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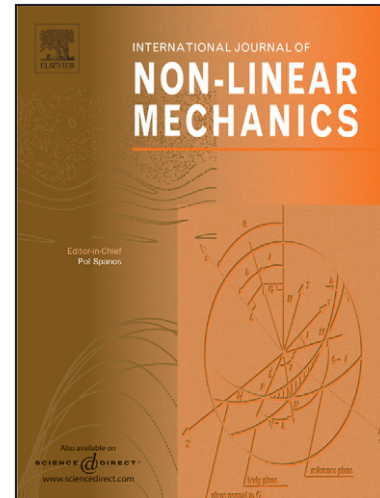
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SHEAR DEFORMATION EFFECT IN NONLINEAR ANALYSIS OF COMPOSITE BEAMS OF VARIABLE CROSS SECTION

E.J. Sapountzakis and D.G. Panagos
cvsapoun@central.ntua.gr¹ dpanagosgr@yahoo.gr²

Institute of Structural Analysis, School of Civil Engineering, National Technical
University of Athens, Zografou Campus, GR – 157 80 Athens, Greece.

Key words: Transverse shear stresses; Shear center; Shear deformation coefficients;
Beam; Nonlinear analysis; Variable Cross Section; Boundary element
method

Abstract

In this paper the nonlinear analysis of a composite Timoshenko beam with arbitrary variable cross section undergoing moderate large deflections under general boundary conditions is presented employing the analog equation method (AEM), a BEM-based method. The composite beam consists of materials in contact, each of which can surround a finite number of inclusions. The materials have different elasticity and shear moduli with same Poisson's ratio and are firmly bonded together. The beam is subjected in an arbitrarily concentrated or distributed variable axial loading, while the shear loading is applied at the shear center of the cross section, avoiding in this way the induction of a twisting moment. To account for shear deformations, the concept of shear deformation coefficients is used. Five boundary value problems are formulated with respect to the transverse displacements, the axial

¹ Associate Professor, School of Civil Engineering, National Technical University, Zografou Campus, GR-157 80 Athens, Greece.

² Doctoral Student, School of Civil Engineering, National Technical University, Zografou Campus, GR-157 80 Athens, Greece.

displacement and to two stress functions and solved using the Analog Equation Method. Application of the boundary element technique yields a system of nonlinear equations from which the transverse and axial displacements are computed by an iterative process. The evaluation of the shear deformation coefficients is accomplished from the aforementioned stress functions using only boundary integration. Numerical examples are worked out to illustrate the efficiency, the accuracy, the range of applications of the developed method and the influence of the shear deformation effect.

1. Introduction

One of the problems often encountered in engineering practice is the analysis of members of structures of variable cross section which except for the transverse loading are also subjected to intense axial loading. Also, composite structural elements consisting of a relatively weak matrix material reinforced by stronger inclusions or of materials in contact are of increasing technological importance. Steel beams or columns totally encased in concrete, fiber-reinforced materials or concrete plates stiffened by steel beams are most common examples. The analysis of these members becomes much more accurate and complex taking into account that the axial force is nonlinearly coupled with the transverse deflections, avoiding in this way the inaccuracies arising from a linearized second – order analysis. Moreover, unless the beam is very “thin” the error incurred from the ignorance of the effect of shear deformation may be substantial, particularly in the case of heavy lateral loading.

Over the past twenty years, many researchers have developed and validated various methods of performing a linearized second-order analysis on members of constant or variable homogeneous or composite cross section [1-5]. Consequently,

due to the demand of more accurate results nonlinear analyses have also been performed for beams of variable cross section either deriving the variations of the total potential and using them as the basis for a FE analysis [6-7] or presenting a BEM-based method [8]. In all these studies shear deformation effect is ignored.

This effect has been taken into account in constant cross section beams. More specifically, Kim et al. presented a practical second-order inelastic static [9] and dynamic [10] analysis for 3-D steel frames, Machado and Cortinez [11] a geometrically non-linear beam theory for the lateral buckling problem of bisymmetric thin-walled composite simply supported or cantilever beams, while the boundary element method has also been employed for first-order [12-14] and nonlinear [15] analyses. To the authors' knowledge Timoshenko beams of variable cross section have been studied employing only first-order analysis [16].

In this paper a BEM-based method is employed for the nonlinear analysis of a composite Timoshenko beam with variable cross section undergoing moderate large deflections under general boundary conditions. The composite beam consists of materials in contact, each of which can surround a finite number of inclusions. The materials have different elasticity and shear moduli with same Poisson's ratio and are firmly bonded together. The beam is subjected in an arbitrarily concentrated or distributed variable axial loading, while the shear loading is applied at the shear center of the cross section, avoiding in this way the induction of a twisting moment. To account for shear deformations, the concept of shear deformation coefficients is used. Five boundary value problems are formulated with respect to the transverse displacements, the axial displacement and to two stress functions and solved using the Analog Equation Method [17], a BEM-based method. Application of the boundary element technique yields a system of nonlinear equations from which the transverse

and axial displacements are computed by an iterative process. The evaluation of the shear deformation coefficients is accomplished from the aforementioned stress functions using only boundary integration. The essential features and novel aspects of the present formulation compared with previous ones are summarized as follows.

- i. For the first time in the literature shear deformation effect is taken into account in the nonlinear analysis of variable cross section beams.
- ii. The beam is subjected in an arbitrarily concentrated or distributed variable axial loading.
- iii. The beam is supported by the most general boundary conditions including elastic support or restrain.
- iv. The analysis is not restricted to a linearized second – order one but is a nonlinear one arising from the fact that the axial force is nonlinearly coupled with the transverse deflections (additional terms are taken into account).
- v. Previous formulations concerning composite beams of thin walled cross sections or laminated cross-sections are analyzing these beams using the ‘refined models’. However, these models do not satisfy the continuity conditions of transverse shear stress at layer interfaces and assume that the transverse shear stress along the thickness coordinate remains constant, leading to the fact that kinematic or static assumptions cannot be always valid [18-20].
- vi. The shear deformation coefficients are evaluated using an energy approach, instead of Timoshenko’s [21] and Cowper’s [22] definitions, for which several authors [23] have pointed out that one obtains unsatisfactory results or definitions given by other researchers [24, 25], for which these factors take negative values.

- vii. The effect of the material's Poisson ratio ν is taken into account.
- viii. The proposed method employs a pure BEM approach (requiring only boundary discretization) resulting in line or parabolic elements instead of area elements of the FEM solutions (requiring the whole cross section to be discretized into triangular or quadrilateral area elements), while a small number of line elements are required to achieve high accuracy.

Numerical examples are worked out to illustrate the efficiency, the accuracy, the range of applications of the developed method and the influence of the shear deformation effect.

2. Statement of the problem

Consider a prismatic beam of length l with a doubly symmetric composite variable cross section of arbitrary shape, consisting of materials in contact, each of which can surround a finite number of inclusions, with modulus of elasticity E_j , shear modulus G_j and common Poisson's ratio ν , occupying the regions Ω_j ($j=1,2,\dots,K$) of the y,z plane (Fig.1b). The materials of these regions are firmly bonded together and are assumed homogeneous, isotropic and linearly elastic. Let also the boundaries of the nonintersecting regions Ω_j be denoted by Γ_j ($j=1,2,\dots,K$). These boundary curves are piecewise smooth, i.e. they may have a finite number of corners. Without loss of generality, it may be assumed that the x -axis of the beam principal coordinate system is the line joining the centroids of the cross sections. The beam is subjected to an arbitrarily distributed axial loading p_x and to torsionless bending arising from arbitrarily distributed transverse loading p_y , p_z and bending moments m_y , m_z along y and z axes, respectively (Fig.1a).

Under the action of the aforementioned loading the displacement field of the beam is given as

$$u(x, y, z) = u(x) + z\theta_y(x) - y\theta_z(x) \quad (1a)$$

$$w(x, y, z) = w(x) \quad (1b)$$

$$v(x, y, z) = v(x) \quad (1c)$$

where u , w , v are the axial and transverse beam displacements with respect to x, z, y axes, respectively and θ_y , θ_z are the angles of rotation of the cross-section due to bending. Employing the strain-displacement relations of the three - dimensional elasticity for moderate displacements [26, 27], the following strain components can be easily obtained

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2} \left[\left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial x} \right)^2 \right] \quad (2a)$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \quad (2b)$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad (2c)$$

$$\varepsilon_{yy} = \varepsilon_{zz} = \gamma_{yz} = 0 \quad (2d)$$

Substituting eqn.(1a) in eqns (2b,c) we obtain the angles of rotation of the cross-section due to bending taking into account shear deformation effect as

$$\theta_y(x) = -\frac{\partial w}{\partial x} + \gamma_{xz} \quad (3a)$$

$$\theta_z(x) = \frac{\partial v}{\partial x} - \gamma_{xy} \quad (3b)$$

Assuming small rotations, the total angles of rotation ω_y , ω_z (Fig.2) of the cross-section in the x - z and x - y planes of the beam, respectively, satisfy the following relations

$$\cos \omega_y \approx 1 \qquad \cos \omega_z \approx 1 \qquad (4a,b)$$

$$\sin \omega_y \approx \omega_y = -\frac{dw}{dx} = \theta_y - \gamma_{xz} \qquad \sin \omega_z \approx \omega_z = -\frac{dv}{dx} = -\theta_z - \gamma_{xy} \qquad (5a,b)$$

while the corresponding curvatures are given as

$$k_y = \frac{d\theta_y}{dx} = -\frac{d^2w}{dx^2} + \frac{d\gamma_{xz}}{dx} \qquad (6a)$$

$$k_z = \frac{d\theta_z}{dx} = \frac{d^2v}{dx^2} - \frac{d\gamma_{xy}}{dx} \qquad (6b)$$

where γ_{xz} , γ_{xy} are the additional angles of rotation of the cross-section due to shear deformation (Fig.2a).

Noting that the direction of the axial force N is the tangential to the deformed centroidal axis and its normal in the x - y and x - z planes of the beam gives the direction of the shear forces Q_y , Q_z , respectively (Fig.2), employing eqns. (4a,b), (5a,b) and referring to Fig. 2b, the stress resultants R_x , R_y , R_z acting in the x , y , z directions (undeformed beam directions), respectively, are related to the axial N and shear Q_y , Q_z forces as

$$R_x = N - Q_z \frac{dw}{dx} - Q_y \frac{dv}{dx} \qquad R_y = Q_y + N \frac{dv}{dx} \qquad R_z = Q_z + N \frac{dw}{dx} \qquad (7a,b,c)$$

The second and the third term in the right hand side of eqn. (7a), express the influence of the shear forces Q_z and Q_y on the horizontal stress resultant R_x . However, these terms can be neglected since $Q_z w'$ and $Q_y v'$ are much smaller than N and thus eqn. (7a) can be written as

$$R_x \approx N \quad (8)$$

Moreover, the axial force N , the shear forces Q_y , Q_z and the bending moments M_y , M_z are given from the well-known relations [26]

$$N = E_I A \varepsilon_{xx} \quad (9a)$$

$$Q_y = G_I A_y \gamma_{xy} \quad Q_z = G_I A_z \gamma_{xz} \quad (9b,c)$$

$$M_y = E_I I_{yy} k_y \quad M_z = E_I I_{zz} k_z \quad (9d,e)$$

where the first material is considered as reference material; $G_I A_z$, $G_I A_y$ are the cross-section's shear rigidities of the Timoshenko's beam theory, with

$$A_y = \kappa_y A = \frac{I}{\alpha_y} A \quad A_z = \kappa_z A = \frac{I}{\alpha_z} A \quad (10a,b)$$

the shear areas with respect to y , z axes, respectively; κ_y , κ_z are the shear correction factors; α_y , α_z the shear deformation coefficients and A , I_{yy} , I_{zz} are the area, and the moments of inertia with respect to y , z axes, respectively of the composite cross section given as

$$A = A(x) = \sum_{j=1}^K \frac{G_j}{G_I} \int_{\Omega_j} d\Omega_j \quad (11a)$$

$$I_{yy} = I_{yy}(x) = \sum_{j=1}^K \frac{E_j}{E_1} \int_{\Omega_j} z^2 d\Omega_j \quad (11b)$$

$$I_{zz} = I_{zz}(x) = \sum_{j=1}^K \frac{E_j}{E_1} \int_{\Omega_j} y^2 d\Omega_j \quad (11c)$$

It is worth here noting that the reduction of eqns.(9d,e) and (11b,c) using the modulus of elasticity E_1 and of eqns.(9b,c) and (11a) using the shear modulus G_1 of the first material, could be achieved using any other material, considering it as reference material.

The governing equation for the beam transverse displacement $w = w(x)$ will be derived by considering the equilibrium of the deformed element in the x - z plane. Thus, referring to Fig. 2b we obtain

$$\frac{dR_x}{dx} + p_x = 0 \quad \frac{dR_z}{dx} + p_z = 0 \quad \frac{dM_y}{dx} - Q_z + m_y = 0 \quad (12a,b,c)$$

Employing equations (9c) and (10b), equation (6a) can be written as

$$k_y = -\frac{d^2 w}{dx^2} + \frac{\alpha_z}{G_1 A} Q_{z,x} - \frac{\alpha_z}{G_1 A^2} Q_z A_{,x} + \frac{\alpha_{z,x}}{G_1 A} Q_z \quad (13)$$

Substituting eqns. (8), (7c) into eqns. (12a,b), using eqn. (12c) to eliminate Q_z , employing the relation (9d) and utilizing eqn. (13) we obtain the expressions of the angle of rotation due to bending θ_y and the stress resultants M_y , R_z , as these are given in Appendix A and the governing differential equation as

$$C_4 \frac{d^4 w}{dx^4} = C_3 \frac{d^3 w}{dx^3} + C_2 \frac{d^2 w}{dx^2} + C_1 \frac{dw}{dx} + C_0 \quad \text{inside the beam} \quad (14)$$

where the functions B_i ($i=1,2$) are given as

$$B_1 = a_z E_l A I_{yy,x} A_{,x} - 2a_z E_l I_{yy} A_{,x}^2 + a_z E_l I_{yy} A A_{,xx} + 2E_l I_{yy} A a_{z,x} A_{,x} + G_l A^3 - E_l A^2 I_{yy,x} a_{z,x} - E_l I_{yy} A^2 a_{z,xx} \quad (15)$$

$$B_2 = G_l B_1 \quad (16)$$

the subscript $,x$ denotes differentiation with respect to x axis, and the coefficients C_i ($i=0,1,2,3,4$) are given in Appendix B. Moreover, the pertinent boundary conditions of the problem are given as

$$\alpha_1^z w(x) + \alpha_2^z R_z(x) = \alpha_3^z \quad (17a)$$

$$\beta_1^z \theta_y(x) + \beta_2^z M_y(x) = \beta_3^z \quad \text{at the beam ends } x=0,l \quad (17b)$$

where α_i^z, β_i^z ($i=1,2,3$) are given constants, while the angle of rotation θ_y and the stress resultants M_y, R_z at the beam ends $x=0,l$ are given as

$$\theta_y = -\frac{d^3 w}{dx^3} \frac{\alpha_z E_l}{B_2} \left(I_{yy} G_l A^2 + \alpha_z I_{yy} A N \right) - \frac{dw}{dx} - \frac{d^2 w}{dx^2} \frac{\alpha_z E_l}{B_2} \left(G_l A^2 I_{yy,x} + \alpha_z A I_{yy,x} N - 2\alpha_z I_{yy} A_{,x} N + 2I_{yy} A \alpha_{z,x} N \right) \quad (18a)$$

$$M_y = \frac{d^3 w}{dx^3} \frac{E_l^2 I_{yy}^2}{B_2} \left(a_z G_l A A_{,x} + a_z^2 A_{,x} N - G_l A^2 a_{z,x} - a_z A a_{z,x} N \right) + \frac{d^2 w}{dx^2} \frac{E_l}{B_2} \left(\alpha_z E_l G_l A I_{yy} I_{yy,x} A_{,x} + \alpha_z^2 E_l I_{yy} I_{yy,x} A_{,x} N - 2\alpha_z^2 E_l I_{yy}^2 A^{-1} A_{,x}^2 N - I_{yy} B_2 - \alpha_z I_{yy} (G_l A)^{-1} N B_2 - 2E_l I_{yy}^2 A a_{z,x}^2 N + 4a_z E_l I_{yy}^2 a_{z,x} A_{,x} N - E_l I_{yy} G_l A^2 a_{z,x} I_{yy,x} - a_z E_l I_{yy} A a_{z,x} I_{yy,x} N \right) \quad (18b)$$

$$R_z = -\frac{d^3 w}{dx^3} \frac{E_I A}{B_I} \left(I_{yy} G_I A^2 + a_z I_{yy} A N \right) + \frac{dw}{dx} N - \frac{d^2 w}{dx^2} \frac{E_I A}{B_I} \left(G_I A^2 I_{yy,x} + a_z A I_{yy,x} N - 2a_z I_{yy} A_{,x} N + 2I_{yy} A a_{z,x} N \right) \quad (18c)$$

Eqns. (17) describe the most general boundary conditions associated with the problem at hand and can include elastic support or restrain. It is apparent that all types of the conventional boundary conditions (clamped, simply supported, free or guided edge) can be derived from these equations by specifying appropriately the functions α_i^z and β_i^z (e.g. for a clamped edge it is $\alpha_1^z = \beta_1^z = 1$, $\alpha_2^z = \alpha_3^z = \beta_2^z = \beta_3^z = 0$).

Similarly, considering the beam in the x - y plane we obtain the boundary value problem of the beam transverse displacement $v = v(x)$ as

$$C_9 \frac{d^4 v}{dx^4} = C_8 \frac{d^3 v}{dx^3} + C_7 \frac{d^2 v}{dx^2} + C_6 \frac{dv}{dx} + C_5 \quad \text{inside the beam} \quad (19)$$

$$\alpha_1^y v(x) + \alpha_2^y R_y(x) = \alpha_3^y \quad (20a)$$

$$\beta_1^y \theta_z(x) + \beta_2^y M_z(x) = \beta_3^y \quad \text{at the beam ends } x = 0, l \quad (20b)$$

where α_i^y, β_i^y ($i = 1, 2, 3$) are given constants and the expressions of the angle of rotation θ_z and the stress resultants M_z, R_y at the beam ends $x = 0, l$ are given as

$$\theta_z = -\frac{d^3 v}{dx^3} \frac{\alpha_y E_I}{B_4} \left(I_{zz} G_I A^2 + \alpha_y I_{zz} A N \right) + \frac{dv}{dx} - \frac{d^2 v}{dx^2} \frac{\alpha_y E_I}{B_4} \left(G_I A^2 I_{zz,x} + \alpha_y A I_{zz,x} N - 2\alpha_y I_{zz} A_{,x} N + 2I_{zz} A \alpha_{y,x} N \right) \quad (21a)$$

$$\begin{aligned}
 M_z = & \frac{d^3 v}{dx^3} \frac{E_l I_{zz}^2}{B_4} \left(a_y G_l A A_{,x} + a_y^2 A_{,x} N - G_l A^2 a_{y,x} - a_y A a_{y,x} N \right) \\
 & + \frac{d^2 v}{dx^2} \frac{E_l I_{zz}}{B_4} \left(\alpha_y E_l G_l A I_{zz,x} A_{,x} + \alpha_y^2 E_l I_{zz,x} A_{,x} N - 2 E_l I_{zz} A a_{y,x}^2 N + \alpha_y (G_l A)^{-1} N B_4 \right. \\
 & \left. - 2 \alpha_y^2 E_l I_{zz} A^{-1} A_{,x}^2 N + 4 a_y E_l I_{zz} a_{y,x} A_{,x} N + B_4 - E_l G_l A^2 a_{y,x} I_{zz,x} - a_y E_l A a_{y,x} I_{zz,x} N \right)
 \end{aligned} \quad (21b)$$

$$\begin{aligned}
 R_y = & \frac{d^3 v}{dx^3} \frac{E_l A}{B_3} \left(I_{zz} G_l A^2 + a_y I_{zz} A N \right) + \frac{dv}{dx} N \\
 & + \frac{d^2 v}{dx^2} \frac{E_l A}{B_3} \left(G_l A^2 I_{zz,x} + a_y A I_{zz,x} N - 2 a_y I_{zz} A_{,x} N + 2 I_{zz} A a_{y,x} N \right)
 \end{aligned} \quad (21c)$$

the functions B_i ($i=3,4$) are given as

$$\begin{aligned}
 B_3 = & 2 a_y E_l I_{zz} A_{,x}^2 - a_y E_l A I_{zz,x} A_{,x} - a_y E_l I_{zz} A A_{,xx} + G_l A^3 \\
 & - 2 E_l I_{zz} A a_{y,x} A_{,x} + E_l A^2 I_{zz,x} a_{y,x} + E_l I_{zz} A^2 a_{y,xx}
 \end{aligned} \quad (22a)$$

$$B_4 = G_l B_3 \quad (22b)$$

and the coefficients C_i ($i=5,6,7,8,9$) are given in Appendix B.

In both of the aforementioned boundary value problems the axial force N inside the beam or at its boundary is obtained after substituting eqn.(2a) in eqn.(9a) as

$$N = E_l A \left[\frac{du}{dx} + \frac{I}{2} \left(\left(\frac{dw}{dx} \right)^2 + \left(\frac{dv}{dx} \right)^2 \right) \right] \quad (23)$$

where $u = u(x)$ is the beam axial displacement, which can be evaluated from the solution of the boundary value problem arising after substituting eqns (8), (23) in eqn.(12a) as

$$E_1 A \frac{d^2 u}{dx^2} + E_1 A_{,x} \frac{du}{dx} = -p_x - E_1 \left[\frac{A_{,x}}{2} \left(\left(\frac{dw}{dx} \right)^2 + \left(\frac{dv}{dx} \right)^2 \right) + A \left(\frac{dw}{dx} \frac{d^2 w}{dx^2} + \frac{dv}{dx} \frac{d^2 v}{dx^2} \right) \right]$$

inside the beam (24)

$$c_1 u(x) + c_2 N(x) = c_3$$

at the beam ends $x = 0, l$ (25)

where c_i ($i = 1, 2, 3$) are given constants.

The solution of the boundary value problems prescribed from eqns (14), (17a,b) and (19), (20a,b) presumes the evaluation of the shear deformation coefficients a_z , a_y corresponding to the principal centroidal system of axes Syz . These coefficients are established equating the approximate formula of the shear strain energy per unit length [23]

$$U_{appr.} = \frac{a_y Q_y^2}{2AG_1} + \frac{a_z Q_z^2}{2AG_1} \quad (26)$$

with the exact one given from

$$U_{exact} = \sum_{j=1}^K \frac{E_1}{E_j} \int_{\Omega_j} \frac{(\tau_{xz})_j^2 + (\tau_{xy})_j^2}{2G_1} d\Omega_j \quad (27)$$

and are obtained as [28]

$$a_y = \frac{l}{\kappa_y} = \frac{A}{E_1 \Delta^2} \sum_{j=1}^K \int_{\Omega_j} E_j \left((\nabla \Theta)_j - \mathbf{e} \right) \cdot \left((\nabla \Theta)_j - \mathbf{e} \right) d\Omega_j \quad (28a)$$

$$a_z = \frac{l}{\kappa_z} = \frac{A}{E_1 \Delta^2} \sum_{j=1}^K \int_{\Omega_j} E_j \left((\nabla \Phi)_j - \mathbf{d} \right) \cdot \left((\nabla \Phi)_j - \mathbf{d} \right) d\Omega_j \quad (28b)$$

where $(\tau_{xz}), (\tau_{xy})$ are the transverse (direct) shear stress components,

$(\nabla) \equiv \mathbf{i}_y (\partial/\partial y) + \mathbf{i}_z (\partial/\partial z)$ is a symbolic vector with $\mathbf{i}_y, \mathbf{i}_z$ the unit vectors along y

and z axes, respectively, Δ is given from

$$\Delta = 2(I + \nu)I_{yy}I_{zz} \quad (29)$$

ν is the Poisson ratio of the cross section material, \mathbf{e} and \mathbf{d} are vectors defined as

$$\mathbf{e} = e_y \mathbf{i}_y + e_z \mathbf{i}_z = \left(\nu I_{yy} \frac{y^2 - z^2}{2} \right) \mathbf{i}_y + (\nu I_{yy} yz) \mathbf{i}_z \quad (30a)$$

$$\mathbf{d} = d_y \mathbf{i}_y + d_z \mathbf{i}_z = (\nu I_{zz} yz) \mathbf{i}_y - \left(\nu I_{zz} \frac{y^2 - z^2}{2} \right) \mathbf{i}_z \quad (30b)$$

and $\Theta(y, z)_j, \Phi(y, z)_j$ are stress functions, which are evaluated from the solution of the following Neumann type boundary value problems [28]

$$\nabla^2 \Theta_j = -2I_{yy}y \quad \text{in } \Omega_j \quad (j = 1, 2, \dots, K) \quad (31a)$$

$$E_j \left(\frac{\partial \Theta}{\partial n} \right)_j - E_i \left(\frac{\partial \Theta}{\partial n} \right)_i = (E_j - E_i) \mathbf{n} \cdot \mathbf{e} \quad \text{on } \Gamma_j \quad (j = 1, 2, \dots, K) \quad (31b)$$

$$\nabla^2 \Phi_j = -2I_{zz}z \quad \text{in } \Omega_j \quad (j = 1, 2, \dots, K) \quad (32a)$$

$$E_j \left(\frac{\partial \Phi}{\partial n} \right)_j - E_i \left(\frac{\partial \Phi}{\partial n} \right)_i = (E_j - E_i) \mathbf{n} \cdot \mathbf{d} \quad \text{on } \Gamma_j \quad (j = 1, 2, \dots, K) \quad (32b)$$

where E_i is the modulus of elasticity of the Ω_i region at the common part of the boundaries of Ω_j and Ω_i regions, or $E_i = 0$ at the free part of the boundary of Ω_j region, while $(\partial/\partial n)_j \equiv n_y (\partial/\partial y)_j + n_z (\partial/\partial z)_j$ denotes the directional derivative

normal to the boundary Γ_j . The vector \mathbf{n} normal to the boundary Γ_j is positive if it points to the exterior of the Ω_j region. It is worth here noting that the normal derivatives across the interior boundaries vary discontinuously. In the case of negligible shear deformations $a_z = a_y = 0$. The boundary conditions (31b), (32b) have been derived from the physical consideration that the traction vector in the direction of the normal vector \mathbf{n} on the interfaces separating the j and i different materials are equal in magnitude and opposite in direction, while it vanishes on the free surface of the beam.

3. Integral Representations – Numerical Solution

According to the precedent statement of the problem, the nonlinear analysis of a beam of variable composite cross section including shear deformation reduces in establishing the transverse displacements $w = w(x)$, $v = v(x)$ having continuous derivatives up to the fourth order with respect to x , the axial displacement $u = u(x)$ having continuous derivatives up to the second order with respect to x and the stress functions $\Phi(y, z)_j$, $\Theta(y, z)_j$ having continuous partial derivatives up to the second order with respect to y, z .

3.1. For the transverse displacements w, v .

The numerical solution of the boundary value problems described by eqns (14), (17a,b) and (19), (20a,b) is similar. For this reason, in the following we will analyze the solution of the problem of eqns (14), (17a,b) noting any alteration or addition for the problem of eqns (19), (20a,b). Eqn (14) is solved using the Analog Equation

Method [17], as it has been employed for the beam equation including axial forces in Sapountzakis and Tsiatas [5].

Thus, if w is the sought solution of the boundary value problem described by eqns (14) and (17a,b), differentiating this function four times yields

$$\frac{d^4 w}{dx^4} = q_z(x) \quad (33)$$

Eqn (33) indicates that the solution of the original problem can be obtained as the deflection of a constant cross section beam with unit flexural rigidity and infinite shear rigidity subjected to a flexural fictitious load $q_z(x)$ under the same boundary conditions. The fictitious load is unknown and is established using BEM as follows.

The integral representation of the deflection w can be written as

$$w(x) = \int_0^l q_z A_4(r) dx - \left[A_4(r) \frac{d^3 w}{dx^3} + A_3(r) \frac{d^2 w}{dx^2} + A_2(r) \frac{dw}{dx} + A_1(r) w \right]_0^l \quad (34)$$

where the kernels $A_i(r)$, ($i=1,2,3,4$) are given as

$$A_1(r) = -\frac{l}{2} \operatorname{sgn} \rho \quad (35a)$$

$$A_2(r) = -\frac{l}{2} l(1 - |\rho|) \quad (35b)$$

$$A_3(r) = -\frac{l^2}{4} |\rho| (|\rho| - 2) \operatorname{sgn} \rho \quad (35c)$$

$$A_4(r) = \frac{l^3}{12} (2 + |\rho|^3 - 3|\rho|^2) \quad (35d)$$

where $\rho = r/l$, $r = x - \xi$, x , ξ points of the beam. Differentiating eqn (34) results in the integral representations of the derivatives of the deflection w as

$$\frac{dw(x)}{dx} = \int_0^l q_z A_3(r) dx - \left[A_3(r) \frac{d^3 w}{dx^3} + A_2(r) \frac{d^2 w}{dx^2} + A_1(r) \frac{dw}{dx} \right]_0^l \quad (36a)$$

$$\frac{d^2 w(x)}{dx^2} = \int_0^l q_z A_2(r) dx - \left[A_2(r) \frac{d^3 w}{dx^3} + A_1(r) \frac{d^2 w}{dx^2} \right]_0^l \quad (36b)$$

$$\frac{d^3 w(x)}{dx^3} = \int_0^l q_z A_1(r) dx - \left[A_1(r) \frac{d^3 w}{dx^3} \right]_0^l \quad (36c)$$

The integral representations (34), (36a) written for the beam ends 0 , l together with the boundary conditions (17a,b) can be employed to express the unknown boundary quantities w , w' , w'' and w''' in terms of q_z . Elimination of these quantities from the discretized counterpart of eqn (34) applied to all nodal points in the interior of the beam yields

$$\{W\} = [B_z] \{q_z\} \quad (37)$$

where $[B_z]$ is an $L \times L$ matrix, with L the number of the nodal points along the beam axis and $\{q_z\}$ is a vector including the L unknown nodal values of the fictitious load. Moreover, the discretized counterpart of eqns (36a,b,c) when applied to all nodal points in the interior of the beam, after elimination of the boundary quantities yields

$$\{W'\} = [B'_z] \{q_z\} \quad \{W''\} = [B''_z] \{q_z\} \quad \{W'''\} = [B'''_z] \{q_z\} \quad (38a,b,c)$$

where $[B'_z]$, $[B''_z]$, $[B'''_z]$ are known $L \times L$ coefficient matrices. Note that eqns (37) and (38a,b,c) are valid for homogeneous boundary conditions ($a_3^z = \beta_3^z = 0$). For non-homogeneous boundary conditions, an additive constant vector will appear in the right hand side of these equations.

Finally, applying eqn (14) to the L nodal points in the interior of the beam we obtain the following linear system of equations with respect to q_z

$$[[C_4] - [C_3][B'''_z] - [C_2][B''_z] - [C_1][B'_z]]\{q_z\} = \{C_0\} \quad (39)$$

while following the same procedure for the boundary value problem (19), (20a,b) we obtain the corresponding linear system of equations with respect to q_y as

$$[[C_9] - [C_8][B'''_y] - [C_7][B''_y] - [C_6][B'_y]]\{q_y\} = \{C_5\} \quad (40)$$

where $[C_i]$, ($i=1,2,3,4,6,7,8,9$) are diagonal $L \times L$ matrices whose elements are given from the coefficients C_i ($i=1,2,3,4,6,7,8,9$) given in Appendix B, at the corresponding nodal point in the interior of the beam; $\{C_0\}$, $\{C_5\}$ are vectors with L elements including the values of the coefficients C_0 , C_5 given in Appendix B, at the L nodal points. It is worth here noting that the values of the derivatives of the cross section area, of the moments of inertia, of the shear deformation coefficients etc. in the aforementioned coefficients result after approximating the corresponding derivatives with appropriate central, forward or backward finite differences.

3.2 For the axial displacement u .

Let u be the sought solution of the boundary value problem described by eqns (24) and (25). Differentiating this function two times yields

$$\frac{d^2 u}{dx^2} = q_x(x) \quad (41)$$

Eqn (41) indicates that the solution of the original problem can be obtained as the axial displacement of a beam with unit axial rigidity subjected to a flexural fictitious load $q_x(x)$ under the same boundary conditions. The fictitious load is unknown.

The solution of eqn (41) is given in integral form as

$$u(x) = -\int_0^l u^* q_x ds + \left[u^* \frac{du}{dx} - \frac{du^*}{dx} u \right]_0^l \quad (42)$$

where u^* is the fundamental solution, which is given as

$$u^* = \frac{l}{2} |r| \quad (43)$$

Following the same procedure as in 3.1, the discretized counterpart of eqn (42) and its first derivative with respect to x , when applied to all nodal points in the interior of the beam yields

$$\{U\} = [B_x] \{q_x\} \quad (44a)$$

$$\{U'\} = [B'_x] \{q_x\} \quad (44b)$$

where $[B_x]$, $[B'_x]$ are known matrices with dimensions $L \times L$, similar with those mentioned before for the deflection w and the following system of equations with respect to q_x , q_y , q_z is obtained

$$\begin{aligned} [D_x'']\{q_x\} + [D_x'] [B'_x]\{q_x\} = -\{p_x\} \\ - [D_x'] \left([[B'_z]\{q_z\}]_{dg.} [B'_z]\{q_z\} + [[B'_y]\{q_y\}]_{dg.} [B'_y]\{q_y\} \right) \\ - [D_x''] \left([[B''_z]\{q_z\}]_{dg.} [B'_z]\{q_z\} + [[B''_y]\{q_y\}]_{dg.} [B'_y]\{q_y\} \right) \end{aligned} \quad (45)$$

where the symbol $[]_{dg.}$ indicates a diagonal matrix with the elements of the included column matrix, $\{q_x\}$, $\{q_y\}$, $\{p_x\}$ are vectors with L elements, similar with those mentioned before for the deflection w , while the axial force at the neutral axis of the beam can be expressed as follows

$$\{N\} = [D_x''] [B'_x]\{q_x\} + \frac{I}{2} [D_x''] \left([[B'_z]\{q_z\}]_{dg.} [B'_z]\{q_z\} + [[B'_y]\{q_y\}]_{dg.} [B'_y]\{q_y\} \right) \quad (46)$$

In eqns. (45), (46) $[D_x'']$, $[D_x']$ are diagonal $L \times L$ matrices whose elements are given from

$$(D_x'')_{ii} = E_I A_i \quad (47a)$$

$$(D_x')_{ii} = \frac{E_I}{2} \left(\frac{dA}{dx} \right)_i \quad (47b)$$

at the L nodal points in the interior of the beam. Eqns. (39), (40), (45) and (46) constitute a nonlinear coupled system of equations with respect to q_x , q_y , q_z and N

quantities. The solution of this system is accomplished iteratively by employing the two term acceleration method [29, 30].

3.3 For the stress functions $(\Phi(y, z))_j$ and $(\Theta(y, z))_j$.

The evaluation of the stress functions $(\Phi(y, z))_j$ and $(\Theta(y, z))_j$ is accomplished using BEM as this is presented in Mokos and Sapountzakis [28].

Moreover, since the torsionless bending problem of beams is solved by the BEM, the domain integrals for the evaluation of the area, the bending moments of inertia and the shear deformation coefficients (eqns 11a,b,c, 28a,b) have to be converted to boundary line integrals, in order to maintain the pure boundary character of the method. This can be achieved using integration by parts, the Gauss theorem and the Green identity. Thus, the moments of inertia and the cross section area can be written as

$$I_{yy} = \frac{I}{E_I} \sum_{j=1}^K \int_{\Gamma_j} (E_j - E_i) (yz^2 \cos \beta) ds \quad (48a)$$

$$I_{zz} = \frac{I}{E_I} \sum_{j=1}^K \int_{\Gamma_j} (E_j - E_i) (zy^2 \sin \beta) ds \quad (48b)$$

$$A = \frac{I}{2G_I} \sum_{j=1}^K \int_{\Gamma_j} (G_j - G_i) (y \cos \beta + z \sin \beta) ds \quad (48c)$$

while the shear deformation coefficients a_y and a_z are obtained from the relations

$$a_y = \frac{A}{E_I \Delta^2} \left((4\nu + 2) I_{yy} I_{\Theta y} + \frac{I}{4} \nu^2 I_{yy}^2 I_{ed} - I_{\Theta e} \right) \quad (49a)$$

$$a_z = \frac{A}{E_I \Delta^2} \left((4\nu + 2) I_{zz} I_{\Phi z} + \frac{1}{4} \nu^2 I_{zz}^2 I_{ed} - I_{\Phi d} \right) \quad (49b)$$

where

$$I_{\Theta e} = \sum_{j=1}^K \int_{\Gamma_j} (E_j - E_i) (\Theta)_j (\mathbf{n} \cdot \mathbf{e}) ds \quad (50a)$$

$$I_{\Phi d} = \sum_{j=1}^K \int_{\Gamma_j} (E_j - E_i) (\Phi)_j (\mathbf{n} \cdot \mathbf{d}) ds \quad (50b)$$

$$I_{ed} = \sum_{j=1}^K \int_{\Gamma_j} (E_j - E_i) \left(y^4 z \sin \beta + z^4 y \cos \beta + \frac{2}{3} y^2 z^3 \sin \beta \right) ds \quad (50c)$$

$$I_{\Theta y} = \frac{1}{6} \sum_{j=1}^K \int_{\Gamma_j} (E_j - E_i) \left[-2 I_{yy} y^4 z \sin \beta + \left(3 (\Theta)_j \cos \beta - y (\mathbf{n} \cdot \mathbf{e}) \right) y^2 \right] ds \quad (50d)$$

$$I_{\Phi z} = \frac{1}{6} \sum_{j=1}^K \int_{\Gamma_j} (E_j - E_i) \left[-2 I_{zz} z^4 y \cos \beta + \left(3 (\Phi)_j \sin \beta - z (\mathbf{n} \cdot \mathbf{d}) \right) z^2 \right] ds \quad (50e)$$

4. Numerical examples

On the basis of the analytical and numerical procedures presented in the previous sections, a computer program has been written and representative examples have been studied to demonstrate the efficiency, the accuracy and the range of applications of the developed method.

Example 1

A cantilever beam of a hollow rectangular composite cross section $b \times h$ of constant width $b = 0.12m$, consisting of two columns ($E_{ref} = E_I = 2.1 \times 10^8 kN / m^2$) of $t_I = 4mm$ thickness and two plates ($E_2 = 2.1 \times 10^7 kN / m^2$) of $t_2 = 20mm$ thickness, having linearly varying height of maximum value at $x = 0$: $h_0 = 0.20m$ loaded both

axially by a distributed and a concentrated load and transversely by a distributed load, as shown in Fig. 3 has been studied. In Fig. 4 the transverse deflection v and in Fig. 5 the axial displacement u along the axis of the beam with $h_l = 0.12m$ (29 boundary elements, 58 nodal points) are presented as compared with those obtained from a linear analysis and from a FEM solution using 60-beam elements [31] taking into account or ignoring shear deformation effect, while in Table 1 the convergence of the results for various numbers of boundary elements for both the transverse deflection v and the axial displacement u at the mid and end points of the beam are presented. From both figures the coincidence of the results between BEM and FEM when shear deformation effect is ignored is observed, while the small deviation of the results of these methods when shear deformation effect is included is attributed to the employed cubic shape functions in the FEM solution for the approximation of the deformation of the beam [32]. From Table 1 the convergence of the proposed method for few boundary elements is observed, while from Fig.5 it comes up that even for tensile axial loading the results of the nonlinear analysis show a remarkable beam shortening coming from the intense transverse load. Moreover, the increment of both the deflection and the axial displacement due to the influence of the shear deformation effect are observed. In Table 2 the influence of the shear deformation effect is demonstrated by presenting the discrepancy of the transverse deflection v and the axial displacement u at the $l/6$ of the beam for various beam heights h_l at $x=l$. Finally, in Table 3 the same displacement components of the beam with $h_l = 0.08m$ are presented as compared with those obtained from a linear analysis and from a FEM solution using 60-beam elements [31] taking into account or ignoring shear deformation effect. The conclusions already drawn are verified.

Example 2

A clamped beam of a composite cross section consisting of an I-type section ($E_{ref} = E_I = 2.0 \times 10^8 \text{ kN/m}^2$) of constant thickness $t = 4 \text{ mm}$ in contact with two rectangular stiffeners ($E_2 = 3.0 \times 10^7 \text{ kN/m}^2$) of thickness $d = 4 \text{ cm}$, having constant width $b = 0.14 \text{ m}$ and linearly varying total height of maximum value at $x = 0, x = l$: $h_0 = h_l = 0.18 \text{ m}$ loaded by a concentrated load P_z at its midspan, as shown in Fig. 6 has been studied (79 boundary elements, 158 nodal points). In Fig. 7 the transverse deflection w at $x = l/2$ of the beam with total height at midspan $h_{mid} = 0.12 \text{ m}$, for various values of the concentrated load are presented as compared with those obtained from a linear analysis and from a FEM solution using 160-beam elements [31] taking into account or ignoring shear deformation effect. Moreover, in Table 4 the influence of the shear deformation effect is demonstrated by presenting the discrepancy of the transverse deflection w at $x = l/2$ of the beam for various beam heights h_{mid} at the midspan. Finally, in Table 5 the same displacement component of the beam with midspan height $h_{mid} = 0.14 \text{ m}$ is presented as compared with those obtained from a linear analysis and from a FEM solution using 160-beam elements [31] taking into account or ignoring shear deformation effect, while in Table 6 the convergence of the results for various numbers of boundary elements for the transverse deflection w at $x = l/2$ of the beam with total height at midspan $h_{mid} = 0.12 \text{ m}$ and for $P_z = 1500 \text{ kN}$ are presented. From the aforementioned figure and tables the conclusions drawn from example 1 concerning the comparison of the results between BE and FE methods, the influence of the shear deformation effect, the discrepancy of the results arising from the linear or nonlinear analysis and the convergence of the method for few boundary elements are once more verified.

Example 3

A beam clamped at $x = 0$ and simply supported at $x = l$ of a composite cross section consisting of an I-type section ($E_I = 2.1 \times 10^8 \text{ kN/m}^2$) of constant thickness $t = 4 \text{ mm}$ totally encased in a rectangular matrix ($E_{ref} = E_2 = 3.0 \times 10^7 \text{ kN/m}^2$) of constant width $b = 0.10 \text{ m}$ and linearly varying height of maximum value at $x = 0$: $h_0 = 0.20 \text{ m}$ loaded both axially by a distributed and a concentrated load and transversely by a concentrated load, as shown in Fig. 8 has been studied (49 boundary elements, 98 nodal points). In Fig. 9 the transverse deflection w and in Fig. 10 the axial displacement u along the axis of the beam with $h_l = 0.10 \text{ m}$ are presented as compared with those obtained from a linear analysis and from a FEM solution using 100-beam elements [31] taking into account or ignoring shear deformation effect. In Table 7 the influence of the shear deformation effect is demonstrated by presenting the discrepancy of the transverse deflection w and the axial displacement u at the $l/10$ of the beam for various beam heights h_l at $x = l$. Finally, in Table 8 the same displacement components of the beam with $h_l = 0.15 \text{ m}$ are presented as compared with those obtained from a linear analysis and from a FEM solution using 100-beam elements [31] taking into account or ignoring shear deformation effect. As it was also notified in example 1 even for tensile axial loading the results of the nonlinear analysis show a remarkable beam shortening coming from the intense transverse load, which is increased with the cross section's slope. Moreover, the conclusions already drawn from the previous examples concerning the comparison of the results between BE and FE methods, the influence of the shear deformation effect and the discrepancy of the results arising from the linear or nonlinear analysis are once more verified.

5. Concluding remarks

In this paper a BEM-based method, is employed for the nonlinear analysis of a composite Timoshenko beam with variable cross section undergoing moderate large deflections under general boundary conditions. Five boundary value problems are formulated with respect to the transverse displacements, the axial displacement and to two stress functions and solved using the Analog Equation Method. The evaluation of the shear deformation coefficients is accomplished from the aforementioned stress functions using only boundary integration. The main conclusions that can be drawn from this investigation are

- a. The numerical technique presented in this investigation is well suited for computer aided analysis for beams of arbitrary doubly symmetric composite variable cross section.
- b. The significant influence of geometrical nonlinear analysis in beam elements subjected in intense transverse loading is verified.
- c. The discrepancy between the results of the linear and the nonlinear analysis demonstrates the significant influence of the axial loading.
- d. The convergence of the results is achieved employing few boundary elements. Moreover, coincidence of the results arising from BE and FE methods is observed when shear deformation effect is ignored, while the small deviation of the results of these methods when shear deformation effect is included is attributed to the employed cubic shape functions in the FEM solution for the approximation of the deformation of the beam.
- e. The increment of all the deflections and the axial displacements due to the influence of the shear deformation effect is observed.

- f. The developed procedure retains the advantages of a BEM solution over a pure domain discretization method since it requires only boundary discretization.

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APPENDIX A

Angle of rotation due to bending θ_y and stress resultants M_y , R_z in the interior of the beam.

$$\begin{aligned} \theta_y = & -\frac{d^3 w}{dx^3} \frac{\alpha_z E_I}{B_2} \left(I_{yy} G_I A^2 + \alpha_z I_{yy} A N \right) \\ & -\frac{d^2 w}{dx^2} \frac{\alpha_z E_I}{B_2} \left(G_I A^2 I_{yy,x} + \alpha_z A I_{yy,x} N - 2\alpha_z I_{yy} A p_x - 2\alpha_z I_{yy} A_{,x} N + 2I_{yy} A \alpha_{z,x} N \right) \\ & +\frac{dw}{dx} \frac{E_I}{B_2} \left(\alpha_z^2 A I_{yy,x} p_x + \alpha_z^2 I_{yy} A p_{x,x} + 2\alpha_z I_{yy} A \alpha_{z,x} p_x - E_I^{-1} B_2 - 2\alpha_z^2 I_{yy} A_{,x} p_x \right) \\ & -\frac{\alpha_z E_I}{B_2} \left(\alpha_z A I_{yy,x} p_z + \alpha_z I_{yy} A p_{z,x} - 2\alpha_z I_{yy} A_{,x} p_z - E_I^{-1} G_I A^2 m_y + 2I_{yy} A \alpha_{z,x} p_z \right) \end{aligned} \quad (A1)$$

$$\begin{aligned} M_y = & \frac{d^3 w}{dx^3} \frac{E_I^2 I_{yy}^2}{B_2} \left(a_z G_I A A_{,x} + a_z^2 A_{,x} N - G_I A^2 a_{z,x} - a_z A a_{z,x} N \right) \\ & +\frac{d^2 w}{dx^2} \frac{E_I}{B_2} \left(\alpha_z E_I G_I A I_{yy} I_{yy,x} A_{,x} + \alpha_z^2 E_I I_{yy} I_{yy,x} A_{,x} N - 2\alpha_z^2 E_I I_{yy}^2 A_{,x} p_x - I_{yy} B_2 \right. \\ & \quad - 2\alpha_z^2 E_I I_{yy}^2 A^{-1} A_{,x}^2 N - \alpha_z I_{yy} (G_I A)^{-1} N B_2 - 2E_I I_{yy}^2 A a_{z,x}^2 N + 4a_z E_I I_{yy}^2 a_{z,x} A_{,x} N \\ & \quad \left. - E_I I_{yy} G_I A^2 a_{z,x} I_{yy,x} + 2a_z E_I I_{yy}^2 A a_{z,x} p_x - a_z E_I I_{yy} A a_{z,x} I_{yy,x} N \right) \\ & +\frac{dw}{dx} \frac{E_I}{B_2} \left(2\alpha_z^2 E_I I_{yy}^2 A^{-1} A_{,x}^2 p_x - \alpha_z^2 E_I I_{yy} I_{yy,x} A_{,x} p_x - \alpha_z^2 E_I I_{yy}^2 A_{,x} p_{x,x} \right. \\ & \quad - 4\alpha_z E_I I_{yy}^2 \alpha_{z,x} A_{,x} p_x + \alpha_z I_{yy} (G_I A)^{-1} p_x B_2 + \alpha_z E_I I_{yy} A \alpha_{z,x} I_{yy,x} p_x \\ & \quad \left. + 2E_I I_{yy}^2 A a_{z,x}^2 p_x + \alpha_z E_I I_{yy}^2 A \alpha_{z,x} p_{x,x} \right) \\ & +\frac{E_I}{B_2} \left(\alpha_z^2 E_I I_{yy} I_{yy,x} A_{,x} p_z + \alpha_z^2 E_I I_{yy}^2 A_{,x} p_{z,x} - 2\alpha_z^2 E_I I_{yy}^2 A^{-1} A_{,x}^2 p_z - \alpha_z I_{yy} G_I A A_{,x} m_y \right. \\ & \quad - \alpha_z I_{yy} (G_I A)^{-1} p_z B_2 - \alpha_z E_I I_{yy}^2 A \alpha_{z,x} p_{z,x} + I_{yy} G_I A^2 \alpha_{z,x} m_y - 2E_I I_{yy}^2 A \alpha_{z,x}^2 p_z \\ & \quad \left. + 4\alpha_z E_I I_{yy}^2 \alpha_{z,x} A_{,x} p_z - \alpha_z E_I I_{yy} A \alpha_{z,x} I_{yy,x} p_z \right) \end{aligned} \quad (A2)$$

$$\begin{aligned} R_z = & -\frac{d^3 w}{dx^3} \frac{E_I A}{B_1} \left(I_{yy} G_I A^2 + a_z I_{yy} A N \right) \\ & -\frac{d^2 w}{dx^2} \frac{E_I A}{B_1} \left(G_I A^2 I_{yy,x} + a_z A I_{yy,x} N - 2a_z I_{yy} A p_x - 2a_z I_{yy} A_{,x} N + 2I_{yy} A a_{z,x} N \right) \\ & +\frac{dw}{dx} \frac{E_I A}{B_1} \left(a_z A I_{yy,x} p_x + a_z I_{yy} A p_{x,x} - 2a_z I_{yy} A_{,x} p_x + 2I_{yy} A a_{z,x} p_x + (E_I A)^{-1} N B_1 \right) \\ & -\frac{E_I A}{B_1} \left(a_z A I_{yy,x} p_z + a_z I_{yy} A p_{z,x} - 2a_z I_{yy} A_{,x} p_z - E_I^{-1} G_I A^2 m_y + 2I_{yy} A a_{z,x} p_z \right) \end{aligned} \quad (A3)$$

APPENDIX B

Coefficients C_i ($i=0,1,2,3,4,5,6,7,8,9$) of the governing differential equations concerning the nonlinear analysis of a Composite Beam of Variable Cross Section Subjected in Variable Axial Loading including Shear Deformation Effect (eqns. 15, 19)

$$\begin{aligned}
C_0 = & m_{y,x} + p_z - 4 \frac{E_l I_{yy}}{G_l A} a_{z,xx} p_z + 2a_z \frac{E_l^2}{G_l^2 A^2} a_{z,x} I_{yy,x}^2 p_{z,x} - 3a_z^2 \frac{E_l^2 I_{yy}}{G_l^2 A^4} I_{yy,x} A_{,x} A_{,xx} p_z \\
& - 6 \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} a_{z,x}^2 A_{,x} p_{z,x} + 3a_z \frac{E_l I_{yy}}{G_l A^2} A_{,x} p_{z,x} + 3 \frac{E_l^2 I_{yy}^2}{G_l^2 A^2} a_{z,x} a_{z,xx} p_{z,x} - \frac{E_l}{G_l A} a_{z,x} I_{yy,x} m_{y,x} \\
& + a_z \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,xx} I_{yy,xx} p_z - a_z \frac{E_l}{G_l A} I_{yy,xx} p_z - 2 \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,x}^2 I_{yy,xx} p_z + \frac{E_l}{G_l A} a_{z,x} I_{yy,xx} m_y \\
& + 3 \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,x} a_{z,xx} I_{yy,x} p_{z,x} + 4a_z \frac{E_l I_{yy}}{G_l A^2} A_{,xx} p_z - \frac{E_l I_{yy}}{G_l A} a_{z,xx} m_{y,x} + 3a_z^2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^4} A_{,xx}^2 p_z \\
& + 3 \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,x}^2 I_{yy,x} p_z - 2a_z \frac{E_l}{G_l A} I_{yy,x} p_{z,x} - 6a_z \frac{E_l I_{yy}}{G_l A^4} A_{,x}^3 m_y - a_z \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,xxx} I_{yy,x} p_z \\
& + 5a_z \frac{E_l}{G_l A^2} I_{yy,x} A_{,x} p_z + 4 \frac{E_l^2}{G_l^2 A^2} a_{z,x}^2 I_{yy,x}^2 p_z + 2 \frac{E_l}{G_l A} a_{z,xx} I_{yy,x} m_y + 8 \frac{E_l I_{yy}}{G_l A^2} a_{z,x} A_{,x} p_z \\
& + 6a_z \frac{E_l^2 I_{yy}}{G_l^2 A^4} a_{z,x} I_{yy,x} A_{,x}^2 p_z - 3 \frac{E_l I_{yy}}{G_l A} a_{z,x} p_{z,x} - 3 \frac{E_l I_{yy}}{G_l A^2} a_{z,x} A_{,xx} m_y - 5 \frac{E_l}{G_l A} a_{z,x} I_{yy,x} p_z \\
& + 2 \frac{E_l I_{yy}}{G_l A^2} a_{z,x} A_{,x} m_{y,x} + a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^2} a_{z,xx} p_{z,xx} - 2a_z \frac{E_l}{G_l A^2} I_{yy,x} A_{,xx} m_y - a_z \frac{E_l I_{yy}}{G_l A^2} A_{,xxx} m_y \\
& + 2a_z^2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^4} A_{,x}^2 p_{z,xx} - a_z^2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} A_{,xx} p_{z,xx} + 2a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} a_{z,x} A_{,xxx} p_z - a_z \frac{E_l I_{yy}}{G_l A} p_{z,xx} \\
& - 6a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^4} a_{z,x} A_{,x} A_{,xx} p_z + 2a_z \frac{E_l^2 I_{yy}}{G_l^2 A^3} a_{z,x} I_{yy,xx} A_{,x} p_z - 4a_z \frac{E_l^2}{G_l^2 A^3} a_{z,x} I_{yy,x}^2 A_{,x} p_z \\
& - a_z \frac{E_l}{G_l A^2} I_{yy,xx} A_{,x} m_y + \frac{E_l I_{yy}}{G_l A} a_{z,xxx} m_y - 2a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} a_{z,x} A_{,x} p_{z,xx} - 2a_z \frac{E_l I_{yy}}{G_l A^3} A_{,x}^2 m_{y,x} \\
& + 6 \frac{E_l I_{yy}}{G_l A^3} a_{z,x} A_{,x}^2 m_y - 8a_z \frac{E_l I_{yy}}{G_l A^3} A_{,x}^2 p_z - 3 \frac{E_l I_{yy}}{G_l A^2} a_{z,xx} A_{,x} m_y + 2a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} a_{z,xxx} A_{,x} p_z \\
& + a_z \frac{E_l I_{yy}}{G_l A^2} A_{,xx} m_{y,x} + 3 \frac{E_l^2 I_{yy}^2}{G_l^2 A^2} a_{z,xx}^2 p_z + 6 \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} a_{z,x}^2 A_{,xx} p_z + 6a_z \frac{E_l I_{yy}}{G_l A^3} A_{,x} A_{,xx} m_y
\end{aligned}$$

$$\begin{aligned}
 & -6 \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} a_{z,x} a_{z,xx} A_{,x} p_z - 6 \frac{E_l^2 I_{yy}}{G_l^2 A^3} a_{z,x}^2 I_{yy,x} A_{,x} p_z - 6 a_z \frac{E_l^2 I_{yy}}{G_l^2 A^3} a_{z,x} I_{yy,x} A_{,x} p_{z,x} \\
 & - a_z^2 \frac{E_l^2 I_{yy}}{G_l^2 A^3} I_{yy,x} A_{,x} p_{z,xx} + a_z^2 \frac{E_l^2 I_{yy}}{G_l^2 A^3} I_{yy,xx} A_{,x} p_{z,x} - 6 a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} a_{z,xx} A_{,xx} p_z \\
 & - a_z \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,x} I_{yy,xx} p_{z,x} - 2 a_z^2 \frac{E_l^2}{G_l^2 A^3} I_{yy,x}^2 A_{,x} p_{z,x} - a_z^2 \frac{E_l^2 I_{yy}}{G_l^2 A^3} I_{yy,x} A_{,xx} p_z \\
 & - 2 a_z \frac{E_l^2}{G_l^2 A^2} a_{z,xx} I_{yy,x}^2 p_z + a_z^2 \frac{E_l^2 I_{yy}}{G_l^2 A^3} I_{yy,x} A_{,xxx} p_z - 3 a_z^2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^4} A_{,x} A_{,xx} p_{z,x} \\
 & + 2 a_z^2 \frac{E_l^2}{G_l^2 A^3} I_{yy,x}^2 A_{,xx} p_z + 6 a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^4} a_{z,xx} A_{,x}^2 p_z + 6 a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^4} a_{z,x} A_{,x}^2 p_{z,x} \\
 & + 4 a_z \frac{E_l}{G_l A^3} I_{yy,x} A_{,x}^2 m_y + 3 a_z^2 \frac{E_l^2 I_{yy}}{G_l^2 A^4} I_{yy,x} A_{,x}^2 p_{z,x} + a_z \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,x} I_{yy,x} p_{z,xx} \\
 & + a_z \frac{E_l}{G_l A^2} I_{yy,x} A_{,x} m_{y,x} - 4 \frac{E_l}{G_l A^2} a_{z,x} I_{yy,x} A_{,x} m_y - 2 a_z^2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^4} A_{,x} A_{,xxx} p_z \\
 & + a_z^2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} A_{,xxx} p_{z,x} - 2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^2} a_{z,x} a_{z,xxx} p_z
 \end{aligned} \tag{B0}$$

$$\begin{aligned}
 C_l = & -p_x - 2 a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} a_{z,x} A_{,xxx} p_x - 2 a_z^2 \frac{E_l^2}{G_l^2 A^3} I_{yy,x}^2 A_{,xx} p_x + 2 a_z^2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^4} A_{,x} A_{,xxx} p_x \\
 & - 8 \frac{E_l I_{yy}}{G_l A^2} a_{z,x} A_{,x} p_x - 3 a_z \frac{E_l I_{yy}}{G_l A^2} A_{,x} p_{x,x} - 5 a_z \frac{E_l}{G_l A^2} I_{yy,x} A_{,x} p_x + 6 \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} a_{z,x}^2 A_{,x} p_{x,x} \\
 & + 2 a_z \frac{E_l}{G_l A} I_{yy,x} p_{x,x} + 5 \frac{E_l}{G_l A} a_{z,x} I_{yy,x} p_x - 4 \frac{E_l^2}{G_l^2 A^2} a_{z,x}^2 I_{yy,x}^2 p_x - 3 \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,x}^2 I_{yy,x} p_{x,x}
 \end{aligned}$$

$$\begin{aligned}
 & + 2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^2} a_{z,x} a_{z,xxx} p_x + a_z \frac{E_l I_{yy}}{G_l A} p_{x,xx} - 6 a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^4} a_{z,xx} A_{,x}^2 p_x - a_z^2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} A_{,xxx} p_{x,x} \\
 & - 3 \frac{E_l^2 I_{yy}^2}{G_l^2 A^2} a_{z,xx}^2 p_x - 2 a_z^2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^4} A_{,x}^2 p_{x,xx} + a_z^2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} A_{,xx} p_{x,xx} - a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^2} a_{z,xx} p_{x,xx} \\
 & + 2 \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,x}^2 I_{yy,xx} p_x - 2 a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} a_{z,xxx} A_{,x} p_x + a_z \frac{E_l}{G_l A} I_{yy,xx} p_x + 3 \frac{E_l I_{yy}}{G_l A} a_{z,x} p_{x,x} \\
 & + 4 \frac{E_l I_{yy}}{G_l A} a_{z,xx} p_x - 6 \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} a_{z,x}^2 A_{,xx} p_x + a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^2} a_{z,xxx} p_{x,x} - 3 \frac{E_l^2 I_{yy}^2}{G_l^2 A^2} a_{z,x} a_{z,xx} p_{x,x} \\
 & - 3 a_z^2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^4} A_{,xx}^2 p_x - 4 a_z \frac{E_l I_{yy}}{G_l A^2} A_{,xx} p_x + 6 \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} a_{z,x} a_{z,xx} A_{,x} p_x + 8 a_z \frac{E_l I_{yy}}{G_l A^3} A_{,x}^2 p_x \\
 & + 6 a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^4} a_{z,x} A_{,x} A_{,xx} p_x + a_z^2 \frac{E_l^2 I_{yy}}{G_l^2 A^3} I_{yy,x} A_{,x} p_{x,xx} - 2 a_z \frac{E_l^2 I_{yy}}{G_l^2 A^3} a_{z,x} I_{yy,xx} A_{,x} p_x \\
 & + 6 \frac{E_l^2 I_{yy}}{G_l^2 A^3} a_{z,x}^2 I_{yy,x} A_{,x} p_x - 6 a_z \frac{E_l^2 I_{yy}}{G_l^2 A^4} a_{z,x} I_{yy,x} A_{,x}^2 p_x + 4 a_z \frac{E_l^2}{G_l^2 A^3} a_{z,x} I_{yy,x}^2 A_{,x} p_x \\
 & - 6 a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^4} a_{z,x} A_{,x}^2 p_{x,x} + 6 a_z \frac{E_l^2 I_{yy}}{G_l^2 A^3} a_{z,x} I_{yy,x} A_{,x} p_{x,x} + 2 a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} a_{z,xx} A_{,x} p_{x,xx} \\
 & - 3 a_z^2 \frac{E_l^2 I_{yy}}{G_l^2 A^4} I_{yy,x} A_{,x}^2 p_{x,x} - 2 a_z \frac{E_l^2}{G_l^2 A^2} a_{z,x} I_{yy,x}^2 p_{x,x} + 3 a_z^2 \frac{E_l^2 I_{yy}}{G_l^2 A^4} I_{yy,x} A_{,x} A_{,xx} p_x \\
 & + 3 a_z^2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^4} A_{,x} A_{,xx} p_{x,x} - 3 \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,x} a_{z,xx} I_{yy,x} p_x + 6 a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} a_{z,xx} A_{,xx} p_x \\
 & - a_z \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,x} I_{yy,x} p_{x,xx} + a_z \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,x} I_{yy,xx} p_{x,x} - a_z^2 \frac{E_l^2 I_{yy}}{G_l^2 A^3} I_{yy,x} A_{,xxx} p_x \\
 & + 2 a_z \frac{E_l^2}{G_l^2 A^2} a_{z,xx} I_{yy,x}^2 p_x - a_z^2 \frac{E_l^2 I_{yy}}{G_l^2 A^3} I_{yy,xx} A_{,x} p_{x,x} - a_z \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,xx} I_{yy,xx} p_x \\
 & + a_z \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,xxx} I_{yy,x} p_x + 2 a_z^2 \frac{E_l^2}{G_l^2 A^3} I_{yy,x}^2 A_{,x} p_{x,x} + a_z^2 \frac{E_l^2 I_{yy}}{G_l^2 A^3} I_{yy,xx} A_{,xx} p_x \quad (B1)
 \end{aligned}$$

$$\begin{aligned}
 C_2 = & N - E_l I_{yy,xx} + 2a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^2} a_{z,xxx} p_x + 3 \frac{E_l^2 I_{yy}}{G_l A^2} a_{z,xx} I_{yy,x} A_{,x} - 2 \frac{E_l^2 I_{yy}}{G_l A^2} a_{z,x} I_{yy,xx} A_{,x} \\
 & + 4 \frac{E_l^2}{G_l^2 A^2} a_{z,x}^2 I_{yy,x}^2 N - 2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^2} a_{z,x} a_{z,xxx} N + 6 \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} a_{z,x}^2 A_{,xx} N - 6 a_z^2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^4} A_{,x}^2 p_{x,x} \\
 & + a_z \frac{E_l^2 I_{yy}}{G_l A^2} I_{yy,x} A_{,xxx} - 2 a_z \frac{E_l^2}{G_l^2 A^2} a_{z,xx} I_{yy,x}^2 N + 6 \frac{E_l I_{yy}}{G_l A} a_{z,x} p_x + 4 \frac{E_l^2}{G_l A^2} a_{z,x} I_{yy,x}^2 A_{,x} \\
 & - 6 a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^4} a_{z,x} A_{,x} A_{,xx} N + 12 a_z \frac{E_l^2 I_{yy}}{G_l^2 A^3} a_{z,x} I_{yy,x} A_{,x} p_x + 2 a_z \frac{E_l^2 I_{yy}}{G_l^2 A^3} a_{z,x} I_{yy,xx} A_{,x} N \\
 & + 8 \frac{E_l I_{yy}}{G_l A^2} a_{z,x} A_{,x} N + a_z^2 \frac{E_l^2 I_{yy}}{G_l^2 A^3} I_{yy,x} A_{,xxx} N - a_z \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,xxx} I_{yy,x} N - 4 \frac{E_l I_{yy}}{G_l A} a_{z,xx} N \\
 & + 2 a_z \frac{E_l^2}{G_l A^2} I_{yy,x}^2 A_{,xx} - 3 a_z^2 \frac{E_l^2 I_{yy}}{G_l^2 A^4} I_{yy,x} A_{,x} A_{,xx} N + 3 \frac{E_l^2 I_{yy}^2}{G_l^2 A^2} a_{z,xx}^2 N - 6 a_z \frac{E_l I_{yy}}{G_l A^2} A_{,x} p_x \\
 & + 4 a_z^2 \frac{E_l^2}{G_l^2 A^3} I_{yy,x}^2 A_{,x} p_x + 3 a_z^2 \frac{E_l^2 I_{yy}}{G_l^2 A^3} I_{yy,x} A_{,x} p_{x,x} - a_z \frac{E_l}{G_l A} I_{yy,xx} N + 3 a_z \frac{E_l I_{yy}}{G_l A} p_{x,x} \\
 & - 12 a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^4} a_{z,x} A_{,x}^2 p_x + 2 a_z \frac{E_l^2 I_{yy}}{G_l A^3} I_{yy,xx} A_{,x}^2 + \frac{E_l^2 I_{yy}}{G_l A} a_{z,xx} I_{yy,xx} - 8 a_z \frac{E_l I_{yy}}{G_l A^3} A_{,x}^2 N \\
 & - a_z \frac{E_l^2 I_{yy}}{G_l A^2} I_{yy,xx} A_{,xx} + 4 a_z \frac{E_l I_{yy}}{G_l A^2} A_{,xx} N - 4 a_z \frac{E_l}{G_l^2 A^2} a_{z,x} I_{yy,x}^2 p_x - 4 a_z \frac{E_l^2}{G_l A^3} I_{yy,x}^2 A_{,x}^2 \\
 & - 5 \frac{E_l}{G_l A} a_{z,x} I_{yy,x} N - \frac{E_l^2 I_{yy}}{G_l A} a_{z,xxx} I_{yy,x} + 3 \frac{E_l^2 I_{yy}}{G_l A^2} a_{z,x} I_{yy,x} A_{,xx} + 5 a_z \frac{E_l}{G_l A^2} I_{yy,x} A_{,x} N \\
 & - 2 \frac{E_l^2}{G_l A} a_{z,xx} I_{yy,x}^2 + 12 \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} a_{z,x} A_{,x} p_x - 2 a_z^2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} A_{,xxx} p_x - 2 \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,x}^2 I_{yy,xx} N \\
 & + 3 a_z^2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^4} A_{,xx}^2 N + 3 a_z^2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} A_{,xx} p_{x,x} + 6 a_z \frac{E_l^2 I_{yy}}{G_l A^4} A_{,x}^3 I_{yy,x} + 4 a_z \frac{E_l}{G_l A} I_{yy,x} p_x \\
 & - 4 a_z \frac{E_l^2}{G_l^2 A^3} a_{z,x} I_{yy,x}^2 A_{,x} N + 6 a_z \frac{E_l^2 I_{yy}}{G_l^2 A^4} a_{z,x} I_{yy,x} A_{,x}^2 N - 3 a_z \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,x} I_{yy,x} p_{x,x} \\
 & + 2 a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} a_{z,x} A_{,xxx} N - 2 a_z^2 \frac{E_l^2 I_{yy}}{G_l^2 A^3} I_{yy,xx} A_{,x} p_x + 3 \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,x} a_{z,xx} I_{yy,x} N
 \end{aligned}$$

$$\begin{aligned}
 & -a_z^2 \frac{E_l^2 I_{yy}}{G_l^2 A^3} I_{yy,xx} A_{,xx} N + 6a_z^2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^4} A_{,x} A_{,xx} p_x + 2a_z \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,x} I_{yy,xx} p_x \\
 & -6 \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} a_{z,x} a_{z,xx} A_{,x} N - 6a_z^2 \frac{E_l^2 I_{yy}}{G_l^2 A^4} I_{yy,x} A_{,x}^2 p_x + 2a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} a_{z,xxx} A_{,x} N \\
 & + 2a_z^2 \frac{E_l^2}{G_l^2 A^3} I_{yy,x}^2 A_{,xx} N + a_z \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,xx} I_{yy,xx} N - 6 \frac{E_l^2 I_{yy}}{G_l^2 A^3} a_{z,x}^2 I_{yy,x} A_{,x} N \\
 & -6a_z \frac{E_l^2 I_{yy}}{G_l^2 A^3} I_{yy,x} A_{,x} A_{,xx} + 6a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} a_{z,x} A_{,x} p_{x,x} - 6a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} a_{z,xx} A_{,xx} N \\
 & -6 \frac{E_l^2 I_{yy}}{G_l^2 A^3} a_{z,x} I_{yy,x} A_{,x}^2 - 2a_z^2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^4} A_{,x} A_{,xxx} N + 6a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^4} a_{z,xx} A_{,x}^2 N \\
 & -6 \frac{E_l^2 I_{yy}^2}{G_l^2 A^2} a_{z,x} a_{z,xx} p_x - 3a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^2} a_{z,xx} p_{x,x} - 6 \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,x}^2 I_{yy,x} p_x \quad (B2)
 \end{aligned}$$

$$\begin{aligned}
 C_3 = & -2E_l I_{yy,x} + 3a_z^2 \frac{E_l^2 I_{yy}}{G_l^2 A^3} I_{yy,x} A_{,x} p_x - 3a_z \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,x} I_{yy,x} p_x + 6a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^4} a_{z,x} A_{,x}^2 N \\
 & + 3 \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,x}^2 I_{yy,x} N - 6a_z^2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^4} A_{,x}^2 p_x + 3 \frac{E_l^2 I_{yy}^2}{G_l^2 A^2} a_{z,x} a_{z,xx} N - 3a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^2} a_{z,xx} p_x \\
 & + 3a_z^2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} A_{,xx} p_x + a_z^2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} A_{,xxx} N - a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^2} a_{z,xxx} N - 3a_z^2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^4} A_{,x} A_{,xx} N \\
 & - 2a_z^2 \frac{E_l^2}{G_l^2 A^3} I_{yy,x}^2 A_{,x} N + 3a_z \frac{E_l I_{yy}}{G_l A^2} A_{,x} N + 6a_z \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} a_{z,x} A_{,x} p_x - 2a_z \frac{E_l}{G_l A} I_{yy,x} N \\
 & - 6 \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} a_{z,x} A_{,x}^2 + a_z \frac{E_l^2 I_{yy}}{G_l A^2} I_{yy,xx} A_{,x} - 2a_z \frac{E_l^2}{G_l A^2} I_{yy,x}^2 A_{,x} + a_z^2 \frac{E_l^2 I_{yy}}{G_l^2 A^3} I_{yy,xx} A_{,x} N \\
 & - 3 \frac{E_l I_{yy}}{G_l A} a_{z,x} N + 3a_z \frac{E_l I_{yy}}{G_l A} p_x - 6a_z \frac{E_l^2 I_{yy}}{G_l^2 A^3} a_{z,x} I_{yy,x} A_{,x} N + 3a_z^2 \frac{E_l^2 I_{yy}}{G_l^2 A^4} I_{yy,x} A_{,x}^2 N
 \end{aligned}$$

$$\begin{aligned}
 & -\frac{E_l^2 I_{yy}}{G_l A} a_{z,x} I_{yy,xx} - 6 \frac{E_l^2 I_{yy}^2}{G_l^2 A^3} a_{z,x}^2 A_{,x} N + 6 a_z \frac{E_l^2 I_{yy}^2}{G_l A^4} A_{,x}^3 + 2 a_z \frac{E_l^2}{G_l^2 A^2} a_{z,x} I_{yy,x}^2 N \\
 & - a_z \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,x} I_{yy,xx} N + 3 \frac{E_l^2 I_{yy}^2}{G_l A^2} a_{z,x} A_{,xx} - 6 a_z \frac{E_l^2 I_{yy}}{G_l A^3} A_{,x} A_{,xx} + 3 \frac{E_l^2 I_{yy}}{G_l A^2} a_{z,xx} A_{,x} \\
 & - \frac{E_l^2 I_{yy}^2}{G_l A} a_{z,xxx} + 2 \frac{E_l^2}{G_l A} a_{z,x} I_{yy,x}^2 + a_z \frac{E_l^2 I_{yy}^2}{G_l A^2} A_{,xxx} \quad (B3)
 \end{aligned}$$

$$\begin{aligned}
 C_4 = & E_l I_{yy} - 2 a_z \frac{E_l^2 I_{yy}}{G_l A^3} A_{,x}^2 - \frac{E_l^2 I_{yy}^2}{G_l A} a_{z,xx} - a_z \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,x} I_{yy,x} N + a_z^2 \frac{E_l^2 I_{yy}}{G_l^2 A^3} I_{yy,x} A_{,x} N \\
 & + 2 \frac{E_l^2 I_{yy}^2}{G_l A^2} a_{z,x} A_{,x} + a_z \frac{E_l^2 I_{yy}^2}{G_l A^2} A_{,xx} - \frac{E_l^2 I_{yy}}{G_l A} a_{z,x} I_{yy,x} + a_z \frac{E_l I_{yy}}{G_l A} N - 2 a_z^2 \frac{E_l^2 I_{yy}^2}{G_l^2 A^4} A_{,x}^2 N \\
 & - 3 a_z \frac{E_l^2 I_{yy}}{G_l^2 A^2} a_{z,xx} N + a_z^2 \frac{E_l^2 I_{yy}}{G_l^2 A^3} A_{,xx} N + 2 a_z \frac{E_l^2 I_{yy}}{G_l^2 A^3} a_{z,x} A_{,x} N + a_z \frac{E_l^2 I_{yy}}{G_l A^2} I_{yy,x} A_{,x} \quad (B4)
 \end{aligned}$$

$$\begin{aligned}
 C_5 = & m_{z,x} + p_y - 3 a_y \frac{E_l I_{zz}}{G_l A^2} A_{,x} p_{y,x} + 2 a_y \frac{E_l^2}{G_l^2 A^2} a_{y,x} I_{zz,x}^2 p_{y,x} - a_y^2 \frac{E_l^2 I_{yy}}{G_l^2 A^3} I_{zz,xx} A_{,xx} p_y \\
 & + 4 \frac{E_l I_{zz}}{G_l A} a_{y,xx} p_y - 6 \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} a_{y,x}^2 A_{,x} p_{y,x} - 2 \frac{E_l^2 I_{zz}}{G_l^2 A^2} a_{y,x}^2 I_{zz,xx} p_y + a_y \frac{E_l}{G_l A^2} I_{zz,xx} A_{,x} m_z \\
 & + 2 a_y \frac{E_l}{G_l A} I_{zz,x} p_{y,x} - 8 \frac{E_l I_{zz}}{G_l A^2} a_{y,x} A_{,x} p_y - 4 a_y \frac{E_l^2}{G_l^2 A^3} a_{y,x} I_{zz,x}^2 A_{,x} p_y + 3 \frac{E_l I_{zz}}{G_l A} a_{y,x} p_{y,x} \\
 & + a_y \frac{E_l}{G_l A} I_{zz,xx} p_y + a_y \frac{E_l^2 I_{zz}}{G_l^2 A^2} a_{y,xx} I_{zz,xx} p_y - \frac{E_l}{G_l A} a_{y,x} I_{zz,xx} m_z + 3 \frac{E_l^2 I_{zz}}{G_l^2 A^2} a_{y,x}^2 I_{zz,x} p_{y,x} \\
 & + 3 \frac{E_l I_{zz}}{G_l A^2} a_{y,xx} A_{,x} m_z - 2 \frac{E_l I_{zz}}{G_l A^2} a_{y,x} A_{,x} m_{z,x} + 2 a_y \frac{E_l}{G_l A^2} I_{zz,x} A_{,xx} m_z - 2 \frac{E_l}{G_l A} a_{y,xx} I_{zz,x} m_z \\
 & + 4 \frac{E_l}{G_l A^2} a_{y,x} I_{zz,x} A_{,x} m_z - 6 a_y \frac{E_l I_{zz}}{G_l A^3} A_{,x} A_{,xx} m_z + a_y \frac{E_l I_{zz}}{G_l A^2} A_{,xxx} m_z + 3 \frac{E_l I_{zz}}{G_l A^2} a_{y,x} A_{,xx} m_z \\
 & - 2 a_y^2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^4} A_{,x} A_{,xxx} p_y - \frac{E_l I_{zz}}{G_l A} a_{y,xxx} m_z + 6 a_y \frac{E_l I_{zz}}{G_l A^4} A_{,x}^3 m_z - 6 a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} a_{y,xx} A_{,xx} p_y \\
 & + 3 a_y^2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^4} A_{,xx}^2 p_y - a_y^2 \frac{E_l^2 I_{zz}}{G_l^2 A^3} I_{zz,x} A_{,x} p_{y,xx} + \frac{E_l I_{zz}}{G_l A} a_{y,xx} m_{z,x} - a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^2} a_{y,xxx} p_{y,x}
 \end{aligned}$$

$$\begin{aligned}
 & + 5 \frac{E_l}{G_l A} a_{y,x} I_{zz,x} p_y + 4 \frac{E_l^2}{G_l^2 A^2} a_{y,x}^2 I_{zz,x}^2 p_y + a_y \frac{E_l I_{zz}}{G_l A} p_{y,xx} + 3 \frac{E_l^2 I_{zz}}{G_l^2 A^2} a_{y,x} a_{y,xx} I_{zz,x} p_y \\
 & + a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^2} a_{y,xx} p_{y,xx} + 3 \frac{E_l^2 I_{zz}^2}{G_l^2 A^2} a_{y,xx}^2 p_y - 5 a_y \frac{E_l}{G_l A^2} I_{zz,x} A_{,x} p_y - a_y^2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} A_{,xx} p_{y,xx} \\
 & + 2 a_y^2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^4} A_{,x}^2 p_{y,xx} + 3 \frac{E_l^2 I_{zz}^2}{G_l^2 A^2} a_{y,x} a_{y,xx} p_{y,x} - 4 a_y \frac{E_l I_{zz}}{G_l A^2} A_{,xx} p_y + \frac{E_l}{G_l A} a_{y,x} I_{zz,x} m_{z,x} \\
 & + 8 a_y \frac{E_l I_{zz}}{G_l A^3} A_{,x}^2 p_y - a_y \frac{E_l I_{zz}}{G_l A^2} A_{,xx} m_{z,x} - 4 a_y \frac{E_l}{G_l A^3} I_{zz,x} A_{,x}^2 m_z + a_y^2 \frac{E_l^2 I_{zz}}{G_l^2 A^3} I_{zz,x} A_{,xxx} p_y \\
 & - 6 \frac{E_l I_{zz}}{G_l A^3} a_{y,x} A_{,x}^2 m_z + 6 \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} a_{y,x}^2 A_{,xx} p_y - 2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^2} a_{y,x} a_{y,xxx} p_y + 2 a_y \frac{E_l I_{zz}}{G_l A^3} A_{,x}^2 m_{z,x} \\
 & - 3 a_y^2 \frac{E_l^2 I_{zz}}{G_l^2 A^4} I_{zz,x} A_{,x} A_{,xx} p_y - 6 \frac{E_l^2 I_{zz}}{G_l^2 A^3} a_{y,x}^2 I_{zz,x} A_{,x} p_y - 6 a_y \frac{E_l I_{zz}}{G_l^2 A^3} a_{y,x} I_{zz,x} A_{,x} p_{y,x} \\
 & + 2 a_y \frac{E_l^2 I_{zz}}{G_l^2 A^3} a_{y,x} I_{zz,xx} A_{,x} p_y + a_y \frac{E_l^2 I_{zz}}{G_l^2 A^2} a_{y,x} I_{zz,x} p_{y,xx} - a_y \frac{E_l^2 I_{zz}}{G_l^2 A^2} a_{y,x} I_{zz,xx} p_{y,x} \\
 & + 6 a_y \frac{E_l^2 I_{zz}}{G_l^2 A^4} a_{y,x} I_{zz,x} A_{,x}^2 p_y - 2 a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} a_{y,x} A_{,x} p_{y,xx} - 2 a_y^2 \frac{E_l^2}{G_l^2 A^3} I_{zz,x}^2 A_{,x} p_{y,x} \\
 & - 6 a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^4} a_{y,x} A_{,x} A_{,xx} p_y - 6 \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} a_{y,x} a_{y,xx} A_{,x} p_y + 2 a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} a_{y,xxx} A_{,x} p_y \\
 & + 6 a_z \frac{E_l^2 I_{zz}^2}{G_l^2 A^4} a_{y,x} A_{,x}^2 p_{y,x} + 3 a_y^2 \frac{E_l^2 I_{zz}}{G_l^2 A^4} I_{zz,x} A_{,x}^2 p_{y,x} - 3 a_y^2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^4} A_{,x} A_{,xx} p_{y,x} \\
 & + a_y^2 \frac{E_l^2 I_{zz}}{G_l^2 A^3} I_{zz,xx} A_{,x} p_{y,x} - 2 a_y \frac{E_l^2}{G_l^2 A^2} a_{y,xx} I_{zz,x}^2 p_y + 2 a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} a_{y,x} A_{,xxx} p_y \\
 & + 6 a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^4} a_{y,xx} A_{,x}^2 p_y - a_y \frac{E_l^2 I_{zz}}{G_l^2 A^2} a_{y,xxx} I_{zz,x} p_y - 2 a_y^2 \frac{E_l^2}{G_l^2 A^3} I_{zz,x}^2 A_{,xx} p_y \\
 & + a_y^2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} A_{,xxx} p_{y,x} - a_y \frac{E_l}{G_l A^2} I_{zz,x} A_{,x} m_{z,x}
 \end{aligned} \tag{B5}$$

$$\begin{aligned}
C_6 = & p_x + a_y^2 \frac{E_l^2 I_{zz}}{G_l^2 A^3} I_{yy,x} A_{,xxx} p_x - a_y \frac{E_l^2 I_{zz}}{G_l^2 A^2} a_{y,x} I_{zz,xx} p_{x,x} - 3a_y^2 \frac{E_l^2 I_{zz}}{G_l^2 A^4} I_{zz,x} A_{,x} A_{,xx} p_x \\
& - 6 \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} a_{y,x}^2 A_{,x} p_{x,x} + 3 \frac{E_l^2 I_{zz}^2}{G_l^2 A^2} a_{y,xx}^2 p_x - 3a_y \frac{E_l I_{zz}}{G_l A^2} A_{,x} p_{x,x} - a_y \frac{E_l^2 I_{zz}}{G_l^2 A^2} a_{y,xxx} I_{zz,x} p_x \\
& + a_y \frac{E_l I_{zz}}{G_l A} p_{x,xx} + 2a_y^2 \frac{E_l^2}{G_l^2 A^3} I_{zz,x}^2 A_{,xx} p_x + 3 \frac{E_l I_{zz}}{G_l A} a_{y,x} p_{x,x} + a_y \frac{E_l^2 I_{zz}}{G_l^2 A^2} a_{y,x} I_{zz,x} p_{x,xx} \\
& + 2a_y \frac{E_l^2 I_{zz}}{G_l^2 A^3} a_{y,xxx} A_{,x} p_x + 6 \frac{E_l^2 I_{zz}}{G_l^2 A^3} a_{y,x}^2 A_{,xx} p_x + a_y \frac{E_l}{G_l A} I_{zz,xx} p_x - 8 \frac{E_l I_{zz}}{G_l A^2} a_{y,x} A_{,x} p_x \\
& + 3 \frac{E_l^2 I_{zz}}{G_l^2 A^2} a_{y,x} a_{y,xx} I_{zz,x} p_x + 2a_y \frac{E_l^2 I_{zz}}{G_l^2 A^3} a_{y,x} I_{zz,xx} A_{,x} p_x - 6a_y \frac{E_l^2 I_{zz}}{G_l^2 A^4} a_{y,x} A_{,x} A_{,xx} p_x \\
& + 4 \frac{E_l^2}{G_l^2 A^2} a_{y,x}^2 I_{zz,x}^2 p_x + 2a_y \frac{E_l}{G_l A} I_{zz,x} p_{x,x} + 5 \frac{E_l}{G_l A} a_{y,x} I_{zz,x} p_x - 5a_y \frac{E_l}{G_l A^2} I_{zz,x} A_{,x} p_x \\
& + a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^2} a_{y,xx} p_{x,xx} + 2a_y^2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^4} A_{,x}^2 p_{x,xx} + 4 \frac{E_l I_{zz}}{G_l A} a_{y,xx} p_x + a_y^2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} A_{,xxx} p_{x,x} \\
& + 3a_y^2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^4} A_{,xx}^2 p_x + 8a_y \frac{E_l I_{zz}}{G_l A^3} A_{,x}^2 p_x - 4a_y \frac{E_l I_{zz}}{G_l A^2} A_{,xx} p_x - 6 \frac{E_l^2 I_{zz}}{G_l^2 A^3} a_{y,x} a_{y,xx} A_{,x} p_x \\
& - 6a_y \frac{E_l^2 I_{zz}}{G_l^2 A^3} a_{y,x} I_{zz,x} A_{,x} p_{x,x} - 6 \frac{E_l^2 I_{zz}}{G_l^2 A^3} a_{y,x}^2 I_{zz,x} A_{,x} p_x - 4a_y \frac{E_l^2}{G_l^2 A^3} a_{y,x} I_{zz,x}^2 A_{,x} p_x \\
& + 3a_y^2 \frac{E_l^2 I_{zz}}{G_l^2 A^4} I_{zz,x} A_{,x}^2 p_{x,x} + 2a_y \frac{E_l^2}{G_l^2 A^2} a_{y,x} I_{zz,x}^2 p_{x,x} - 2a_y^2 \frac{E_l^2}{G_l^2 A^3} I_{zz,x}^2 A_{,x} p_{x,x} \\
& - 3a_y^2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^4} A_{,x} A_{,xx} p_{x,x} + 2a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} a_{y,x} A_{,xxx} p_x - 2a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} a_{y,xx} A_{,x} p_{x,xx} \\
& - 2a_y \frac{E_l^2}{G_l^2 A^2} a_{y,xx} I_{zz,x}^2 p_x + 6a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^4} a_{y,xx} A_{,x}^2 p_x + 6a_y \frac{E_l^2 I_{zz}}{G_l^2 A^4} a_{y,x} I_{zz,x} A_{,x}^2 p_x \\
& + a_y^2 \frac{E_l^2 I_{zz}}{G_l^2 A^3} I_{zz,xx} A_{,x} p_{x,x} + a_y \frac{E_l^2 I_{zz}}{G_l^2 A^2} a_{y,xx} I_{zz,xx} p_x - 2a_y^2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^4} A_{,x} A_{,xxx} p_x \\
& - a_y^2 \frac{E_l^2 I_{zz}}{G_l^2 A^3} I_{zz,xx} A_{,xx} p_x - a_y^2 \frac{E_l^2 I_{zz}}{G_l^2 A^3} I_{zz,x} A_{,x} p_{x,xx} + 3 \frac{E_l^2 I_{zz}^2}{G_l^2 A^2} a_{y,x} a_{y,xx} p_{x,x}
\end{aligned}$$

$$\begin{aligned}
 & -2 \frac{E_l^2 I_{zz}}{G_l^2 A^2} a_{y,x}^2 I_{zz,xx} p_x + 6a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^4} a_{y,x} A_{,x}^2 p_{x,x} - a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^2} a_{y,xxx} p_{x,x} \\
 & -2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^2} a_{y,x} a_{y,xxx} p_x + 3 \frac{E_l^2 I_{zz}}{G_l^2 A^2} a_{y,x}^2 I_{zz,x} p_{x,x} - 6a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} a_{y,xx} A_{,xx} p_x \\
 & -a_y^2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} A_{,xx} p_{x,xx}
 \end{aligned} \tag{B6}$$

$$\begin{aligned}
 C_7 = & -E_l I_{zz,xx} - N - 2a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^2} a_{y,xxx} p_x - 3 \frac{E_l^2 I_{zz}}{G_l A^2} a_{y,xx} I_{zz,x} A_{,x} + 2 \frac{E_l^2 I_{zz}}{G_l A^2} a_{y,x} I_{zz,xx} A_{,x} \\
 & - 3a_y^2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} A_{,xx} p_{x,x} - 4 \frac{E_l^2}{G_l^2 A^2} a_{y,x}^2 I_{zz,x}^2 N + 3a_y \frac{E_l I_{zz}}{G_l A} p_{x,x} + 2a_y^2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^4} A_{,x} A_{,xxx} N \\
 & + 4a_y \frac{E_l I_{zz}}{G_l A^2} A_{,xx} N + 2a_y^2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} A_{,xxx} p_x - 6 \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} a_{y,x}^2 A_{,xx} N - 6a_y^2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^4} A_{,x} A_{,xx} p_x \\
 & + 2 \frac{E_l^2}{G_l A} a_{y,xx} I_{zz,x}^2 + 6a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} a_{y,xx} A_{,xx} N - 12 \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} a_{y,x}^2 A_{,x} p_x + \frac{E_l^2 I_{zz}}{G_l A} a_{y,xxx} I_{zz,x} \\
 & + 5a_y \frac{E_l}{G_l A^2} I_{zz,x} A_{,x} N + 6 \frac{E_l^2 I_{zz}}{G_l A^3} a_{y,x} I_{zz,x} A_{,x}^2 - 3 \frac{E_l^2 I_{zz}^2}{G_l^2 A^2} a_{y,xx}^2 N - 4 \frac{E_l^2}{G_l A^2} a_{y,x} I_{zz,x}^2 A_{,x} \\
 & - 5 \frac{E_l}{G_l A} a_{y,x} I_{zz,x} N + 6a_y^2 \frac{E_l^2 I_{zz}}{G_l^2 A^4} I_{zz,x} A_{,x}^2 p_x - a_y^2 \frac{E_l^2 I_{zz}}{G_l^2 A^3} I_{zz,x} A_{,xxx} N + 6 \frac{E_l I_{zz}}{G_l A} a_{y,x} p_x \\
 & - 3a_y^2 \frac{E_l^2 I_{zz}}{G_l^2 A^3} I_{zz,x} A_{,x} p_{x,x} - 4 \frac{E_l I_{zz}}{G_l A} a_{y,xx} N - 6a_y \frac{E_l^2 I_{zz}}{G_l A^4} A_{,x}^3 I_{zz,x} + 8 \frac{E_l I_{zz}}{G_l A^2} a_{y,x} A_{,x} N \\
 & - \frac{E_l^2 I_{zz}}{G_l A} a_{y,xx} I_{zz,xx} - 2a_y \frac{E_l^2 I_{zz}}{G_l A^3} I_{zz,xx} A_{,x}^2 + 4a_y \frac{E_l^2}{G_l A^3} I_{zz,x} A_{,x}^2 + a_y^2 \frac{E_l^2 I_{zz}}{G_l^2 A^3} I_{zz,xx} A_{,xx} N \\
 & - a_y \frac{E_l^2 I_{zz}}{G_l A^2} I_{zz,x} A_{,xxx} + a_y \frac{E_l^2 I_{zz}}{G_l A^2} I_{zz,xx} A_{,xx} + 6a_y^2 \frac{E_l^2 I_{zz}}{G_l^2 A^4} A_{,x}^2 p_{x,x} - a_y \frac{E_l}{G_l A} I_{zz,xx} N
 \end{aligned}$$

$$\begin{aligned}
 & -6a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^4} a_{y,xx} A_{,x}^2 N + 4a_y \frac{E_l}{G_l A} I_{zz,x} p_x + 2 \frac{E_l^2 I_{zz}}{G_l^2 A^2} a_{y,x}^2 I_{zz,xx} N - 6a_y \frac{E_l I_{zz}}{G_l A^2} A_{,x} p_x \\
 & + 4a_y \frac{E_l^2}{G_l^2 A^3} a_{y,x} I_{zz,xx}^2 A_{,x} N - 6a_y \frac{E_l^2 I_{zz}}{G_l^2 A^4} a_{y,x} I_{zz,x} A_{,x}^2 N + a_y \frac{E_l^2 I_{zz}}{G_l^2 A^2} a_{y,xxx} I_{zz,x} N \\
 & - 2a_y \frac{E_l^2}{G_l A^2} I_{zz,x}^2 A_{,xx} - 8a_y \frac{E_l I_{zz}}{G_l A^3} A_{,x}^2 N + 6 \frac{E_l^2 I_{zz}}{G_l^2 A^2} a_{y,x}^2 I_{zz,x} p_x - 3a_y^2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^4} A_{,xx}^2 N \\
 & - 2a_y \frac{E_l^2 I_{zz}}{G_l^2 A^3} a_{y,x} I_{zz,xx} A_{,x} N - 12a_y \frac{E_l^2 I_{zz}}{G_l^2 A^3} a_{y,x} I_{zz,x} A_{,x} p_x - 4a_y^2 \frac{E_l^2}{G_l^2 A^3} I_{zz,x}^2 A_{,x} p_x \\
 & + 3a_y^2 \frac{E_l^2 I_{zz}}{G_l^2 A^4} I_{zz,x} A_{,x} A_{,xx} N - 3 \frac{E_l^2 I_{zz}}{G_l^2 A^2} a_{y,x} a_{y,xx} I_{zz,x} N + 6 \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} a_{y,x} a_{y,xx} A_{,x} N \\
 & + 6a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^4} a_{y,x} A_{,x} A_{,xx} N + 6a_y \frac{E_l^2 I_{zz}}{G_l A^3} I_{zz,x} A_{,x} A_{,xx} - 2a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} a_{y,xxx} A_{,x} N \\
 & - 2a_y \frac{E_l^2 I_{zz}}{G_l^2 A^2} a_{y,x} I_{zz,xx} p_x - a_y \frac{E_l^2 I_{zz}}{G_l^2 A^2} a_{y,xx} I_{zz,xx} N + 2a_y^2 \frac{E_l^2 I_{zz}}{G_l^2 A^3} I_{zz,xx} A_{,x} p_x \\
 & + 6 \frac{E_l^2 I_{zz}}{G_l^2 A^3} a_{y,x}^2 I_{zz,x} A_{,x} N - 6a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} a_{y,x} A_{,x} p_{x,x} + 2a_y \frac{E_l^2}{G_l^2 A^2} a_{y,xx} I_{zz,x}^2 N \\
 & - 2a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} a_{y,x} A_{,xxx} N - 2a_y^2 \frac{E_l^2}{G_l^2 A^3} I_{zz,x}^2 A_{,xx} N + 12a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^4} a_{y,x} A_{,x}^2 p_x \\
 & + 6 \frac{E_l^2 I_{zz}^2}{G_l^2 A^2} a_{y,x} a_{y,xx} p_x + 4a_y \frac{E_l^2}{G_l^2 A^2} a_{y,x} I_{zz,x}^2 p_x + 3a_y \frac{E_l^2 I_{zz}}{G_l^2 A^2} a_{y,x} I_{zz,x} p_{x,x} \\
 & + 3a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^2} a_{y,xx} p_{x,x} + 2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^2} a_{y,x} a_{y,xxx} N - 3 \frac{E_l^2 I_{zz}}{G_l A^2} a_{y,x} I_{zz,x} A_{,xx} \quad (B7)
 \end{aligned}$$

$$C_8 = -2E_l I_{zz,x} + 2a_y^2 \frac{E_l^2}{G_l^2 A^3} I_{zz,x}^2 A_{,x} N + 3a_y \frac{E_l^2 I_{zz}}{G_l^2 A^2} a_{y,x} I_{zz,x} p_x - 6a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^4} a_{y,x} A_{,x}^2 N$$

$$\begin{aligned}
 & + 6 \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} a_{y,x}^2 A_{,x} N + 3 a_y \frac{E_l I_{zz}}{G_l A} p_x + \frac{E_l^2 I_{zz}^2}{G_l A} a_{y,xxx} - 6 a_y \frac{E_l^2 I_{zz}^2}{G_l A^4} A_{,x}^3 - 2 a_y \frac{E_l}{G_l A} I_{zz,x} N \\
 & - 6 a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} a_{y,x} A_{,x} p_x - 2 a_y \frac{E_l^2}{G_l^2 A^2} a_{y,x} I_{zz,x}^2 N + 6 a_y \frac{E_l^2 I_{zz}^2}{G_l A^3} A_{,x} A_{,xx} - 3 \frac{E_l I_{zz}}{G_l A} a_{y,x} N \\
 & + 6 a_y \frac{E_l^2 I_{zz}}{G_l^2 A^3} a_{y,x} I_{zz,x} A_{,x} N - 2 \frac{E_l^2}{G_l A} a_{y,x} I_{zz,x}^2 - a_y^2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} A_{,xxx} N + a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^2} a_{y,xxx} N \\
 & + 6 \frac{E_l^2 I_{zz}^2}{G_l A^3} a_{y,x} A_{,x}^2 - a_y \frac{E_l^2 I_{zz}}{G_l A^2} I_{zz,xx} A_{,x} + 3 a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^2} a_{y,xx} p_x + a_y \frac{E_l^2 I_{zz}}{G_l^2 A^2} a_{y,x} I_{zz,xx} N \\
 & - 3 a_y^2 \frac{E_l^2 I_{zz}}{G_l^2 A^3} I_{zz,x} A_{,x} p_x + 6 a_y^2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^4} A_{,x}^2 p_x + \frac{E_l^2 I_{zz}}{G_l A} a_{y,x} I_{zz,xx} - 3 \frac{E_l^2 I_{zz}}{G_l^2 A^2} a_{y,x}^2 I_{zz,x} N \\
 & - 3 \frac{E_l^2 I_{zz}^2}{G_l^2 A^2} a_{y,x} a_{y,xx} N + 3 a_y^2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^4} A_{,x} A_{,xx} N - 3 a_y^2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} A_{,xx} p_x + 3 a_y \frac{E_l I_{zz}}{G_l A^2} A_{,x} N \\
 & - 3 a_y^2 \frac{E_l^2 I_{zz}}{G_l^2 A^4} I_{yy,x} A_{,x}^2 N - 3 \frac{E_l^2 I_{zz}^2}{G_l A^2} a_{y,x} A_{,xx} - 3 \frac{E_l^2 I_{zz}^2}{G_l A^2} a_{y,xx} A_{,x} - a_y^2 \frac{E_l^2 I_{zz}}{G_l^2 A^3} I_{zz,xx} A_{,x} N \\
 & - a_y \frac{E_l^2 I_{zz}^2}{G_l A^2} A_{,xxx} + 2 a_y \frac{E_l^2}{G_l A^2} I_{zz,x}^2 A_{,x}
 \end{aligned} \tag{B8}$$

$$\begin{aligned}
 C_9 & = E_l I_{zz} - a_y^2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} I_{zz,x} A_{,x} N + a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^2} a_{y,xx} N + a_y \frac{E_l I_{zz}}{G_l A} N - a_y^2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} A_{,xx} N \\
 & + 2 a_y^2 \frac{E_l^2 I_{zz}^2}{G_l^2 A^4} A_{,x}^2 N + a_y \frac{E_l^2 I_{zz}}{G_l^2 A^2} a_{y,x} I_{zz,x} N - 2 a_y \frac{E_l^2 I_{zz}^2}{G_l^2 A^3} a_{y,x} A_{,x} N + \frac{E_l^2 I_{zz}}{G_l A} a_{y,x} I_{zz,x} \\
 & - a_y \frac{E_l^2 I_{zz}}{G_l A^2} I_{zz,x} A_{,x} + 2 a_y \frac{E_l^2 I_{zz}^2}{G_l A^3} A_{,x}^2 + \frac{E_l^2 I_{zz}^2}{G_l A} a_{y,xx} - a_y \frac{E_l^2 I_{zz}^2}{G_l A^2} A_{,xx} - 2 \frac{E_l^2 I_{zz}^2}{G_l A^2} a_{y,x} A_{,x}
 \end{aligned} \tag{B9}$$

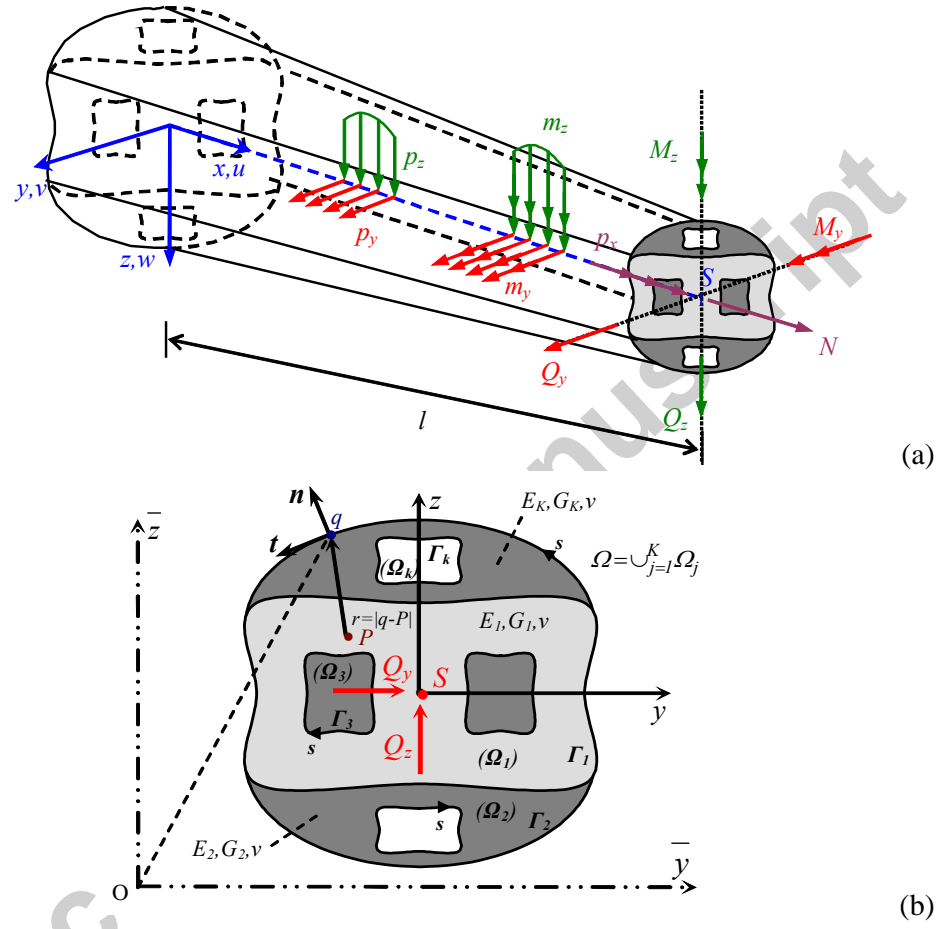


Fig.1. Beam of variable cross section (a) occupying the two dimensional region Ω (b).

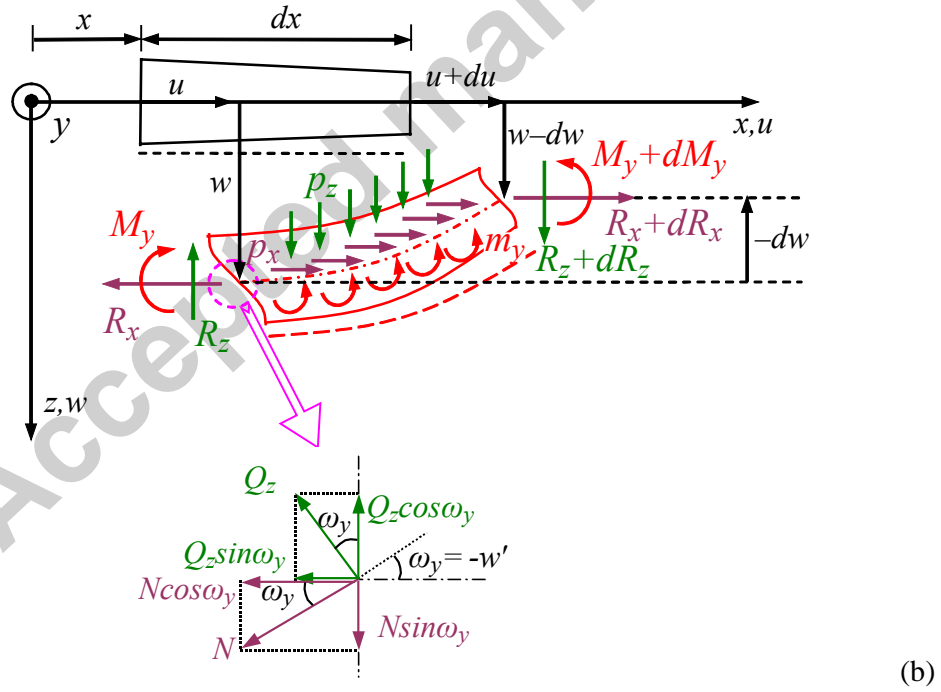
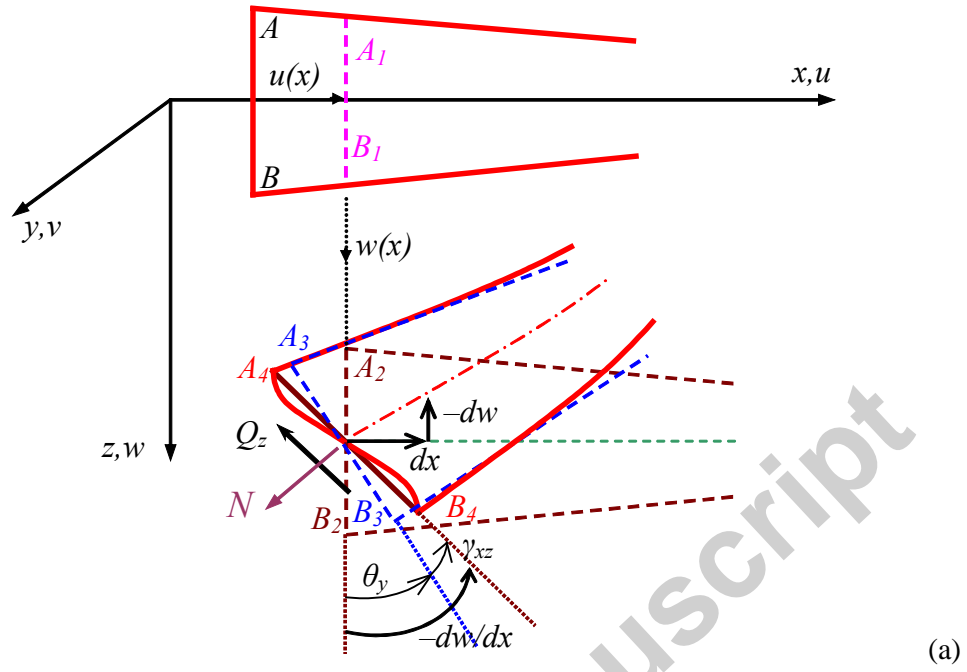


Fig.2. Displacements (a) and forces (b) acting on the deformed element in the xz plane.

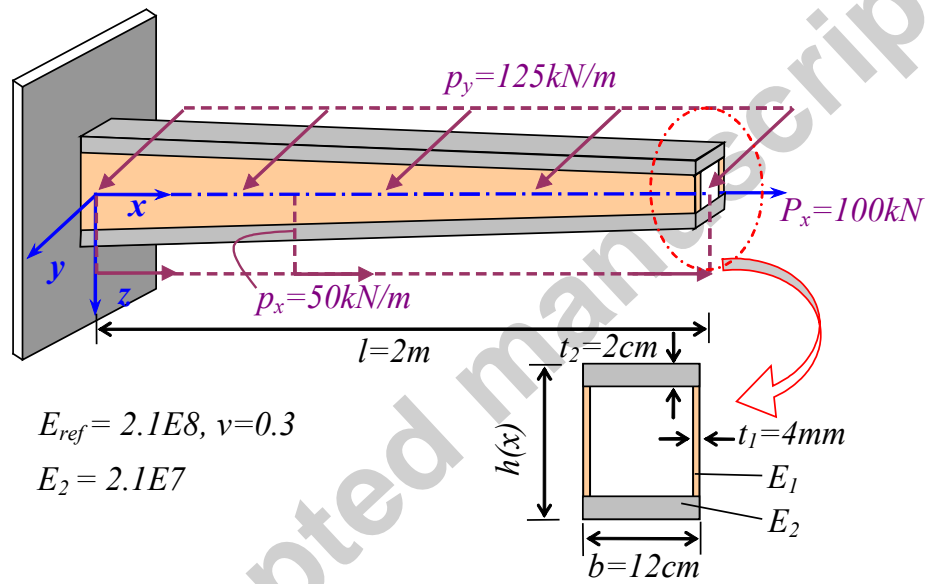


Fig.3. Cantilever beam of a composite hollow variable cross section of Example 1.

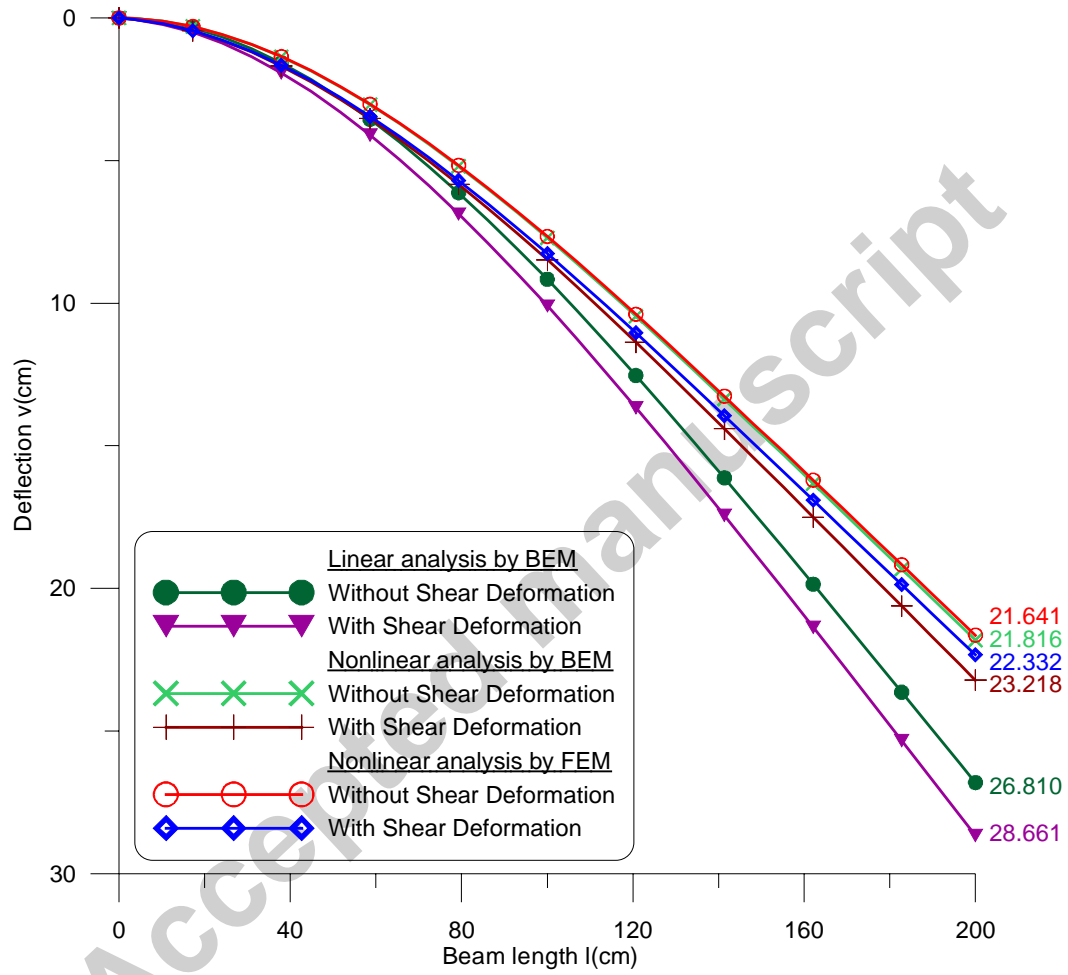


Fig. 4. Deflections v of the composite beam of Example 1.

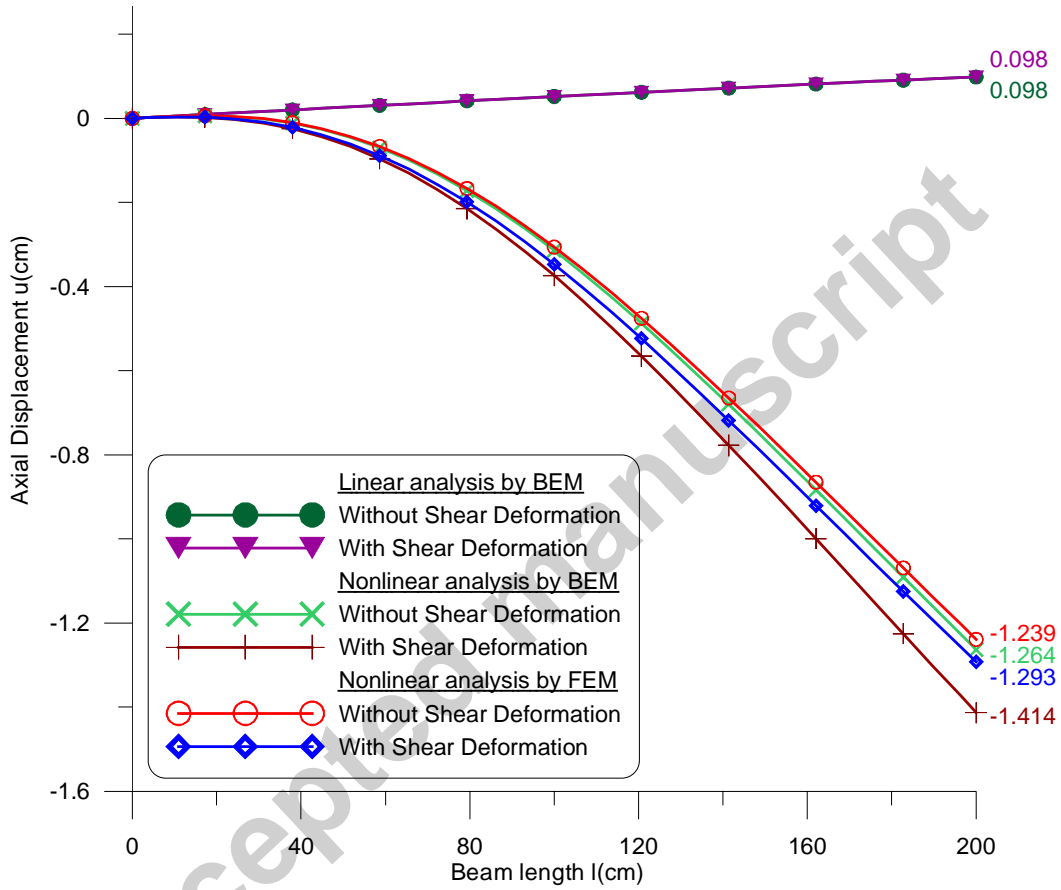


Fig. 5. Axial displacement u of the composite beam of Example 1.

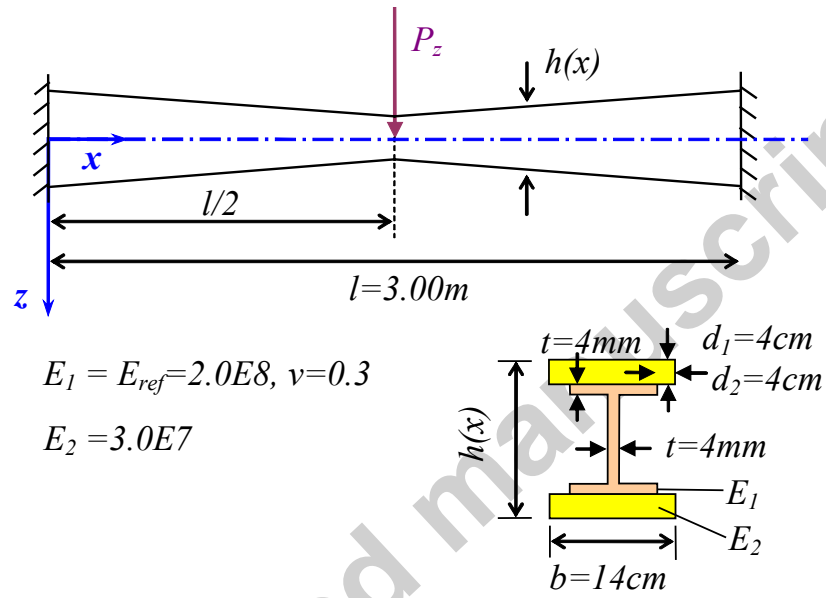


Fig.6. Clamped beam of an I-type variable cross section of Example 2.

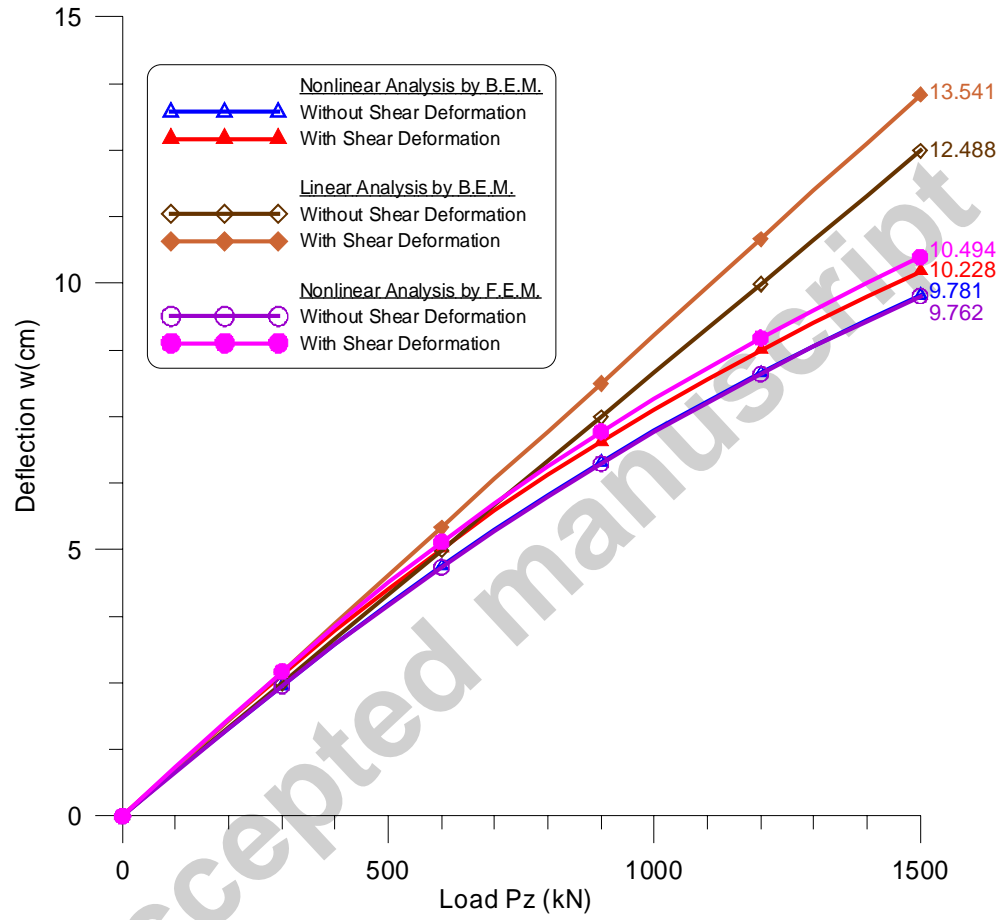


Fig. 7. Deflections $w(\text{cm})$ at midspan of the clamped beam of example 2 ($h_{mid} = 0.12\text{m}$).

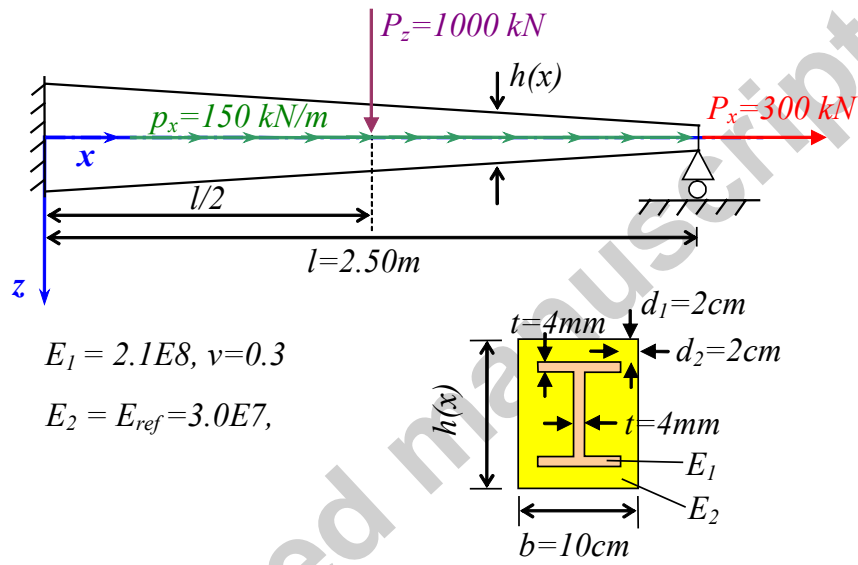


Fig.8. Beam of a linearly varying composite cross section of Example 3.

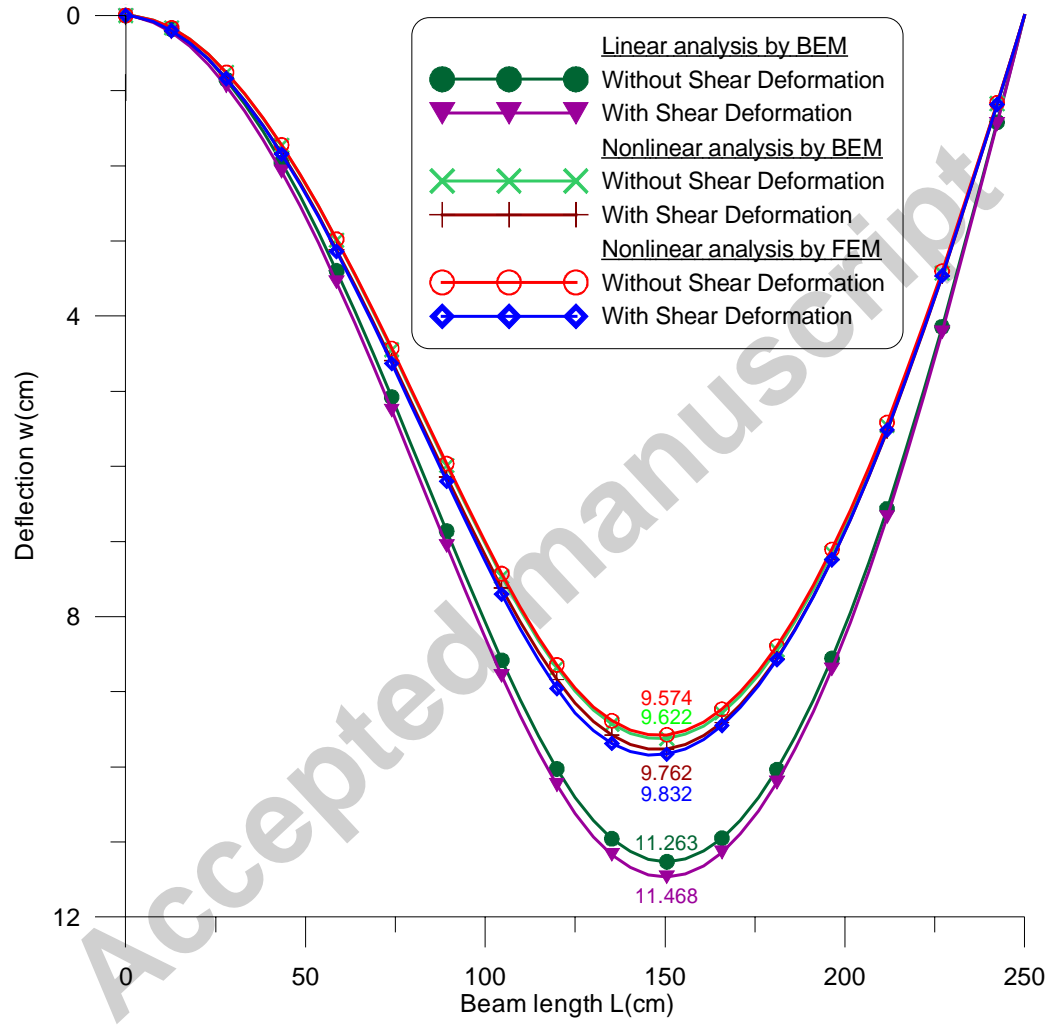


Fig. 9. Deflections w of the composite beam of Example 3.

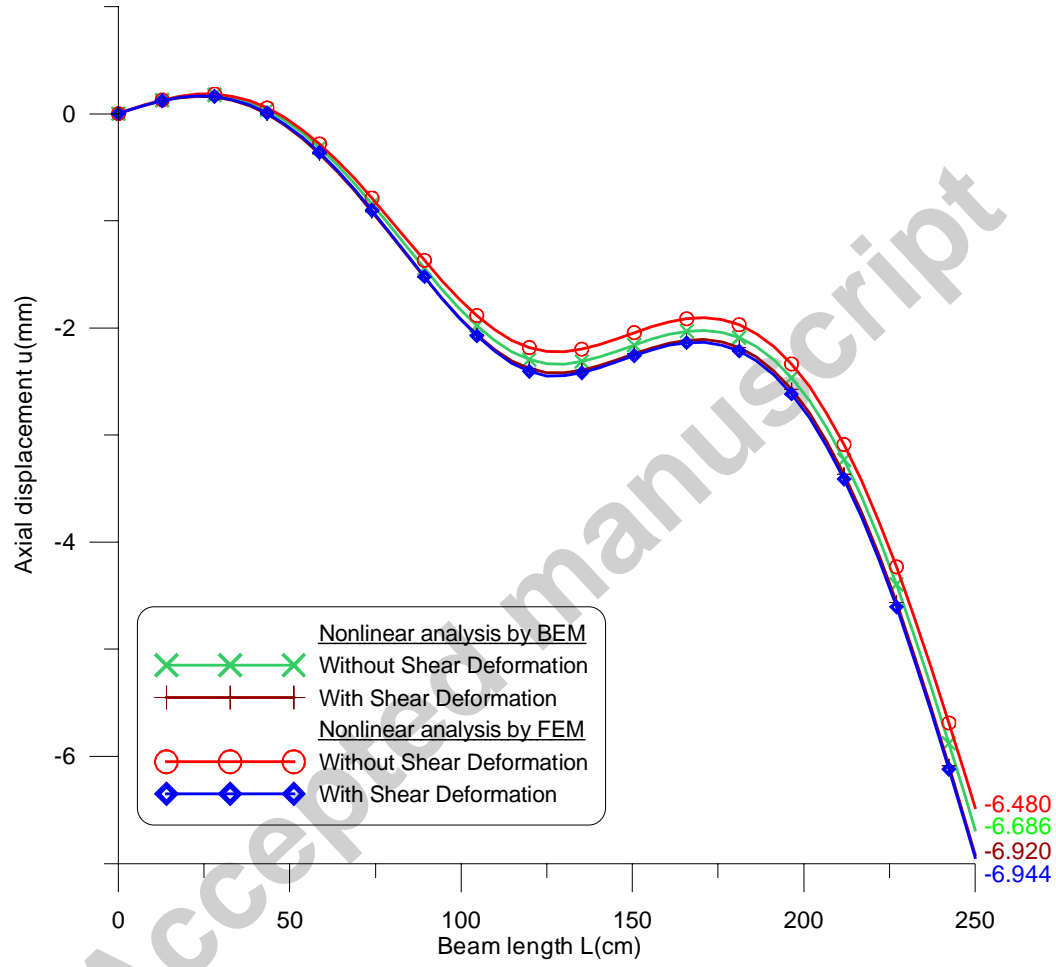


Fig. 10. Axial displacement u of the composite beam of Example 3.

Table 1. Transverse deflection v and axial displacement u at $l/2$, l of the beam of Example 1, for various numbers of boundary elements.

Transverse deflection v (cm)				
No. of B.E.	Nonlinear Analysis with shear deformation		Nonlinear Analysis without shear deformation	
	$x=l/2$	$x=l$	$x=l/2$	$x=l$
13	8.476	23.183	7.705	21.793
17	8.480	23.196	7.708	21.805
21	8.483	23.206	7.709	21.811
25	8.485	23.214	7.710	21.814
29	8.486	23.218	7.711	21.816
Axial displacement u (cm)				
No. of B.E.	Nonlinear Analysis with shear deformation		Nonlinear Analysis without shear deformation	
	$x=l/2$	$x=l$	$x=l/2$	$x=l$
13	-0.372	-1.401	-0.315	-1.261
17	-0.372	-1.405	-0.314	-1.262
21	-0.373	-1.409	-0.314	-1.263
25	-0.373	-1.412	-0.314	-1.263
29	-0.374	-1.414	-0.314	-1.264

Table 2. Transverse deflection v and axial displacement u at the $l/6$ of the beam of Example 1.

Transverse deflection v (cm)									
a/a	x(cm)	$h_l=0.08m$		$h_l=0.12m$		$h_l=0.16m$		$h_l=0.20m$	
		Ignoring Shear Deform.	With Shear Deform.	Ignoring Shear Deform.	With Shear Deform.	Ignoring Shear Deform.	With Shear Deform.	Ignoring Shear Deform.	With Shear Deform.
1	31.034	0.936	1.204	0.932	1.203	0.927	1.203	0.922	1.196
2	65.517	3.774	4.299	3.715	4.251	3.657	4.208	3.601	4.148
3	100.000	7.913	8.668	7.711	8.486	7.522	8.326	7.347	8.145
4	134.483	12.819	13.782	12.383	13.381	11.994	13.036	11.644	12.680
5	168.966	18.052	19.212	17.328	18.541	16.702	17.975	16.149	17.416
6	200.000	22.804	24.138	21.816	23.218	20.971	22.450	20.233	21.705

Axial displacement u (cm)									
a/a	x(cm)	$h_l=0.08m$		$h_l=0.12m$		$h_l=0.16m$		$h_l=0.20m$	
		Ignoring Shear Deform.	With Shear Deform.	Ignoring Shear Deform.	With Shear Deform.	Ignoring Shear Deform.	With Shear Deform.	Ignoring Shear Deform.	With Shear Deform.
1	31.034	-0.002	-0.011	-0.002	-0.011	-0.002	-0.011	-0.002	-0.011
2	65.517	-0.103	-0.134	-0.099	-0.130	-0.095	-0.127	-0.092	-0.123
3	100.000	-0.334	-0.393	-0.314	-0.374	-0.297	-0.358	-0.281	-0.341
4	134.483	-0.663	-0.753	-0.614	-0.705	-0.572	-0.665	-0.536	-0.626
5	168.966	-1.041	-1.161	-0.953	-1.075	-0.880	-1.005	-0.818	-0.939
6	200.000	-1.386	-1.533	-1.264	-1.414	-1.163	-1.317	-1.078	-1.227

Table 3. Transverse deflection v and axial displacement u at the $1/6$ of the beam of Example 1 for $h_f = 0.08m$.

a/a	x(cm)	Linear Analysis by BEM		Nonlinear Analysis			
		Without Shear Deform.	With Shear Deform.	Without Shear Deform. by BEM	With Shear Deform. by FEM [31]	Without Shear Deform. by BEM	With Shear Deform. by FEM [31]
Transverse deflection v (cm)							
1	31.034	1.089	1.377	0.936	0.933	1.204	1.188
2	65.517	4.461	5.062	3.774	3.754	4.299	4.213
3	100.000	9.503	10.411	7.913	7.862	8.668	8.454
4	134.483	15.629	16.840	12.819	12.723	13.782	13.389
5	168.966	22.321	23.834	18.052	17.905	19.212	18.595
6	200.000	28.502	30.286	22.804	22.611	24.138	23.283
Axial displacement u (cm)							
1	31.034	0.017	0.017	-0.002	-0.001	-0.011	-0.009
2	65.517	0.036	0.036	-0.103	-0.099	-0.134	-0.124
3	100.000	0.055	0.055	-0.334	-0.324	-0.393	-0.366
4	134.483	0.074	0.074	-0.663	-0.648	-0.753	-0.699
5	168.966	0.094	0.094	-1.041	-1.018	-1.161	-1.073
6	200.000	0.112	0.112	-1.386	-1.358	-1.533	-1.411

Table 4. Transverse deflection $w(\text{cm})$ at $x=l/2$ of the beam of Example 2 for various values of the concentrated load and of the total midspan height.

a/a	$P_z (\text{kN})$	$h_{mid}=0.12\text{m}$		$h_{mid}=0.14\text{m}$		$h_{mid}=0.16\text{m}$		$h_{mid}=0.18\text{m}$	
		Ignoring Shear Deform.	With Shear Deform.	Ignoring Shear Deform.	With Shear Deform.	Ignoring Shear Deform.	With Shear Deform.	Ignoring Shear Deform.	With Shear Deform.
1	200	1.652	1.788	1.500	1.628	1.369	1.473	1.254	1.337
2	400	3.232	3.479	2.949	3.186	2.700	2.897	2.480	2.639
3	600	4.694	5.020	4.311	4.631	3.966	4.236	3.657	3.878
4	800	6.024	6.402	5.568	5.946	5.152	5.475	4.772	5.040
5	1000	7.228	7.639	6.722	7.139	6.253	6.613	5.818	6.119
6	1200	8.320	8.751	7.780	8.222	7.271	7.656	6.796	7.120
7	1400	9.315	9.758	8.751	9.210	8.215	8.616	7.709	8.047

Table 5. Transverse deflection $w(\text{cm})$ at $x=l/2$ of the beam of Example 2 with total midspan height $h_{mid}=0.14\text{m}$ for various values of the concentrated load.

a/a	$P_z (\text{kN})$	Linear Analysis by BEM		Nonlinear Analysis			
		Without Shear Deform.	With Shear Deform.	Ignoring Shear Deform.		With Shear Deform.	
				by BEM	by FEM [31]	by BEM	by FEM [31]
1	100	0.755	0.820	0.754	0.751	0.819	0.844
2	300	2.264	2.461	2.234	2.227	2.419	2.494
3	500	3.774	4.102	3.642	3.632	3.924	4.044
4	700	5.284	5.743	4.953	4.941	5.305	5.464
5	900	6.793	7.383	6.158	6.146	6.557	6.752
6	1100	8.303	9.024	7.262	7.252	7.693	7.921
7	1300	9.812	10.665	8.275	8.267	8.727	8.984
8	1500	11.322	12.305	9.208	9.202	9.672	9.959

Table 6. Transverse deflection $w(\text{cm})$ at $x = l / 2$ of the beam of Example 2 with total midspan height $h_{\text{mid}} = 0.12\text{m}$, for $P_z = 1500\text{kN}$ and for various numbers of boundary elements.

No of B.E.	Nonlinear Analysis with shear deformation	Nonlinear Analysis without shear deformation
9	10.607	10.248
19	10.451	9.924
29	10.396	9.856
39	10.332	9.826
49	10.284	9.808
59	10.259	9.796
69	10.230	9.787
79	10.228	9.781

Table 7. Transverse deflection w and axial displacement u at the $l/10$ of the beam of Example 3.

Transverse deflection w (cm)							
a/a	x(cm)	$h_l=0.10m$		$h_l=0.15m$		$h_l=0.20m$	
		Ignoring Shear Deform.	With Shear Deform.	Ignoring Shear Deform.	With Shear Deform.	Ignoring Shear Deform.	With Shear Deform.
1	22.959	0.516	0.579	0.438	0.497	0.378	0.433
2	48.469	2.120	2.233	1.734	1.837	1.442	1.537
3	73.980	4.450	4.593	3.496	3.624	2.805	2.920
4	99.490	6.990	7.146	5.265	5.402	4.077	4.198
5	125.000	8.994	9.147	6.499	6.631	4.868	4.982
6	150.510	9.622	9.762	6.697	6.815	4.877	4.978
7	176.020	8.778	8.895	5.903	5.999	4.198	4.278
8	201.531	6.629	6.713	4.321	4.388	3.014	3.069
9	227.041	3.430	3.474	2.180	2.214	1.500	1.527
10	250.000	0.000	0.000	0.000	0.000	0.000	0.000

Axial displacement u (mm)							
a/a	x(cm)	$h_l=0.10m$		$h_l=0.15m$		$h_l=0.20m$	
		Ignoring Shear Deform.	With Shear Deform.	Ignoring Shear Deform.	With Shear Deform.	Ignoring Shear Deform.	With Shear Deform.
1	22.959	0.177	0.163	0.196	0.184	0.207	0.198
2	48.469	-0.058	-0.104	0.132	0.098	0.245	0.219
3	73.980	-0.841	-0.914	-0.211	-0.261	0.132	0.096
4	99.490	-1.818	-1.903	-0.564	-0.621	0.053	0.013
5	125.000	-2.333	-2.417	-0.621	-0.676	0.149	0.112
6	150.510	-2.160	-2.241	-0.402	-0.456	0.353	0.315
7	176.020	-2.041	-2.130	-0.300	-0.362	0.459	0.416
8	201.531	-2.675	-2.792	-0.568	-0.647	0.371	0.316
9	227.041	-4.395	-4.564	-1.246	-1.354	0.099	0.027
10	250.000	-6.686	-6.920	-2.089	-2.229	-0.242	-0.331

Table 8. Transverse deflection w and axial displacement u at the $l/5$ of the beam of Example 3 for $h_l = 0.15m$.

a/a	x(cm)	Linear Analysis by BEM		Nonlinear Analysis			
		Without Shear Deform.	With Shear Deform.	Without Shear Deform. by BEM	With Shear Deform. by FEM [31]	Without Shear Deform. by BEM	With Shear Deform. by FEM [31]
Transverse deflection w (cm)							
1	48.469	1.888	2.003	1.734	1.732	1.837	1.860
2	99.490	5.768	5.935	5.265	5.257	5.402	5.499
3	150.510	7.394	7.549	6.697	6.685	6.815	6.922
4	201.531	4.814	4.905	4.321	4.310	4.388	4.425
5	250.000	0.000	0.000	0.000	0.000	0.000	0.000
Axial displacement u (mm)							
1	48.469	0.523	0.523	0.132	0.148	0.098	0.104
2	99.490	1.049	1.049	-0.564	-0.513	-0.621	-0.637
3	150.510	1.548	1.548	-0.402	-0.339	-0.456	-0.486
4	201.531	2.016	2.016	-0.568	-0.494	-0.647	-0.698
5	250.000	2.430	2.430	-2.089	-1.990	-2.229	-2.298

LIST OF FIGURE LEGENDS

Fig.1. Beam of variable cross section (a) occupying the two dimensional region Ω
(b).

Fig.2. Displacements (a) and forces (b) acting on the deformed element in the xz
plane.

Fig.3. Cantilever beam of a composite hollow variable cross section of Example 1.

Fig. 4. Deflections v of the composite beam of Example 1.

Fig. 5. Axial displacement u of the composite beam of Example 1.

Fig.6. Clamped beam of an I-type variable cross section of Example 2.

Fig. 7. Deflections $w(\text{cm})$ at midspan of the clamped beam of example 2
($h_{mid} = 0.12\text{m}$).

Fig.8. Beam of a linearly varying composite cross section of Example 3

Fig. 9. Deflections w of the composite beam of Example 3.

Fig. 10. Axial displacement u of the composite beam of Example 3.