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Nalini Anantharaman

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EXPONENTIAL DECAY FOR PRODUCTS OF FOURIER INTEGRAL OPERATORS

NALINI ANANTHARAMAN

Abstract. This text contains an alternative presentation, and in certain cases an improvement, of the "hyperbolic dispersive estimate" proved in [1, 3], where it was used to make progress towards the quantum unique ergodicity conjecture. The main statement is a sufficient condition to have exponential decay of the norm of a product of sub-unitary Fourier integral operators. The improved version presented here is needed in the two papers [5] and [6].

1. Introduction

On a Hilbert space $\mathcal{H}$, consider the product $\hat{P}_n\hat{P}_{n-1}\cdots\hat{P}_1$ of a large number of operators $\hat{P}_j$, with $\|\hat{P}_j\| = 1$. Think, for instance, of the case where each operator $\hat{P}_j$ is an orthogonal projector, or a product of an orthogonal projector and a unitary operator. What kind of geometric considerations can be helpful to prove that the norm $\|\hat{P}_n\hat{P}_{n-1}\cdots\hat{P}_1\|$ is strictly less than 1 ? or better, that it decays exponentially fast with n ? In Section 2, we will describe a situation in which $\mathcal{H} = L^2(\mathbb{R}^d)$, and the operators $\hat{P}_j$ are Fourier integral operators associated to a sequence of canonical transformations κ_j. We will give a "hyperbolicity" condition, on the sequence of transformations κ_j and on the symbols of the operators $\hat{P}_j$, under which we can prove exponential decay of the norm $\|\hat{P}_n\hat{P}_{n-1}\cdots\hat{P}_1\|$.

This technique was introduced in [1, 3], and is used in [1, 3, 4, 18, 19, 6] to prove results related to the quantum unique ergodicity conjecture. In [1, 3], the proofs are written on a riemannian manifold of negative curvature, for $\hat{P}_n = e^{\frac{i\tau\hbar\triangle}{2}}\hat{\chi}_n$, where the operators $\hat{\chi}_n$ belong to a finite family of pseudodifferential operators, supported inside compact sets of small diameters, and where $\triangle$ is the laplacian and $\tau > 0$ is fixed. The exponential decay is then used to prove a lower bound on the "entropy" of eigenfunctions, answering by the negative the long-standing question : can a sequence of eigenfunctions concentrate on a closed geodesic, as the eigenvalue goes to infinity ? An expository paper can be found in [15], see also the forthcoming paper [2]. We give here an alternative presentation, based on the use of local adapted symplectic coordinates, which leads in certain cases to an improvement, needed in the two papers [5] and [6].

Let us also mention the work of Nonnenmacher-Zworski [16, 17], Christianson [8, 9, 10], Datchev [11], and Burq-Guillarmou-Hassell [7], who showed how to use these techniques in scattering situations, to prove the existence of a gap below the real axis in the resolvent spectrum, and to get local smoothing estimates with loss, as well as Strichartz estimates.

In this context, the idea of proving exponential decay for Fourier integral operators was also present, although in an implicit form, in Doi's work [12].

The technique is presented in the first four sections, and the applications needed in [5, 6] are stated in section 5.

2. A hyperbolic dispersion estimate

In this section, $\mathbb{R}^d \times (\mathbb{R}^d)^*$ is endowed with the canonical symplectic form $\omega_o = \sum_{j=1}^d dx_j \wedge d\xi_j$, where dx_j denotes the projection on the j-th vector of the canonical basis in $\mathbb{R}^d$, and $d\xi_j$ is the projection on the j-th vector of the dual basis in $(\mathbb{R}^d)^*$. The space $\mathbb{R}^d$ will also be endowed with its usual scalar product, denoted $\langle .,. \rangle$, and we use it to systematically identify $\mathbb{R}^d$ with $(\mathbb{R}^d)^*$.

We consider a sequence of smooth ($\mathcal{C}^\infty$) canonical transformations $\kappa_n : \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d \times \mathbb{R}^d$, preserving ω_o ($n \in \mathbb{N}$). We will only be interested in the restriction of κ_1 to a fixed relatively compact neighbourhood Ω of 0, and it is actually sufficient for us to assume that the product $\kappa_n \circ \kappa_{n-1} \circ \cdots \circ \kappa_1$ is well defined, for all n, on Ω. The Darboux-Lie theorem ensures that every lagrangian foliation can be mapped, by a symplectic change of coordinates, to the foliation of $\mathbb{R}^d \times \mathbb{R}^d$ by the "horizontal" leaves $\mathcal{L}_{\xi_0} = \{(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d, \xi = \xi_0\}$. For our purposes (section 5), there is no loss of generality if we make the simplifying assumption that each symplectic transformation κ_n preserves this horizontal foliation. It means that κ_n is of the form $(x, \xi) \mapsto (x', \xi' = p_n(\xi))$ where $p_n : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a smooth function. In more sophisticated words, κ_n has a generating function of the form $S_n(x, x', \theta) = \langle p_n(\theta), x' \rangle - \langle \theta, x \rangle + \alpha_n(\theta)$ (where $x, x', \theta \in \mathbb{R}^d$, and $\alpha_n : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a smooth function). We have the equivalence

$$\big[(x', \xi') = \kappa_n(x, \xi)\big] \Longleftrightarrow \big[\xi = -\partial_x S_n(x, x', \theta),\ \xi' = \partial_{x'} S_n(x, x', \theta),\ \partial_\theta S_n(x, x', \theta) = 0\big].$$

The product $\kappa_n \circ \ldots \circ \kappa_2 \circ \kappa_1$ also preserves the horizontal foliation, and it admits the generating function

$$\begin{aligned}\langle p_n \circ \ldots \circ p_1(\theta), x' \rangle - \langle \theta, x \rangle + \alpha_1(\theta) + \alpha_2(p_1(\theta)) + \ldots + \alpha_n(p_{n-1} \circ \ldots \circ p_1(\theta)) \\ = \langle p_n \circ \ldots \circ p_1(\theta), x' \rangle - \langle \theta, x \rangle + A_n(\theta),\end{aligned}$$

where the equality defines $A_n(\theta)$.

If p is a map $\mathbb{R}^d \longrightarrow \mathbb{R}^d$, we will denote ∇p the matrix $(\frac{\partial p_i}{\partial \theta_j})_{ij}$, which represents its differential in the canonical basis.

Assumptions (H) : We shall be interested in the following operators, acting on $L^2(\mathbb{R}^d)$:

$$\hat{P}_n f(x') = \frac{1}{(2\pi\hbar)^d} \int_{x \in \mathbb{R}^d, \theta \in \mathbb{R}^d} e^{\frac{iS_n(x,x',\theta)}{\hbar}} a^{(n)}(x, x', \theta, \hbar) f(x) dx d\theta,$$

where $\hbar > 0$ is a parameter destined to go to 0. We will assume the following :

(H1) The functions p_n are smooth diffeomorphisms, and all the derivatives of p_n, of p_n^{-1} and of α_n are bounded uniformly in n.

(H2) For a given $\hbar > 0$, the function $(x, x', \theta) \mapsto a^{(n)}(x, x', \theta, \hbar)$ is of class $\mathcal{C}^\infty$;

(H3) The function $a^{(1)}(x, x', \theta, \hbar)$ is supported in Ω with respect to the variable x;
(H4) With respect to the variables (x', θ), the functions $a^{(n)}(x, x', \theta, \hbar)$ have a compact support $x' \in \Omega_1, \theta \in \Omega_2$, independent of n and $\hbar$;
(H5) When $\hbar \longrightarrow 0$, each $a^{(n)}(x, x', \theta, \hbar)$ has an asymptotic expansion

$$a^{(n)}(x, x', \theta, \hbar) \sim (\det \nabla p_n(\theta))^{1/2} \sum_{k=0}^{\infty} \hbar^k a_k^{(n)}(x, x', \theta),$$

valid up to any order and in all the $\mathcal{C}^\ell$ norms on compact sets. Besides, these asymptotic expansions are *uniform with respect to* n;
(H6) If $(x', \theta') = \kappa_n(x, \theta)$, we have $|a_0^{(n)}(x, x', \theta)| \leq 1$. This condition ensures that $\|\hat{P}_n\|_{L^2 \longrightarrow L^2} \leq 1 + \mathcal{O}(\hbar)$.

The operators $\hat{P}_n$ are (semiclassical) *Fourier integral operators* associated with the transformations κ_n (section §6).

2.1. **Propagation of a single plane wave.** The following theorem is essentially proved in [1]. We denote $e_{\xi_0,\hbar}$ the function $e_{\xi_0,\hbar}(x) = e^{\frac{i\langle \xi_0, x\rangle}{\hbar}}$.

Theorem 2.1. *Fix $\xi_0 \in \mathbb{R}^d$. Denote $\xi_n = p_n \circ \ldots \circ p_1(\xi_0)$.*

In addition to the assumptions (H) above, assume that

$$\limsup_{k \longrightarrow +\infty} \frac{1}{k} \log \|\nabla(p_{n+k} \circ p_{n+k-1} \circ \ldots \circ p_{n+1})(\xi_n)\| \leq 0,$$

uniformly in n.

Fix $\mathcal{K} > 0$ arbitrary, and an integer $M \in \mathbb{N}$. Then we have, for $n = \mathcal{K}|\log \hbar|$, and $\tilde{\epsilon} > 0$ arbitrary,

$$\hat{P}_n \circ \ldots \circ \hat{P}_2 \circ \hat{P}_1 e_{\xi_0,\hbar}(x) = e^{i\frac{A_n(\xi_0)}{\hbar}} e_{\xi_n,\hbar}(x) (\det \nabla p_n \circ \ldots \circ p_1(\xi_0))^{1/2} \left[\sum_{k=0}^{M-1} \hbar^k b_k^{(n)}(x, \xi_n) \right] + \mathcal{O}(\hbar^{M(1-\tilde{\epsilon})}).$$

The functions $b_k^{(n)}$, defined on $\mathbb{R}^d \times \mathbb{R}^d$, are smooth, supported in $\Omega_1 \times \Omega_2$, and

$$b_0^{(n)}(x_n, \xi_n) = \prod_{j=1}^{n} a_0^{(j)}(x_j, x_{j+1}, \xi_j),$$

where we denote $\xi_n = p_n \circ \ldots \circ p_1(\xi_0)$, $x_n = x$ and the other terms are defined by the relations $(x_j, \xi_j) = \kappa_j \circ \ldots \circ \kappa_1(x_0, \xi_0)$.

The functions $b_k^{(n)}$, for $k > 0$, have the same support as $b_0^{(n)}$. We have $|b_0^{(n)}(x_n, \xi_n)| \leq 1$, and besides, we have bounds

$$\|d_x^j b_k^{(n)}\|_\infty \leq C(k, j, \epsilon) n^{j+3k} e^{\epsilon(j+2k)n}$$

valid for arbitrary $\epsilon > 0$, where the prefactor $C(k, j, \epsilon)$ does not depend on n.

If n is fixed, and if we write $\hat{P}_n \circ \ldots \circ \hat{P}_2 \circ \hat{P}_1 e_{\xi_0,\hbar}(x)$ explicitly as an integral over $(\mathbb{R}^d)^{2n}$, this theorem is a straightforward application of the stationary phase method. If n is allowed to go to infinity as $\hbar \longrightarrow 0$, our result amounts, in some sense, to applying the method of stationary phase on a space whose dimension goes to ∞, and this is known to be very delicate. The theorem was first proved this way, in an unpublished version (available on request or on my webpage) of the paper [1]. A nicer proof, written with the collaboration of Stéphane Nonnenmacher, is available in [1], and has also appeared under different forms in [3, 16]. In these papers, the proofs are written on a riemannian manifold, for $\hat{P}_n = e^{\frac{i\tau\hbar\triangle}{2}}\hat{\chi}_n$, where the operators $\hat{\chi}_n$ belong to a finite family of pseudodifferential operators, whose symbols are supported inside compact sets of small diameters, and where $\triangle$ is the laplacian and $\tau > 0$ is fixed. In local coordinates, and on a manifold of constant negative sectional curvature, the calculations done in [1, 3] amount to the simpler statement presented here (see section 5).

In all the papers cited above, the dynamical systems under study satisfy a *uniform hyperbolicity* (or Anosov) property, ensuring an exponential decay

$$\sup_{\xi\in\Omega_2}\|\nabla(p_n \circ \ldots \circ p_2 \circ p_1)(\xi)\| \leq Ce^{-\lambda n}, \tag{2.1}$$

with uniform constants $C, \lambda > 0$. This is why, following [16], we call our result a *hyperbolic dispersion estimate.*

2.2. **Estimating the norm of $\hat{P}_n \circ \ldots \circ \hat{P}_2 \circ \hat{P}_1$.** We use the $\hbar$-Fourier transform

$$\mathcal{F}_\hbar u(\xi) = \frac{1}{(2\pi\hbar)^{d/2}}\int_{\mathbb{R}^d} u(x)e^{-\frac{i\langle\xi,x\rangle}{\hbar}}dx,$$

the inversion formula

$$u(x) = \frac{1}{(2\pi\hbar)^{d/2}}\int_{\mathbb{R}^d} \mathcal{F}_\hbar u(\xi)e^{\frac{i\langle\xi,x\rangle}{\hbar}}d\xi,$$

and the Plancherel formula $\|u\|_{L^2(\mathbb{R}^d)} = \|\mathcal{F}_\hbar u\|_{L^2(\mathbb{R}^d)}$.

We denote by $\widetilde{\Omega}_2$ an open, relatively compact subset of $\mathbb{R}^d$, that contains the closure $\overline{\Omega_2}$. Using the Fourier inversion formula, Theorem 2.1 implies, in a straightforward manner, the following

Theorem 2.2. *In addition to the assumptions (H) above, assume that*

$$\limsup_{k\longrightarrow+\infty}\frac{1}{k}\log\|\nabla(p_{n+k}\circ p_{n+k-1}\circ\ldots\circ p_{n+1})(\xi_n)\| \leq 0,$$

uniformly in n and $\xi \in \widetilde{\Omega}_2$ (with $\xi_n = p_n \circ \ldots \circ p_1(\xi)$).

Fix $\mathcal{K} > 0$ arbitrary. Then, for $n = \mathcal{K}|\log\hbar|$,

$$\|\hat{P}_n \circ \ldots \circ \hat{P}_2 \circ \hat{P}_1\|_{L^2\longrightarrow L^2} \leq \frac{|\widetilde{\Omega}_2|^{1/2}}{(2\pi\hbar)^{d/2}}\sup_{\xi\in\widetilde{\Omega}_2}|\det\nabla p_n \circ \ldots \circ p_1(\xi)|^{1/2}(1+\mathcal{O}(\hbar n^3 e^{\epsilon n})),$$

where $|\widetilde{\Omega}_2|$ denotes the volume of $\widetilde{\Omega}_2$ (and $\epsilon > 0$ is arbitrary).

Of course, since $\|\hat{P}_j\|_{L^2\longrightarrow L^2} \leq 1+\mathcal{O}(\hbar)$, we always have the trivial bound $\|\hat{P}_n \circ \ldots \circ \hat{P}_2 \circ \hat{P}_1\|_{L^2\longrightarrow L^2} \leq 1+\mathcal{O}(\hbar|\log\hbar|)$. Since we are working in the limit $\hbar \longrightarrow 0$, our estimate can only have an interest if we have an upper bound of the form

$$\sup_{\xi\in\widetilde{\Omega}_2} |\det \nabla p_n \circ \ldots \circ p_1(\xi)|^{1/2} \leq Ce^{-\lambda n}, \qquad \lambda > 0, \tag{2.2}$$

and if $\mathcal{K}$ is large enough. Note that (2.2) is weaker than the condition (2.1).

We now state a refinement of Theorem 2.2. We consider the same family $\hat{P}_i$, satisfying assumptions (H). The multiplicative constants in our estimate have no importance, and in what follows we will often omit them.

Theorem 2.3. *Assume as above that*

$$\limsup_{k\longrightarrow+\infty} \frac{1}{k}\log\|\nabla(p_{n+k}\circ p_{n+k-1}\circ\ldots\circ p_{n+1})(\xi_n)\| \leq 0,$$

uniformly in n and $\xi \in \widetilde{\Omega}_2$.

Let $r \leq d$, and assume that the coisotropic foliation by the leaves $\{\xi_{r+1}=c_{r+1},\ldots,\xi_d=c_d\}$ is invariant by each canonical transformation κ_n. In other words, the map p_n is of the form

$$p_n((\xi_1,\ldots,\xi_r),(\xi_{r+1},\ldots,\xi_d)) = (m_n(\xi_1,\ldots,\xi_d),\tilde{p}_n(\xi_{r+1},\ldots,\xi_d)),$$

where $m_n : \mathbb{R}^d \longrightarrow \mathbb{R}^r$ and $\tilde{p}_n : \mathbb{R}^{d-r} \longrightarrow \mathbb{R}^{d-r}$.

Fix $\mathcal{K} > 0$ arbitrary. Then there exists $\hbar_{\mathcal{K}} > 0$ such that, for $n = \mathcal{K}|\log\hbar|$, and for $\hbar < \hbar_{\mathcal{K}}$,

$$\|\hat{P}_n\circ\ldots\circ\hat{P}_2\circ\hat{P}_1\|_{L^2\longrightarrow L^2} \leq \frac{1}{(2\pi\hbar)^{(r-\epsilon)/2}}\frac{\sup_{\xi\in\widetilde{\Omega}_2}|(\det\nabla p_n\circ\ldots\circ p_1(\xi))|^{1/2}}{\inf_{\xi\in\widetilde{\Omega}_2}|(\det\nabla\tilde{p}_n\circ\ldots\circ\tilde{p}_1(\xi))|^{1/2}}(1+\mathcal{O}(n^3\hbar e^{\epsilon n}))$$

for $\epsilon > 0$ arbitrary.

In addition, if we make the stronger assumption that $\|\nabla(p_{n+k}\circ p_{n+k-1}\circ\ldots\circ p_{n+1})(\xi_n)\|$ is bounded above, uniformly in n,k and for $\xi\in\tilde{\Omega}_2$, we have

$$\|\hat{P}_n\circ\ldots\circ\hat{P}_2\circ\hat{P}_1\|_{L^2\longrightarrow L^2} \leq \frac{1}{(2\pi\hbar)^{r/2}}\frac{\sup_{\xi\in\widetilde{\Omega}_2}|(\det\nabla p_n\circ\ldots\circ p_1(\xi))|^{1/2}}{\inf_{\xi\in\widetilde{\Omega}_2}|(\det\nabla\tilde{p}_n\circ\ldots\circ\tilde{p}_1(\xi))|^{1/2}}(1+\mathcal{O}(n^3\hbar)).$$

Theorem 2.3 is an improvement of Theorem 2.2 in the case where we have

$$\frac{1}{(2\pi\hbar)^{d/2}}\sup_{\xi\in\Omega_2}|(\det\nabla p_n\circ\ldots\circ p_1(\xi_0))^{1/2}| \gg 1$$

but

$$\frac{1}{(2\pi\hbar)^{r/2}}\frac{\sup_{\xi\in\Omega_2}|(\det\nabla p_n\circ\ldots\circ p_1(\xi))|^{1/2}}{\inf_{\xi\in\Omega_2}|(\det\nabla\tilde{p}_n\circ\ldots\circ\tilde{p}_1(\xi))|^{1/2}} \ll 1.$$

As a trivial example, when each κ_n is the identity, Theorem 2.2 gives a non-optimal bound, whereas we can take $r=0$ in Theorem 2.3, and recover the (almost) optimal bound $\|\hat{P}_n\circ\ldots\circ\hat{P}_2\circ\hat{P}_1\|_{L^2\longrightarrow L^2} \leq 1+\mathcal{O}(\hbar|\log\hbar|^3)$. A less trivial example will be given in section 5.

3. Proof of Theorem 2.1

The ideas below are contained in [1, 3]; however, our notations here are quite different, and we recall (without giving all details) the main steps. In all this section, M is a fixed integer, and all the calculations are done modulo remainders of order $\hbar^M$ (with explicit control of the constants).

In all that follows, it is useful to keep in mind the following : if $(x', \xi') = \kappa_n \circ \ldots \circ \kappa_2 \circ \kappa_1(x, \xi)$, we have $\xi' = p_n \circ \ldots \circ p_2 \circ p_1(\xi)$, and $x = \nabla(p_n \circ \ldots \circ p_2 \circ p_1)^\intercal x' + \nabla A_n(\xi)$.

3.1. **One step of the iteration.** Let us first fix $\xi \in \mathbb{R}^d$, and look at the action of the operator $\hat{P}_n$ on a function of the form

$$b_\xi(x) = e^{\frac{i\langle \xi, x\rangle}{\hbar}} b(x)$$

where

$$b(x) = \sum_{k=0}^{M-1} \hbar^k b_k(x),$$

and where the functions b_k are of class $\mathcal{C}^\infty$.

We introduce the following notation :

$$(T_n^\xi a)(x') = a_0^{(n)}(x, x', \xi) a(x)$$

where x is the point such that $(x', p_n(\xi)) = \kappa_n(x, \xi)$ (in other words, $x = \nabla p_n(\xi)^\intercal x' + \nabla \alpha_n(\xi)$). In the case $a_0^{(n)} \equiv 1$, we note that the operator $U_n^\xi : a \mapsto (\det \nabla p_n(\xi))^{1/2} T_n^\xi a$ is unitary on $L^2(\mathbb{R}^d)$. If we assume (as above) that $|a_0^{(n)}(x, x', \xi)| \leq 1$, it defines a bounded operator on $L^2(\mathbb{R}^d)$, of norm ≤ 1.

A standard application of the stationary phase method yields :

Proposition 3.1.

$$\hat{P}_n b_\xi(x') = e^{i\frac{\alpha_n(\xi) + \langle p_n(\xi), x'\rangle}{\hbar}} (\det \nabla p_n(\xi))^{1/2} \left[\sum_{k=0}^{M-1} \hbar^k b'_k(x') \right] + \hbar^M R_M(x'),$$

where :

- $b'_0(x') = (T_n^\xi b_0)(x')$;
- $b'_k(x') = \sum_{0 \leq l \leq k-1} D_n^{2(k-l)} b_l(x') + (T_n^\xi b_k)(x')$, *where the operator* $D_n^{2(k-l)}$ *is a differential operator of order* $2(k-l)$ *(whose expression also depends on* ξ*, although it does not appear in our notations). Its coefficients can be expressed in terms of the derivatives of order* $\leq 2(k-l)$ *of* $a_l^{(n)}$*, and of order* $\leq 2(k-l)+3$ *of* p_n, p_n^{-1} *and* α_n*, at the point* (x, x', ξ)*, where* $(x', p_n(\xi)) = \kappa_n(x, \xi)$*).*
- *There exists an integer* N_d *(depending only on the dimension* d*), and a positive real number* C *such that*

$$\|R_M\|_{L^2(\mathbb{R}^d)} \leq C \sum_{k=0}^{M-1} \|b_k\|_{\mathcal{C}^{2(M-k)+N_d}}.$$

The constant C can be expressed in terms of a fixed finite number of derivatives of the functions $a_l^{(n)}$ ($l \leq M-1$), p_n, p_n^{-1} and α_n at the point (x, x', ξ). Under our assumptions (H1) and (H5), C is uniformly bounded for all n. Also note that, under (H4), the functions b'_k are always supported inside the relatively compact set Ω_1.

3.2. **After many iterations.** We can now describe the action of the product $\hat{P}_n \circ \ldots \circ \hat{P}_2 \circ \hat{P}_1$ on $e_{\xi_0,\hbar}$. We will give an approximate expression of $\hat{P}_n \circ \ldots \circ \hat{P}_2 \circ \hat{P}_1 e_{\xi_0,\hbar}(x)$, in the form

$$e^{i\frac{A_n(\xi_0)}{\hbar}} e_{\xi_n,\hbar}(x)(\det \nabla p_n \circ \ldots \circ p_1(\xi_0))^{1/2} \left[\sum_{k=0}^{M-1} \hbar^k b_k^{(n)}(x)\right],$$

as announced in the theorem. This expression will approximate $\hat{P}_n \circ \ldots \circ \hat{P}_2 \circ \hat{P}_1 e_{\xi_0,\hbar}$ up to an error of order $\hbar^{M(1-\tilde{\epsilon})}$ for any $\tilde{\epsilon} > 0$. The function $b_k^{(n)}(x)$ depends, of course, on ξ_0, and in the final statement of the theorem we indicated this dependence by writing $b_k^{(n)}(x, \xi_n)$ (with $\xi_n = p_n \circ \ldots \circ p_1(\xi_0)$).

The method consists in iterating the method described in Section 3.1, controlling carefully how the remainders grow with n in the L^2 norm. We recall that $\|\hat{P}_n\|_{L^2(\mathbb{R}^d)} \leq 1 + \mathcal{O}(\hbar)$, uniformly in n.

Suppose that, after n iterations, we have proved that

$$\hat{P}_n \circ \ldots \circ \hat{P}_2 \circ \hat{P}_1 e_{\xi_0,\hbar}(x) = e^{i\frac{A_n(\xi_0)}{\hbar}} e_{\xi_n,\hbar}(x)(\det \nabla p_n \circ \ldots \circ p_1(\xi_0))^{1/2} \left[\sum_{k=0}^{M-1} \hbar^k b_k^{(n)}(x)\right] + \hbar^M \mathcal{R}_M^{(n)}(x).$$

The calculations done in Section 3.1 allows to describe the action of $\hat{P}_{n+1}$ on $e_{\xi_n,\hbar}\left[\sum_{k=0}^{M-1} \hbar^k b_k^{(n)}\right]$:

$$\begin{aligned} (3.1) \quad & \hat{P}_{n+1} e_{\xi_n,\hbar} \left[\sum_{k=0}^{M-1} \hbar^k b_k^{(n)}\right](x) \\ & = e^{i\frac{\alpha_{n+1}(\xi_n) + \langle p_{n+1}(\xi_n), x\rangle}{\hbar}} (\det \nabla p_{n+1}(\xi_n))^{1/2} \left[\sum_{k=0}^{M-1} \hbar^k b_k^{(n+1)}(x)\right] + \hbar^M R_M^{(n+1)}(x). \end{aligned}$$

Note that

$$\prod_{\ell=1}^{n} (\det \nabla p_\ell(\xi_{\ell-1}))^{1/2} = (\det \nabla p_n \circ \ldots \circ p_1(\xi_0))^{1/2},$$

and

$$A_n(\xi_0) = \alpha_1(\xi_0) + \alpha_2(\xi_1) + \cdots + \alpha_n(\xi_{n-1}),$$

so that

$$\begin{aligned} \hat{P}_{n+1} \hat{P}_n \circ \ldots \circ \hat{P}_2 \circ \hat{P}_1 e_{\xi_0,\hbar}(x) = e^{i\frac{A_{n+1}(\xi_0)}{\hbar}} e_{\xi_{n+1},\hbar}(x)(\det \nabla p_{n+1} \circ \ldots \circ p_1(\xi_0))^{1/2} \left[\sum_{k=0}^{M-1} \hbar^k b_k^{(n+1)}(x)\right] \\ + \hbar^M \mathcal{R}_M^{(n+1)}(x), \end{aligned}$$

with the relation $\mathcal{R}_M^{(n+1)} = e^{i\frac{A_n(\xi_0)}{\hbar}}(\det \nabla p_n \circ \ldots \circ p_1(\xi_0))^{1/2} R_M^{(n+1)} + \hat{P}_{n+1}\mathcal{R}_M^{(n)}$.

We need to control how each term in these expansions will grow with n, and in particular, to control the remainder terms. We form an array $B^{(n)}$ that contains all the functions $b_k^{(n)}$, and a certain number of higher order differentials :

$$B_{j,k}^{(n)} = d^j b_k^{(n)},$$

with $0 \leq k \leq M-1$ and $0 \leq j \leq 2(M-k) + N_d$. The index k indicates the power of $\hbar$, and the index j indicates the number of differentials. Note that $d^j b_k^{(n)}$ is a (symmetric) covariant tensor field of order j on $\mathbb{R}^d$. If σ is a covariant tensor field of order j on $\mathbb{R}^d$, we define $\|\sigma\|_\infty = \sup_{x\in\mathbb{R}^d} |\sigma_x|$, where $|\sigma_x|$ is the norm of the j-linear form σ_x. By assumption (H4), the forms $d^j b_k^{(n)}$ all vanish outside the compact set Ω_1.

There is a linear relation between $B^{(n)}$ and $B^{(n+1)}$, that we can make a little more explicit. We extend the definition of the operators T_n^ξ (previously defined on functions) to covariant tensor fields, by letting (if σ is of order j)

$$(T_n^\xi \sigma)_{x'}(v_1, \ldots, v_j) = a_0^{(n)}(x, x', \xi)\sigma_x(\nabla p_n(\xi)^\intercal v_1, \ldots, \nabla p_n(\xi)^\intercal v_j),$$

where $x = \nabla p_n(\xi)^\intercal x'$. Taking successive derivatives of the relation

$$b_k^{(n+1)} = \sum_{0\leq l\leq k-1} D_{n+1}^{2(k-l)} b_l^{(n)} + T_{n+1}^\xi b_k^{(n)},$$

which appears in Proposition 3.1, we obtain a linear relation of the form :

$$B^{(n+1)} = K_{n+1}B^{(n)} + L_{n+1}B^{(n)} + T_{n+1}^\xi B^{(n)},$$

where T_{n+1}^ξ acts "diagonally", meaning that $[T_{n+1}^\xi B^{(n)}]_{j,k} = T_{n+1}^\xi \left(B_{j,k}^{(n)}\right)$. The only information we need about the other terms is that $[K_{n+1}B^{(n)}]_{j,k}$ depends only on the components $B_{j',l}^{(n)}$, for $l \leq k-1$ and $j' \leq 2(k-l)+j$; and $[L_{n+1}B^{(n)}]_{j,k}$ depends only on the components $B_{j',k}^{(n)}$, with $j' \leq j-1$. Besides, we have

$$\max_{j,k} \|[K_{n+1}B^{(n)}]_{j,k}\|_\infty \leq C \max_{j,k} \|B_{j,k}^{(n)}\|_\infty,$$

where C does not depend on n by our assumptions (H4) (and the same holds with K_{n+1} replaced by L_{n+1}).

By induction, we see that $B^{(n)}$ can be expressed as

$$B^{(n)} = \sum_{A_\ell \in \{T_\ell^\xi, K_\ell, L_\ell\}} A_n \circ A_{n-1} \circ \cdots \circ A_1 B^{(0)}.$$

In a product of the form $A_n \circ A_{n-1} \circ \cdots \circ A_1$ (where $A_\ell \in \{T_\ell^\xi, K_\ell, L_\ell\}$ for all $\ell = 1, \ldots, n$), we see that there can be at most M indices ℓ for which $A_\ell = K_\ell$, and $2M + N_d$ indices k such that $A_\ell = L_\ell$ (otherwise the product $A_n \circ A_{n-1} \circ \cdots \circ A_1$ vanishes). Even more

precisely, when we write

$$B_{j,k}^{(n)} = \left[\sum_{A_\ell \in \{T_\ell^\xi, K_\ell, L_\ell\}} A_n \circ A_{n-1} \circ \cdots \circ A_1 B^{(0)} \right]_{j,k}, \tag{3.2}$$

in the right-hand side there can be at most k indices ℓ with $A_\ell = K_\ell$, and $2k+j$ indices ℓ with $A_\ell = L_\ell$. Hence, the sum has at most $2^{3k+j} C_n^{3k+j} \sim C(k,j) n^{3k+j}$ terms.

We now use our assumption that

$$\limsup_{k \longrightarrow +\infty} \frac{1}{k} \log \|\nabla (p_{n+k} \circ p_{n+k-1} \circ \ldots \circ p_{n+1})(\xi_n)\| \leq 0,$$

uniformly in n. Combined with (3.2), this implies that, for any $\epsilon > 0$, we have

$$\|B_{j,k}^{(n)}\|_\infty \leq C(k,j,\epsilon) n^{3k+j} e^{\epsilon n (j+2k)}.$$

These estimates (combined with Proposition 3.1) imply that

$$\|R_M^{(n+1)}\|_{L^2(\mathbb{R}^d)} \leq C \sum_{k=0}^{M-1} \sum_{j=0}^{2(M-k)+N_d} \|B_{j,k}^{(n)}\|_\infty \leq C(M,j,\epsilon) n^{3M+N_d} e^{\epsilon n (2M+N_d)}.$$

Remember the induction relation

$$\mathcal{R}_M^{(n+1)} = e^{i \frac{A_n(\xi_0)}{\hbar}} (\det \nabla p_n \circ \ldots \circ p_1(\xi_0))^{1/2} R_M^{(n+1)} + \hat{P}_{n+1} \mathcal{R}_M^{(n)}.$$

We have $\|\hat{P}_{n+1} \mathcal{R}_M^{(n)}\|_{L^2(\mathbb{R}^d)} \leq (1 + \mathcal{O}(\hbar)) \|\mathcal{R}_M^{(n)}\|_{L^2(\mathbb{R}^d)}$. If we restrict our attention to $n \leq \mathcal{K} |\log \hbar|$ (where $\mathcal{K}$ is fixed), this induction relation implies that $\|\mathcal{R}_M^{(n)}\|_{L^2(\mathbb{R}^d)} = \mathcal{O}(\hbar^{-\tilde{\epsilon} M})$ for any $\tilde{\epsilon} > 0$.

4. Proof of Theorems 2.2 and 2.3

4.1. **Theorem 2.2.** The proof of Theorem 2.2 is now very easy. Let $u \in L^2(\mathbb{R}^d)$. We know that

$$u(x) = \frac{1}{(2\pi\hbar)^{d/2}} \int_{\mathbb{R}^d} \mathcal{F}_\hbar u(\xi) e^{\frac{i\langle \xi, x \rangle}{\hbar}} d\xi.$$

Let $\widetilde{\Omega}_2$ be an open set containing the closure of Ω_2. We decompose $u = u_1 + u_2$, where

$$u_1(x) = \frac{1}{(2\pi\hbar)^{d/2}} \int_{\widetilde{\Omega}_2} \mathcal{F}_\hbar u(\xi) e^{\frac{i\langle \xi, x \rangle}{\hbar}} d\xi$$

and

$$u_2(x) = \frac{1}{(2\pi\hbar)^{d/2}} \int_{\mathbb{R}^d \setminus \widetilde{\Omega}_2} \mathcal{F}_\hbar u(\xi) e^{\frac{i\langle \xi, x \rangle}{\hbar}} d\xi.$$

Since $\hat{P}_1^* \hat{P}_1$ is a pseudodifferential operator, whose complete symbol is supported in $\Omega \times \Omega_2$, we have $\|\hat{P}_1 u_2\|_{L^2(\mathbb{R}^d)} = \mathcal{O}(\hbar^\infty) \|u_2\|_{L^2(\mathbb{R}^d)}$.

Concerning u_1, we apply Theorem 2.1 for each $\xi \in \widetilde{\Omega}_2$. We take $n = \mathcal{K}|\log \hbar|$ and choose M accordingly, large enough so that

$$\mathcal{O}(\hbar^{M(1-\tilde{\epsilon})}) \ll \sup_{\xi \in \widetilde{\Omega}_2} |\det \nabla p_n \circ \ldots \circ p_1(\xi)|^{1/2}.$$

From Theorem 2.1, we know that

$$\|\hat{P}_n \circ \ldots \circ \hat{P}_2 \circ \hat{P}_1 e_{\xi,\hbar}\|_{L^2(\mathbb{R}^d)} \leq |\det \nabla p_n \circ \ldots \circ p_1(\xi)|^{1/2}(1 + \mathcal{O}(\hbar n^3 e^{2\epsilon n}))$$

(for $\epsilon > 0$ arbitrary). By a direct application of the triangular inequality, it follows that

$$\|\hat{P}_n \circ \ldots \circ \hat{P}_2 \circ \hat{P}_1 u_1\|_{L^2(\mathbb{R}^d)} \leq \frac{1}{(2\pi\hbar)^{d/2}} \sup_{\xi \in \widetilde{\Omega}_2} |\det \nabla p_n \circ \ldots \circ p_1(\xi)|^{1/2}(1 + \mathcal{O}(\hbar n^3 e^{2\epsilon n}))\|\mathcal{F}_\hbar u\|_{L^1(\widetilde{\Omega}_2)}$$

$$\leq \frac{1}{(2\pi\hbar)^{d/2}} \sup_{\xi \in \widetilde{\Omega}_2} |\det \nabla p_n \circ \ldots \circ p_1(\xi)|^{1/2}(1 + \mathcal{O}(\hbar n^3 e^{2\epsilon n}))|\widetilde{\Omega}_2|^{1/2}\|\mathcal{F}_\hbar u\|_{L^2(\mathbb{R}^d)}$$

and our result follows.

4.2. **Theorem 2.3.**

4.2.1. *The Cotlar-Stein lemma.*

Lemma 4.1. *Let E, F be two Hilbert spaces. Let $(A_\alpha) \in \mathcal{L}(E, F)$ be a countable family of bounded linear operators from E to F. Assume that for some $R > 0$ we have*

$$\sup_\alpha \sum_\beta \|A_\alpha^* A_\beta\|^{\frac{1}{2}} \leq R$$

and

$$\sup_\alpha \sum_\beta \|A_\alpha A_\beta^*\|^{\frac{1}{2}} \leq R$$

Then $A = \sum_\alpha A_\alpha$ converges strongly and A is a bounded operator with $\|A\| \leq R$.

The Cotlar-Stein lemma is often used to bound in a precise manner the norm of pseudodifferential operators.

4.2.2. Remember that we assume everywhere that $n = \mathcal{K}|\log \hbar|$, with $\mathcal{K}$ fixed. In order to bound the norm of $\hat{P}_n \circ \ldots \circ \hat{P}_1$ (modulo $\hbar^N$ for arbitrary N), the results of the previous sections show that it is enough to bound the norm of the operator A defined by

$$Af(x') \tag{4.1}$$

$$= \frac{1}{(2\pi\hbar)^d} \int_{\xi \in \widetilde{\Omega}_2, x \in \mathbb{R}^d} (\det \nabla p_n \circ \ldots \circ p_1(\xi))^{1/2} \left[\sum_{k=0}^{M-1} \hbar^k b_k^{(n)}(x', \xi_n)\right] e^{\frac{i}{\hbar}(\langle \xi_n, x'\rangle + A_n(\xi) - \langle \xi, x\rangle)} f(x) dx d\xi$$

$$= \frac{1}{(2\pi\hbar)^{d/2}} \int_{\xi \in \widetilde{\Omega}_2, x \in \mathbb{R}^d} (\det \nabla p_n \circ \ldots \circ p_1(\xi))^{1/2} \left[\sum_{k=0}^{M-1} \hbar^k b_k^{(n)}(x', \xi_n)\right] e^{\frac{i}{\hbar}(\langle \xi_n, x'\rangle + A_n(\xi))} \mathcal{F}_\hbar f(\xi) d\xi$$

for a suitable choice of M, large. We denote everywhere $\xi_n = p_n \circ \ldots \circ p_1(\xi)$.

We decompose $\mathbb{R}^d = \mathbb{R}^r \times \mathbb{R}^{d-r}$, and write any $\xi \in \mathbb{R}^d$ as $\xi = (\xi_{(r)}, \tilde{\xi})$ where $\xi_{(r)} \in \mathbb{R}^r$ and $\tilde{\xi} \in \mathbb{R}^{d-r}$. Under our current assumptions, ξ_n decomposes as $\xi_n = (\xi_{n(r)}, \tilde{\xi}_n)$, where $\tilde{\xi}_n = \tilde{p}_n \circ \ldots \circ \tilde{p}_1(\tilde{\xi})$.

We now introduce a (real-valued) smooth compactly supported χ on $\mathbb{R}^{d-r}$, such that $0 \leq \chi \leq 1$, and having the property that

$$\sum_{\ell \in \mathbb{Z}^{d-r}} \chi(\tilde{\xi} - \ell) = 1$$

for all $\tilde{\xi} \in \mathbb{R}^{d-r}$. For $\hbar > 0$, $\ell \in \mathbb{Z}^{d-r}$ and $\tilde{\xi} \in \mathbb{R}^{d-r}$, we denote $\chi_{\hbar,\ell}(\tilde{\xi}) = \chi\left(\frac{\tilde{\xi}}{2\pi\hbar} - \ell\right)$. Using the same notation as in (4.1), we define

$$\begin{aligned}
&(4.2) \quad A_\ell f(x') \\
&= \frac{1}{(2\pi\hbar)^d} \int (\det \nabla p_n \circ \ldots \circ p_1(\xi))^{1/2} \left[\sum_{k=0}^{M-1} \hbar^k b_k^{(n)}(x', \xi_n)\right] \chi_{\hbar,\ell}(\tilde{\xi}_n) e^{\frac{i}{\hbar}(\langle \xi_n, x'\rangle + A_n(\xi) - \langle \xi, x\rangle)} f(x) dx d\xi \\
&= \frac{1}{(2\pi\hbar)^{d/2}} \int_{\xi \in \widetilde{\Omega}_2} (\det \nabla p_n \circ \ldots \circ p_1(\xi))^{1/2} \left[\sum_{k=0}^{M-1} \hbar^k b_k^{(n)}(x', \xi_n)\right] \chi_{\hbar,\ell}(\tilde{\xi}_n) e^{\frac{i}{\hbar}(\langle \xi_n, x'\rangle + A_n(\xi))} \mathcal{F}_\hbar f(\xi) d\xi.
\end{aligned}$$

It is clear that $A = \sum_{\ell \in \mathbb{Z}^{d-r}} A_\ell$. A crucial remark is that the function $\xi \mapsto \chi_{\hbar,\ell}(\tilde{\xi}_n)$, defined on Ω_2, is supported in a set of volume $\leq (2\pi\hbar)^{d-r} \frac{1}{\inf_{\xi \in \widetilde{\Omega}_2} |(\det \nabla \tilde{p}_n \circ \ldots \circ \tilde{p}_1(\xi))|}$.

We are going to apply the Cotlar-Stein lemma to this decomposition. Let us write explicitly the expression for the adjoint :

$$\begin{aligned}
&(4.3) \quad A_\ell^* f(x) \\
&= \frac{1}{(2\pi\hbar)^d} \int (\det \nabla p_n \circ \ldots \circ p_1(\xi))^{1/2} \left[\sum_{k=0}^{M-1} \hbar^k \overline{b_k^{(n)}}(x', \xi_n)\right] \chi_{\hbar,\ell}(\tilde{\xi}_n) e^{-\frac{i}{\hbar}(\langle \xi_n, x'\rangle + A_n(\xi) - \langle \xi, x\rangle)} f(x') dx' d\xi.
\end{aligned}$$

We shall evaluate the norm of $A_m^* A_\ell$ and $A_\ell A_m^*$, for all $m, \ell \in \mathbb{Z}^{d-r}$.

4.2.3. *Norm of $A_m^* A_\ell$.* We evaluate the norm of $A_m^* A_\ell$ acting on $L^2(\mathbb{R}^d)$ by studying the scalar product $\langle A_\ell f, A_m f\rangle$ for $f \in L^2(\mathbb{R}^d)$. Using expression (4.2) and bilinearity of the scalar product, we will bound the scalar product $\langle A_\ell f, A_m f\rangle$ by studying separately each bracket

$$(4.4) \qquad \chi_{\hbar,\ell}(\tilde{\xi}_n)\chi_{\hbar,m}(\tilde{\xi}'_n) \left\langle \left[\sum_{k=0}^{M-1} \hbar^k b_k^{(n)}(x', \xi_n)\right] e^{\frac{i}{\hbar}\langle \xi_n, x'\rangle}, \left[\sum_{k=0}^{M-1} \hbar^k b_k^{(n)}(x', \xi'_n)\right] e^{\frac{i}{\hbar}\langle \xi'_n, x'\rangle} \right\rangle_{L^2_{x'}}.$$

Using the notation of §4.2.2, we decompose the complex phase $\langle \xi_n, x'\rangle - \langle \xi'_n, x'\rangle$ into $\langle \xi_{n(r)}, x'_{(r)}\rangle - \langle \xi'_{n(r)}, x'_{(r)}\rangle + \langle \tilde{\xi}_n, \tilde{x}'\rangle - \langle \tilde{\xi}'_n, \tilde{x}'\rangle$. In the integral defining the scalar product (4.4), we perform an integration by parts with respect to $\tilde{x}' \in \mathbb{R}^{d-r}$: we integrate N times the function $e^{\frac{i}{\hbar}\langle \tilde{\xi}_n, \tilde{x}'\rangle - \langle \tilde{\xi}'_n, \tilde{x}'\rangle}$ and differentiate the functions $b_k^{(n)}(x', \xi_n)$. Using the estimates of Theorem 2.1, we obtain

Proposition 4.2.

$$\chi_{\hbar,\ell}(\tilde{\xi}_n)\chi_{\hbar,m}(\tilde{\xi}'_n)\left|\left\langle\left[\sum_{k=0}^{M-1}\hbar^k b_k^{(n)}(x',\xi_n)\right]e^{\frac{i}{\hbar}\langle\xi_n,x'\rangle},\left[\sum_{k=0}^{M-1}\hbar^k b_k^{(n)}(x',\xi'_n)\right]e^{\frac{i}{\hbar}\langle\xi'_n,x'\rangle}\right\rangle\right| \leq C(\epsilon)e^{\epsilon Nn}\frac{1}{(\|m-\ell\|+1)^N} \tag{4.5}$$

for $\epsilon > 0$ arbitrary.

The integer N will be chosen soon. We now use the bilinearity of the scalar product, and the fact that

$$\|\chi_{\hbar,\ell}(\tilde{\xi}_n)\mathcal{F}_\hbar f(\xi)\|_{L^1(\widetilde{\Omega}_2)} \leq (2\pi\hbar)^{(d-r)/2}\frac{1}{\inf_{\xi\in\widetilde{\Omega}_2}|(\det\nabla\tilde{p}_n\circ\ldots\circ\tilde{p}_1(\xi))|^{1/2}}\|\mathcal{F}_\hbar f(\xi)\|_{L^2(\mathbb{R}^d)}.$$

Combined with expression (4.2), this yields that

$$\|A_m^*A_\ell\| \leq C(\epsilon)e^{\epsilon Nn}\frac{1}{(\|m-\ell\|+1)^N}\frac{1}{(2\pi\hbar)^r}\frac{\sup_{\xi\in\widetilde{\Omega}_2}|(\det\nabla p_n\circ\ldots\circ p_1(\xi))|}{\inf_{\xi\in\widetilde{\Omega}_2}|(\det\nabla\tilde{p}_n\circ\ldots\circ\tilde{p}_1(\xi))|}. \tag{4.6}$$

Looking at the statement of the Cotlar-Stein lemma, we see that we must choose N large enough such that $\sum_{\ell\in\mathbb{Z}^{d-r}}\frac{1}{(\|\ell\|+1)^{N/2}} < +\infty$.

Remark 4.3. If we make the assumption that $\|\nabla(p_{n+k}\circ p_{n+k-1}\circ\ldots\circ p_{n+1})(\xi_n)\|$ is bounded above, uniformly in n, k and $\xi\in\widetilde{\Omega}_2$, we see that we can take $\epsilon = 0$ in all the statements made above.

4.2.4. *Norm of $A_\ell A_m^*$.* This step is actually shorter than the previous one. We now have to evaluate the scalar product $\langle A_\ell^* f, A_m^* f\rangle$ for $f\in L^2(\mathbb{R}^d)$, and we use the expression (4.3) of the adjoint. We do not need integration by parts, as we see directly that $\langle A_\ell^* f, A_m^* f\rangle$ vanishes as soon as $\|m-\ell\|$ is too large (in fact, the supports of $\chi_{\hbar,\ell}$ and $\chi_{\hbar,m}$ are disjoint if $\|m-\ell\| > C$, where C is fixed and depends only on the support of χ). In what follows we consider the case $\|m-\ell\|\leq C$. We see that $A_\ell^* f$ is the $\mathcal{F}_\hbar$-transform of

$$F_\ell : \xi\mapsto\frac{1}{(2\pi\hbar)^{d/2}}\int(\det\nabla p_n\circ\ldots\circ p_1(\xi))^{1/2}\left[\sum_{k=0}^{M-1}\hbar^k\overline{b_k^{(n)}}(x',\xi_n)\right]\chi_{\hbar,\ell}(\tilde{\xi}_n)e^{-\frac{i}{\hbar}(\langle\xi_n,x'\rangle+A_n(\xi))}f(x')dx'.$$

We recall that each $b_k^{(n)}(x',\xi_n)$ is supported in $\{x'\in\Omega_1\}$, and we bound

$$\|F_\ell\|_{L^2(\mathbb{R}^d)} \leq \frac{1}{(2\pi\hbar)^{d/2}}\sup_{\xi\in\widetilde{\Omega}_2}|(\det\nabla p_n\circ\ldots\circ p_1(\xi))|^{1/2}(2\pi\hbar)^{(d-r)/2}\frac{1}{\inf_{\xi\in\widetilde{\Omega}_2}|(\det\nabla\tilde{p}_n\circ\ldots\circ\tilde{p}_1(\xi))|^{1/2}}\|f\|_{L^1(\Omega_1)},$$

and $\|f\|_{L^1(\Omega_1)}\leq|\Omega_1|^{1/2}\|f\|_{L^2(\mathbb{R}^d)}$. We again obtain the bound

$$\|A_\ell A_m^*\| \leq \frac{1}{(2\pi\hbar)^r}\frac{\sup_{\xi\in\widetilde{\Omega}_2}|(\det\nabla p_n\circ\ldots\circ p_1(\xi))|}{\inf_{\xi\in\widetilde{\Omega}_2}|(\det\nabla\tilde{p}_n\circ\ldots\circ\tilde{p}_1(\xi))|}, \tag{4.7}$$

and $\|A_\ell A_m^*\| = 0$ if $\|\ell - m\| > C$. Estimates (4.6) and (4.7), combined with the Cotlar-Stein lemma, yield Theorem 2.3. The last statement of the theorem comes from Remark 4.3.

5. Examples

We now give examples of application of Theorems 2.2 and 2.3. These results are needed in [6] and [5].

Let $\mathbf{Y}$ be a d-dimensional $\mathcal{C}^\infty$ manifold. The cotangent bundle $T^*\mathbf{Y}$ is endowed with its canonical symplectic form, denoted ω. Let $H : T^*\mathbf{Y} \longrightarrow \mathbb{R}$ be a smooth function (hamiltonian), and let $\Phi_H^t : T^*\mathbf{Y} \longrightarrow T^*\mathbf{Y}$ be the corresponding hamiltonian flow. We assume for simplicity that (Φ_H^t) is complete. We fix a time step $\tau > 0$, arbitrary. Before specifying the operators $\hat{P}_n$ to which we will apply the previous results, we have to make several assumptions concerning the underlying geometric situation.

We assume that we have a smooth foliation $\mathcal{F}$ of $T^*\mathbf{Y}$ by lagrangian leaves (in the sequel we shall simply speak about a "lagrangian foliation"), and will denote $\mathcal{F}^{(n)} = \Phi_H^{n\tau}(\mathcal{F})$. Let $O \subset T^*\mathbf{Y}$ be an open, relatively compact subset of $T^*\mathbf{Y}$; we assume that we have a finite open covering of O, $O \subset O_1 \cup O_2 \cup \ldots \cup O_K$, with the following properties :

> for any $n > 0$ and any sequence $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \in \{1, \ldots, K\}^n$ such that $O_{\alpha_0} \cap \Phi_H^{-\tau}(O_{\alpha_1}) \cap \ldots \cap \Phi_H^{-(n-1)\tau}(O_{\alpha_{n-1}}) \neq \emptyset$, we can find for all $k \leq n-1$ a smooth symplectic coordinate chart $\Psi_k : (O_{\alpha_k}, \omega) \longrightarrow (\mathbb{R}^{2d}, \omega_o)$ which maps O_{α_k} to a ball in $\mathbb{R}^{2d}$, and the foliation $\mathcal{F}^{(k)}\rceil_{O_{\alpha_k}}$ to the horizontal foliation of that ball; the collection of coordinate charts (Ψ_k) may depend on the sequence $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$, however, it can be chosen so that all the derivatives of Ψ_k and Ψ_k^{-1} are bounded, independently of n, $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$, and k.

We can now give some details about the operators $\hat{P}_k$ to which we shall apply the main results. We fix a family $\hat{\chi}_1, \ldots, \hat{\chi}_K$ of $\hbar$-pseudodifferential operators (see the appendix), such that the full symbol of $\hat{\chi}_k$ is compactly supported inside O_k. We also assume that its principal symbol χ_k (which is a smooth function on $T^*\mathbf{Y}$) satisfies $\|\chi_k\|_{\mathcal{C}^0} \leq 1$.

Let $\hat{H}$ be a self-adjoint $\hbar$-pseudodifferential operator with principal symbol H.

Fix, finally, a sequence $(\alpha_0, \alpha_2, \ldots, \alpha_{n-1}) \in \{1, \ldots, K\}^n$. We shall use Theorems 2.2 and 2.3 to estimate the norm of the product $\prod_{k=0}^{n-1} \hat{\chi}_{\alpha_{k+1}} e^{-\frac{i\tau\hbar\hat{H}}{\hbar}} \hat{\chi}_{\alpha_k}$. The operator $\hat{P}_k$ will be $\hat{\chi}_{\alpha_{k+1}} e^{-\frac{i\tau\hbar\hat{H}}{\hbar}} \hat{\chi}_{\alpha_k}$, read in an adapted coordinate system :

Once the sequence $(\alpha_0, \alpha_2, \ldots, \alpha_{n-1})$ is fixed, we consider the family of coordinates Ψ_k described in our assumptions. We fix a collection of Fourier integral operators $U_k : L^2(\mathbf{Y}) \longrightarrow L^2(\mathbb{R}^d)$, associated with the canonical transformation Ψ_k $(k = 0, \ldots n-1)$, and such that the pseudodifferential operator $U_k^* U_k$ satisfies $U_k^* U_k \hat{\chi}_k = \hat{\chi}_k + \mathcal{O}(\hbar^\infty)$ and $\hat{\chi}_k = \hat{\chi}_k U_k^* U_k + \mathcal{O}(\hbar^\infty)$(where the $\mathcal{O}$ is to be understood in the $L^2(\mathbf{Y})$-operator norm). Note that the operators U_k depend on the sequence $(\alpha_0, \alpha_2, \ldots, \alpha_{n-1})$, but (in the geometric situation described above) we can assume that their symbols have derivatives of all orders bounded independently of $(\alpha_0, \alpha_2, \ldots, \alpha_{n-1})$ and of k. We take $\hat{P}_k = U_{k+1} \hat{\chi}_{\alpha_{k+1}} e^{-\frac{i\tau\hbar H}{\hbar}} \hat{\chi}_{\alpha_k} U_k^*$. It is

a Fourier integral operator $L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$, associated with the canonical transformation $\kappa_k = \Psi_{k+1}\Phi_H^\tau\Psi_k^{-1}$, which by construction preserves the horizontal foliation. These operators also satisfy all assumptions (H), hence we can apply to them Theorems 2.2 and 2.3.

We give a concrete example of application, used in [6] and [5]. Let G denote a non-compact connected simple Lie group with finite center. We choose a Cartan involution Θ for G, and let $K < G$ be the Θ-fixed maximal compact subgroup. Let $\mathfrak{g} = Lie(G)$, and let θ denote the differential of Θ, giving the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathfrak{k} = Lie(K)$. Let $\mathbf{S} = G/K$ be the associated symmetric space. For a lattice $\Gamma < G$ we write $\mathbf{X} = \Gamma\backslash G$ and $\mathbf{Y} = \Gamma\backslash G/K$, the latter being a locally symmetric space of non-positive curvature.

Fix now a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$. The dimension of $\mathfrak{a}$ is called the real rank of G, and will be denoted by r in the sequel. We denote by $\mathfrak{a}^*$ the real dual of $\mathfrak{a}$. Let $\mathfrak{g}_\alpha = \{X \in \mathfrak{g}, \forall H \in \mathfrak{a} : ad(H)X = \alpha(H)X\}$, $\Delta = t\Delta(\mathfrak{a} : \mathfrak{g}) = \{\alpha \in \mathfrak{a}^* \setminus \{0\}, \mathfrak{g}_\alpha \neq \{0\}\}$ and call the latter the (restricted) roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. For $\alpha \in \Delta$, we denote by m_α the dimension of $\mathfrak{g}_\alpha$. The subalgebra $\mathfrak{g}_0$ is θ-invariant, and hence $\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{p}) \oplus (\mathfrak{g}_0 \cap \mathfrak{k})$. By the maximality of $\mathfrak{a}$ in $\mathfrak{p}$, we must then have $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$ where $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$, the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$.

A subset $\Pi \subset \Delta(\mathfrak{a} : \mathfrak{g})$ will be called a system of simple roots if every root can be uniquely expressed as an integral combination of elements of Π with either all coefficients non-negative or all coefficients non-positive. Fixing a simple system Π we get a notion of positivity. We will denote by Δ^+ the set of positive roots, by $\Delta^- = -\Delta^+$ the set of negative roots. For $\mathfrak{n} = \oplus_{\alpha>0}\mathfrak{g}_\alpha$ and $\bar{\mathfrak{n}} = \Theta\mathfrak{n} = \oplus_{\alpha<0}\mathfrak{g}_\alpha$ we have $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{m} \oplus \bar{\mathfrak{n}}$.

Let $N, A < G$ be the connected subgroups corresponding to the subalgebras $\mathfrak{n}, \mathfrak{a} \subset \mathfrak{g}$ respectively, and let $M = Z_K(\mathfrak{a})$. Then $\mathfrak{m} = Lie(M)$, though M is not necessarily connected.

On $T^*\mathbf{S}$, consider the algebra $\mathcal{H}$ of smooth G-invariant hamiltonians, that are polynomial in the fibers of the projection $T^*\mathbf{S} \longrightarrow \mathbf{S}$. The structure theory of semisimple Lie algebras shows that $\mathcal{H}$ is isomorphic to a polynomial ring in r generators. Moreover, the elements of $\mathcal{H}$ commute under the Poisson bracket. Thus, we have on $T^*\mathbf{S}$ a family of r independent commuting Hamiltonian flows $H_1, ..., H_r$. Since all these flows are G-equivariant, they descend to the quotient $T^*\mathbf{Y}$.

We apply the discussion above to $\mathbf{Y}$ and $H \in \mathcal{H}$. We assume that O is such that the differentials $(dH_1, \ldots, dH_r)$ are everywhere independent on O. It is known that any given regular common energy layer $\{H_1 = E_1, \ldots, H_r = E_r\} \subset T^*\mathbf{Y}$ may naturally be identified (in a G-equivariant way) with G/M. We thus have an equivariant map $O \longrightarrow \mathbb{R}^r \times G/M$ which is a diffeomorphism onto its image. In all that follows, we identify O with an open subset of $\mathbb{R}^r \times G/M$. Under this identification, the action of Φ_H^t is transported to

$$(E_1, \ldots, E_r, \rho M) \mapsto (E_1, \ldots, E_r, \rho e^{ta_{E_1,\cdots,E_r}} M),$$

where $a_{E_1,\ldots,E_r} \in \mathfrak{a}$ depends smoothly on $E_1, \ldots, E_r$, and linearly on H – see [14, 6] for more explanations. The foliation $\mathcal{F}$ is invariant under Φ_H^t, and can be described as follows : the leaf of $(E_1, \ldots, E_r, \rho M) \in \mathbb{R}^r \times G/M$ is $(E_1, \ldots, E_r) \times \{\rho a\bar{n}M, a \in A, \bar{n} \in \bar{N}\}$.

We assume that each O_k is small enough so that, for any given $(E_1, \dots, E_r, \rho M) \in O_k$, the map

$$\begin{array}{rcl} \mathbb{R}^r \times \mathfrak{n} \times \mathfrak{a} \times \bar{\mathfrak{n}} & \longrightarrow & \mathbb{R}^r \times G/M \\ (\varepsilon_1, \dots, \varepsilon_r, X, Y, Z) & \mapsto & (\varepsilon_1, \dots, \varepsilon_r, \rho e^X e^Y e^Z M) \end{array}$$

is a local diffeomorphism from a neighbourhood of $(E_1, \dots, E_r, 0, 0, 0)$ onto O_k. In such coordinates, the leaves of the foliation $\mathcal{F}$ are then given by the equations $(\varepsilon_1, \dots, \varepsilon_r, X) = cst$.

Let $(\alpha_0, \dots, \alpha_{n-1})$ be such that $O_{\alpha_0} \cap \Phi_H^{-\tau}(O_{\alpha_1}) \cap \dots \cap \Phi_H^{-(n-1)\tau}(O_{\alpha_{n-1}}) \neq \emptyset$. We can then take $\rho = \rho_k$ and $(E_1, \dots, E_r)$ such that $(E_1, \dots, E_r, \rho_k M) \in O_{\alpha_k}$ and $(E_1, \dots, E_r, \rho_{k+1} M) = \Phi_H^\tau(E_1, \dots, E_r, \rho_k M)$. As explained in the previous paragraph, fixing ρ_k allows to identify O_{α_k} with a subset of $\mathbb{R}^r \times \mathfrak{n} \times \mathfrak{a} \times \bar{\mathfrak{n}}$; and we denote $(\varepsilon_1, \dots, \varepsilon_r, X, Y, Z)$ these coordinates.

Denote by d the dimension of $\mathbf{S}$ (note that $d = r + \dim \mathfrak{n} = r + \sum_{\alpha \in \Delta^+} m_\alpha$). By the Darboux-Lie theorem (see [20]), we can find some coordinate system $\Psi_k = (x_1^k, \dots, x_d^k, \xi_1^k, \dots, \xi_d^k) : O_{\alpha_k} \longrightarrow \mathbb{R}^d$ mapping ω to ω_o, and such that $(\xi_1^k, \dots, \xi_d^k) = (\varepsilon_1, \cdots, \varepsilon_r, X)$. The canonical transformations $\kappa_k = \Psi_{k+1} \Phi_H^\tau \Psi_k^{-1}$ preserve the horizontal foliation of $\mathbb{R}^d$, hence they are of the form $\kappa_k \ : (x, \xi) \mapsto (x', \xi' = p_k(\xi))$. It turns out, in this particular case, that the maps p_k are all the same, and of the particular form

$$p_k(\varepsilon_1, \cdots, \varepsilon_r, X) = (\varepsilon_1, \dots, \varepsilon_r, Ad(e^{\tau a_{\varepsilon_1, \cdots, \varepsilon_r}}).X),$$

where $a_{\varepsilon_1, \dots, \varepsilon_r} \in \mathfrak{a}$ depends smoothly on $\varepsilon_1, \dots, \varepsilon_r$. The linear maps $Ad(e^{\tau a_{\varepsilon_1, \dots, \varepsilon_r}})$ acting on $\mathfrak{n}$ are all simultaneously diagonalizable, the eigenspaces being the root spaces $\mathfrak{g}_\alpha$, with eigenvalue $e^{\tau \alpha(a_{\varepsilon_1, \dots, \varepsilon_r})}$. We are thus in a case of application of Theorem 2.3 provided that $\alpha(a_{\varepsilon_1, \dots, \varepsilon_r}) \leq 0$ for all $\alpha \in \Delta^+$. If we fix $J \subset \Delta^+$ arbitrarily, the map κ_k preserves the coisotropic foliation by the leaves $(\varepsilon_1, \dots, \varepsilon_r, X_J) = cst$ (where any $X \in \mathfrak{n}$ is decomposed into $X = \sum_{\alpha \in \Delta^+} X_\alpha$, $X_\alpha \in \mathfrak{g}_\alpha$, and X_J is defined by $X_J = \sum_{\alpha \in J} X_\alpha$).

Corollary 5.1. *Assume that H and O are such that $\alpha(a_{\varepsilon_1, \dots, \varepsilon_r}) \leq \delta \leq 0$ for all $(\varepsilon_1, \dots, \varepsilon_r, \rho M) \in O$ and all $\alpha \in \Delta^+$. Fix a subset $J \subset \Delta^+$ of the sets of roots.*

Fix $\mathcal{K} > 0$ arbitrary. Then, for $n = \mathcal{K} |\log \hbar|$, and for every sequence $(\alpha_0, \dots, \alpha_{n-1})$,

$$\| \prod_{k=0}^{n-1} \hat{\chi}_{\alpha_{k+1}} e^{-\frac{i\tau\hbar H}{\hbar}} \hat{\chi}_{\alpha_k} \| \leq \frac{1}{(2\pi\hbar)^{(d-r-\sum_{\alpha \in J} m_\alpha)/2}} \prod_{\alpha \in \Delta^+ \setminus J} e^{\frac{n\delta}{2}}$$

for $\hbar > 0$ small enough.

Remark 5.2. In the situation of [6], we actually do not have $\alpha(a_{\varepsilon_1, \dots, \varepsilon_r}) \leq \delta \leq 0$, but $\alpha(a_{\varepsilon_1, \dots, \varepsilon_r}) \leq \delta$, where $\delta > 0$ can be made arbitrarily small by conveniently choosing the set O. Once $\mathcal{K}$ is given, we can choose δ small enough (and O) so that the proof of Section 4.2 still works for $n = \mathcal{K} |\log \hbar|$.

In the special case $G = SO_o(d, 1)$, $\mathbf{Y}$ is a hyperbolic manifold of dimension d. We have $r = 1$, and $\mathcal{H}$ is generated by the laplacian $\triangle$. Taking $H = -\frac{\triangle}{2}$, $J = \emptyset$, and keeping the

same notations as before, we obtain in this case

$$\|\prod_{k=0}^{n-1}\hat{\chi}_{\alpha_{k+1}}e^{-\frac{i\tau\hbar H}{\hbar}}\hat{\chi}_{\alpha_k}\| \leq \frac{1}{(2\pi\hbar)^{(d-1)/2}}e^{-n\tau\frac{(d-1)}{2}(1-\eta)}$$

if we assume that the symbols of the pseudodifferential operators $\hat{\chi}_k$ are all supported in $\{\|\xi\| \in [1-\eta, 1+\eta]\}$ for some small $\eta > 0$. The result proved in [1, 3], which was only based on the idea of Theorem 2.2, was

$$\|\prod_{k=0}^{n-1}\hat{\chi}_{\alpha_{k+1}}e^{-\frac{i\tau\hbar H}{\hbar}}\hat{\chi}_{\alpha_k}\| \leq \frac{1}{(2\pi\hbar)^{d/2}}e^{-n\tau\frac{(d-1)}{2}(1-\eta)}.$$

We see that Theorem 2.3 allows to improve the prefactor $\frac{1}{(2\pi\hbar)^{d/2}}$ to $\frac{1}{(2\pi\hbar)^{(d-1)/2}}$, as needed in [5].

Remark 5.3. Versions of the hyperbolic dispersion estimate have also been proved for more general uniformly hyperbolic dynamical systems [1, 3, 16, 18], and even for certain non-uniformly hyperbolic systems [19]. We refer the reader to [15] for an expository paper. It is not clear to me whether the new presentation (and improvement) introduced here can be used for those systems. Indeed, there is in general no *smooth* lagrangian foliations preserved by the hamiltonian flow, and so one cannot hope that the symplectic changes of coordinates Ψ_k used above will have uniformly bounded derivatives. On the other hand, control of high order derivatives is crucial when one applies the techniques of semiclassical analysis (method of stationary phase, integration by parts,...) It is a drawback of semiclassical analysis that it cannot deal with symplectic transformations of low regularity : I don't know if this obstacle can be overcome.

6. A few definitions

For the reader's convenience, we clarify the terminology used in this paper. This is of course not an exhaustive tutorial on semiclassical analysis : for this we refer the reader to [13].

Let M and N be two smooth manifolds of the same dimension d. Their cotangent bundles T^*M and T^*N are respectively equipped with the canonical symplectic forms ω_M and ω_N. Let κ be a diffeomorphism from an open relatively compact subset $O_M \subset T^*M$ onto an open subset $O_N \subset T^*N$, sending ω_M to ω_N (such a κ is called a symplectic diffeomorphism, of a canonical transformation).

In this paper, we say that an operator[1] $\hat{P} = \hat{P}_\hbar : L^2(M) \longrightarrow L^2(N)$ is a (semiclassical) Fourier integral operator associated with κ, if it is a finite sum of operators of the form

$$\hat{Q}f(x') = \frac{1}{(2\pi\hbar)^{\frac{d+m}{2}}}\int_{x\in\mathbb{R}^d,\theta\in\mathbb{R}^m} e^{\frac{iS(x,x',\theta)}{\hbar}}a(x,x',\theta,\hbar)f(x)dxd\theta,$$

where

- $m \geq 0$ is an integer and S is a smooth function on $M \times N \times \mathbb{R}^m$.

[1]More properly, a family of operators depending on $\hbar > 0$

- For a given $\hbar > 0$, the function $(x, x', \theta) \mapsto a(x, x', \theta, \hbar)$ is of class $\mathcal{C}^\infty$ and has compact support, independent of $\hbar$;
- When $\hbar \longrightarrow 0$, $a(x, x', \theta, \hbar)$ has an asymptotic expansion
$$a(x, x', \theta, \hbar) \sim \sum_{k=0}^{\infty} \hbar^k a_k(x, x', \theta),$$
valid up to any order and in all the $\mathcal{C}^\ell$ norms.
- for any (x, x', θ) in the support of a, and for any $(x, \xi) \in T^*_x M$, $(x', \xi') \in T^*_{x'} N$, we have

$$\big[\xi = -\partial_x S_n(x, x', \theta),\ \xi' = \partial_{x'} S_n(x, x', \theta),\ \partial_\theta S_n(x, x', \theta) = 0\big] \Longrightarrow \big[(x, \xi) \in O_M, (x', \xi') = \kappa(x, \xi)\big].$$

If $\hat{P}$ is a Fourier integral operator associated with κ and $\hat{P}'$ is a Fourier integral operator associated with κ', then $\hat{P}' \circ \hat{P}$ is a Fourier integral operator associated with $\kappa' \circ \kappa$. Also, the adjoint $\hat{P}^*$ is a Fourier integral operator associated with κ^{-1}.

In the case $M = N$, a pseudodifferential operator is a Fourier integral operator associated with $\kappa =$ the identity.

If $\hat{P}$ is a pseudodifferential operator on $M = \mathbb{R}^d$, we define its full symbol as the function $a_\hbar(x, \xi)$, defined on $\mathbb{R}^d \times \mathbb{R}^d$, by

$$a_\hbar(x_0, \xi_0) = e^{\frac{-i\langle \xi_0, x_0\rangle}{\hbar}} \hat{P}(e^{\frac{i\langle \xi_0, \bullet\rangle}{\hbar}})\rceil_{x_0}.$$

By the stationary phase method, the function $a_\hbar$ can be shown to have an asymptotic expansion, valid in all $\mathcal{C}^k$-norms on $\mathbb{R}^d$,

$$a_\hbar(x, \xi) \sim \sum_{j \in \mathbb{N}} \hbar^j a_j(x, \xi).$$

The term a_0 is called the principal symbol. We say that the full symbol of $\hat{P}$ vanishes on $\Omega \subset \mathbb{R}^d \times \mathbb{R}^d$ if all the a_j vanish on Ω.

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Laboratoire de Mathématique, Université d'Orsay Paris XI, 91405 Orsay Cedex
E-mail address: `Nalini.Anantharaman@math.u-psud.fr`