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Tools for parsimonious edge-colouring of graphs with maximum degree three

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Abstract

In a graph G of maximum degree Δ let γ denote the largest fraction of edges that can be Δ -edge-coloured. Albertson and Haas showed that $\gamma \geq \frac{13}{15}$ when G is cubic [1]. The notion of δ -minimum edge colouring was introduced in [3] in order to extend the so called *parsimonious edge-colouring* to graphs with maximum degree 3. We propose here an english translation of some structural properties already present in [2, 3] (in French) for δ -minimum edge colourings of graphs with maximum degree 3.

Keywords : Cubic graph; Edge-colouring;

1 Introduction

Throughout this note, we shall be concerned with connected graphs with maximum degree 3. We know by Vizing's theorem [7] that these graphs can be edge-coloured with 4 colours. Let $\phi : E(G) \rightarrow \{\alpha, \beta, \gamma, \delta\}$ be a proper edge-colouring of G . It is often of interest to try to use one colour (say δ) as few as possible. When an edge colouring is optimal, following this constraint, we shall say that ϕ is δ -*minimum*. In [2] we gave without proof (in French) results on δ -*minimum* edge-colourings of cubic graphs. Some of them have been obtained later and independently by Steffen [5] and [6]. The purpose of Section 2 is to give with their proofs those results as structural properties of δ -minimum edge-colourings.

An edge colouring of G using colours $\alpha, \beta, \gamma, \delta$ is said to be δ -*improper* provided that adjacent edges having the same colours (if any) are coloured with δ . It is clear that a proper edge colouring (and hence a δ -minimum edge-colouring) of G is a particular δ -improper edge colouring. For a proper or δ -improper edge colouring ϕ of G , it will be convenient to denote $E_\phi(x)$ ($x \in \{\alpha, \beta, \gamma, \delta\}$) the set of edges coloured with x by ϕ . For $x, y \in \{\alpha, \beta, \gamma, \delta\}, x \neq y$, $\phi(x, y)$ is the partial subgraph of G spanned by these two colours, that is $E_\phi(x) \cup E_\phi(y)$ (this subgraph being a union of paths and even cycles where the colours x and y alternate). Since any two δ -minimum edge-colourings of G have the same number of edges coloured δ we shall denote by $s(G)$ this number (the *colour number* as defined in [5]).

As usual, for any undirected graph G , we denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges and we suppose that $|V(G)| = n$ and

$|E(G)| = m$. A *strong matching* C in a graph G is a matching C such that there is no edge of $E(G)$ connecting any two edges of C , or, equivalently, such that C is the edge-set of the subgraph of G induced on the vertex-set $V(C)$.

2 Structural properties of δ -minimum edge-colourings

The graph G considered in the following series of Lemmas will have maximum degree 3.

Lemma 1 [2, 3] *Any 2-factor of G contains at least $s(G)$ disjoint odd cycles.*

Proof Assume that we can find a 2-factor of G with $k < s(G)$ odd cycles. Then let us colour the edges of this 2-factor with α and β , except one edge (coloured δ) on each odd cycle of our 2-factor and let us colour the remaining edges by γ . We get hence a new edge colouring ϕ with $E_\phi(\delta) < s(G)$, impossible. \square

Lemma 2 [2, 3] *Let ϕ be a δ -minimum edge-colouring of G . Any edge in $E_\phi(\delta)$ is incident to α, β and γ . Moreover each such edge has one end of degree 2 and the other of degree 3 or the two ends of degree 3.*

Proof Any edge of $E_\phi(\delta)$ is certainly adjacent to α, β and γ . Otherwise this edge could be coloured with the missing colour and we should obtain an edge colouring ϕ' with $|E_{\phi'}(\delta)| < |E_\phi(\delta)|$. \square

Lemma 3 *Let ϕ be a δ -improper colouring of G then there exists a proper colouring of G ϕ' such that $E_{\phi'}(\delta) \subseteq E_\phi(\delta)$*

Proof Let ϕ be a δ -improper edge colouring of G . If ϕ is a proper colouring, we are done. Hence, assume that uv and uw are coloured δ . If $d(u) = 2$ we can change the colour of uv to α, β or γ since v is incident to at most two colours in this set.

If $d(u) = 3$ assume that the third edge uz incident to u is also coloured δ , then we can change the colour of uv for the same reason as above.

If uz is coloured with α, β or γ , then v and w are incident to the two remaining colours of the set $\{\alpha, \beta, \gamma\}$ otherwise one of the edges uv, uw can be recoloured with the missing colour. W.l.o.g., consider that uz is coloured α then v and w are incident to β and γ . Since u has degree 1 in $\phi(\alpha, \beta)$ let P be the path of $\phi(\alpha, \beta)$ which ends on u . We can assume that v or w (say v) is not the other end vertex of P . Exchanging α and β along P does not change the colours incident to v . But now uz is coloured α and we can change the colour of uv with β .

In each case, we get hence a new δ -improper edge colouring ϕ_1 with $E_{\phi_1}(\delta) \subsetneq E_\phi(\delta)$. Repeating this process leads us to construct a proper edge colouring of G with $E_{\phi'}(\delta) \subseteq E_\phi(\delta)$ as claimed. \square

Lemma 4 [2, 3] *Let ϕ be a δ -minimum edge-colouring of G . For any edge $e = uv \in E_\phi(\delta)$ there are two colours x and y in $\{\alpha, \beta, \gamma\}$ such that the connected component of $\phi(x, y)$ containing the two ends of e is an even path joining these two ends.*

Proof Without loss of generality assume that u is incident to α and β and v is incident to γ (see Lemma 2). In any case (v has degree 3 or degree 2) u and v are contained in paths of $\phi(\alpha, \gamma)$ or $\phi(\beta, \gamma)$. Assume that they are contained in paths of $\phi(\alpha, \gamma)$. If these paths are disjoint then we can exchange the two colours on the path containing u , e will be incident hence to only two colours β and γ in this new edge-colouring and e could be recoloured with α , a contradiction since we consider a δ -minimum edge-colouring. \square

Lemma 5 [3] *If G is a cubic graph then $|A_\phi| \equiv |B_\phi| \equiv |C_\phi| \equiv s(G) \pmod{2}$.*

Proof $\phi(\alpha, \beta)$ contains $2|A_\phi| + |B_\phi| + |C_\phi|$ vertices of degree 1 and must be even. Hence we get $|B_\phi| \equiv |C_\phi| \pmod{2}$. In the same way we get $|A_\phi| \equiv |B_\phi| \pmod{2}$ leading to $|A_\phi| \equiv |B_\phi| \equiv |C_\phi| \equiv s(G) \pmod{2}$. \square

Remark 6 An edge of $E_\phi(\delta)$ is in A_ϕ when its ends can be connected by a path of $\phi(\alpha, \beta)$, B_ϕ by a path of $\phi(\beta, \gamma)$ and C_ϕ by a path of $\phi(\alpha, \gamma)$. It is clear that A_ϕ , B_ϕ and C_ϕ are not necessarily pairwise disjoint since an edge of $E_\phi(\delta)$ with one end of degree 2 is contained in 2 such sets. Assume indeed that $e = uv \in E_\phi(\delta)$ with $d(u) = 3$ and $d(v) = 2$ then, if u is incident to α and β and v is incident to γ we have an alternating path whose ends are u and v in $\phi(\alpha, \gamma)$ as well as in $\phi(\beta, \gamma)$. Hence e is in $A_\phi \cap B_\phi$. When $e \in A_\phi$ we can associate to e the odd cycle $C_{A_\phi}(e)$ obtained by considering the path of $\phi(\alpha, \beta)$ together with e . We define in the same way $C_{B_\phi}(e)$ and $C_{C_\phi}(e)$ when e is in B_ϕ or C_ϕ . In the following lemma we consider an edge in A_ϕ , an analogous result holds true whenever we consider edges in B_ϕ or C_ϕ as well.

Lemma 7 below was proven in [4], we give its proof for sake of completeness.

Lemma 7 [4] *Let ϕ be a δ -minimum edge-colouring of G and let e be an edge in A_ϕ (B_ϕ , C_ϕ) then for any edge $e' \in C_{A_\phi}(e)$ there is a δ -minimum edge-colouring ϕ' such that $E_{\phi'}(\delta) = E_\phi(\delta) - \{e\} \cup \{e'\}$, $e' \in A_{\phi'}$ and $C_{A_\phi}(e) = C_{A_{\phi'}}(e')$. Moreover, each edge outside $C_{A_\phi}(e)$ but incident with this cycle is coloured γ (α , β respectively), ϕ and ϕ' only differ on the edges of $C_{A_\phi}(e)$.*

Proof By exchanging colours δ and α and δ and β successively along the cycle $C_{A_\phi}(e)$, we are sure to obtain an edge colouring preserving the number of edges coloured δ . Since we have supposed that ϕ is δ -minimum, at each step, the resulting edge colouring is proper and δ -minimum (Lemma 3). Hence, there is no edge coloured δ incident with $C_{A_\phi}(e)$, which means that every such edge is coloured with γ .

We can perform these exchanges until e' is coloured δ . In the δ -minimum edge-colouring ϕ' hence obtained, the two ends of e' are joined by a path of

$\phi(\alpha, \beta)$. Which means that e' is in A_ϕ and $C_{A_\phi}(e) = C_{A_{\phi'}}(e')$. \square

For each edge $e \in E_\phi(\delta)$ (where ϕ is a δ -minimum edge-colouring of G) we can associate one or two odd cycles following the fact that e is in one or two sets among A_ϕ , B_ϕ or C_ϕ . Let \mathcal{C} be the set of odd cycles associated to edges in $E_\phi(\delta)$.

Lemma 8 [2, 3] *For each cycle $C \in \mathcal{C}$, there are no two consecutive vertices with degree two.*

Proof Otherwise, we exchange colours along C in order to put the colour δ on the corresponding edge and, by Lemma 2, this is impossible in a δ -minimum edge-colouring. \square

Lemma 9 [2, 3] *Let $e_1, e_2 \in E_\phi(\delta)$ and let $C_1, C_2 \in \mathcal{C}$ be such that $C_1 \neq C_2$, $e_1 \in E(C_1)$ and $e_2 \in E(C_2)$ then C_1 and C_2 are (vertex) disjoint.*

Proof If e_1 and e_2 are contained in the same set A_ϕ , B_ϕ or C_ϕ , we are done since their respective ends are joined by an alternating path of $\phi(x, y)$ for some two colours x and y in $\{\alpha, \beta, \gamma\}$.

Without loss of generality assume that $e_1 \in A_\phi$ and $e_2 \in B_\phi$. Assume moreover that there exists an edge e such that $e \in C_1 \cap C_2$. We have hence an edge $f \in C_1$ with exactly one end on C_2 . We can exchange colours on C_1 in order to put the colour δ on f . Which is impossible by Lemma 7. \square

Lemma 10 [2, 3] *Let $e_1 = uv_1$ be an edge of $E_\phi(\delta)$ such that v_1 has degree 2 in G . Then v_1 is the only vertex in $N(u)$ of degree 2 and for any edge $e_2 = u_2v_2 \in E_\phi(\delta)$, $\{e_1, e_2\}$ induces a $2K_2$.*

Proof We have seen in Lemma 2 that uv_1 has one end of degree 3 while the other has degree 2 or 3. Hence, we have $d(u) = 3$ and $d(v_1) = 2$. Let v_2 and v_3 the other neighbours of u . From Remark 6, we know that v_2 and v_3 are not pendant vertices. Assume that $d(v_2) = 2$ and uv_2 is coloured α , uv_3 is coloured β and, finally v_1 is incident to an edge coloured γ . The alternating path of $\phi(\beta, \gamma)$ using the edge uv_3 ends with the vertex v_1 (see Lemma 4), then, exchanging the colours along the component of $\phi(\beta, \gamma)$ containing v_2 allows us to colour uv_2 with γ and uv_1 with α . The new edge colouring ϕ' so obtained is such that $|E_{\phi'}(\delta)| \leq |E_\phi(\delta)| - 1$, impossible. \square

Lemma 11 [2, 3] *Let e_1 and e_2 be two edges of $E_\phi(\delta)$. If e_1 and e_2 are contained in two distinct sets of A_ϕ, B_ϕ or C_ϕ then $\{e_1, e_2\}$ induces a $2K_2$ otherwise e_1, e_2 are joined by at most one edge.*

Proof Assume in a first stage that $e_1 \in A_\phi$ and $e_2 \in B_\phi$. Since $C_{e_1}(\phi)$ and $C_{e_2}(\phi)$ are disjoint by Lemma 9, we know that e_1 and e_2 have no common vertex. The edges having exactly one end in $C_{e_1}(\phi)$ are coloured γ while those having exactly one end in $C_{e_2}(\phi)$ are coloured α . Hence there is no edge between e_1 and e_2 as claimed.

Assume in a second stage that $e_1 = u_1v_1, e_2 = u_2v_2 \in A_\phi$. Since $C_{e_1}(\phi)$ and $C_{e_2}(\phi)$ are disjoint by Lemma 9, we can consider that e_1 and e_2 have no common vertex. The edges having exactly one end in $C_{e_1}(\phi)$ (or $C_{e_2}(\phi)$) are coloured γ . Assume that u_1u_2 and v_1v_2 are edges of G . We may suppose without loss of generality that u_1 and u_2 are incident to α while v_1 and v_2 are incident to β (if necessary, colours α and β can be exchanged on $C_{e_1}(\phi)$ and $C_{e_2}(\phi)$). We know that u_1u_2 and v_1v_2 are coloured γ . Let us colour e_1 and e_2 with γ and u_1u_2 with β and v_1v_2 with α . We get a new edge colouring ϕ' where $|E_{\phi'}(\delta)| \leq |E_\phi(\delta)| - 2$, contradiction since ϕ is a δ -minimum edge-colouring. \square

Lemma 12 [2, 3] *Let e_1, e_2 and e_3 be three distinct edges of $E_\phi(\delta)$ contained in the same set A_ϕ, B_ϕ or C_ϕ . Then $\{e_1, e_2, e_3\}$ induces a subgraph with at most four edges.*

Proof Without loss of generality assume that $e_1 = u_1v_1, e_2 = u_2v_2$ and $e_3 = u_3v_3 \in A_\phi$. From Lemma 11 we have just to suppose that (up to the names of vertices) $u_1u_3 \in E(G)$ and $v_1v_2 \in E(G)$. Possibly, by exchanging the colours α and β along the 3 disjoint paths of $\phi(\alpha, \beta)$ joining the ends of each edge e_1, e_2 and e_3 , we can suppose that u_1 and u_3 are incident to β while v_1 and v_2 are incident to α . Let ϕ' be obtained from ϕ when u_1u_3 is coloured with α , v_1v_2 with β and u_1v_1 with γ . It is easy to check that ϕ' is a proper edge-colouring with $|E_{\phi'}(\delta)| \leq |E_\phi(\delta)| - 1$, contradiction since ϕ is a δ -minimum edge-colouring. \square

Lemma 13 [2, 3] *Let $e_1 = u_1v_1$ be an edge of $E_\phi(\delta)$ such that v_1 has degree 2 in G . Then for any edge $e_2 = u_2v_2 \in E_\phi(\delta)$ $\{e_1, e_2\}$ induces a $2K_2$.*

Proof From Lemma 11 we have to consider that e_1 and e_2 are contained in the same set A_ϕ, B_ϕ or C_ϕ . Assume without loss of generality that they are contained in A_ϕ . From Lemma 11 again we have just to consider that there is a unique edge joining these two edges and we can suppose that $u_1u_2 \in E(G)$. Possibly, by exchanging the colours α and β along the 2 disjoint paths of $\phi(\alpha, \beta)$ joining the ends of each edge e_1, e_2 , we can suppose that u_1 and u_2 are incident to β while v_1 and v_2 are incident to α . We know that u_1u_2 is coloured γ . Let ϕ' be obtained from ϕ when u_1u_2 is coloured with α and u_1v_1 is coloured with γ . It is easy to check that ϕ' is a proper edge-colouring with $|E_{\phi'}(\delta)| \leq |E_\phi(\delta)| - 1$, contradiction since ϕ is a δ -minimum edge-colouring. \square

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