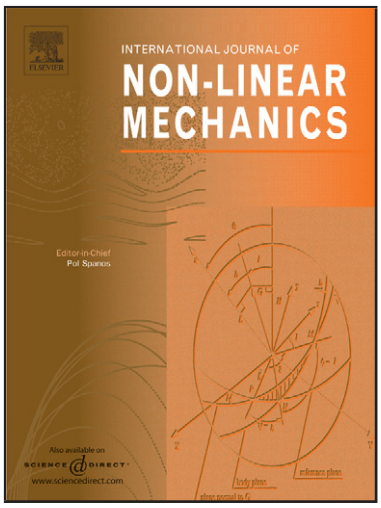
Post-bifurcation analysis of a thin-walled hyperelastic tube under inflation

Y.B. Fu, S.P. Pearce, K.K. Liu

Author's Accepted Manuscript



www.elsevier.com/locate/nlm

Post-bifurcation analysis of a thin-walled hyperelastic tube under inflation

Y.B. Fu, S.P. Pearce, K.K. Liu

PII: S0020-7462(08)00050-4
DOI: doi:10.1016/j.ijnonlinmec.2008.03.003
Reference: NLM 1460

To appear in: *International Journal of Non-Linear Mechanics*

Received date: 10 January 2008
Revised date: 11 March 2008
Accepted date: 11 March 2008

Cite this article as: Y.B. Fu, S.P. Pearce and K.K. Liu, Post-bifurcation analysis of a thin-walled hyperelastic tube under inflation, *International Journal of Non-Linear Mechanics* (2008), doi:10.1016/j.ijnonlinmec.2008.03.003

Post-bifurcation analysis of a thin-walled hyperelastic tube under inflation

Y.B. Fu[1], S.P. Pearce[1], and K.K. Liu[2]

[1]Department of Mathematics, Keele University, ST5 5BG, U.K.

[2]Institute for Science and Technology in Medicine, Keele University, ST4 7QB, U.K.

Abstract

We consider the problem of bulging, or necking, of an infinite thin-walled hyperelastic tube that is inflated by an internal pressure, with the axial stretch at infinity maintained at unity. We present a simple procedure that can be used to derive the bifurcation condition and to determine the near-critical behaviour analytically. It is shown that there is a bifurcation with zero mode number and that the associated axial variation of near-critical bifurcated configurations is governed by a first-order differential equation that admits a locally bulging or necking solution. This result suggests that the corresponding bifurcation pressure can be identified with the so-called initiation pressure which featured in recent experimental studies. This is supported by good agreement between our theoretical predications and one set of experimental data. It is also shown that the Gent material model can support both bulging and necking solutions whereas the Varga and Ogden material models can only support bulging solutions. Relevance of the present method to the study of nonlinear wave propagation in a fluid-filled distensible tube is also discussed.

1 Introduction

Inflation of membrane tubes and the associated problem of bifurcation and instability is a classical subject that has been studied in many books and papers. For a membrane tube sealed by a rigid plug at each end and modeled by the Mooney-Rivlin material model, Kydoniefs and Spencer [1] showed that as the radius of the middle section increased, the pressure passed through a maximum (the *limiting pressure*) and then decreased, a phenomenon that had previously been observed for spherical membranes by Green and Shield [2], Adkins and Rivlin [3], and Green and Adkins [4]. Kydoniefs and Spencer [1] also observed the tendency, in the case of a neo-Hookean membrane tube, for a nearly spherical bulge to form at the centre of the cylinder. Determination of the limiting pressure for other material models has also received some attention; see, for instance, Benedict *et al* [5] and Alexander [6]. Yin

[7] addressed the question whether a material could support a fully nonlinear bifurcation mode consisting of two uniform sections joined by a non-uniform transitional section, a phenomenon commonly observed in party balloons. He found the conditions that needed to be imposed on the strain-energy function for the material. The mechanisms for the initiation and propagation of bulges in an inflated membrane tube was made more transparent through the analysis of Chater and Hutchinson [8], and the experimental and numerical studies of Kyriakides and Chang [9, 10]. In particular, these authors observed that this problem shared the same features as a family of other problems such as propagating buckles in long metal tubes under external pressure [11, 12], propagating necks in some polymeric materials when pulled in tension [13], and phase transition problems [14, 15, 16]. To this family, we may also add the problem of kink band formation and propagation in layered structures [17] and fibre-reinforced composites [18]. We observe, however, that whereas the determination of the propagation pressure through the equal-area rule is now universally accepted, there is still some uncertainty concerning the determination of the pressure, termed the *initiation pressure*, at which bulging first occurs. Kyriakides and Chang [9] took this pressure to be the limiting pressure corresponding to the first turning point in the pressure-volume diagram for uniform inflation, but this value was found to be slightly higher than what had been observed in the experiments (see their figure 9). Various other numerical schemes have also been used to determine the pressure-volume curves beyond the initiation pressure; see, for instance, Duffet and Reddy [19], Shi and Moita [20], Verron and Marckmann [21], and the references therein. Additional experimental results are given recently by Pamplona *et al* [22] and Goncalves *et al* [23].

The bifurcation and stability properties of the uniform inflation state are expected to be relevant to the determination of the initiation pressure. A thorough bifurcation study of the uniform inflation state was carried out by Haughton and Ogden [24], with some additional results given more recently by Haughton [25]. Stability of the uniform inflation state was studied by Corneliussen and Shield [26], Shield [27], and Chen [28] using the energy method. However, although Shield's stability results were discussed by Kyriakides and Chang [10], the results were not found to be relevant to the determination of the initiation pressure. This is probably because neither the bifurcation nor the stability analysis mentioned above is able to describe correctly the bifurcation mode having zero wave number, and it is in fact this mode that is relevant to the determination of the initiation pressure. This special mode is not sinusoidal or constant — its correct variation can only be found from a near-critical nonlinear analysis (see, for instance, Fu [29]). It will be shown that this mode is localized so that it can also be supported by a finite tube. In fact, this localized mode is governed by a first-order differential equation that is an integral of the classical Korteweg-de Vries evolution equation.

The Gent material model [30] has recently emerged as a popular and possibly realistic model for rubber and bio-materials. This is a so-called finite-chain model that imposes

a maximum stretch that the material (or more precisely the polymeric molecular chains composing the material) can experience. This model has been discussed in a number of recent papers; see Horgan and Saccomandi [31, 32], and Ogden *et al* [33]. Gent [34] showed, for the case of closed ends, that the pressure-volume curve for the Gent material has a maximum followed by a minimum, which implies that for each pressure value in a certain interval there are two corresponding stable volume values (a property which is not found in the Varga or neo-Hookean material model). Gent [35, p.167] further stated that this is consistent with the experimental result that "thin-walled rubber tubes undergo a strikingly non-uniform deformation at a critical inflation pressure. One portion of the tube becomes highly distended as a bubble or aneurysm while the rest remains lightly inflated". This is exactly what has already been observed by Kyriakides and Chang [10] for other material models.

The initial motivation for the current study is to extend Gent's [35] own study and discover more about what the Gent model can predict with regard to the formation of an aneurysm (a localized, blood-filled dilation of a blood vessel caused by disease or weakening of the vessel wall). We focus on the simplest case when an infinite cylindrical membrane is inflated by an internal pressure with the remote axial stretch maintained at unity all the time (i.e. both before and after any bulging has taken place; this case differs from the closed-end case in that the pressure-volume curve for the Gent material would now be monotonically increasing). In the course of our study we find that analysis can be carried out much further than previously known before having to resort to numerical calculations.

The rest of this paper proceeds as follows. After formulating the problem in the next section, we show in Section 3 that the bifurcation condition can in fact be deduced from two simple equations that determine the two principal stretches at a point where the radius is a local maximum or minimum. In Section 4 we carry out a near-critical post-bifurcation analysis and derive the leading-order amplitude equation for the bifurcation solution. The amplitude equation is then used to determine under what conditions a localized solution is possible and, if it is, whether it is of the bulging or necking type. In Section 5 we present some numerical results for the fully nonlinear post-bifurcation states, performed with the aid of Mathematica, and compare them with the analytical near-critical results obtained in Section 4. In the final section, we discuss implications of our findings to the determination of the initiation pressure, and the relevance of our methodology to the study of nonlinear wave propagation in fluid-filled distensible tubes.

2 Governing equations

We consider the problem of inflation of an infinite cylindrical membrane tube that is incompressible, isotropic, and hyperelastic. The tube is assumed to have uniform thickness H and uniform inner radius R before inflation. To simplify analysis, we assume that the remote

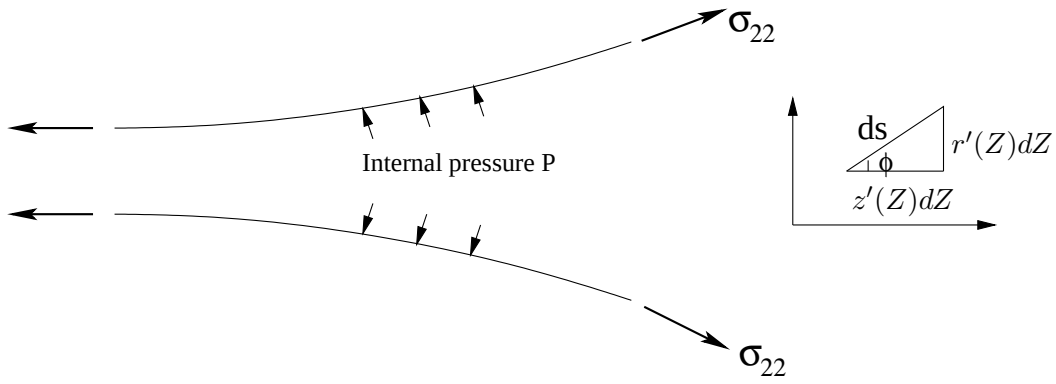


Figure 1: Axisymmetric deformation of a thin-walled rubber tube

axial stretch is maintained at unity all the time, but our results can easily be modified for the case when the remote axial stretch takes any other value. We also assume that when the tube is inflated by an internal pressure, the inflated configuration maintains axial symmetry, but its radius can be constant or varying along the axial direction. We shall refer to the configuration with a constant radius the *primary state* and all other possible configurations *bifurcated states*. Thus, in general, the axisymmetric deformed configuration may be described by

$$r = r(Z), \qquad \theta = \Theta, \qquad z = z(Z), \tag{2.1}$$

where Z and z are the axial coordinates of a representative material particle before and after inflation, respectively, and r is the inner radius after inflation; see Fig. 1.

Since the deformation is axially symmetric, the principal directions of stretch coincide with the lines of latitude, the meridian and the normal to the deformed surface. Thus, the principal stretches are given by

$$\lambda_1 = \frac{r}{R}, \qquad \lambda_2 = \sqrt{r'^2 + z'^2}, \qquad \lambda_3 = \frac{h}{H}, \tag{2.2}$$

where the indices $(1, 2, 3)$ are used for the latitudinal, meridional and normal directions respectively and the primes indicate differentiation with respect to Z.

The principal Cauchy stresses $\sigma_1, \sigma_2, \sigma_3$ in the deformed configuration for an incompressible material are given by

$$\sigma_i = \lambda_i W_i - p, \qquad i = 1, 2, 3 \text{ (no summation)}, \tag{2.3}$$

where $W = W(\lambda_1, \lambda_2, \lambda_3)$ is the strain-energy function, $W_i = \partial W/\partial \lambda_i$, and p is the pressure associated with the constraint of incompressibility; see, for instance, Ogden [36]. Utilising the incompressibility constraint $\lambda_1\lambda_2\lambda_3 = 1$ and the membrane assumption $\sigma_3 = 0$ we find

$$\sigma_i = \lambda_i \hat{W}_i, \qquad i = 1, 2 \text{ (no summation)}, \tag{2.4}$$

where $\hat{W}(\lambda_1,\lambda_2)=W(\lambda_1,\lambda_2,\lambda_1^{-1}\lambda_2^{-1})$ and $\hat{W}_1=\partial\hat{W}/\partial\lambda_1$ etc.

From the fact that the resultant in the Z-direction at any cross section (balancing the tension in the membrane and the inflation pressure) must be a constant, we obtain

$$\frac{\hat{W}_2 z'}{\lambda_2}-\frac{1}{2}P\lambda_1^2 R=C_2, \tag{2.5}$$

where P is the internal pressure scaled by H and C_2 is a constant. By considering equilibrium in the normal direction of an infinitesimal volume element, we obtain

$$\frac{\sigma_2(r''z'-r'z'')}{\lambda_2^3}-\frac{z'\sigma_1}{r\lambda_2}+\frac{P}{\lambda_3}=0. \tag{2.6}$$

With the use of (2.4) and the differentiated form of (2.5), we may integrate (2.6) once to obtain

$$\hat{W}-\lambda_2\hat{W}_2=C_1, \tag{2.7}$$

where C_1 is another constant of integration. This integral was first noted by Pipkin [37]. Equations (2.5) and (2.7) correspond to Haughton's [25] equations (3.29) and (3.27) respectively.

Without loss of generality, we take $R=1$ (which is equivalent to scaling all the length variables by R). To find the constants C_1 and C_2 we assume that the cylinder is uniform far away from the origin with a constant radius r_∞:

$$\lim_{Z\to\pm\infty} r(Z)=r_\infty, \quad \lim_{Z\to\pm\infty} z(Z)=Z,$$

and hence

$$r'\to 0, \quad z'\to 1, \quad \lambda_1\to r_\infty, \quad \lambda_2\to 1 \ \text{ as } Z\to\pm\infty.$$

Thus, from the outset we focus on localized bifurcation solutions by excluding sinusoidal-type solutions. On taking the limit $Z\to\infty$ in (2.5), (2.6) and (2.7), we obtain

$$C_2=\hat{W}_2^{(\infty)}-\frac{1}{2}Pr_\infty^2, \quad P=\frac{\hat{W}_1^{(\infty)}}{r_\infty}, \quad C_1=\hat{W}^{(\infty)}-\hat{W}_2^{(\infty)}, \tag{2.8}$$

where the superscript (∞) signifies evaluation at $\lambda_1=r_\infty,\lambda_2=1$.

We shall consider three types of materials: Varga, Ogden and Gent, for which the strain-energy functions are given, respectively, by

$$W=2\mu(\lambda_1+\lambda_2+\lambda_3-3), \tag{2.9}$$

$$W=\sum_{r=1}^{3}\mu_r(\lambda_1^{\alpha_r}+\lambda_2^{\alpha_r}+\lambda_3^{\alpha_r}-3)/\alpha_r, \tag{2.10}$$

$$W=-\frac{1}{2}\mu J_m\ln(1-\frac{J_1}{J_m}), \quad J_1=\lambda_1^2+\lambda_2^2+\lambda_3^2-3, \tag{2.11}$$

where μ is the shear modulus for infinitesimal deformations, J_m is a material constant and

$$\alpha_1 = 1.3, \quad \alpha_2 = 5.0, \quad \alpha_3 = -2.0, \quad \mu_1 = 1.491, \quad \mu_2 = 0.003, \quad \mu_3 = -0.023$$

are material constants given by Ogden [38] (see also Ogden [36]). For rubbers, a typical value for J_m is 97.2 which corresponds to a maximum stretch of 10 in uniaxial tension [34]. Horgan and Saccomandi [31] give, from experimental data, values of J_m between 0.422 and 3.93 for human arteries. In the limit $J_m \to \infty$, (2.11) reduces to

$$W = \frac{1}{2}\mu(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3), \tag{2.12}$$

the energy function for the neo-Hookean material.

For the problem under consideration, the volume of the tube is given by $v = \pi\lambda_1^2$ per unit length in the z-direction and so turning points of the (P, v)-curve coincide with those of the (P, λ_1)-curve. It is easy to show with the aid of $(2.8)_2$ that the (P, λ_1)-curve has a maximum at $\lambda_1 = r_\infty = 1.7321$ and no minimum for the Varga material, has a maximum at $r_\infty = 1.9582$ and a minimum at $r_\infty = 2.8380$ for the Ogden material and is monotone increasing for the Gent material (for all $J_m > 0$).

3 Bifurcation condition

With C_1 and C_2 known, equations (2.5) and (2.7) are two first-order coupled nonlinear differential equations for $r(Z)$ and $z(Z)$. For a bulging/necking solution that is symmetric about the origin $Z = 0$, we impose the conditions

$$r'(0) = 0, \quad z(0) = 0. \tag{3.1}$$

In order to integrate (2.5) and (2.7) from $Z = 0$ towards $Z \to \infty$ to obtain a bulging/necking solution, we need to know $r(0)$ as well as $z(0)$. To find $r(0)$, we evaluate (2.5) and (2.7) at $Z = 0$ and obtain respectively

$$\hat{W}_2(r_0, z_0') - \frac{\hat{W}_1^{(\infty)}}{2r_\infty}(r_0^2 - r_\infty^2) - \hat{W}_2^{(\infty)} = 0, \tag{3.2}$$

$$\hat{W}(r_0, z_0') - z_0'\,\hat{W}_2(r_0, z_0') - \hat{W}^{(\infty)} + \hat{W}_2^{(\infty)} = 0, \tag{3.3}$$

where $r_0 = r(0), z_0' = z'(0) \geq 0$. These two equations can be solved to find r_0 (and z_0') as a function of r_∞. Obviously $r_0 = r_\infty$ is always a solution which corresponds to the uniform solution $r \equiv r_\infty$ of (2.5) and (2.7). A necessary condition for a non-uniform bifurcated solution to exist is that (3.2) and (3.3) have a solution other than $r_0 = r_\infty$, $z_0' = 1$. In Figures 2(a), 3(a) and 4(a), we have shown such (non-trivial) solutions of (3.2) and (3.3) for the Varga material, the Gent material with $J_m = 97.2$ and the Ogden material, respectively. The $\omega(r_\infty)$ in these figures is defined by (4.8) and $\omega(r_\infty) = 0$ determines the bifurcation values

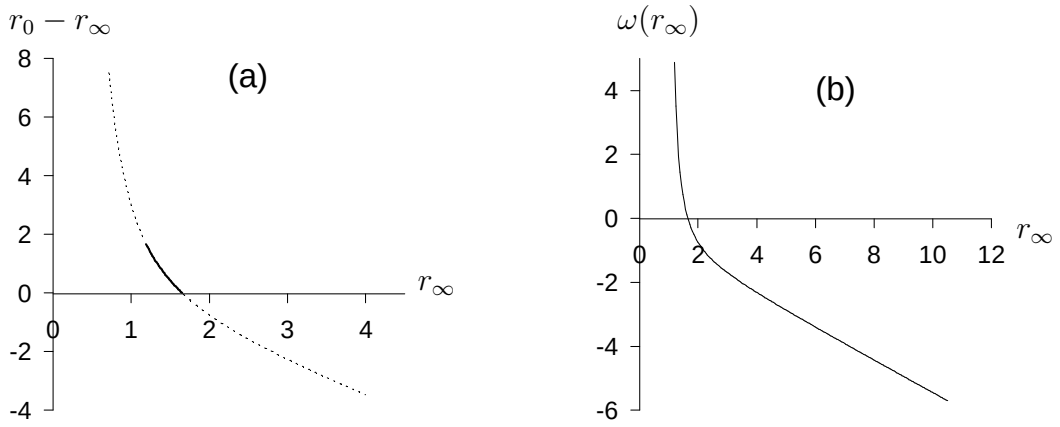


Figure 2: Amplitude $r_0 - r_\infty$ and $\omega(r_\infty)$ as functions of r_∞ for the Varga material. The dotted lines in (a) indicate the fact that the corresponding solution is either periodic or unbounded.

of r_∞. As expected, such solutions only exist for certain values of r_∞. Of course, existence of such a non-trivial value of r_0 does not guarantee that it corresponds to a localized solution of $r(Z)$. This will be discussed in more details in Sections 4 and 5.

Now suppose that after the trivial solution $r_0 = r_\infty$ has been factorized out, (3.2) and (3.3) have a solution given by

$$r_0 = h(r_\infty). \tag{3.4}$$

Then any bifurcation values are given by the roots of $r_\infty = h(r_\infty)$, i.e. the intersection of the trivial solution with the nontrivial solution. This suggests a method to derive the bifurcation condition as follows.

We assume that the solution of (3.2) for z_0' in terms of r_0 is given by

$$z_0' = k(r_0), \quad \text{with } k(r_\infty) = 1. \tag{3.5}$$

We may alternatively choose (3.3) for this purpose. Of course, in general an explicit expression for $k(r_0)$ does not exist, but we may differentiate (3.2) once and twice with respect to r_0 and then evaluating the resulting expressions at $r_0 = r_\infty$ to obtain

$$\hat{W}_1^{(\infty)} - \hat{W}_{12}^{(\infty)} - \hat{W}_{22}^{(\infty)}\, k'(r_\infty) = 0, \tag{3.6}$$

$$\hat{W}_{112}^{(\infty)} + 2k'(r_\infty)\hat{W}_{122}^{(\infty)} + (k'(r_\infty))^2\hat{W}_{222}^{(\infty)} + k''(r_\infty)\hat{W}_{22}^{(\infty)} - \frac{\hat{W}_1^{(\infty)}}{r_\infty} = 0. \tag{3.7}$$

These two equations can be solved to obtain an explicit expression for $k'(r_\infty)$ and $k''(r_\infty)$.

Next, with the use of (3.5), the left hand side of the second equation (3.3) is a function of r_0 only and we know that it having r_∞ as a non-trivial root defines the bifurcation condition. Differentiating the left hand side of (3.3) with respect to r_0 and then evaluating at $r_0 = r_\infty$ followed by the use of (3.6), we find that the left hand side becomes identically zero, indicating

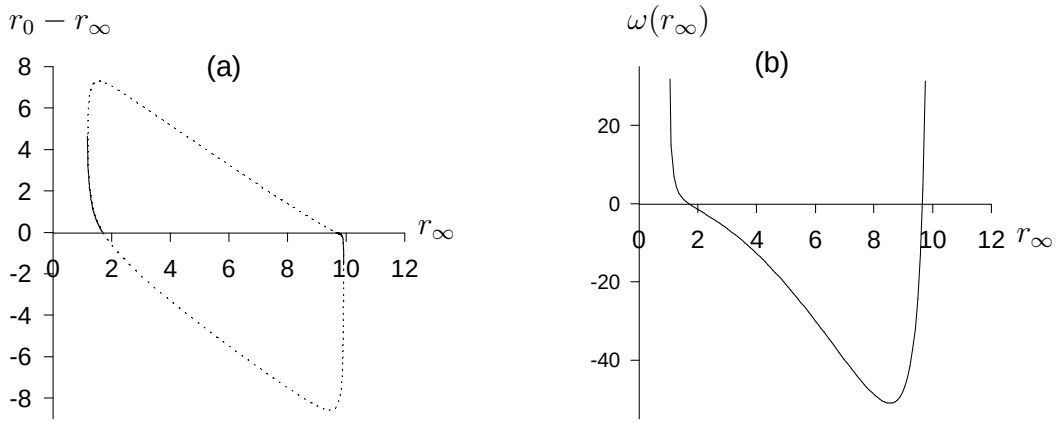


Figure 3: Amplitude $r_0 - r_\infty$ and $\omega(r_\infty)$ as functions of r_∞ for the Gent material with $J_m = 97.2$. The dotted lines in (a) indicate the fact that the corresponding solution is either periodic or unbounded.

that the trivial solution $r_0 = r_\infty$ is in fact a double root. Thus, near $r_0 = r_\infty$ the left hand side of (3.3), as a function of r_0, expands like $(r_0 - r_\infty)^2 S(r_\infty) + O((r_0 - r_\infty)^3)$, and the bifurcation condition corresponds to $S(r_\infty) = 0$. Therefore, the bifurcation condition can be derived by differentiating (3.3) twice with respect to r_0 and then evaluating at $r_0 = r_\infty$. We then obtain for the bifurcation condition,

$$r_\infty(\hat{W}_1^{(\infty)} - \hat{W}_{12}^{(\infty)})^2 + \hat{W}_{22}^{(\infty)}(\hat{W}_1^{(\infty)} - r_\infty \hat{W}_{11}^{(\infty)}) = 0, \tag{3.8}$$

which agrees with Haughton's [25] equation (3.50) (note that the only plus sign in his equation (3.43) should be a minus). With the use of $(2.8)_2$ we may rewrite the above bifurcation condition as

$$r_\infty(\hat{W}_1^{(\infty)} - \hat{W}_{12}^{(\infty)})^2 - r_\infty^2 \hat{W}_{22}^{(\infty)} \frac{dP}{dr_\infty} = 0. \tag{3.9}$$

Assuming that $\hat{W}_{22}^{(\infty)} > 0$ (see discussion in Section 6), we see from (3.9) that a bifurcation can never take place on the section of the (P, r_∞)-curve where $dP/dr_\infty < 0$ (Haughton and Ogden [24]). A likely scenario is that as inflation proceeds, a bifurcation will take place before the limiting pressure is reached.

To illustrate the above idea, we consider the Varga material defined by (2.9). Equation (3.3) can be solved explicitly to express z_0' in terms of r_0. Equation (3.2) then becomes

$$(r_0 - r_\infty)^2 \left\{ r_0 - \frac{2(1 + r_\infty^2)}{r_\infty^3} \right\} = 0, \tag{3.10}$$

and so the bifurcation condition is given by $r_\infty = 2(1 + r_\infty^2)/r_\infty^3$, which has a unique positive real root $r_\infty = \sqrt{1 + \sqrt{3}} \approx 1.6529$, smaller than the value 1.7321 of r_∞ corresponding to the turning point of the (P, r_∞)-curve.

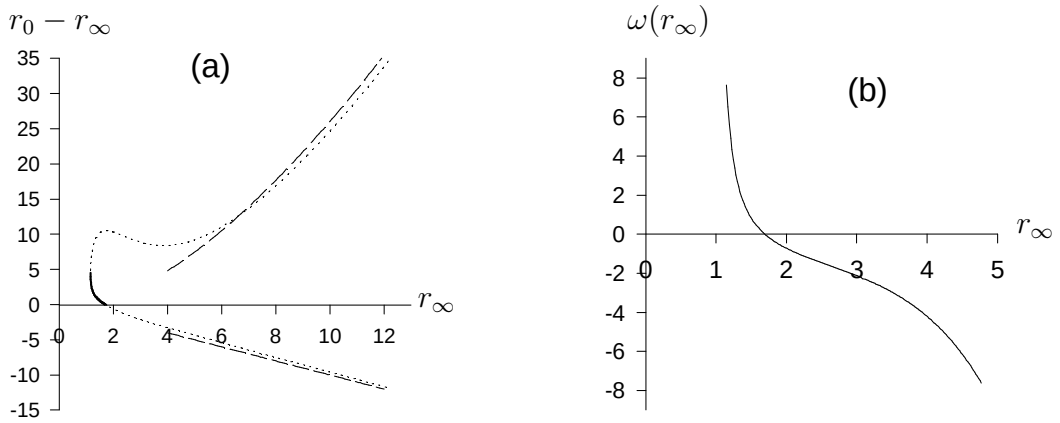


Figure 4: Amplitude $r_0 - r_\infty$ and $\omega(r_\infty)$ as functions of r_∞ for the Ogden material. The dotted lines in (a) indicate the fact that the corresponding solution is either periodic or unbounded.

4 Near-critical post-bifurcation analysis

In this section we show that the bifurcation condition (3.8) also emerges naturally in our near-critical post-bifurcation analysis. The latter analysis also yields the conditions under which a localized bifurcation solution is possible.

We first rewrite (2.7) and (2.5) as

$$f(\lambda_1, \lambda_2) \equiv \hat{W} - \lambda_2 \hat{W}_2 - C_1 = 0, \tag{4.1}$$

$$g(\lambda_1, \lambda_2) \equiv \frac{\lambda_2}{\hat{W}_2}(C_2 + \frac{P}{2}\lambda_1^2) = z', \tag{4.2}$$

which define the two functions f and g. We write

$$\lambda_1 = r_\infty + y, \tag{4.3}$$

where y is assumed to be small. Correspondingly, we expect λ_2 to have the expansion

$$\lambda_2 = 1 + d_1 y + \frac{1}{2} d_2 y^2 + \mathcal{O}(y^3), \tag{4.4}$$

where the coefficients can be obtained by substituting (4.3) and (4.4) into (4.1), expanding the latter as a power series in y, and then equating the coefficients of y and y^2. This can best be carried out with the aid of a symbolic manipulation package such as *Mathematica*. Likewise, by substituting (4.3) and (4.4) into (4.2) and expanding in terms of y, we obtain

$$z' = 1 + g_1 y + \frac{1}{2} g_2 y^2 + \mathcal{O}(y^3), \tag{4.5}$$

where

$$g_1 = \left.\frac{dg}{dy}\right|_{y=0}, \quad g_2 = \left.\frac{d^2 g}{dy^2}\right|_{y=0}.$$

On substituting (4.4) and (4.5) into $\lambda_2 = \sqrt{r'^2 + z'^2}$, we obtain $d_1 = g_1$ to leading order, and

$$y'^2 = \omega(r_\infty) y^2 + \mathcal{O}(y^3) \tag{4.6}$$

to the next order, where

$$\omega(r_\infty) = d_2 - g_2. \tag{4.7}$$

In terms of the various derivatives of $\hat{W}$, we have

$$\omega(r_\infty) = \frac{-1}{r_\infty \hat{W}_2^{(\infty)} \hat{W}_{22}^{(\infty)}} \left\{ r_\infty (\hat{W}_1^{(\infty)} - \hat{W}_{12}^{(\infty)})^2 + \hat{W}_{22}^{(\infty)} (\hat{W}_1^{(\infty)} - r_\infty \hat{W}_{11}^{(\infty)}) \right\}. \tag{4.8}$$

The behaviour of solution of the differential equation (4.6) changes when $\omega(r_\infty) = 0$. This gives the bifurcation condition consistent with the earlier expression (3.8).

For the Varga and Gent materials, $\omega(r_\infty)$ is given by

$$\omega(r_\infty) = \frac{2 + 2r_\infty^2 - r_\infty^4}{2\,(r_\infty - 1)\, r_\infty^2}, \tag{4.9}$$

and

$$\omega(r_\infty) = \frac{J_m r_\infty^2 (r_\infty^8 - 6r_\infty^4 - 4r_\infty^2 - 3) - (r_\infty^2 - 1)^2 (r_\infty^8 + 6r_\infty^6 + 8r_\infty^4 - 2r_\infty^2 - 1)}{r_\infty^2 (r_\infty^2 - 1)(r_\infty^6 - (J_m + 1) r_\infty^4 - (3J_m + 1) r_\infty^2 + 1)}, \tag{4.10}$$

respectively. The corresponding expression for the Ogden material is too long and is not written out here. In Figures 2(b), 3(b) and 4(b) we have shown $\omega(r_\infty)$ for the three materials. We observe that the Varga and Ogden materials each predict one bifurcation value, given by $r_\infty = 1.6529$ and $r_\infty = 1.6955$, respectively, but the Gent material predicts two bifurcation values: 1.74261 and 5.0850 when $J_m = 30$, and 1.6907 and 9.6523 when $J_m = 97.2$. We also find that as J_m decreases the two bifurcation values move towards each other and coalesce at $r_\infty = 2.2545$ when $J_m = 11.0693$. When J_m is decreased further no bifurcation values exist. Thus, the type of bifurcation addressed in the present paper cannot occur in human arteries for which J_m ranges between 0.422 and 3.93.

The above expansion (4.6) can in fact be extended to higher orders with the aid of *Mathematica*. We have, in place of (4.6),

$$y'^2 = \omega(r_\infty)\, y^2 + \gamma(r_\infty)\, y^3 + \mathcal{O}(y^4), \tag{4.11}$$

where the expression for the coefficient $\gamma(r_\infty)$ is easily obtained using *Mathematica* but is not written out here for brevity.

In the near-critical post-bifurcation analysis, we write

$$r_\infty = r_{cr} + \epsilon r_1, \tag{4.12}$$

where r_{cr} is a root of the bifurcation condition $\omega(r_\infty) = 0$, ϵ is a positive small parameter and r_1 is a constant. On substituting (4.12) into (4.11) and expanding ω around r_{cr}, we obtain

$$y'^2 = \omega'(r_{cr}) \epsilon r_1 y^2 + \gamma(r_{cr}) y^3, \tag{4.13}$$

where we have neglected the higher-order terms. We assume that y' does not change sign for $0 \leq Z < \infty$. Then when $r_0 > r_\infty$ (a bulging solution), we have $y' < 0$ and $y > 0$; whereas when $r_0 < r_\infty$ (a necking solution), we have $y' > 0$ and $y < 0$. Thus, for both cases we obtain from (4.13)

$$y' = -y\sqrt{\omega'(r_{cr})\epsilon r_1 + \gamma(r_{cr})y}. \tag{4.14}$$

The amplitude equation (4.14) indicates that y is of order ϵ and that its variation takes place on an $O(1/\sqrt{\epsilon})$ lengthscale.

We now consider the solution of the following four prototypal equations:

$$y_1' = -ay_1\sqrt{1+by_1}, \quad y_2' = -ay_2\sqrt{1-by_2}, \tag{4.15}$$

$$y_3' = -ay_3\sqrt{-1+by_3}, \quad y_4' = -ay_4\sqrt{-1-by_4}, \tag{4.16}$$

where y_1, y_2, y_3, y_4 are functions of Z and a, b are positive constants given by

$$a = \sqrt{|\omega'(r_{cr})\epsilon r_1|}, \quad b = |\gamma(r_{cr})|/a^2.$$

The relevance of these equations to (4.14) is obvious and we have

$$y = \begin{cases} y_1 & \text{if } \omega'(r_{cr})(r_\infty - r_{cr}) > 0,\ \gamma(r_{cr}) > 0, \\ y_2 & \text{if } \omega'(r_{cr})(r_\infty - r_{cr}) > 0,\ \gamma(r_{cr}) < 0, \\ y_3 & \text{if } \omega'(r_{cr})(r_\infty - r_{cr}) < 0,\ \gamma(r_{cr}) > 0, \\ y_4 & \text{if } \omega'(r_{cr})(r_\infty - r_{cr}) < 0,\ \gamma(r_{cr}) < 0. \end{cases} \tag{4.17}$$

It can be shown that

$$y_1 = \begin{cases} \frac{1}{b}[\sinh(\frac{1}{2}aZ + A)]^{-2}, & \text{if } y_1 > 0 \\ -\frac{1}{b}[\cosh(\frac{1}{2}aZ + A)]^{-2}, & \text{if } y_1 < 0 \end{cases} \tag{4.18}$$

$$y_2 = \begin{cases} -\frac{1}{b}[\sinh(\frac{1}{2}aZ + A)]^{-2}, & \text{if } y_2 < 0 \\ \frac{1}{b}[\cosh(\frac{1}{2}aZ + A)]^{-2}, & \text{if } y_2 > 0 \end{cases} \tag{4.19}$$

$$y_3 = \frac{1}{b[\cos(\frac{1}{2}aZ + A)]^2}, \quad y_4 = -\frac{1}{b[\cos(\frac{1}{2}aZ + A)]^2}, \tag{4.20}$$

where A is an integration constant which can be set to zero since the origin of Z can be arbitrarily chosen.

By inspection, we may immediately dismiss y_3, y_4, $y_1 > 0$ and $y_2 < 0$ as we are only looking for localized solutions and they all blow up at finite values of Z. We are then left with

$$y = \begin{cases} -\frac{1}{b}[\cosh(\frac{1}{2}aZ)]^{-2}, & \text{if } \omega'(r_{cr})(r_\infty - r_{cr}) > 0,\ \gamma(r_{cr}) > 0, \\ \frac{1}{b}[\cosh(\frac{1}{2}aZ)]^{-2}, & \text{if } \omega'(r_{cr})(r_\infty - r_{cr}) > 0,\ \gamma(r_{cr}) < 0. \end{cases} \tag{4.21}$$

It follows that near-critical localized solutions exist only if

$$\omega'(r_{cr})(r_\infty - r_{cr}) > 0, \tag{4.22}$$

and that when such a solution exists it is of the bulging type if $\gamma(r_{cr}) < 0$ and of the necking type if $\gamma(r_{cr}) > 0$. Thus, only a super-critical localized solution is possible if

$$\omega'(r_{cr}) > 0, \tag{4.23}$$

that is only if the curve of $\omega(r_\infty)$ at the bifurcation point has a positive slope. Similarly, only a sub-critical localized solution is possible $\omega'(r_{cr}) < 0$.

When the condition (4.22) is not satisfied, our analysis above would predict a solution that blows up at a finite value of Z, but we should bear in mind the fact that our analysis has been conducted under the small-amplitude assumption so that terms of order y^4 in (4.11) could be neglected. Thus, when y grows and reaches a sufficient amplitude the higher order terms may no longer be neglected and they may act to suppress further growth. Our numerical integration to be explained in Section 5 does indeed show that whenever (4.22) is not satisfied the corresponding solution is usually periodic and hence does not satisfy our conditions at infinity.

We note that y and $r_0 - r_\infty$ have the same sign and that a bulging/aneurysm solution corresponds to $y > 0$ whereas a necking solution corresponds to $y < 0$. Thus, referring to Figures 2(a), 3(a) and 4(a), we see that for the Gent material with $J_m = 97.2$, near the first bifurcation point, $r_\infty = 1.6907$, the post-critical solution must necessarily be of the bulging type, whereas near the second bifurcation point, $r_\infty = 9.6523$, the post-critical solution must necessarily be of the necking type. For the other two materials, the post-critical solution associated with the only bifurcation point must necessarily be of the bulging type.

We also observe that the maximum or minimum of y occurs when $y' = 0$ and it occurs at $Z = 0$. It then follows from (4.14) that the maximum/minimum is given by

$$y_{\max/\min} = -\omega'(r_{cr})(r_\infty - r_{cr})/\gamma(r_{cr}).$$

The relation

$$r_0 \approx r_\infty + y_{\max/\min} = r_\infty - \omega'(r_{cr})(r_\infty - r_{cr})/\gamma(r_{cr})$$

then provides a consistency check on our numerical values of r_0.

We conclude this section by remarking that the bifurcation problem addressed in this section corresponds to a bifurcation with zero mode number. Such bifurcations are usually associated with infinite domains but the corresponding spatially localized solutions are also valid for a finite domain since the finite boundaries (such as the ends of the tube) are not felt by the localized solutions. For such bifurcation problems, linear bifurcation analysis would incorrectly predict the eigenfunction as a constant; the correct shape of the eigenfunction can only be deduced from a weakly nonlinear, multiple-scale analysis. Such an analysis would usually yield an amplitude equation of the form

$$A_{XX} + c_1 A + c_2 A^2 = 0, \tag{4.24}$$

where X is a far-distance variable, A is the amplitude function and c_1 and c_2 are constants; see Fu [29, p.362]. Multiplying (4.24) by A_X and then integrating, we would obtain

$$(A_X)^2 + c_1 A^2 + \frac{2}{3} c_2 A^3 = \text{const}, \tag{4.25}$$

which is seen to be of the same form as (4.13) derived earlier. It is due to the simplicity of the current problem that we were able to derive (4.13) without having to conduct a multiple-scale analysis in the standard manner.

5 Numerical solutions

Following Kyriakides and Chang [10], we may write down the following system of differential equations for $\lambda_1, \lambda_2, \phi$:

$$\begin{aligned} \lambda_1' &= \lambda_2 \sin\phi, \\ \lambda_2' &= \frac{\hat{W}_1 - \lambda_2 \hat{W}_{12}}{\hat{W}_{22}} \sin\phi, \\ \phi' &= \frac{\hat{W}_1}{\hat{W}_2} \cos\phi - \frac{\hat{W}_1^\infty}{r_\infty \hat{W}_2} \lambda_1 \lambda_2, \end{aligned} \tag{5.1}$$

where ϕ is defined in Fig. 1. These equations are solved subject to the initial conditions

$$\lambda_1(0) = r(0), \quad \lambda_2(0) = z'(0), \quad \phi(0) = 0. \tag{5.2}$$

We could also use λ_1, λ_2, r' as dependent variables (Yang and Feng [39]), in which case we solve

$$\begin{aligned} \lambda_1' &= r', \\ \lambda_2' &= \frac{\hat{W}_1 - \lambda_2 \hat{W}_{12}}{\hat{W}_{22}} \frac{r'}{\lambda_2}, \\ (r')' &= \frac{\hat{W}_1 \lambda_2}{\hat{W}_2} + \left(\frac{\hat{W}_1 - \lambda_2 \hat{W}_{12}}{\hat{W}_{22}} - \frac{\hat{W}_1 \lambda_2}{\hat{W}_2} \right) \left(\frac{r'}{\lambda_2} \right)^2 - \frac{\hat{W}_1^\infty}{r_\infty \hat{W}_2} \lambda_1 \lambda_2^2 \sqrt{1 - \left(\frac{r'}{\lambda_2} \right)^2} \end{aligned} \tag{5.3}$$

subject to the conditions

$$\lambda_1(0) = r(0), \quad \lambda_2(0) = z'(0), \quad r'(0) = 0. \tag{5.4}$$

For the Varga material, (2.7) and (2.5) reduce to

$$\sqrt{r'^2 + z'^2} = \frac{2 r_\infty}{r(r_\infty^2 + 2 - r r_\infty)}, \tag{5.5}$$

and

$$\frac{z'}{\sqrt{r'^2 + z'^2}} \left(1 - \frac{1}{r(r'^2 + z'^2)} \right) - \frac{P r^2}{2H} = 1 - \frac{1}{r_\infty} - \frac{P r_\infty^2}{2H}, \tag{5.6}$$

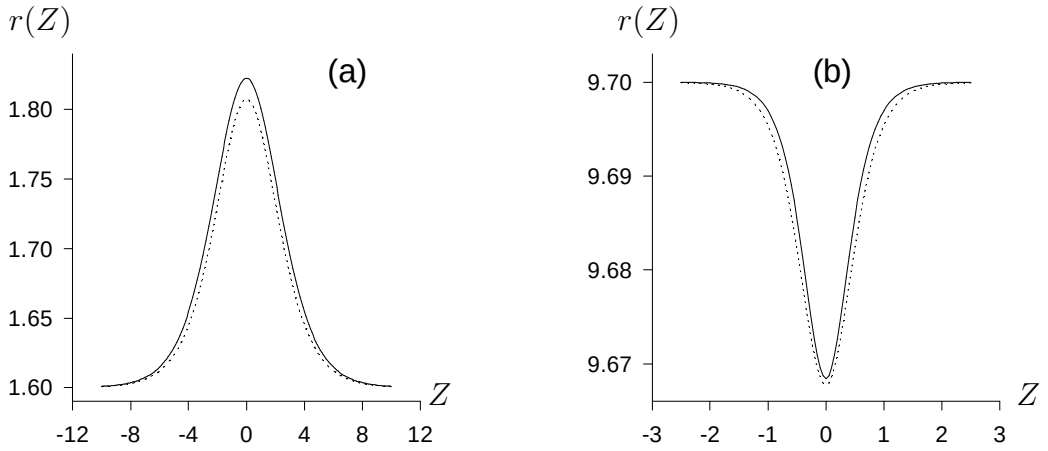


Figure 5: Near-critical bulging/necking solutions corresponding to the Gent material with $J_m = 97.2$. (a) $r_\infty = 1.6$; (b) $r_\infty = 9.7$. Solid lines: numerical solution; dotted lines: asymptotic solution given by (4.21).

respectively. We then obtain from (5.5) a single first-order differential equation

$$r' = \pm\sqrt{\frac{4r_\infty^2}{r^2(r_\infty^2 + 2 - rr_\infty)^2} - z'^2}, \tag{5.7}$$

where z' can be expressed explicitly in terms of r with the use of (5.5) and (5.6). We note that this simplification is possible because r' appears in (5.6) only through $\sqrt{r'^2 + z'^2}$ which can be eliminated with the aid of (5.5).

We cannot solve (5.7) subject to the condition $r(0) = r_0$ directly since the latter is a fixed point of (5.7), but locking to the fixed point can be avoided by slightly perturbing the initial condition. For instance, we may use $r(0) = r_0 + 0.0001$ as the initial condition and the resulting solution provides a good enough approximation to the actual solitary wave type solution. This provides a useful check on our numerical solutions.

In Fig. 5(a, b), we have shown two post-critical solutions for the Gent material corresponding to $r_\infty = 1.6$ and $r_\infty = 9.7$, which are close to the two bifurcation values 1.6907 and 9.6523, respectively. We see good agreement between the numerical solutions and the asymptotic solution given by (4.21).

Earlier in Section 3 we have already indicated that not every initial condition at $Z = 0$ will correspond to a localized solution. The criterion (4.22), together with the plots in Figures 2(b), 3(b) and 4(b), enables us to deduce whether a solution will be localized when r_∞ is sufficiently close to a bifurcation value. For other values of r_∞, we have to resort to numerical integrations.

We first discuss the Varga material and refer to Fig. 2. The criterion (4.22) indicates that the bifurcated solution is a localized bulge for r_∞ close to and less than r_{cr}. As r_∞ is reduced from r_{cr}, numerical integration of (5.1) shows that the solution remains localized

only for $r^* < r_\infty < r_{cr}$, where, as pointed out by Haughton [25], the lower bound r^* is the value of r_∞ at which $r(0)$, calculated according to (3.10), makes the right hand side of (5.5) blow up, that is $r(0) = 2(1+r_\infty^2)/r_\infty^3 = r_\infty + 2/r_\infty$ from which we obtain $r^* = 2^{1/4} \approx 1.1892$ (Haughton's value of $r^* \approx 1.22$ seems to be in error). For values of r_∞ outside the interval $[r^*, r_{cr}]$ the solution is either unbounded or periodic.

For the Gent material with $J_m = 97.2$, our numerical calculation yields the results shown in Fig. 3, where again we have used solid lines to indicate association with localized solutions and dashed lines with either unbounded or periodic solutions. We find that the solid line segment originating from the first bifurcation point terminates at the nearest turning point. However, contrary to our expectation, the other solid line segment originating from the second bifurcation point terminates at $r_\infty = 9.8046$, before the nearest turning point at $r_\infty = 9.91015$ is reached. We have also carried out calculations for $J_m = 30$ and have found qualitatively similar results.

Finally, for the Ogden material the amplitude diagram of which is given by Fig. 4(a), it is found that the end points that divide localized and unbounded/periodic solutions correspond to the turning point and the bifurcation value of r_∞ in Fig. 4(a). Thus, the bifurcated solution is localized only for $1.1525 < r_\infty < 1.6955$. The upper and lower branches of the amplitude curve in Fig. 4(a) have asymptotic representations given by $r_0 \sim 1.23 r_\infty^{1.48} - 0.39 r_\infty^{0.52}$ and $r_0 \sim 0$, respectively. These asymptotic expressions are plotted in Fig. 4(a) in dashed lines in order to validate the numerical results.

6 Initiation pressure and further discussions

Stability analysis for the uniform inflation solution has been performed by Chen [28]. For the case of pressure control and free ends, he found that for a stable solution the following conditions must be satisfied:

$$\begin{aligned} &(\lambda_1^2 \hat{W}_{11} - \lambda_1 \hat{W}_1) \geq 0, \quad \hat{W}_{22} \geq 0, \quad \hat{W}_2 \geq 0, \\ &\lambda_2^2 \hat{W}_{22}(\lambda_1^2 \hat{W}_{11} - \lambda_1 \hat{W}_1) - (\lambda_1 \lambda_2 \hat{W}_{12} - \lambda_1 \hat{W}_1)^2 \geq 0, \end{aligned} \tag{6.1}$$

whereas a strict inequality version of these conditions gives a sufficient condition for (local) stability.

We note that $\hat{W}_{22} > 0$ can be interpreted as one of the conditions required by strong ellipticity for 2D deformations. The resultant axial force (in the membrane) required to maintain a constant λ_2 at infinity is $\hat{F} = 2\pi r h \sigma_2 = 2\pi H \hat{W}_2$, which has the same sign as $\hat{W}_2$. It is known [26, 24] that, independently of the form of the strain energy function, $\hat{F} \geq 0$ if $\lambda_1 \lambda_2^2 \geq 1$ and $\hat{F} \leq 0$ if $\lambda_1 \lambda_2^2 \leq 1$. In particular, for $\lambda_2 = 1$ we must have $\hat{F} > 0$ during inflation. Thus, we assume that the second and third inequalities in (6.1) are satisfied. The first inequality in (6.1) is then implied by the last inequality, and the latter, when evaluated at $\lambda_1 = r_\infty, \lambda_2 = 1$, is simply $\omega(r_\infty) \geq 0$. Thus, the bifurcation condition $\omega(r_\infty) = 0$ marks

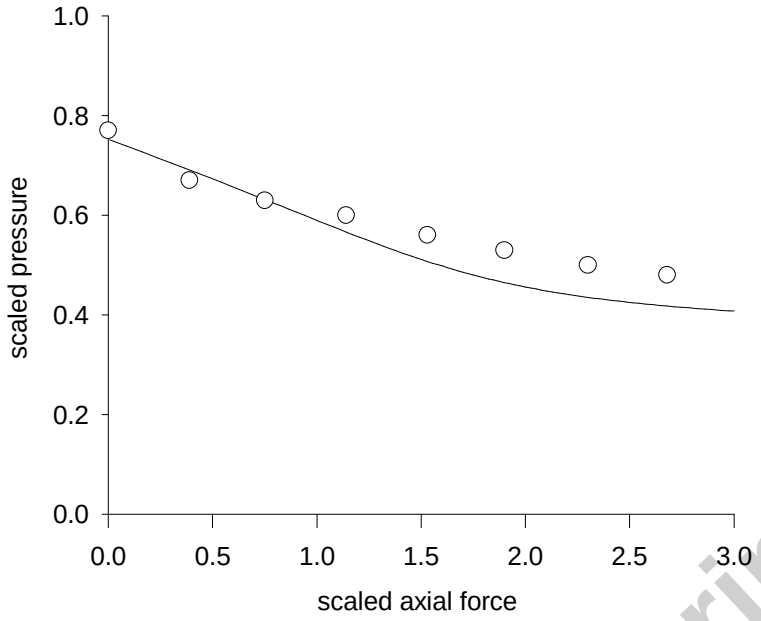


Figure 6: Comparison of the bifurcation pressure (solid line) determined by the bifurcation condition (6.2) and the initiation pressure (circles) determined experimentally by Kyriakides and Chang [9] and given in their figure 9.

the boundary of stability, as has been shown for many other cases. The stability of the uniform inflation state can then be read off straightaway from our Figures 2(b), 3(b) and 4(b).

We now argue that the first bifurcation pressure determined from $\omega(r_\infty) = 0$ can be used as the so-called initiation pressure that has featured in recent experimental studies. This is supported by the following observations: (i) it is at this pressure that the uniform inflation solution ceases to be stable; (ii) the near-critical bifurcated deformation is a bulge, of precisely the same form as observed experimentally; (iii) Figures 2(a), 3(a) and 4(a) show that the initial bulging is associated with a reduction in r_∞ from r_{cr}, again just as observed experimentally. As further support, we have shown in Fig.6 this pressure against the experimental data taken from figure 9 of Kyriakides and Chang [9]. In Kyriakides and Chang's experiments, the tube was modeled as an Ogden material and the membrane tube was closed at one end at which an extra axial force F is also applied. For this case, our bifurcation condition $\omega(r_\infty) = 0$ would take the form

$$\lambda_2^2 \hat{W}_{22}(\lambda_1^2 \hat{W}_{11} - \lambda_1 \hat{W}_1) - (\lambda_1 \lambda_2 \hat{W}_{12} - \lambda_1 \hat{W}_1)^2 = 0, \tag{6.2}$$

and the scaled pressure and axial force would be given by

$$\frac{PR}{\mu H} = \frac{1}{\lambda_1 \lambda_2} \cdot \frac{\hat{W}_1}{\mu}, \qquad \frac{F}{2\pi \mu R H} = \frac{\hat{W}_2}{\mu} - \frac{\lambda_1}{2\lambda_2} \cdot \frac{\hat{W}_1}{\mu}, \tag{6.3}$$

where the shear modulus μ is calculated according to $2\mu = \sum_{n=1}^{3} \mu_n \alpha_n$. For each specified λ_1, we solve (6.2) to find the corresponding λ_2, and hence the scaled pressure and axial force. We vary λ_1 in the interval $(1, 4)$ to generate the curve in Fig. 6. We see excellent agreement between our theoretical predications and the first four experimental results. It is not yet clear why the agreement becomes poorer for higher values of F, but we observe that the eight experimental results correspond to an axial stretch approximately equal to 1.12, 1.28, 1.49, 1.80, 2.23, 2.68, 3.13 and 3.49, respectively. Thus, the poor agreement corresponds to fairly high axial stretches.

Finally, we note that the methodology presented in the present paper can also be used to study nonlinear wave propagation in fluid-filled distensible tubes. Our amplitude equation (4.13) is in fact an integral of the well-known Kortewed-de Vries (KdV) equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{6.4}$$

when u is assumed to take the form $u = (\gamma/2)y(x - ct),\ c = \omega'(r_{cr})\epsilon r_1$. The KdV equation is the prototypal evolution equation for waves where weak nonlinearity and weak dispersion operate simultaneously, and is known to have solitary-wave solutions. In a recent study by Epstein and Johnston [40], the authors found that no solitary waves could propagate if the fluid was initially stationary prior to wave propagation. This is in contrast with previous studies by Demiray [41], Demiray and Dost [42], and Erbay *et al* [43] where (6.4) was derived and the contrary was found in the context of a small-amplitude and long-wavelength approximation. We note that Epstein and Johnston's [40] equations (6.4) and (6.6) can be cast in our form (4.1) and (4.2). Our bifurcation condition $\omega(r_\infty) = 0$ would be replaced by the leading-order dispersion relation $\omega(c) = 0$ which determines the linear wave speed c corresponding to infinite wavelength. Our near-critical post-bifurcation analysis would then correspond to a small-amplitude and long-wavelength analysis, thus enabling a direct comparison with previous studies. This will be carried out in a separate study.

Acknowledgements

This work is supported by a joint grant from BBSRC and EPSRC (BB/D014786/1) under their Stem Cell Science and Engineering Initiative. We thank Professor Andrej Ilichev for some stimulating discussions.

References

1. A.D. Kydoniefs, A.J.M. Spencer, Finite axisymmetric deformations of an initially cylindrical elastic membrane, Q. Jl. Mech. Appl. Math. 22 (1969) 87-97.

2. A.E. Green, R.T. Shield, Finite elastic deformation of incompressible isotropic bodies, Proc. R. Soc. A202 (1950) 407-419.

3. J.E. Adkins, R.W. Rivlin, Large elastic deformations of isotropic materials. IX. The deformation of thin shells, Phil. Trans. R. Soc. A 244 (1952) 505-531.

4. A.E. Green, J.E. Adkins, *Large elastic deformations and non-linear continuum mechanics*. Clarendon Press, Oxford (1960).

5. R. Benedict, A. Wineman, W.H. Yang, The determination of limiting pressure in simultaneous elongation and inflation of nonlinear elastic tubes, Int. J. Solids Strut. 15 (1979) 241-249.

6. H. Alexander, The tensile instability of an inflated cylindrical membrane as affected by an axial load, Int. J. Mech. Sci. 13 (1971) 87-95.

7. W.-L. Yin, Non-uniform inflation of a cylindrical elastic membrane and direct determination of the strain energy function, J. Elast. 7 (1977) 265-282.

8. E. Chater, J.W. Hutchinson, On the propagation of bulges and buckles, ASME J. Appl. Mech. 51 (1984) 269-277.

9. S. Kyriakides, Y.-C. Chang, On the inflation of a long elastic tube in the presence of axial load, Int. J. Solids Struct. 26 (1990) 975-991.

10. S. Kyriakides, Y.-C. Chang, The initiation and propagation of a localized instability in an inflated elastic tube, Int. J. Solids Struct. 27 (1991) 1085-1111.

11. S. Kyriakides, C.D. Babcock, Experimental determination of the propagation pressure of circular pipes, ASME J. Press. Vessel Tech. 103 (1981) 328-336.

12. S. Kyriakides, Propagating buckles in long confined cylindrical shells, Int. J. Solids Struct. 22 (1986) 1579-1597.

13. J.W. Hutchinson, K.W. Neale, Neck propagation, J. Mech. Phys. Solids 31 (1983) 405-426.

14. J.L. Ericksen, Equilibrium of bars, J. Elast. 5 (1975) 191-201.

15. K. Bhattacharya, *Microstructure of martensite: why it forms and how it gives rise to the shape-memory effect*. Oxford University Press, Oxford (2003).

16. Y.B. Fu, A.B. Freidin, Characterization and stability of two-phase piecewise-homogeneous deformations, Proc. R. Soc. Lond. A 460 (2004) 3065-3094.

17. M.A. Wadee, G.W. Hunt, M.A. Peletier, Kink band instability in layered structures, J. Mech. Phys. Solids 52 (2004) 1071-1091.

18. Y.B. Fu, Y.T. Zhang, Continuum-mechanical modelling of kink-band formation in fibre-reinforced composites, Int. J. Solids Strut. 43 (2006) 3306-3323.

19. G.A. Duffett, B.D. Reddy, The solution of multi-parameter systems of equations with application to problems in nonlinear elasticity, Comput. Methods Appl. Mech. Engrg. 59 (1986) 179-213.

20. J. Shi, G.F. Moita, The post-critical analysis of axisymmetric hyper-elastic membranes by the finite element method, Comput. Methods Appl. Mech. Engrg. 135 (1996) 265-281.

21. E. Verron, G. Marckmann, An axisymmetric B-spline model for the non-linear inflation of rubberlike membrane, Comput. Methods Appl. Mech. Engrg. 190 (2001) 6271-6289.

22. D.C. Pamplona, P.B. Goncalves, S.R.X. Lopes, Finite deformations of cylindrical membrane under internal pressure, Int. J. Mech. Sci. 48 (2006) 683-696.

23. P.B. Goncalves, D. Pamplona, S.R.X. Lopes, Finite deformations of an initially stressed cylindrical shell under internal pressure, Int. J. Mech. Sci. 50 (2008) 92-103.

24. D.M. Haughton, R.W. Ogden, Bifurcation of inflated circular cylinders of elastic material under axial loading. I. Membrane theory for thin-walled tubes, J. Mech. Phys. Solids 27 (1979) 179-212.

25. D.M. Haughton, Elastic Membranes, chapter 7 in *Nonlinear Elasticity: Theory and Applications* (eds Y.B. Fu and R.W. Ogden) Cambridge University Press, Cambridge (2001).

26. A.H. Corneliussen, R.T. Shield, Finite deformation of elastic membranes with application to the stability of an inflated and extended tube, Arch. ration. Mech. Anal. 7 (1961) 273-304.

27. R.T. Shield, On the stability of finitely deformed elastic membranes; Part II: Stability of inflated cylindrical and spherical membranes, ZAMP 23 (1972) 16-34.

28. Y.-C. Chen, Stability and bifurcation of finite deformations of elastic cylindrical membranes - part I. stability analysis, Int. J. Solids Structures 34 (1997) 1735-1749.

29. Y.B. Fu, Perturbation methods and nonlinear stability analysis, chapter 10 in *Nonlinear Elasticity: Theory and Applications* (eds Y.B. Fu and R.W. Ogden) Cambridge University Press, Cambridge (2001).

30. A.N. Gent, A new constitutive relation for rubber, Rubber Chem. Technol. 69 (1996) 59-61.

31. C.O. Horgan, G. Saccomandi, A description of arterial wall mechanics using limiting chain extensibility constitutive models, Biomechanics and Modeling in Mechanobiology 1 (2003) 251-266.

32. C.O. Horgan, G. Saccomandi, Phenomenological hyperelastic strain-stiffening constitutive models for rubber, Rubber Chem. Technol. 79 (2006) 152-169.

33. R.W. Ogden, G. Saccomandi, I. Sgura, On worm-like chain models within the three-dimensional continuum mechanics framework, Proc. R. Soc. Lond. A 462 (2006) 749-768.

34. A.N. Gent, Elastic instabilities of inflated rubber shells, Rubber Chem. Technol. 72 (1999) 263 - 268.

35. A.N. Gent, Elastic instabilities in rubber, Int. J. Non-Linear Mech. 40 (2005) 165

36. R.W. Ogden, *Non-linear elastic deformations*. Dover Publications, New York (1997).

37. A.C. Pipkin, Integration of an equation in membranes theory, ZAMP 19 (1968) 818-819.

38. R.W. Ogden, Large deformation isotropic elasticity: on the correlation of theory and experiment for incompressible rubberlike solids, Proc. R. Soc. Lond. A 326 (1972) 565-584.

39. W.H. Yang, W.W. Feng, On Axisymmetrical Deformations of Nonlinear Membranes, ASME J. Appl. Mech. 37 (1970) 1002-1011.

40. M. Epstein, C.R. Johnston, On the exact speed and amplitude of solitary waves in fluid-filled elastic tubes, Proc. R. Soc. Lond. A 457 (2001) 1195-1213.

41. H. Demiray, Solitary waves in prestressed elastic tubes, Bull. Math. Biol. 58 (1996) 939-955.

42. H. Demiray, S. Dost, Axial and transverse solitary waves in prestressed thin elastic tubes, ARI 50 (1998) 201-210.

43. S. Erbay, H.A. Erbay, S. Dost, Wave propagation in fluid-filled non-linear viscoelastic tubes, Acta Mech. 95 (1992) 87-102.