi$  as well.

**Lemma 7** [2, 3] *Let  $\phi$  be a  $\delta$ -minimum edge-colouring of  $G$  and let  $e$  be an edge in  $A_\phi$  ( $B_\phi$ ,  $C_\phi$ ) then for any edge  $e' \in C_{A_\phi}(e)$  there is a  $\delta$ -minimum edge-colouring  $\phi'$  such that  $E_{\phi'}(\delta) = E_\phi(\delta) - \{e\} \cup \{e'\}$ ,  $e' \in A_{\phi'}$  and  $C_{A_\phi}(e) = C_{A_{\phi'}}(e')$ . Moreover, each edge outside  $C_{A_\phi}(e)$  but incident with this cycle is coloured  $\gamma$  ( $\alpha$ ,  $\beta$  respectively),  $\phi$  and  $\phi'$  only differ on the edges of  $C_{A_\phi}(e)$ .*

**Proof** By exchanging colours  $\delta$  and  $\alpha$  and  $\delta$  and  $\beta$  successively along the cycle  $C_{A_\phi}(e)$ , we are sure to obtain an edge colouring preserving the number of edges coloured  $\delta$ . Since we have supposed that  $\phi$  is  $\delta$ -minimum, at each step, the resulting edge colouring is proper and  $\delta$ -minimum (Lemma 3). Hence, there is no edge coloured  $\delta$  incident with  $C_{A_\phi}(e)$ , which means that every such edge is coloured with  $\gamma$ .

We can perform these exchanges until  $e'$  is coloured  $\delta$ . In the  $\delta$ -minimum edge-colouring  $\phi'$  hence obtained, the two ends of  $e'$  are joined by a path of  $\phi(\alpha, \beta)$ . Which means that  $e'$  is in  $A_{\phi'}$  and  $C_{A_\phi}(e) = C_{A_{\phi'}}(e')$ .  $\square$

For each edge  $e \in E_\phi(\delta)$  (where  $\phi$  is a  $\delta$ -minimum edge-colouring of  $G$ ) we can associate one or two odd cycles following the fact that  $e$  is in one or two

sets among  $A_\phi$ ,  $B_\phi$  or  $C_\phi$ . Let  $\mathcal{C}$  be the set of odd cycles associated to edges in  $E_\phi(\delta)$ .

**Lemma 8** [2, 3] *For each cycle  $C \in \mathcal{C}$ , there are no two consecutive vertices with degree two.*

**Proof** Otherwise, we exchange colours along  $C$  in order to put the colour  $\delta$  on the corresponding edge and, by Lemma 2, this is impossible in a  $\delta$ -minimum edge-colouring.  $\square$

**Lemma 9** [2, 3] *Let  $e_1, e_2 \in E_\phi(\delta)$  and let  $C_1, C_2 \in \mathcal{C}$  be such that  $C_1 \neq C_2$ ,  $e_1 \in E(C_1)$  and  $e_2 \in E(C_2)$  then  $C_1$  and  $C_2$  are (vertex) disjoint.*

**Proof** If  $e_1$  and  $e_2$  are contained in the same set  $A_\phi$ ,  $B_\phi$  or  $C_\phi$ , we are done since their respective ends are joined by an alternating path of  $\phi(x, y)$  for some two colours  $x$  and  $y$  in  $\{\alpha, \beta, \gamma\}$ .

Without loss of generality assume that  $e_1 \in A_\phi$  and  $e_2 \in B_\phi$ . Assume moreover that there exists an edge  $e$  such that  $e \in C_1 \cap C_2$ . We have hence an edge  $f \in C_1$  with exactly one end on  $C_2$ . We can exchange colours on  $C_1$  in order to put the colour  $\delta$  on  $f$ . Which is impossible by Lemma 7.  $\square$

**Lemma 10** [2, 3] *Let  $e_1 = uv_1$  be an edge of  $E_\phi(\delta)$  such that  $v_1$  has degree 2 in  $G$ . Then  $v_1$  is the only vertex in  $N(u)$  of degree 2 and for any edge  $e_2 = u_2v_2 \in E_\phi(\delta)$ ,  $\{e_1, e_2\}$  induces a  $2K_2$ .*

**Proof** We have seen in Lemma 2 that  $uv_1$  has one end of degree 3 while the other has degree 2 or 3. Hence, we have  $d(u) = 3$  and  $d(v_1) = 2$ . Let  $v_2$  and  $v_3$  the other neighbours of  $u$ . From Remark 6, we know that  $v_2$  and  $v_3$  are not pendant vertices. Assume that  $d(v_2) = 2$  and  $uv_2$  is coloured  $\alpha$ ,  $uv_3$  is coloured  $\beta$  and, finally  $v_1$  is incident to an edge coloured  $\gamma$ . The alternating path of  $\phi(\beta, \gamma)$  using the edge  $uv_3$  ends with the vertex  $v_1$  (see Lemma 4), then, exchanging the colours along the component of  $\phi(\beta, \gamma)$  containing  $v_2$  allows us to colour  $uv_2$  with  $\gamma$  and  $uv_1$  with  $\alpha$ . The new edge colouring  $\phi'$  so obtained is such that  $|E_{\phi'}(\delta)| \leq |E_\phi(\delta)| - 1$ , impossible.  $\square$

**Lemma 11** [2, 3] *Let  $e_1$  and  $e_2$  be two edges of  $E_\phi(\delta)$ . If  $e_1$  and  $e_2$  are contained in two distinct sets of  $A_\phi, B_\phi$  or  $C_\phi$  then  $\{e_1, e_2\}$  induces a  $2K_2$  otherwise  $e_1, e_2$  are joined by at most one edge.*

**Proof** Assume in a first stage that  $e_1 \in A_\phi$  and  $e_2 \in B_\phi$ . Since  $C_{e_1}(\phi)$  and  $C_{e_2}(\phi)$  are disjoint by Lemma 9, we know that  $e_1$  and  $e_2$  have no common vertex. The edges having exactly one end in  $C_{e_1}(\phi)$  are coloured  $\gamma$  while those having exactly one end in  $C_{e_2}(\phi)$  are coloured  $\alpha$ . Hence there is no edge between  $e_1$  and  $e_2$  as claimed.

Assume in a second stage that  $e_1 = u_1v_1, e_2 = u_2v_2 \in A_\phi$ . Since  $C_{e_1}(\phi)$  and  $C_{e_2}(\phi)$  are disjoint by Lemma 9, we can consider that  $e_1$  and  $e_2$  have no common vertex. The edges having exactly one end in  $C_{e_1}(\phi)$  (or  $C_{e_2}(\phi)$ ) are coloured  $\gamma$ . Assume that  $u_1u_2$  and  $v_1v_2$  are edges of  $G$ . We may suppose

without loss of generality that  $u_1$  and  $u_2$  are incident to  $\alpha$  while  $v_1$  and  $v_2$  are incident to  $\beta$  (if necessary, colours  $\alpha$  and  $\beta$  can be exchanged on  $C_{e_1}(\phi)$  and  $C_{e_2}(\phi)$ ). We know that  $u_1u_2$  and  $v_1v_2$  are coloured  $\gamma$ . Let us colour  $e_1$  and  $e_2$  with  $\gamma$  and  $u_1u_2$  with  $\beta$  and  $v_1v_2$  with  $\alpha$ . We get a new edge colouring  $\phi'$  where  $|E_{\phi'}(\delta)| \leq |E_{\phi}(\delta)| - 2$ , contradiction since  $\phi$  is a  $\delta$ -minimum edge-colouring.  $\square$

**Lemma 12** [2, 3] *Let  $e_1, e_2$  and  $e_3$  be three distinct edges of  $E_{\phi}(\delta)$  contained in the same set  $A_{\phi}, B_{\phi}$  or  $C_{\phi}$ . Then  $\{e_1, e_2, e_3\}$  induces a subgraph with at most four edges.*

**Proof** Without loss of generality assume that  $e_1 = u_1v_1, e_2 = u_2v_2$  and  $e_3 = u_3v_3 \in A_{\phi}$ . From Lemma 11 we have just to suppose that (up to the names of vertices)  $u_1u_3 \in E(G)$  and  $v_1v_2 \in E(G)$ . Possibly, by exchanging the colours  $\alpha$  and  $\beta$  along the 3 disjoint paths of  $\phi(\alpha, \beta)$  joining the ends of each edge  $e_1, e_2$  and  $e_3$ , we can suppose that  $u_1$  and  $u_3$  are incident to  $\beta$  while  $v_1$  and  $v_2$  are incident to  $\alpha$ . Let  $\phi'$  be obtained from  $\phi$  when  $u_1u_3$  is coloured with  $\alpha$ ,  $v_1v_2$  with  $\beta$  and  $u_1v_1$  with  $\gamma$ . It is easy to check that  $\phi'$  is a proper edge-colouring with  $|E_{\phi'}(\delta)| \leq |E_{\phi}(\delta)| - 1$ , contradiction since  $\phi$  is a  $\delta$ -minimum edge-colouring.  $\square$

**Lemma 13** [2, 3] *Let  $e_1 = u_1v_1$  be an edge of  $E_{\phi}(\delta)$  such that  $v_1$  has degree 2 in  $G$ . Then for any edge  $e_2 = u_2v_2 \in E_{\phi}(\delta)$   $\{e_1, e_2\}$  induces a  $2K_2$ .*

**Proof** From Lemma 11 we have to consider that  $e_1$  and  $e_2$  are contained in the same set  $A_{\phi}, B_{\phi}$  or  $C_{\phi}$ . Assume without loss of generality that they are contained in  $A_{\phi}$ . From Lemma 11 again we have just to consider that there is a unique edge joining these two edges and we can suppose that  $u_1u_2 \in E(G)$ . Possibly, by exchanging the colours  $\alpha$  and  $\beta$  along the 2 disjoint paths of  $\phi(\alpha, \beta)$  joining the ends of each edge  $e_1, e_2$ , we can suppose that  $u_1$  and  $u_2$  are incident to  $\beta$  while  $v_1$  and  $v_2$  are incident to  $\alpha$ . We know that  $u_1u_2$  is coloured  $\gamma$ . Let  $\phi'$  be obtained from  $\phi$  when  $u_1u_2$  is coloured with  $\alpha$  and  $u_1v_1$  is coloured with  $\gamma$ . It is easy to check that  $\phi'$  is a proper edge-colouring with  $|E_{\phi'}(\delta)| \leq |E_{\phi}(\delta)| - 1$ , contradiction since  $\phi$  is a  $\delta$ -minimum edge-colouring.  $\square$

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