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Stationary Solutions to the Vlasov-Poisson System

Fahd KARAMI ^{*} Simon LABRUNIE [†] Bruno PINÇON [‡]

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Abstract

In this paper, we present an existence and uniqueness result for stationary Vlasov-Poisson system and we study the asymptotic behavior of the solution near the corner.

Keywords: Vlasov-Poisson, Plasma physics, Polygonal domains, Asymptotic analysis.

1 Introduction.

We consider a gas of charged particles. These particles are described by their distribution function $f(x, v)$ where $x \in \Omega \subset \mathbb{R}^N$ is the position of the particle and v its velocity. We consider the Stationary Vlasov-Poisson system which governs the motion of charged particles under the influence of an electrostatic field. The stationary Vlasov equation reads

$$v \cdot \nabla_x f - E(x) \cdot \nabla_v f = 0, \quad (x, v) \in \Omega \times \mathbb{R}^N, \quad (1)$$

where the electric field $E(x)$ is given by the equations

$$E(x) = -\nabla_x (\phi[f] - \phi_e), \quad -\Delta \phi[f] = \int f(x, v) dv := \rho[f] \quad \text{and} \quad -\Delta \phi_e = \rho_e.$$

where $\rho[f]$ is a spatial density of particles, ϕ the self-consistent electrostatic potential, ρ_e is the density of the fixed charges and ϕ_e is the external potential. Denoted by $\nu(x)$ the outward normal vector of Ω at a point $x \in \partial\Omega$ and we assume that $\partial\Omega := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ where

$$\begin{aligned} \phi[f] &= \phi_e = 0 && \text{on } \Gamma_1 \\ \phi[f] &= 0, \quad \phi_e = \phi_{in} && \text{on } \Gamma_2 \\ \frac{\partial \phi[f]}{\partial \nu} &= \frac{\partial \phi_e}{\partial \nu} = 0 && \text{on } \Gamma_3. \end{aligned} \quad (2)$$

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1.1 Assumptions

Throughout this paper, we assume that

(H₁) $\rho_e \in L^p(\Omega)$ and $\phi_{in} \in L^\infty(\Gamma_2) \cap W^{1-1/p, p}(\Gamma_2)$ for some $p > N$.

(H₂) $\gamma \geq 0$ is a strictly decreasing function such that $\forall \delta > 0$

$$s^{\frac{N}{2}-1} \gamma(s - \delta) \in L^1(0, +\infty).$$

(H₃) Assume that $\sigma(r) = - \int_0^r \gamma^{-1}(s) ds$ with $\sigma \in \mathcal{C}^2(\mathbb{R}^+) \cup \mathcal{C}^0(\mathbb{R}_0^+)$ bounded from below and strictly convex

$$\sup_{x \in \Omega} \int_0^\infty \sigma(h(x, s)) ds < \infty \quad \text{and} \quad \lim_{s \rightarrow \infty} \frac{\sigma(s)}{s} = +\infty$$

where $h(x, s) = s^{\frac{N}{2}-1} \gamma(s - \phi_e(x))$.

1.2 Preliminaries

We consider the spaces V and W of functions defined by:

$$V = \{w \in H^1(\Omega) \quad \text{such that} \quad w = 0 \text{ on } \Gamma_1 \cup \Gamma_2\}$$

$$W = \{w \in H^1(\Omega) \quad \text{such that} \quad w = 0 \text{ on } \Gamma_1 \quad \text{and} \quad w = \phi_{in} \text{ on } \Gamma_2\}.$$

Let $f \in L^1(\Omega \times \mathbb{R}^N)$. If $\rho[f] \in V'$ then there exists $\phi \in V$ solution of $-\Delta \phi = \rho[f]$ satisfying

$$\int_\Omega \nabla \phi \cdot \nabla \xi dx = \int_\Omega \rho[f] \xi dx, \quad \text{for any } \xi \in V.$$

We define a mapping Φ and ϕ by $\Phi[\rho] = \Phi[\rho[f]] = \phi$ and $\phi[f] = \phi$, we have

$$\int_\Omega |\nabla \phi[f]|^2 < \infty.$$

If $\rho[f] \in L^1(\Omega) \setminus V'$ then, the solution $\phi \in V$ does not exist, and in this case, we agree that

$$\int_\Omega |\nabla \phi[f]|^2 = \infty.$$

Indeed, by density there exists a sequence $\rho_n \in L^1(\Omega) \cap V'$ such that ρ_n converges to ρ in $L^1(\Omega)$. Then, as $n \rightarrow \infty$ we have

$$\int_\Omega |\nabla \Phi[\rho_n]|^2 \sim \|\rho_n\|_{V'}^2 \rightarrow \infty,$$

otherwise there exists a subsequence will be noted by ρ_n which converges weakly to ρ in V' and this is contradictory because $\rho \notin V'$.

Similarly, we agree that

$$\int_{\Omega \times \mathbb{R}^N} \sigma(f) = \infty \quad \text{if } \sigma(f) \notin L^1(\Omega \times \mathbb{R}^N).$$

It is clear that the operator ϕ is a linear and satisfies

$$\int_{\Omega \times \mathbb{R}^N} f\phi[g] d(x, v) = \int_{\Omega \times \mathbb{R}^N} g\phi[f] d(x, v) \quad \text{and} \quad \int_{\Omega \times \mathbb{R}^N} f\phi[f] d(x, v) = \int_{\Omega} |\nabla\phi[f]|^2 dx.$$

In the next section, we prove existence and uniqueness of the solution of the stationary problem (1-2). For that, we use some techniques for nonlinear optimization in L^1 and we give also some preliminary properties that will be used afterwards. In the third section, we study the asymptotic of the solution near the corner. Then, we establish some interesting properties for the singularity coefficient. The last section is devoted to the asymptotic of the solution of Maxwell-Boltzmann problem with respect to the total mass.

2 Existence and uniqueness

Thanks to (H_1) , there exists a unique $\phi_e \in W^{1,p}(\Omega)$ solution to problem

$$\begin{cases} -\Delta\phi_e = \rho_e & \text{in } \Omega \\ \phi_e = 0 & \text{on } \Gamma_1 \\ \phi_e = \phi_{in} & \text{on } \Gamma_2 \\ \frac{\partial\phi_e}{\partial\nu} = 0 & \text{on } \Gamma_3. \end{cases} \quad (3)$$

Now, any function f satisfying

$$f(x, v) = \gamma \left(\frac{1}{2} |v|^2 + \phi[f](x) - \phi_e(x) - \beta \right) \quad \text{and} \quad \int_{\Omega \times \mathbb{R}^N} f(x, v) = M \quad (4)$$

is a stationary solution to problem (1-2), where β is unknown constant and M is the total mass. Now, we consider the space $L_M^1(\Omega \times \mathbb{R}^N)$ defined by:

$$L_M^1(\Omega \times \mathbb{R}^N) := \{f \in L^1(\Omega \times \mathbb{R}^N) : \int_{\Omega \times \mathbb{R}^N} f dx dv = M\}.$$

To prove the existence of a solution to problem (4), we use some similar argument in [1], [2] and [3].

Proposition 2.1 *Assume that the Hypothesis $(H_1 - H_3)$ are satisfied. There exists a unique function f solution to problem (4). The function f is the minimum on $L_M^1(\Omega \times \mathbb{R}^N)$ of the functional:*

$$J[f] = \int_{\Omega \times \mathbb{R}^N} \left(\sigma(f) + \left(\frac{1}{2} |v|^2 - \phi_e \right) f \right) dx dv + \frac{1}{2} \int_{\Omega} |\nabla\phi[f]|^2 dx.$$

Proof: It is not difficult to see that \mathcal{J} is convex and lower semi-continuous, then \mathcal{J} is weakly *l.s.c.* Taking $g(x, v) = \gamma(\frac{1}{2}|v|^2 - \phi_e(x))$ by (H_2) , we have $g(x, v) \in L^1(\Omega \times \mathbb{R}^N)$. Thanks to the fact that $\sigma(s) - \sigma(r) - (s - r)\sigma'(r) \geq 0$ for any $r, s \geq 0$, we have

$$\begin{aligned} \mathcal{J}[f] &\geq \int_{\Omega \times \mathbb{R}^N} \left(\sigma(g) + \left(\frac{1}{2}|v|^2 - \phi_e \right) g \right) d(x, v) \\ &\geq 2^{\frac{N}{2}-1} |S^{N-1}| \int_{\Omega} \int_0^{\infty} \sigma(s^{\frac{N}{2}-1} \gamma(s - \phi_e)) d(s, x) - \|\phi_e\|_{L^\infty(\Omega)} \|g\|_{L^1(\Omega \times \mathbb{R}^N)} \end{aligned}$$

where $|S^{N-1}|$ is the measure of the unit sphere in \mathbb{R}^N . Using (H_2) and (H_3) we have $\mathcal{J}[f]$ is bounded from below on $L_M^1(\Omega \times \mathbb{R}^N)$. Let $(f_n) \in L_M^1(\Omega \times \mathbb{R}^N)$ is a minimizing sequence for \mathcal{J} . Now, using the Dunford-Pettis compactness criterion, we can extract a subsequence still noted (f_n) such that

$$f_n \rightharpoonup f \quad \text{weakly in } L^1(\Omega \times \mathbb{R}^N) \quad \text{as } n \rightarrow \infty.$$

Since \mathcal{J} is weakly *l.s.c.*, then

$$\mathcal{J}[f] \leq \liminf_{n \rightarrow \infty} \mathcal{J}[f_n] = \inf_{L_M^1} \mathcal{J}.$$

Then the function f is a global minimum of \mathcal{J} , that satisfies $\int_{\Omega} |\nabla \phi[f]|^2 dx < \infty$ and

$$-\gamma^{-1}(f) + \phi[f] + \frac{1}{2}|v|^2 - \phi_e(x) - \beta = 0,$$

where β is the Lagrange multiplier associated to the constraint $\int_{\Omega \times \mathbb{R}^N} f(x, v) = M$. Finally, the uniqueness of the solution follows by the strict convexity of the function σ . ■

Now, let the function G defined by

$$G(r) = \int_{\mathbb{R}^N} \gamma\left(\frac{1}{2}|v|^2 + r\right) dv = 2^{\frac{N}{2}-1} |S^{N-1}| \int_0^{+\infty} s^{\frac{N}{2}-1} \gamma(s + r) ds$$

then the problem (4) is equivalent to solving:

$$\begin{cases} -\Delta \phi = \rho, \quad \rho = G(\phi - \phi_e - \beta) & \text{in } \Omega \\ \phi = 0 & \text{on } \Gamma_1 \cup \Gamma_2 \\ \frac{\partial \phi}{\partial \nu} = 0 & \text{on } \Gamma_3 \end{cases} \quad (5)$$

where β is defined implicitly by:

$$\int_{\Omega} G(\phi - \phi_e - \beta) = M.$$

Proposition 2.2 *For any fixed β , there exists a unique solution to Problem (5) satisfying*

$$\int_{\Omega} \nabla \phi \cdot \nabla \xi = \int_{\Omega} G(\phi - \phi_e - \beta) \xi \quad \text{for all } \xi \in V.$$

Proof: We consider the nonlinear operator $\mathcal{B} : V \rightarrow V'$ defined by

$$\langle \mathcal{B}\phi, v \rangle = \int_{\Omega} \nabla \phi \cdot \nabla v - \int_{\Omega} G(\phi - \phi_e - \beta) v + \int_{\Omega} G(-\phi_e - \beta) v \quad \text{for all } v \in V$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between V and its dual V' . The operator \mathcal{B} is bounded and weakly continuous. Moreover, since G is a decreasing function then $\int_{\Omega} (G(-\phi_e - \beta) - G(\phi - \phi_e - \beta)) \phi \geq 0$ and

$$\frac{\langle \mathcal{B}\phi, \phi \rangle}{\|\phi\|_V} \geq \|\phi\|_V \rightarrow \infty \quad \text{as } \|\phi\|_V \rightarrow \infty.$$

Then the operator \mathcal{B} is coercive. Thanks to [4], for any element of V' (in particular for $G(-\phi_e - \beta) \in L^\infty(\Omega)$) there exists $\phi \in V$, such that $\langle \mathcal{B}\phi, v \rangle = \langle G(-\phi_e - \beta), v \rangle$ for all $v \in V$. \blacksquare

2.1 Dependence of the solution on the total mass

Now, we study the dependence of the solution with respect to the total mass M , and we derive a comparison principle between M and the Lagrange multiplier β .

Proposition 2.3 *Let ϕ_1 and ϕ_2 be two solutions to Problem (5) corresponding to β_1 and β_2 respectively.*

$$\text{If } \beta_1 \geq \beta_2 \text{ then } \phi_1 \geq \phi_2 \text{ in } \Omega.$$

Proof: Let ϕ_1 et ϕ_2 be two solutions corresponding respectively to β_1 and β_2 in the sense of the proposition 2.2, we have

$$\int_{\Omega} \nabla(\phi_1 - \phi_2) \cdot \nabla \xi = \int_{\Omega} \left(G(\phi_1 - \phi_e - \beta_1) - G(\phi_2 - \phi_e - \beta_2) \right) \xi.$$

Taking $\xi = (\phi_2 - \phi_1)^+$ as a test function in this formulation, then

$$\int_{\Omega} \nabla(\phi_1 - \phi_2) \cdot \nabla(\phi_2 - \phi_1)^+ = \int_{\Omega} \left(G(\phi_1 - \phi_e - \beta_1) - G(\phi_2 - \phi_e - \beta_2) \right) (\phi_2 - \phi_1)^+.$$

We have

$$\int_{\Omega} \nabla(\phi_1 - \phi_2) \cdot \nabla(\phi_2 - \phi_1)^+ = - \int_{\Omega} |\nabla(\phi_2 - \phi_1)^+|^2 \leq 0,$$

and

$$\begin{aligned} G(\phi_1 - \phi_e - \beta_1) - G(\phi_2 - \phi_e - \beta_2) &= \left(G(\phi_1 - \phi_e - \beta_2) - G(\phi_2 - \phi_e - \beta_2) \right) \\ &\quad + \left(G(\phi_1 - \phi_e - \beta_1) - G(\phi_1 - \phi_e - \beta_2) \right). \end{aligned}$$

The second term on the right-hand side is positive because $\beta_1 \geq \beta_2$ and G is a decreasing function. Using again the monotonicity of G , we have

$$\int_{\Omega} \left(G(\phi_1 - \phi_e - \beta_2) - G(\phi_2 - \phi_e - \beta_2) \right) (\phi_2 - \phi_1)^+ \geq 0,$$

then

$$\int_{\Omega} \left(G(\phi_1 - \phi_e - \beta_1) - G(\phi_2 - \phi_e - \beta_2) \right) (\phi_2 - \phi_1)^+ \geq 0.$$

Hence

$$- \int_{\Omega} |\nabla(\phi_2 - \phi_1)^+|^2 \geq 0$$

and therefore

$$\nabla(\phi_2 - \phi_1)^+ = 0$$

then $(\phi_2 - \phi_1)^+ = \text{const.}$ Since $(\phi_2 - \phi_1)^+ = 0$ on $\Gamma_1 \cup \Gamma_2$, we deduce that $(\phi_2 - \phi_1)^+ = 0$ and $\phi_1 \geq \phi_2$. \blacksquare

Theorem 2.4 *Let ϕ and ϕ_e be given by the solution of problem (5). Define the mapping*

$$\begin{aligned} \mu : \mathbb{R}^+ &\longrightarrow \mathbb{R}^+ \\ \beta &\longmapsto M = \int_{\Omega} G(\phi - \phi_e - \beta) dx. \end{aligned}$$

Then, μ is a nondecreasing, one to one and onto mapping.

Proof: Let $\beta_1, \beta_2 \in \mathbb{R}^+$, with $\beta_1 \geq \beta_2$ then for $i = 1, 2$, we have

$$\begin{aligned} -\Delta\phi_i &= \rho_i \\ \rho_i &= G(\phi_i - \phi_e - \beta_i) \quad \text{and} \quad M_i = \int_{\Omega} G(\phi_i - \phi_e - \beta_i) dx. \end{aligned}$$

We subtract two equations and taking $w = \phi_1 - \phi_2$, then

$$\begin{cases} -\Delta w = \rho_1 - \rho_2 & \text{in } \Omega \\ w = 0 & \text{on } \Gamma_1 \cup \Gamma_2 \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } \Gamma_3, \end{cases}$$

in other words, for any $\xi \in V$, we have

$$\int_{\Omega} \nabla w \cdot \nabla \xi = \int_{\Omega} (\rho_1 - \rho_2) \xi.$$

Let us take $\xi = \mathcal{H}_{\varepsilon}^{+}(w)$ as a test function in the previous equality, where $\mathcal{H}_{\varepsilon}^{+}(\cdot)$ is the approximation of the Heaviside function $\mathcal{H}^{+}(\cdot)$ defined as

$$\mathcal{H}_{\varepsilon}^{+}(r) = \begin{cases} 1 & \text{if } r > \varepsilon \\ \frac{1}{\varepsilon}r & \text{if } 0 \leq r \leq \varepsilon \\ 0 & \text{if } r < 0, \end{cases}$$

then

$$- \int_{\Omega} |\nabla w|^2 \mathcal{H}_{\varepsilon}^{+}(w) = \int_{\Omega} (\rho_2 - \rho_1) \mathcal{H}_{\varepsilon}^{+}(w).$$

Now, letting $\varepsilon \rightarrow 0$, we get

$$\int_{\Omega} (\rho_2 - \rho_1) \mathcal{H}^{+}(w) \leq 0.$$

Using the proposition 2.3, we have $\beta_1 \geq \beta_2$ then $w = \phi_1 - \phi_2 \geq 0$ and $\mathcal{H}^{+}(w) = 1$. Finally, we have

$$M_1 := \int_{\Omega} \rho_1 \geq \int_{\Omega} \rho_2 := M_2. \quad \blacksquare$$

Let us introduce the spaces Φ_p defined by

$$\Phi_p = \{v \in W^{1,p}(\Omega); \Delta v \in L^p(\Omega), v = 0 \text{ on } \Gamma_1 \cup \Gamma_2\}$$

endowed with the norm $\|\cdot\|_{\Phi_p}$ where

$$\|u\|_{\Phi_p} \sim \|\Delta u\|_{L^p(\Omega)} \quad \text{for all } u \in \Phi_p.$$

Lemma 2.5 *Let f the solution of the problem (4). Then $\rho[f] \in L^{\infty}(\Omega)$ and $\phi[f] \in \Phi_p$.*

Proof: We know that the function f satisfying

$$f(x, v) = \gamma\left(\frac{1}{2}|v|^2 + \phi[f](x) - \phi_e(x) - \beta\right) \quad \text{and} \quad \int_{\Omega \times \mathbb{R}^2} f(x, v) = M.$$

Using the weak maximum principle, we find $\phi[f](x) \geq 0$ in Ω . On the other hand, thanks to (H_1) we have $\rho_e \in L^p(\Omega)$ and $\phi_{in} \in W^{1-1/p, p}(\Gamma_2) \cap L^{\infty}(\Gamma_2)$, then $\phi_e \in W^{1,p}(\Omega) \subset L^{\infty}(\Omega)$. Using the fact that γ is a decreasing function, we have

$$f(x, v) \leq \gamma\left(\frac{1}{2}|v|^2 - \phi_e(x) - \beta\right) \leq \gamma\left(\frac{1}{2}|v|^2 - \|\phi_e\|_{\infty} - \beta\right).$$

Integrating this inequality of the variable v and using the monotonicity of γ , we have

$$\rho[f](x) \leq 2^{\frac{N}{2}-1} |S^{N-1}| \int_0^{+\infty} s^{\frac{N}{2}-1} \gamma\left(s - \|\phi_e\|_{\infty} - \beta\right) ds.$$

According to (H_2) , we deduce that $\rho[f] \in L^{\infty}(\Omega)$ and $\phi[f] \in \Phi_p$. \blacksquare

3 Corner behavior (N=2)

In this section we consider the case where Ω is a polygonal domain of \mathbb{R}^2 and denote by S the vertex of the re-entrant corner in Γ_1 , let $\frac{\pi}{\alpha}$ be the interior angle of Ω at the vertex and (r, θ) the polar coordinates centered at S . Using Grisvard's results [5], then for all $\phi \in \Phi_p$ with $1 < p < \infty$, there exists λ_m , such that

$$\phi - \sum_{0 < m\alpha < 2-2/p} \lambda_m \chi(r) r^{m\alpha} \sin(m\alpha\theta) \in W^{2,p}(\Omega)$$

with $m \geq 1$ and $\chi \in C^\infty(\mathbb{R}^+)$ is a cut-off function defined by

$$\chi(r) = \begin{cases} 1 & \text{if } 0 \leq r < \eta \\ 0 & \text{if } r \geq \eta, \end{cases}$$

where η is small enough. In particular, for $p \in]\frac{2}{2-\alpha}, \frac{1}{1-\alpha}[$ we have $\phi, \phi_e \in \Phi_p$ then its can be decomposed into the regular part $W^{2,p}(\Omega)$ and a singular part as follows:

$$\phi = \phi_{\mathcal{R}} + \lambda \chi(r) r^\alpha \sin(\alpha\theta)$$

$$\phi_e = \phi_{\mathcal{R}}^e + \lambda_e \chi(r) r^\alpha \sin(\alpha\theta)$$

where $\lambda = \int_{\Omega} \rho P_s dx$ and $\lambda_e = \int_{\Omega} \rho_e P_s dx$ with $-\Delta P_s = 0$ in Ω , $P_s = 0$ on $\Gamma_1 \cup \Gamma_2$ and $\frac{\partial P_s}{\partial \nu} = 0$ on Γ_3 and

$$P_s = P_p + \tilde{P} \quad \text{where } P_p = \frac{1}{\pi} r^{-\alpha} \sin(\alpha\theta),$$

and $\tilde{P} \in H^1(\Omega)$ is the solution of

$$-\Delta \tilde{P} = \Delta P_p = 0, \quad \tilde{P} = 0 \quad \text{on } \Gamma_1 \quad \text{and} \quad \tilde{P} = P_p \quad \text{on } \Gamma_2 \cup \Gamma_3.$$

Let U be a bounded open subset of \mathbb{R}^+ , we denote by $L_\tau^p(U)$ the space

$$L_\tau^p(U) := \left\{ u : \text{measurable such that } \|u\|_{L_\tau^p(U)}^p = \int_U |u(t)|^p t^\tau dt < \infty \right\}.$$

We denote $W_\tau^{1,p}(U)$ the space of functions u of $L_\tau^p(U)$ such that $\frac{\partial u}{\partial t} \in L_\tau^p(U)$ where we denote the norm

$$\|u\|_{W_\tau^{1,p}(U)} = \left(\int_U |u(t)|^p t^\tau dt \right)^{1/p} + \left(\int_U \left| \frac{\partial u}{\partial t} \right|^p t^\tau dt \right)^{1/p}.$$

Let \mathcal{H} and \mathcal{L} the linear operators defined by:

$$\mathcal{H}u(t) = \frac{1}{t} \int_0^t u(s) ds$$

$$\mathcal{L}u(t) = \frac{1}{t} \int_t^\infty u(s) ds$$

Thanks to [8] we have:

If $\tau < p - 1$ then

$$\|\mathcal{H}u\|_{L^p_\tau(U)} \leq C \|u\|_{L^p_\tau(U)}. \quad (6)$$

If $\tau > p - 1$ then

$$\|\mathcal{L}u\|_{L^p_\tau(U)} \leq C \|u\|_{L^p_\tau(U)} \quad (7)$$

where C is a constant which depends only on τ and p .

Taking

$$w = \chi\phi_{\mathcal{R}} \text{ and } F = -\Delta w$$

where $F = -\chi\Delta\phi_{\mathcal{R}} - 2\nabla\chi\nabla\phi_{\mathcal{R}} - \phi_{\mathcal{R}}\Delta\chi$. For all $p \in]\frac{2}{2-\alpha}, \frac{1}{1-\alpha}[$, we have $F \in L^p(\Omega)$ and $w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, then $\nabla w \in [W^{1,p}(\Omega)]^2$ and $\frac{\partial w}{\partial \tau} = 0$, where $\tau(x)$ is a tangent vector at a point $x \in \Gamma_1$.

Theorem 3.1 *Let $p \in]\frac{2}{2-\alpha}, \frac{1}{1-\alpha}[$, ϕ be the solution to Problem (5) and $\phi_{\mathcal{R}} \in W^{2,p}(\Omega)$ the regular part, then there exists $g \in L^\infty(\Omega)$ such that*

$$\chi(r)\phi_{\mathcal{R}}(r, \theta) = r \sin(\alpha\theta)g(r, \theta), \quad \|g(r, \theta)\|_{L^\infty(\Omega)} \leq C \|\phi_{\mathcal{R}}\|_{W^{2,p}(\Omega)}$$

where C is a constant which depends only on α , p and Ω .

Proof: Let $p \in]\frac{2}{2-\alpha}, \frac{1}{1-\alpha}[$, we have $w \in W^{2,p}(\Omega)$ then using the polar coordinates, it is not difficult to see that $\frac{\partial^2 w}{\partial r^2}, \frac{1}{r} \frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial w}{\partial \theta}, \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} - \frac{1}{r} \frac{\partial w}{\partial r} \in L^p(\Omega)$. We have $\nabla w \in [W^{1,p}(\Omega)]^2$ with $p > \frac{2}{2-\alpha} \geq 2$, then $\nabla w \in [C(\bar{\Omega})]^2$. Since $w = 0$ in $\partial\Omega$, then $\frac{\partial w}{\partial \tau} = 0$ and the normal components of the gradient is perpendicular to line segment Γ_1 .

By continuity, the gradient of w vanishes at $r = 0$ and $\frac{\partial w}{\partial r}(0, 0) = 0$, then

$$r^{-1} \frac{\partial w(r, \theta)}{\partial r} = \frac{1}{r} \int_0^r \frac{\partial^2 w(\tilde{r}, \theta)}{\partial r^2} d\tilde{r}.$$

Applying (6) with $\tau = 1$ and we integrate at θ , we obtain

$$\int_0^\omega \int_0^R \left| \frac{\partial w(r, \theta)}{\partial r} \right|^{p r^{1-p}} dr d\theta \leq C \int_0^\omega \int_0^R \left| \frac{\partial^2 w(r, \theta)}{\partial r^2} \right|^{p r} dr d\theta. \quad (8)$$

Similarly, we have $w(0, \theta) = 0$ and thanks to (6) we have

$$\int_0^\omega \int_0^R \left| \frac{w(r, \theta)}{r^2} \right|^{p r} dr d\theta \leq C \int_0^\omega \int_0^R \left| \frac{\partial w(r, \theta)}{\partial r} \right|^{p r^{1-p}} dr d\theta < \infty. \quad (9)$$

Thanks to (8) and (9), we have

$$\frac{\partial}{\partial r}\left(\frac{w}{r}\right) = \frac{1}{r}\frac{\partial w}{\partial r} - \frac{w}{r^2} \in L^p(\Omega) \quad \text{and} \quad \frac{w}{r} \in W_1^{1,p}\left(0, R; L^p(]0, \omega[)\right).$$

Now, we have

$$\Delta w = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} \in L^p(\Omega)$$

then $\frac{\partial^2}{\partial \theta^2}\left(\frac{w}{r^2}\right) = \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} \in L^p(\Omega)$. Thanks to the Nirenberg inequality (see [9]), we have

$$\left\|\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{w}{r}\right)\right\|_{L^p(\Omega)} = \left\|\frac{\partial}{\partial \theta}\left(\frac{w}{r^2}\right)\right\|_{L^p(\Omega)} \leq C(p, \Omega) \left\|\frac{w}{r^2}\right\|_{L^p(\Omega)}^{1/2} \left\|\frac{\partial^2}{\partial \theta^2}\left(\frac{w}{r^2}\right)\right\|_{L^p(\Omega)}^{1/2}$$

then $\frac{w}{r} \in W^{1,p}(\Omega) \subset C(\bar{\Omega})$ and therefore $\frac{w}{r}$ is bounded on $\bar{\Omega}$. Now, let us we prove that

$$\frac{1}{r}\frac{\partial w}{\partial \theta} \in W^{1,p}(\Omega).$$

Indeed, we have $\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right) = \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} \in L^p(\Omega)$. Since $w \in W^{1,p}(\Omega)$, then $\frac{1}{r}\frac{\partial w}{\partial \theta} \in L^p(\Omega)$. On the other hand, we have $w \in W^{2,p}(\Omega)$ then

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right) = \frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta} \in L^p(\Omega)$$

hence $\frac{1}{r}\frac{\partial w}{\partial \theta} \in W^{1,p}(\Omega) \subset C(\bar{\Omega})$ and $\left\|\frac{1}{r}\frac{\partial w}{\partial \theta}\right\|_{L^\infty(\Omega)} \leq C\left\|\frac{1}{r}\frac{\partial w}{\partial \theta}\right\|_{W^{1,p}(\Omega)}$. We have $\frac{w}{r}(r, 0) = 0$, then

$$\frac{w(r, \theta)}{r} = \int_0^\theta \frac{\partial}{\partial \theta}\left(\frac{w(r, \tilde{\theta})}{r}\right) d\tilde{\theta},$$

and

$$\left|\frac{w(r, \theta)}{r\theta}\right| \leq \left\|\frac{\partial}{\partial \theta}\left(\frac{w}{r}\right)\right\|_{L^\infty(\Omega)} \leq C\left\|\frac{1}{r}\frac{\partial w}{\partial \theta}\right\|_{W^{1,p}(\Omega)}.$$

Near $\theta = 0$, we have $\theta \sim \frac{\sin(\alpha\theta)}{\alpha}$ and

$$\left|\frac{w(r, \theta)}{r\sin(\alpha\theta)}\right| \leq \frac{C}{\alpha}\left\|\frac{1}{r}\frac{\partial w}{\partial \theta}\right\|_{W^{1,p}(\Omega)}. \quad (10)$$

In the same manner, we have $\frac{w}{r}(r, \frac{\pi}{\alpha}) = 0$, then $\left|\frac{w(r, \theta)}{r(\alpha\theta - \pi)}\right| \leq \frac{C}{\alpha}\left\|\frac{1}{r}\frac{\partial w}{\partial \theta}\right\|_{W^{1,p}(\Omega)}$. Near $\theta = \frac{\pi}{\alpha}$, we have $\alpha\theta - \pi \sim \sin(\alpha\theta - \pi) = -\sin(\alpha\theta)$ and then (10) is satisfied. For $\theta \in [\theta_0, \frac{\pi}{\alpha} - \theta_0]$ with θ_0 small enough, there exists a constant $c > 0$ such that $|\sin(\alpha\theta)| \geq c|\alpha\theta|$ and the inequality (10) is satisfied. This completes the proof of Theorem. \blacksquare

Theorem 3.2 Let ϕ_1 and ϕ_2 be two solutions of problem (5) corresponding to M_1 and M_2 respectively.

If $M_1 \geq M_2$, then $\lambda_1 \geq \lambda_2$

where λ_1 and λ_2 are the singularity coefficients of ϕ_1 and ϕ_2 respectively.

Proof: Thanks to Theorem 3.1, we have

$$\phi_1 - \phi_2 \sim (\lambda_1 - \lambda_2)r^\alpha \sin(\alpha\theta) \text{ in a neighbourhood of the corner.}$$

Thanks to Theorem 2.4, we have $\beta_1 \geq \beta_2$, then using the proposition 2.2 we obtain $\phi_1 \geq \phi_2$ and therefore $\lambda_1 \geq \lambda_2$. \blacksquare

Remark 3.3 1) Taking $\psi = \phi - \phi_e$ and let λ and λ_e the singularity coefficients of ϕ and ϕ_e respectively. If $\lambda \geq \lambda_e$ then $\psi^- \in H^2(\Omega)$. Indeed,

$$\phi - \phi_e \sim (\lambda - \lambda_e)r^\alpha \sin(\alpha\theta) \quad \forall (r, \theta) \in \mathcal{V},$$

where \mathcal{V} is the neighborhood of the corner. Since $\lambda \geq \lambda_e$, then $\psi^- = (\phi - \phi_e)^- = 0$ in \mathcal{V} . Consequently $\psi^- = (\phi_{\mathcal{R}} - \phi_{\mathcal{R}}^e)^- \in H^2(\Omega)$.

2) For $\rho_e \in L^\infty(\Omega)$ and $\beta \geq -G^{-1}(\|\rho_e\|_\infty)$, we have $\phi \geq \phi_e$ in Ω . Subtracting the two equations verified by ϕ and ϕ_e , we have

$$-\Delta(\phi - \phi_e) - \left(G(\phi(x) - \phi_e(x) - \beta) - G(-\beta) \right) = G(-\beta) - \rho_e.$$

Taking $(\phi_e - \phi)^+$ as a test function in the weak formulation of this equation, using the same arguments as in the proposition 2.2, the monotonicity of the function γ and we find the desired result.

3.1 Behavior of Maxwell-Boltzmann Problem

In the case of Maxwellian velocity distribution e.g. $\gamma(s) = \exp(-s)$ the problem becomes:

$$\begin{cases} -\Delta\phi = \kappa \exp(-\phi + \phi_e) := \rho, & \text{in } \Omega \\ \phi = 0 & \text{on } \Gamma_1 \cup \Gamma_2 \\ \frac{\partial\phi}{\partial\nu} = 0 & \text{on } \Gamma_3 \end{cases} \quad (11)$$

and $\int \rho[f] = M$, where $\kappa = \exp(\beta)$.

Theorem 3.4 Let ϕ , ϕ_e , M and κ be given by the solution of the problem (11) and λ be the singularity coefficient of ϕ . As $M \rightarrow 0$, we have

$$\kappa \sim M \left(\int_{\Omega} \exp(\phi_e) dx \right)^{-1} \quad \text{and} \quad \lambda \sim \kappa \int_{\Omega} \exp(\phi_e) P_s dx.$$

Proof: We recall that $\rho = \kappa \exp(\phi_e - \phi)$ and $\kappa = M \left(\int_{\Omega} \exp(\phi_e - \phi) dx \right)^{-1}$, then

$$M \left(\int_{\Omega} \exp(\phi_e) dx \right)^{-1} \leq \kappa \leq M \exp(\|\phi\|_{\infty}) \left(\int_{\Omega} \exp(\phi_e) dx \right)^{-1}.$$

We have

$$\Phi_2 \subset H^{1+\alpha-\varepsilon}(\Omega) \subset C(\bar{\Omega}) \subset L^{\infty}(\Omega),$$

then

$$\|\phi\|_{\infty} \leq C \|\phi\|_{\Phi_2} \leq C \|\rho\|_{L^2(\Omega)}.$$

On the other hand, we have

$$\|\rho\|_{L^2(\Omega)} \leq \|\rho\|_{L^1(\Omega)}^{1/2} \|\rho\|_{\infty}^{1/2} = M^{1/2} \|\rho\|_{\infty}^{1/2} \quad \text{and} \quad \|\rho\|_{\infty} \leq \kappa \exp(\|\phi_e\|_{\infty}).$$

Then, $\|\phi\|_{\infty} \leq C (M \kappa \exp(\|\phi_e\|_{\infty}))^{1/2}$ and

$$M \left(\int_{\Omega} \exp(\phi_e) dx \right)^{-1} \leq \kappa \leq M \exp(C M \kappa \exp(\|\phi_e\|_{\infty})^{1/2}) \left(\int_{\Omega} \exp(\phi_e) dx \right)^{-1}.$$

Since κ is nondecreasing with respect to M , then for $M \rightarrow 0$, we have $\kappa \rightarrow \kappa_0 \geq 0$ and $C M \kappa \exp(\|\phi_e\|_{\infty}) \rightarrow 0$, therefore

$$\kappa \sim M \left(\int_{\Omega} \exp(\phi_e) dx \right)^{-1}.$$

Now, we prove that $\lambda \sim \kappa \int_{\Omega} \exp(\phi_e) P_s dx$. We have

$$\lambda = \int_{\Omega} \rho P_s dx = \kappa \int_{\Omega} \exp(\phi_e - \phi) P_s dx.$$

As $M \rightarrow 0$, we have $\kappa \rightarrow 0$, $\|\rho\|_{\infty} \rightarrow 0$ and $\phi \rightarrow 0$ a.e. in Ω , then

$$\exp(\phi_e - \phi) \rightarrow \exp(\phi_e).$$

Since $|\exp(\phi_e - \phi)| \leq \exp(\phi_e)$, the Lebesgue's Dominated Convergence Theorem give $\exp(\phi_e - \phi) \rightarrow \exp(\phi_e)$ in $L^2(\Omega)$ and therefore we obtain

$$\frac{\lambda}{\kappa} = \int_{\Omega} \exp(\phi_e - \phi) P_s \rightarrow \int_{\Omega} \exp(\phi_e) P_s \quad \text{as } M \rightarrow 0.$$

Which concludes the proof. ■

Now, We explicit the dependence of ϕ and ρ with respect to the parameter κ and taking

$$w_{\kappa} = \frac{\phi_{\kappa}}{\kappa} \quad \text{and} \quad f_{\kappa} = \frac{\rho_{\kappa}}{\kappa}.$$

Proposition 3.5 *We have $\frac{1}{\sqrt{\kappa}} \frac{\nabla \rho_\kappa}{\rho_\kappa}$ and $\frac{1}{\sqrt{\kappa}} \nabla \phi_\kappa$ are uniformly bounded in $L^2(\Omega)$.*

Proof: The problem (11) can be rewritten in f_κ as:

$$\begin{cases} f_\kappa - \frac{1}{\kappa} \Delta \log(f_\kappa) = \frac{1}{\kappa} \rho_e, & \text{in } \Omega \\ f_\kappa = 1 & \text{on } \Gamma_1 \\ f_\kappa = \exp(\phi_{in}) & \text{on } \Gamma_2 \\ \frac{\partial f_\kappa}{\partial \nu} = 0 & \text{on } \Gamma_3 \end{cases} \quad (12)$$

Taking $\log(f_\kappa)$ as a test function in the weak formulation of (12), then

$$\int_{\Omega} f_\kappa \log(f_\kappa) + \frac{1}{\kappa} \int_{\Omega} \left| \frac{\nabla f_\kappa}{f_\kappa} \right|^2 = \frac{1}{\kappa} \int_{\Omega} \rho_e \log(f_\kappa).$$

Using the Young inequality, we obtain

$$\int_{\Omega} f_\kappa \log(f_\kappa) + \frac{1}{\kappa} \int_{\Omega} \left| \frac{\nabla f_\kappa}{f_\kappa} \right|^2 \leq \frac{C_p}{2\kappa} \int_{\Omega} \rho_e^2 + \frac{1}{2\kappa C_p} \int_{\Omega} |\log(f_\kappa)|^2.$$

Thanks to the Poincaré inequality, we have $\|\log(f_\kappa)\|_{L^2(\Omega)} \leq C \left\| \frac{\nabla f_\kappa}{f_\kappa} \right\|_{L^2(\Omega)}$ and then

$$\frac{1}{\kappa} \left(1 - \frac{C^2}{2C_p}\right) \int_{\Omega} \left| \frac{\nabla f_\kappa}{f_\kappa} \right|^2 \leq \frac{C_p}{2\kappa} \int_{\Omega} \rho_e^2 - \int_{\Omega} \rho_\kappa \log(f_\kappa).$$

Taking $C_p = C^2$, we obtain

$$\frac{1}{\kappa} \int_{\Omega} \left| \frac{\nabla f_\kappa}{f_\kappa} \right|^2 \leq \frac{C^2}{\kappa} \int_{\Omega} \rho_e^2 - 2 \int_{\Omega} \rho_\kappa \log(f_\kappa).$$

On the other hand, we have $\int_{\Omega} f_\kappa \log(f_\kappa) \geq |\Omega| \min_{x \in \mathbb{R}} (x \log(x))$ and consequently

$$\left\| \frac{1}{\sqrt{\kappa}} \frac{\nabla f_\kappa}{f_\kappa} \right\|_{L^2(\Omega)} \leq C_1$$

and this shows the first estimate. Now, we have $\phi_\kappa = \phi_e - \log(f_\kappa)$, then

$$\left\| \frac{1}{\sqrt{\kappa}} \nabla \phi_\kappa \right\|_{L^2(\Omega)} \leq \left\| \frac{1}{\sqrt{\kappa}} \nabla \phi_e \right\|_{L^2(\Omega)} + \left\| \frac{1}{\sqrt{\kappa}} \frac{\nabla f_\kappa}{f_\kappa} \right\|_{L^2(\Omega)} \leq C_2.$$

and the second estimate follows immediately. ■

Theorem 3.6 *As $\kappa \rightarrow \infty$, we have*

$$\begin{aligned} \phi &\rightarrow \infty \quad \text{a.e. in } \Omega, \quad \frac{\phi}{\kappa} \rightarrow 0 \quad \text{in } H^1(\Omega) \cap C(\bar{\Omega}) \text{ and} \\ \frac{\rho}{\kappa} &\rightarrow 0 \quad \text{in } L^p(\Omega), \quad \forall p < \infty \text{ and uniformly in every compact of } \Omega. \end{aligned}$$

Proof: We recall that

$$-\Delta w_\kappa = f_\kappa, \quad f_\kappa = \exp(-\kappa w_\kappa + \phi_e), \quad -\Delta \phi_e = \rho_e.$$

Thanks to proposition 2.3, for all x the sequence $f_\kappa(x)$ is a decreasing and bounded from below by 0, then there exists $f \geq 0$ such that

$$f_\kappa \rightarrow f \quad \text{a.e. in } \Omega.$$

We have $w_\kappa \geq 0$ and $\phi_e \in L^\infty(\Omega)$, then

$$|f_\kappa(x)| \leq \exp(\phi_e(x)) \quad \text{a.e. in } \Omega \quad \text{and} \quad \exp(\phi_e) \in L^\infty(\Omega), \quad (13)$$

using Lebesgue's Dominated Convergence Theorem, we have $f_\kappa \rightarrow f$ in $L^p(\Omega)$ for all $p < \infty$. On the other hand, we have $\|w_\kappa\|_{\Phi_p} = \|f_\kappa\|_{L^p(\Omega)}$ then there exists $w \in \Phi_p$ such that

$$w_\kappa \rightarrow w \quad \text{in } \Phi_p.$$

We have

$$\Phi_p \subset H^{1+\alpha-\varepsilon}(\Omega) \Subset H^{\frac{3}{2}+\varepsilon}(\Omega) \subset C(\bar{\Omega}),$$

then

$$w_\kappa \rightarrow w \quad \text{in } C(\bar{\Omega}) \quad \text{and} \quad -\Delta w = f.$$

Now, we will show that $f = 0$ and $w = 0$. Thanks to proposition 3.5, we have

$$\left\| \nabla w_\kappa \right\|_{L^2(\Omega)} \leq \frac{C}{\sqrt{\kappa}},$$

then $f = w = 0$ in Ω and therefore

$$w_\kappa \rightarrow 0 \quad \text{in } \Phi_p.$$

Consequently, we have $\phi_\kappa \rightarrow +\infty$ a.e. in Ω .

Now, using Dini's Theorem [10], we have $f_\kappa \rightarrow 0$ uniformly in every compact of Ω . ■

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