



HAL
open science

Application of the Krylov-Bogoliubov-Mitropolsky method to weakly damped strongly nonlinear planar hamiltonian systems

Serge Bruno Yamgoué, T.C. Timoléon Crépin Kofané

► **To cite this version:**

Serge Bruno Yamgoué, T.C. Timoléon Crépin Kofané. Application of the Krylov-Bogoliubov-Mitropolsky method to weakly damped strongly nonlinear planar hamiltonian systems. *International Journal of Non-Linear Mechanics*, 2007, 42 (10), pp.1240. 10.1016/j.ijnonlinmec.2007.09.001 . hal-00501759

HAL Id: hal-00501759

<https://hal.science/hal-00501759>

Submitted on 12 Jul 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

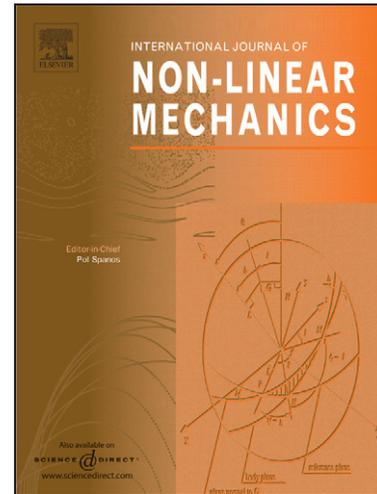
L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Author's Accepted Manuscript

Application of the Krylov-Bogoliubov-Mitropolsky method to weakly damped strongly nonlinear planar hamiltonian systems

Serge Bruno Yamgoué, Timoléon Crépin Kofané

PII: S0020-7462(07)00180-1
DOI: doi:10.1016/j.ijnonlinmec.2007.09.001
Reference: NLM 1395



www.elsevier.com/locate/nlm

To appear in: *International Journal of Non-Linear Mechanics*

Received date: 7 December 2006

Revised date: 3 September 2007

Accepted date: 3 September 2007

Cite this article as: Serge Bruno Yamgoué and Timoléon Crépin Kofané, Application of the Krylov-Bogoliubov-Mitropolsky method to weakly damped strongly nonlinear planar hamiltonian systems, *International Journal of Non-Linear Mechanics* (2007), doi:10.1016/j.ijnonlinmec.2007.09.001

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting galley proof before it is published in its final citable form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.

Application of the Krylov-Bogoliubov-Mitropolsky method to weakly damped strongly nonlinear planar hamiltonian systems

Serge Bruno Yamgoué and Timoléon Crépin Kofané

Laboratoire de Mécanique, Département de Physique, Faculté de Sciences,
Université de Yaoundé I, B.P. 812 Yaoundé Cameroun

Abstract

In this paper, an analytical approximation of damped oscillations of some strongly nonlinear, planar hamiltonian systems is considered. To apply the Krylov-Bogoliubov-Mitropolsky method in this strongly nonlinear case, we mainly provide the formal and exact solutions of the homogeneous part of the variational equations with periodic coefficients resulting from the hamiltonian systems. It is shown that these are simply expressed in terms of the partial derivatives of the solutions, written in action-angle variables, of the hamiltonian systems. Two examples, including a nonlinear harmonic oscillator and the Morse oscillator, are presented to illustrate this extension of the method. The approximate first order solution obtained in each case is observed to be quite satisfactory.

1 Introduction

Oscillations are ubiquitous in all fields of fundamental and applied sciences. The modeling of the involved phenomena leads very often to some ordinary differential equations (ODEs) that, in most of the cases, are nonlinear. Solving nonlinear ODEs is thus of great importance for gaining insights into real-world or engineering problems. Unfortunately, one has long been led to observe that, in general, nonlinearity precludes exact analytical solutions to ODEs. In this context some useful perturbation methods, including the method of multiple scales (MMS), the Lindstedt-Poincaré (LP) method and the Krylov-Bogoliubov-Mitropolsky (KBM) method [1], have been developed to provide approximate analytical solutions for nonlinear ODEs. But these classical perturbation methods apply only to weakly nonlinear problems, i.e., those problems in which (i) there is a linear part and (ii) the magnitude of the nonlinear part is small compared to that of the linear one. This appears to be too restrictive for the nonlinear problems that are currently encountered in various fields.

Therefore, several new approximate methods that overcome one and/or both of the above-mentioned limitations of the classical perturbation methods have been proposed in the literature in recent years. For instance, Liao [2] has described a nonlinear analytical technique which does not require a small parameter. The range of applicability of the LP method has enormously been improved by the technique of expansion of constant [3]. Amore and Aranda [4] have also combined the LP method with the linear delta expansion to develop another interesting approximate method. For systems undergoing symmetric restoring forces, the linearized harmonic balance technique [5, 6] yields very accurate solution.

We remark that these developments are mainly focused to the determination of analytical approximations to periodic or limit-cycle solutions of oscillator equations [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. Comparatively, little attention is granted to the determination of approximate solutions to autonomous damped equations. The few recent works (to our knowledge) in which analytical approximations to damped nonlinear oscillator equations are explicitly considered are by Liao [17], and Chatterjee and collaborators [18, 19, 20]. As a contribution to this effort, we have shown in a previous paper [21] how to merge the idea of expansion of constant [3] and the KBM method to derive better accurate slow flows for damped single degree of freedom oscillators. This contribution is continued in the present paper in which, as in the subharmonic Melnikov method [22], we consider specifically weakly perturbed planar hamiltonian systems. Our motivation stems from the simple observation that, for some systems not directly described by polynomial equations (e.g., the Morse

oscillator considered herein) as assumed in existing works, the application of the available methods might not enable to obtain fully analytical approximate solutions; or would require at least as much work as the direct approach presented here to give the same degree of accuracy.

The remaining part of this paper is organized as follows. In Sec. 2, the applicability of the standard KBM method to strongly nonlinear planar oscillator is formally demonstrated by combining it with the basic ideas of the subharmonic Melnikov method. Then in Sec. 3, two examples, including a damped nonlinear harmonic oscillator and the Morse oscillator, are provided. We end our work in Sec. 4 by some remarks.

2 Formalism

We consider systems of the general form

$$\dot{X} = JDH(X) + \epsilon h(X), \quad X = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2, \quad (2.1)$$

where $0 < \epsilon \ll 1$, $H : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are sufficiently smooth. DH and J are respectively a column vector and a square matrix defined as

$$DH = \begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (2.2)$$

When $\epsilon = 0$, Eq.(2.1) becomes a planar hamiltonian system with hamiltonian function $H(X)$. We assume for this hamiltonian system that in some open set in \mathbb{R}^2 there exists a fixed point of center type, X_c , surrounded by a one-parameter family of periodic orbits. We also assume that the corresponding solutions can be obtained analytically and expressed as functions of two independent variables, $X = X(I, \phi)$, such that $X(I, \phi + 2\pi) = X(I, \phi)$. Here, I is a time-independent parameter (i.e., $\dot{I} = 0$) that determines the amplitude of oscillations for a given periodic orbit. The variable ϕ corresponds to the phase of oscillations. It varies linearly in time: $\phi = \Omega(I)(t - t_0)$, with $\Omega(I)$ the angular frequency of motion and t_0 the initial time.

We remark that in the system coordinate defined by (I, ϕ) , which we assume without loss of generality to be the well-known action-angle coordinate system [22], the hamiltonian is a function of I only, and that the open set mentioned above maps to an open set of $\mathbb{R} \setminus \{0\}$ in which this latter function is strictly monotonic, i.e., $H'(I) \neq 0$. We emphasize however that a similar treatment can be extended even when I is not an action variable.

Several methods exist for the determination of approximate solutions to equations of the form (2.1) when the unperturbed system ($\epsilon = 0$) is linear. However, we are concerned in the present study with the more general situation where the unperturbed system may be strongly nonlinear. We tackle the strong nonlinearity in an approach similar to that of Das and Chatterjee [19]. Here, however, instead of the MMS, we find that the KBM method is most suited for adaptation for strongly nonlinear systems. This method consists in expressing the solution $X(t)$ of Eq.(2.1) as a power series of the small parameter ϵ :

$$X(t) = \sum_{n=0}^{\infty} \epsilon^n X_n(I, \phi), \quad I = I(t), \quad \phi = \phi(t). \quad (2.3)$$

The first coefficient of the series (2.3), $X_0(I, \phi)$, is chosen to be the solution of the unperturbed hamiltonian system. The others coefficients, $X_k(I, \phi)$, $k \geq 1$, are unknown functions to be determined. In addition to the series (2.3), one also assumes in the KBM method that the time evolutions of the action and angle variables are respectively solutions of the following ODEs

$$\dot{I} = \xi(I) = \sum_{n=0}^{\infty} \epsilon^n \xi_n(I), \quad \dot{\phi} = \Omega_{\epsilon}(I) = \sum_{n=0}^{\infty} \epsilon^n \Omega_n(I) \quad (2.4)$$

where the coefficients of both series are also unknown except $\xi_0(I) = 0$ and $\Omega_0(I) = \Omega(I)$. Notice that the approximation (2.3) mainly plays a role as a transformation from (2.4) to (2.1). To obtain the coefficients $X_k(I, \phi)$, $\xi_k(I)$ and $\Omega_k(I)$ for $k \geq 1$, one expands Eq.(2.1) in powers of ϵ after inserting Eq.(2.3) and Eq.(2.4) in it. Then, by equating coefficients of like powers of ϵ in the resulting equation, one is led to solve a sequence of linear ODEs with periodic coefficients of the form

$$\frac{\partial}{\partial \phi} X_k(I, \phi) = A(I, \phi) X_k(I, \phi) + B_k(I, \phi), \quad k \geq 1, \quad (2.5)$$

where $A(I, \phi) = \frac{1}{\Omega(I)} J D^2 H(X_0(I, \phi))$; $D^2 H(X_0(I, \phi))$ being the Hessian matrix of the hamiltonian function H evaluated at the solution $X_0(I, \phi)$ of the unperturbed system and $B_k(I, \phi)$ is a \mathbb{R}^2 column vector whose components depend on $X_j(I, \phi)$, $\xi_j(I)$ and $\Omega_j(I)$, $j = 0, \dots, k-1$. We remark that $B_1(I, \phi)$ is necessarily 2π -periodic with respect to ϕ and that if $X_j(I, \phi)$, $j = 2, \dots, k-1$ are all periodic, so will be $B_k(I, \phi)$. From the linear theory of ordinary differential equations, the general solution of Eq.(2.5), assuming the initial condition $X_k(I, \phi = \phi_0) = X_k^{\phi_0} = (x_k^0, y_k^0)^T$, is formally given by [23]

$$X_k(I, \phi) = K(I, \phi) K(I, \phi_0)^{-1} X_k^{\phi_0} + K(I, \phi) \int_{\phi_0}^{\phi} K(I, \varphi)^{-1} B_k(I, \varphi) d\varphi. \quad (2.6)$$

Here, $K(I, \phi)$ is the fundamental matrix of Eq.(2.5) with $B_k = 0$, i.e., a matrix whose columns comprise the linearly independent solutions of that equation for $B_k \equiv 0$ (homogeneous system); and $K(I, \phi)^{-1}$ denotes its inverse. On the contrary of linear systems with constant matrix, the determination of the fundamental matrix is not a trivial matter when the system's matrix is a function of the independent variable. The case where the given set of linear ODEs derives from an expansion about *the exact solution of a strongly nonlinear hamiltonian system*, such as Eq.(2.5), is particular. We provide here the systematic expression of the fundamental matrix for this case. It is very easy to verify (see **Appendix A**) that the two vectors

$$V_1(I, \phi) = \frac{\partial X_0}{\partial \phi}(I, \phi) \quad \text{and} \quad V_2(I, \phi) = \frac{\partial X_0}{\partial I}(I, \phi) + \frac{\Omega'(I)}{\Omega(I)} \phi V_1(I, \phi) \quad (2.7)$$

are solutions of the homogeneous part of Eq.(2.5). Moreover, as also shown in **Appendix A**, their determinant is the ratio of the derivative $H'(I)$ of the Hamiltonian to the angular frequency $\Omega(I)$; which are all nonzero for periodic orbits. They are thus linearly independent, and so, $K(I, \phi) = [V_1(I, \phi), V_2(I, \phi)]$. We also note that these expressions of $V_1(I, \phi)$ and $V_2(I, \phi)$ are valid for linear systems for which $\Omega'(I) = 0$. The integral in Eq.(2.6) can now be carried out to get the solution of the full equation Eq.(2.5). This solution however still depends on $\xi_k(I)$ and $\Omega_k(I)$ which remain undetermined. To find their expressions, we impose the usual solvability condition, that is, the condition for $X_k(I, \phi)$ to be bounded (in fact, 2π -periodic) in ϕ . Assuming that $B_k(I, \phi)$ is periodic in ϕ , it is straightforward to show that

$$X_k(I, \phi + 2\pi) - X_k(I, \phi) = \lambda_1 V_1(I, \phi) + \lambda_2 V_2(I, \phi), \quad (2.8)$$

with

$$\lambda_1 = 2\pi \frac{\Omega'(I)}{\Omega(I)} \left(\kappa(\phi_0) - \int^{\phi_0} f_{k(2)}(I, \phi) d\phi \right) + \bar{f}_{k(1)} - \pi \frac{\Omega'(I)}{\Omega(I)} \bar{f}_{k(2)}, \quad \lambda_2 = \bar{f}_{k(2)}; \quad (2.9)$$

where $\kappa(\phi_0)$ is the second component of the vector $K(I, \phi_0)^{-1} X_k^{\phi_0}$ while the overbar $(\bar{})$ denotes an average with respect to ϕ , and

$$f_{k(1)}(I, \phi) = B_k(I, \phi) \Lambda \frac{\partial X_k}{\partial I}(I, \phi) \quad \text{and} \quad f_{k(2)}(I, \phi) = \frac{\partial X_k}{\partial \phi}(I, \phi) \Lambda B_k(I, \phi). \quad (2.10)$$

The symbol Λ represents here the usual cross-product of two vectors, defined for $A = [a_1, a_2]^T$ and $B = [b_1, b_2]^T$ as $A \Lambda B = a_1 b_2 - a_2 b_1$. By the linear independence of V_1 and V_2 , the left hand side of Eq.(2.8) vanishes if and only if both λ_1 and λ_2 are zero. This forms a system of two algebraic equations which determine $\xi_k(I)$ and $\Omega_k(I)$. Note that they depend on the initial condition $X_k^{\phi_0}$

which has to be fixed explicitly in order to eliminate unnecessary constants in the final solution Eq.(2.1). Indeed, the solution of a first order planar system of ODEs should mathematically contains exactly two arbitrary constants. These will be provided in our case by the integration of Eq.(2.4). Our choice for the initial condition is: $\phi_0 = 0$ and $X_k^0 = 0$, i.e., $X_k(I, 0) = 0$; other choices are in principle valid.

The approximate solution of the autonomous equation (2.1) is finally given by the truncation at some order, say N , of the series in (2.3), with $I(t)$ and $\phi(t)$ being the solutions of the truncations at the order $N + 1$ (in a strict sense [1]) of the series in (2.4).

3 Examples

In this section, we provide an illustration of the application of the extended KBM method presented above to the determination of approximate solutions to strongly nonlinear planar hamiltonian systems perturbed by viscous damping. We begin with a system which is sufficiently simple for the analytical computations to be achieved quite easily. We then consider the damped Morse oscillator as the second example. In both examples, the center type point for the unperturbed system is the origin. For convenience, we assume that the parameter I is positive and such that $\lim_{I \rightarrow 0} X(I, \phi) = X_c = 0$. From the unperturbed system, we also determine the hamiltonian function H such that $H(X_c) = H(0) = 0$.

3.1 A nonlinear harmonic oscillator

We consider the system

$$\dot{x} = (x^2 + y^2 + 1) y, \quad (3.1a)$$

$$\dot{y} = -(x^2 + y^2 + 1) x - \epsilon \delta y \quad (3.1b)$$

which is similar to but different from an example given by Yagasaki [22] to illustrate his results on the subharmonic Melnikov's theory in the degenerate resonance case. For $\epsilon = 0$, this system admits the Hamiltonian

$$H(x, y) = \frac{1}{4}(x^2 + y^2 + 1)^2. \quad (3.2)$$

The corresponding unperturbed periodic orbits are represented with trigonometric functions as in the case of a linear oscillator:

$$x(t) = I \sin \Omega t, \quad y(t) = I \cos \Omega t. \quad (3.3)$$

Here, however, the pulsation Ω of the motion is a function of the oscillation's amplitude I :

$$\Omega \equiv \Omega(I) = I^2 + 1. \quad (3.4)$$

For our theory, it is important to write down the unperturbed solution as a 2π -periodic function of ϕ . In doing so for (3.3), we choose the simplest parametrization of the amplitude (a possible parametrization is via the action variable ρ : $I = \sqrt{2\rho}$)

$$x(I, \phi) = I \sin \phi, \quad y(I, \phi) = I \cos \phi. \quad (3.5)$$

Inserting these expressions into Eq.(3.2), one gets

$$H(I) = \frac{1}{4}(I^2 + 1)^2 \quad (3.6a)$$

whose derivative

$$H'(I) = I(I^2 + 1) \quad (3.6b)$$

is monotonic for all $I > 0$. In the following, we will constantly drop the argument of Ω in view of simplifying the notation (i.e., we write Ω instead of $\Omega(I)$).

Now, we seek an approximate solution to the perturbed system that includes up to the second order in the perturbation parameter ϵ . So we express $x_\epsilon(t)$ and $y_\epsilon(t)$ as

$$x_\epsilon = I \sin \phi + \sum_{i=1}^3 \epsilon^i x_i(I, \phi) + O(\epsilon^4), \quad (3.7a)$$

$$y_\epsilon = I \cos \phi + \sum_{i=1}^3 \epsilon^i y_i(I, \phi) + O(\epsilon^4); \quad (3.7b)$$

with

$$\dot{I} = \sum_{i=1}^3 \epsilon^i \xi_i(I) + O(\epsilon^4), \quad (3.8a)$$

$$\dot{\phi} = \Omega(I) + \sum_{i=1}^3 \epsilon^i \Omega_i(I) + O(\epsilon^4). \quad (3.8b)$$

Note that Eqs.(3.7) include terms of order $O(\epsilon^3)$ although we are looking for solutions to the order $O(\epsilon^2)$ only. These terms are necessary for the determination of ξ_3 and Ω_3 which, as already stated,

are required in a rigorous sense for a solution at the order $\mathcal{O}(\epsilon^-)$. Substituting Eqs.(3.7)-(3.8) into Eqs.(3.1), and expanding and collecting terms of like powers of ϵ , we find that the matrix A and the first two forcing terms B_i ($i=1,2$) in Eq.(2.5) are respectively given by

$$A(I, \phi) = \frac{1}{\Omega} \begin{pmatrix} I^2 \sin(2\phi) & \Omega + 2I^2 \cos^2 \phi \\ -\Omega - 2I^2 \sin^2 \phi & -I^2 \sin(2\phi) \end{pmatrix}; \quad (3.9)$$

$$B_1(I, \phi) = \frac{1}{\Omega} \begin{pmatrix} \Omega_1 I \cos \phi - \xi_1 \sin \phi \\ \Omega_1 I \sin \phi - (\xi_1 + \delta I) \sin \phi \end{pmatrix}; \quad (3.10)$$

and

$$B_2(I, \phi) = \frac{1}{\Omega} \begin{pmatrix} B_2^{(1)} \\ B_2^{(2)} \end{pmatrix} \quad (3.11)$$

with

$$B_2^{(1)} = (x_1^2 + y_1^2)I \cos \phi + 2Ix_1y_1 \sin \phi + 2Iy_1^2 \cos \phi - \xi_2 \sin \phi - \Omega_2 I \cos \phi - \xi_1 \frac{\partial x_1}{\partial I} - \Omega_1 \frac{\partial x_1}{\partial \phi},$$

$$B_2^{(2)} = \xi_1 \frac{\partial y_1}{\partial I} - (x_1^2 + y_1^2)I \sin \phi - 2Ix_1y_1 \cos \phi - 2Iy_1^2 \sin \phi - \delta y_1 - \xi_2 \cos \phi - \Omega_2 I \sin \phi - \Omega_1 \frac{\partial y_1}{\partial \phi}.$$

The fundamental matrix K is also given by

$$K(I, \phi) = \frac{1}{\Omega} \begin{pmatrix} \Omega \cos \phi & \Omega \sin \phi + 2I^2 \phi \cos \phi \\ -\Omega \sin \phi & \Omega \cos \phi - 2I^2 \phi \sin \phi \end{pmatrix}. \quad (3.12)$$

Performing the algebra involved in the integral of Eq.(2.6) with these expressions of K and B_1 in one hand, and then in the elimination of secular terms in $X_1 = (x_1, y_1)^T$, we find that, corresponding to Eq.(2.9),

$$\lambda_1^{(1)} = -\frac{2\pi I}{\Omega^2} (\Omega_1 \Omega + 2\pi I \xi_1 + \pi \delta I^2)$$

$$\lambda_2^{(1)} = -\frac{\pi}{\Omega} (2\xi_1 + \delta I).$$

Solving the equations $\lambda_1^{(1)} = \lambda_2^{(1)} = 0$ for ξ_1 and Ω_1 , we obtain

$$\xi_1 = -\frac{\delta}{2} I, \quad \Omega_1 = 0. \quad (3.13)$$

Then, the expression of the first order correction is found to be

$$X_1(I, \phi) = -\frac{\delta I}{2\Omega^2} \begin{pmatrix} I^2 \cos \phi \sin^2 \phi \\ [1 + I^2 \cos^2 \phi] \sin \phi \end{pmatrix}. \quad (3.14)$$

Repeating the same operations for κ and B_2 , we also obtain

$$\begin{aligned}\lambda_1^{(2)} &= -\frac{\pi I}{8\Omega^3}(\Omega_2(48I^2 + 16 + 16I^6 + 48I^4) + \delta^2(6I^2 - 5I^4 + 2) + \pi\xi_2 I(64I^2 + 32 + 32I^4)), \\ \lambda_2^{(2)} &= -2\pi\xi_2,\end{aligned}$$

from which

$$\xi_2 = 0, \quad \Omega_2 = \frac{\delta^2}{16\Omega^3}(5I^4 - 6I^2 - 2). \quad (3.15)$$

Whence,

$$X_2(I, \phi) = \frac{\delta^2 I}{16\Omega^4} \begin{pmatrix} [2 + 8I^2 + 4I^4 - (5I^4 + 2I^2) \cos^2 \phi - 4I^4 \cos^4 \phi] \sin \phi \\ I^2[4I^2 \cos^2 \phi + 7I^2 + 2] \cos \phi \sin^2 \phi \end{pmatrix}. \quad (3.16)$$

The final step which involves B_3 (whose cumbersome expression is omitted here) results to

$$\xi_3 = -\frac{\delta^3 I^3}{32\Omega^4}(1 + 10I^2), \quad \Omega_3 = 0. \quad (3.17)$$

Recapitulating, the solution of Eqs.(3.1) to the second order in the perturbation parameter ϵ is given by

$$X_\epsilon(t) = \begin{pmatrix} I \sin \phi(t) \\ I \cos \phi(t) \end{pmatrix} + \epsilon X_1(I(t), \phi(t)) + \epsilon^2 X_2(I(t), \phi(t)) \quad (3.18)$$

where X_1 and X_2 are given respectively by Eq.(3.14) and Eq.(3.16) (in which $\Omega \equiv \Omega(I)$); $I(t)$ and $\phi(t)$ being the solutions of

$$\dot{I} = -\frac{\delta}{2}I - \frac{\delta^3 I^3}{32(1 + I^2)^4}(1 + 10I^2), \quad (3.19a)$$

$$\dot{\phi} = 1 + I^2 + \frac{\delta^2(5I^4 - 6I^2 - 2)}{16(1 + I^2)^3}. \quad (3.19b)$$

This solution is analytic only if the second term in the right hand side of Eq.(3.19a) is neglected. In this case, the solutions of Eqs.(3.19) are given by

$$I(t) = I_0 e^{-\frac{\epsilon\delta}{2}(t-t_0)}, \quad (3.20a)$$

$$\phi(t) = G(I(t)) - G(I_0) + \phi_0; \quad (3.20b)$$

$$G(u) = -\frac{u^2}{\epsilon\delta} - \frac{9\epsilon\delta}{32(1 + u^2)^2} - \frac{\epsilon\delta}{8} \log(u^2 + 1) - \left(\frac{2}{\epsilon\delta} - \frac{\epsilon\delta}{4}\right) \log(|u|) + \frac{7\epsilon\delta}{16(u^2 + 1)}.$$

In Fig. 1, we compare the wave form and the associated trajectory corresponding to the approximate solution of Eq.(3.18) with Eq.(3.20) to the result of direct numerical integration using the fourth order

Runge-Kutta algorithm. The initial conditions have been chosen sufficiently large to ensure that the nonlinearity of the system acts. A very good agreement is observed. It is interesting to remark that such a good result was obtained in [21] (for a different system) starting from an initial approximation which is not the exact solution of the unperturbed hamiltonian system. Both the approach of [21] and the present one exploit the KBM method which essentially performs a transformation to action-angle. One might then expect to still obtain good result by the present approach with an initial approximation different from the exact solution of the hamiltonian system. We recall however that the determination of the linearly independent solutions of the homogeneous part of Eq.(2.5) is crucial for the present procedure. At present, our ability to determine these solutions *heavily* relies on the fact that the initial approximation is the exact solution of the unperturbed system which should furthermore be hamiltonian. Further investigations are therefore necessary to ascertain the applicability of our procedure with different initial approximations.

3.2 The Morse oscillator

In the form of a system of first order ODEs, the equations of the damped but undriven Morse oscillator we now consider reads

$$\dot{x} = y, \quad (3.21a)$$

$$\dot{y} = e^{-2x} - e^{-x} - \epsilon \delta y. \quad (3.21b)$$

Note that the corresponding hamiltonian function is defined as

$$H(x, y) = \frac{1}{2} \left[y^2 + (e^{-x} - 1)^2 \right]. \quad (3.22)$$

In terms of (I, ϕ) , the solutions of Eq.(3.21) with $\epsilon = 0$ are given by

$$x_0(I, \phi) = \ln \left(\frac{1 - \sqrt{1 - \Omega^2 \cos \phi}}{\Omega^2} \right), \quad (3.23a)$$

$$y_0(I, \phi) = \frac{\Omega \sqrt{1 - \Omega^2 \sin \phi}}{1 - \sqrt{1 - \Omega^2 \cos \phi}}; \quad (3.23b)$$

where

$$\Omega \equiv \Omega(I) = 1 - I \quad (3.24)$$

is the angular pulsation of the motion. Inserting Eq.(3.23) into Eq.(3.22), one gets

$$H(I) = I - \frac{I^2}{2}; \quad (3.25a)$$

and

$$H'(I) = \Omega(I) \quad (3.25b)$$

is monotonic for $0 \leq I < 1$.

Letting $f(x) = e^{-2x} - e^{-x}$, the matrix of the linearized system, A , and the two first inhomogeneous vectors $B_1 = (B_1^{(1)}, B_1^{(2)})^T$ and $B_2 = (B_2^{(1)}, B_2^{(2)})^T$ for this example can be expressed in terms of the solutions of the hamiltonian system as

$$\begin{aligned} A &= \begin{pmatrix} 0 & 1 \\ f'(x_0) & 0 \end{pmatrix}; \\ B_1^{(1)} &= -\frac{1}{\Omega} \left(\xi_1 \frac{\partial x_0}{\partial I} + \Omega_1 \frac{\partial x_0}{\partial \phi} \right); \\ B_1^{(2)} &= -\frac{1}{\Omega} \left(\xi_1 \frac{\partial y_0}{\partial I} + \Omega_1 \frac{\partial y_0}{\partial \phi} + \delta y_0 \right); \\ B_2^{(1)} &= -\frac{1}{\Omega} \left(\xi_2 \frac{\partial x_0}{\partial I} + \xi_1 \frac{\partial x_1}{\partial I} + \Omega_2 \frac{\partial x_0}{\partial \phi} + \Omega_1 \frac{\partial x_1}{\partial \phi} \right); \\ B_2^{(2)} &= -\frac{1}{\Omega} \left(\xi_2 \frac{\partial y_0}{\partial I} + \xi_1 \frac{\partial y_1}{\partial I} + \Omega_2 \frac{\partial y_0}{\partial \phi} + \Omega_1 \frac{\partial y_1}{\partial \phi} + \delta y_1 - \frac{1}{2} f''(x_0) x_1^2 \right). \end{aligned}$$

Then, upon evaluating the integrals defining $\lambda_1^{(1)}$ and $\lambda_2^{(1)}$ and subsequently solving for ξ_1 and Ω_1 , we obtain (note that δ is absorbed into ϵ)

$$\xi_1 = -I, \quad \Omega_1 = 0. \quad (3.26)$$

The formal expression of the first correcting term in the approximate solution to the damped oscillator, X_1 , is given by

$$X_1(I, \phi) = Z_\phi(I, \phi) \frac{\partial X_0}{\partial \phi}(I, \phi) + Z_I(I, \phi) \frac{\partial X_0}{\partial I}(I, \phi); \quad (3.27)$$

where

$$Z_\phi(I, \phi) = \int_0^\phi f_1(I, \theta) d\theta + \frac{\Omega'(I)}{\Omega(I)} \phi \int_0^\phi f_2(I, \theta) d\theta - \frac{\Omega'(I)}{\Omega(I)} \int_0^\phi \theta f_2(I, \theta) d\theta \quad (3.28a)$$

$$Z_I(I, \phi) = \int_0^\phi f_2(I, \theta) d\theta; \quad (3.28b)$$

with

$$f_1(I, \theta) = y_0(I, \theta) \frac{\partial x_0}{\partial I}(I, \theta), \quad f_2(I, \theta) = - \left(I + y_0(I, \theta) \frac{\partial x_0}{\partial \theta}(I, \theta) \right). \quad (3.29)$$

It turns out here that the last integral in Eq.(3.28a) cannot be computed analytically unless we use the Fourier series expansion with respect to θ of $f_2(I, \theta)$. We employ the method of residue to

compute the coefficients of the Fourier expansions of $J_1(I, \theta)$ and $J_2(I, \theta)$ and find that

$$f_1(I, \theta) = \sum_{n=1}^{\infty} S_n(I) \sin(n\theta), \quad f_2(I, \theta) = \sum_{n=1}^{\infty} C_n(I) \cos(n\theta) \quad (3.30)$$

with

$$S_n(I) = \frac{1 - n + (2 + n)I - I^2}{I(2 - I)} \left(\frac{I}{2 - I} \right)^{\frac{n}{2}}, \quad C_n(I) = (n - 1 - nI) \left(\frac{I}{2 - I} \right)^{\frac{n}{2}}.$$

Then

$$Z_\phi(I, \phi) = \sum_{n=1}^{\infty} \left(\frac{n\Omega(I)S_n(I) + \Omega'(I)C_n(I)}{n^2\Omega(I)} \right) (1 - \cos(n\phi)), \quad (3.31a)$$

$$Z_I(I, \phi) = \sum_{n=1}^{\infty} \frac{C_n(I)}{n} \sin(n\phi). \quad (3.31b)$$

Using arguments based on the parities of the components of $X_0(I, \phi)$ and $X_1(I, \phi)$ with respect to ϕ , we find that $\xi_2 = 0$. But the fact that the analytic expression of $X_1(I, \phi)$ involves Fourier series renders the algebra involved in the determination of Ω_2 analytically intractable. We are then compelled to neglect this term and, furthermore, to limit ourselves at this level of the approximation for the present example. Therefore, the action and the angle vary approximately as

$$I(t) = I_0 e^{-\epsilon\delta(t-t_0)}, \quad \phi(t) = \phi_0 + t - t_0 + \frac{I_0}{\epsilon\delta} (e^{-\epsilon\delta(t-t_0)} - 1); \quad (3.32)$$

and the approximate solution $X_\epsilon(t) = X_0(t) + \epsilon\delta X_1(t)$ is analytic. As can be observed in Fig. 2, a satisfactory agreement is noticed between this approximation and the direct numerical solution.

4 Concluding remarks

We have been concerned in this paper by the analytical approximation of solutions of weakly perturbed planar hamiltonian systems. For this purpose, we have first provided formal arguments by which the standard KBM method can be applied to such systems. Our primary motivation in this paper has been about strictly dissipative perturbations. Thus the perturbation considered in our examples consists only of viscous damping. However the method can well be applied to conservative perturbations in view of approximating periodic solutions of nonlinear oscillators. In this case, and for hamiltonian systems whose solutions involve jacobian elliptic functions, the method is expected to provide equivalent results to those obtained by the elliptic Lindstedt-Poincaré method [15, 16]. For our examples, the comparison of the obtained approximate analytical solutions to the direct

numerical solutions show quite good agreements. This agreement results from two ingredients: the initiation of the approximation from a solution which is valid in a larger region (the exact solution of the unperturbed hamiltonian system) and the strength of the KBM method in handling the modulation of the amplitude parameter.

5 Acknowledgements

Thanks are due to the anonymous referees whose suggestions have contributed to the improvement of the original manuscript.

Accepted manuscript

We show here, assuming $X_0(I, \phi) = [x_0(I, \phi), y_0(I, \phi)]^T$ is a solution of Eq.(2.1) with $\epsilon = 0$, that $V_1(I, \phi)$ and $V_2(I, \phi)$ defined in Eq.(2.7) are linearly independent solutions of the variational equation (2.5) with $B_k \equiv 0$. The arguments of the various functions are omitted to simplify the notation in what follows.

X_0 being an exact solution of the unperturbed (hamiltonian) equation, we have

$$\begin{aligned} \begin{pmatrix} \frac{\partial H}{\partial y} \\ -\frac{\partial H}{\partial x} \end{pmatrix} &= \dot{X}_0 \\ &= \dot{\phi} \frac{\partial X_0}{\partial \phi} + \dot{I} \frac{\partial X_0}{\partial I} \\ &= \Omega \frac{\partial X_0}{\partial \phi}; \end{aligned} \tag{A.1}$$

since $\dot{\phi} = \Omega$ and $\dot{I} = 0$ for the hamiltonian system. Thus

$$\begin{aligned} V_1 &= \frac{\partial X_0}{\partial \phi} \\ &= \frac{1}{\Omega} \begin{pmatrix} \frac{\partial H}{\partial y} \\ -\frac{\partial H}{\partial x} \end{pmatrix}. \end{aligned} \tag{A.2}$$

Differentiating (A.2) with respect to ϕ , one gets

$$\begin{aligned} \frac{\partial V_1}{\partial \phi} &= \frac{1}{\Omega} \begin{pmatrix} \frac{\partial^2 H}{\partial x \partial y} \frac{\partial x_0}{\partial \phi} + \frac{\partial^2 H}{\partial y^2} \frac{\partial y_0}{\partial \phi} \\ -\frac{\partial^2 H}{\partial x^2} \frac{\partial x_0}{\partial \phi} - \frac{\partial^2 H}{\partial y \partial x} \frac{\partial y_0}{\partial \phi} \end{pmatrix} \\ &= \frac{1}{\Omega} \begin{pmatrix} \frac{\partial^2 H}{\partial x \partial y} & \frac{\partial^2 H}{\partial y^2} \\ -\frac{\partial^2 H}{\partial x^2} & -\frac{\partial^2 H}{\partial y \partial x} \end{pmatrix} \begin{pmatrix} \frac{\partial x_0}{\partial \phi} \\ \frac{\partial y_0}{\partial \phi} \end{pmatrix} \\ &= \frac{1}{\Omega} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial y} \\ \frac{\partial^2 H}{\partial x \partial y} & \frac{\partial^2 H}{\partial y^2} \end{pmatrix} \begin{pmatrix} \frac{\partial x_0}{\partial \phi} \\ \frac{\partial y_0}{\partial \phi} \end{pmatrix} \\ &= \frac{1}{\Omega} JD^2 HV_1 \\ &= AV_1; \end{aligned} \tag{A.3}$$

thus V_1 is a solution of (2.5) with $B_k \equiv 0$. To proceed with V_2 , we first need the expression of $\frac{\partial^2 X_0}{\partial I \partial \phi}$.

It is obtained by differentiating (A.2) with respect to I :

$$\begin{aligned}
\frac{\partial^2 X_0}{\partial I \partial \phi} &= -\frac{\Omega'}{\Omega^2} \begin{pmatrix} \frac{\partial H}{\partial y} \\ -\frac{\partial H}{\partial x} \end{pmatrix} + \frac{1}{\Omega} \begin{pmatrix} \frac{\partial^2 H}{\partial x \partial y} \frac{\partial x_0}{\partial I} + \frac{\partial^2 H}{\partial y^2} \frac{\partial y_0}{\partial I} \\ -\frac{\partial^2 H}{\partial x^2} \frac{\partial x_0}{\partial I} - \frac{\partial^2 H}{\partial y \partial x} \frac{\partial y_0}{\partial I} \end{pmatrix} \\
&= -\frac{\Omega'}{\Omega} \left[\frac{1}{\Omega} \begin{pmatrix} \frac{\partial H}{\partial y} \\ -\frac{\partial H}{\partial x} \end{pmatrix} \right] + \frac{1}{\Omega} \begin{pmatrix} \frac{\partial^2 H}{\partial x \partial y} & \frac{\partial^2 H}{\partial y^2} \\ -\frac{\partial^2 H}{\partial x^2} & -\frac{\partial^2 H}{\partial y \partial x} \end{pmatrix} \begin{pmatrix} \frac{\partial x_0}{\partial I} \\ \frac{\partial y_0}{\partial I} \end{pmatrix} \\
&= -\frac{\Omega'}{\Omega} V_1 + \frac{1}{\Omega} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial y} \\ \frac{\partial^2 H}{\partial x \partial y} & \frac{\partial^2 H}{\partial y^2} \end{pmatrix} \begin{pmatrix} \frac{\partial x_0}{\partial I} \\ \frac{\partial y_0}{\partial I} \end{pmatrix} \\
&= -\frac{\Omega'}{\Omega} V_1 + A \frac{\partial X_0}{\partial I}.
\end{aligned} \tag{A.4}$$

Let us now compute the partial derivative of V_2 with respect to ϕ . We have

$$\begin{aligned}
\frac{\partial V_2}{\partial \phi} &= \frac{\partial}{\partial \phi} \left[\frac{\partial X_0}{\partial I} + \frac{\Omega'(I)}{\Omega(I)} \phi V_1 \right] \\
&= \frac{\partial^2 X_0}{\partial \phi \partial I} + \frac{\Omega'(I)}{\Omega(I)} V_1 + \frac{\Omega'(I)}{\Omega(I)} \phi \frac{\partial V_1}{\partial \phi} \\
&= A \frac{\partial X_0}{\partial I} + \frac{\Omega'(I)}{\Omega(I)} \phi A V_1 \\
&= A \left[\frac{\partial X_0}{\partial I} + \frac{\Omega'(I)}{\Omega(I)} \phi V_1 \right] \\
&= A V_2,
\end{aligned} \tag{A.5}$$

where we have used (A.3) and (A.4) to step from the second to the third of the above equalities.

The determinant of $[V_1, V_2]$ is

$$\begin{aligned}
V_1 \wedge V_2 &= \frac{\partial X_0}{\partial \phi} \Lambda \left[\frac{\partial X_0}{\partial I} + \frac{\Omega'(I)}{\Omega(I)} \phi \frac{\partial X_0}{\partial \phi} \right] \\
&= \frac{\partial X_0}{\partial \phi} \Lambda \frac{\partial X_0}{\partial I} \\
&= \frac{1}{\Omega} \begin{pmatrix} \frac{\partial H}{\partial y} \\ -\frac{\partial H}{\partial x} \end{pmatrix} \Lambda \frac{\partial X_0}{\partial I}, \quad \text{using A.2} \\
&= \frac{1}{\Omega} \left(\frac{\partial H}{\partial y} \frac{\partial y_0}{\partial I} + \frac{\partial H}{\partial x} \frac{\partial x_0}{\partial I} \right) \\
&= \frac{H'}{\Omega} \\
&\neq 0,
\end{aligned} \tag{A.6}$$

since $H' \neq 0$ for periodic orbits as remarked in the text.

- [1] A. H. Nayfeh, D. T. Mook, *Nonlinear Oscillations*, Wiley, New York, 1979.
- [2] S.-J. Liao, An approximate solution technique not depending on small parameters: a special example, *International Journal of Non-Linear Mechanics* **30** (1995) 371–380.
- [3] J.-H. He, Modified Lindstedt-Poincaré methods for some strongly non-linear oscillations, Part I: expansion of a constant, *International Journal of Non-Linear Mechanics* **37** (2002) 309–314.
- [4] P. Amore, A. Aranda, Presenting a new method for the solution of nonlinear problems, *Physics Letters A* **316** (2003) 218–225.
- [5] C. W. Lim, B. S. Wu, A new analytical approach to the Duffing-harmonic oscillator, *Physics Letters A* **311** (2003) 365–373.
- [6] B. S. Wu, C. W. Lim, Large amplitude non-linear oscillations of a general conservative system, *International Journal of Non-Linear Mechanics* **39** (2004) 859–870.
- [7] P. Amore, A. Aranda, Improved Lindstedt-Poincaré method for the solution of nonlinear problems, *Journal of Sound and Vibration* **283** (2005) 1115–1136.
- [8] B. S. Wu, W. P. Sun, C. W. Lim, An analytical approximate technique for a class of strongly non-linear oscillators, *International Journal of Non-Linear Mechanics* **41** (2006) 766–774.
- [9] C. W. Lim, B. S. Wu, W. P. Sun, Higher accuracy analytical approximations to the Duffing-Harmonic oscillator, *Journal of Sound and Vibration* **296** (2006) 1039–1045.
- [10] V. Marinca, N. Herisanu, A modified iteration perturbation method for some nonlinear oscillation problems, *Acta Mechanica* **184** (2006) 231–242.
- [11] H. Hu, Solution of a quadratic nonlinear oscillator by the method of harmonic balance, *Journal of Sound and Vibration* **293** (2006) 462–468.
- [12] H. Hu, Solutions of a quadratic nonlinear oscillator: Iteration procedure, *Journal of Sound and Vibration* **298** (2006) 462–468.
- [13] R. E. Mickens, Iteration method solutions for conservative and limit-cycle $x^{1/3}$ force oscillators, *Journal of Sound and Vibration* **292** (2006) 964–968.

- [14] V. P. Lukomsky, v. B. Bobkov, Asymptotic expansions of the periodic solutions of nonlinear evolution equations, *Nonlinear Dynamics* **16** (1998) 1–21.
- [15] S. H. Chen, X. M. Yang, Y. K. Cheung, Periodic solutions of strongly quadratic non-linear oscillators by the elliptic perturbation method, *Journal of Sound and Vibration* **212** (1998) 771–780.
- [16] S. H. Chen, X. M. Yang, Y. K. Cheung, Periodic solutions of strongly quadratic non-linear oscillators by the elliptic Lindstedt-Poincaré method, *Journal of Sound and Vibration* **227** (1999) 1109–1118.
- [17] S.-J. Liao, An analytic approximate technique for free oscillations of positively damped systems with algebraically decaying amplitude, *International Journal of Non-Linear Mechanics* **38** (2003) 1173–1183.
- [18] A. Chatterjee, Harmonic balance based averaging: approximate realizations of an asymptotic technique, *Nonlinear Dynamics* **32** (2003) 323–343.
- [19] S. L. Das, A. Chatterjee, Multiple scales via Galerkin projections: approximate asymptotics for strongly nonlinear oscillations, *Nonlinear Dynamics* **32** (2003) 161–186.
- [20] K. Nandakumar, A. Chatterjee, Higher-order pseudoaveraging via harmonic balance for strongly nonlinear oscillations, *Journal of Vibrations and Acoustics* **127** (2005) 416–419.
- [21] S. B. Yamgoué, T. C. Kofané, On the analytical approximation of damped oscillations of single degree of freedom oscillators, *International Journal of Non-Linear Mechanics*, **41** (2006)¹ 1248–1254.
- [22] K. Yagasaki, The Melnikov theory for subharmonics and their bifurcations in forced oscillators, *SIAM Journal on Applied Mathematics*, **56** (1996) 1720–1765.
- [23] I. S. Gradshteyn, I. M. Ryzhik, Table of Integrals, Series and Products, Academic Press, 2000.

¹This seems to be an error on the printed article, since it was published in 2007.

Figure 1. Comparison of the approximate analytical (dash-dotted line) waveform and phase trajectory to the corresponding exact numerical (solid line) ones for the nonlinear harmonic oscillator Eq.(3.1) for $\delta = 0.5$. The initial conditions are: $x_0 = 0, y_0 = 5$. Perfect agreement is observed.

Figure 2. Comparison of the approximate analytical (dash-dotted line) waveforms and phase trajectories to the corresponding exact numerical (solid line) ones for the Morse oscillator Eq.(3.21) for $\epsilon\delta = 0.1$ ((a)-(b)) and $\epsilon\delta = 0.2$ ((c)-(d)). The initial conditions are: $x_0 = -0.624, y_0 = 0$. A quite satisfactory agreement is obtained.

Accepted manuscript

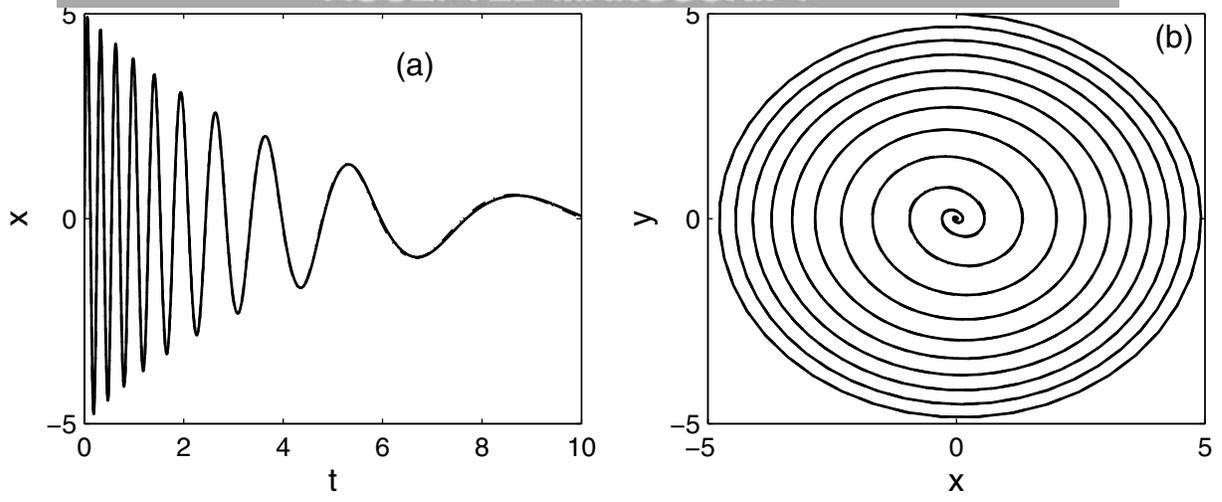


Figure 1: S. B. Yamgoué and T. C. Kofané

Accepted manuscript

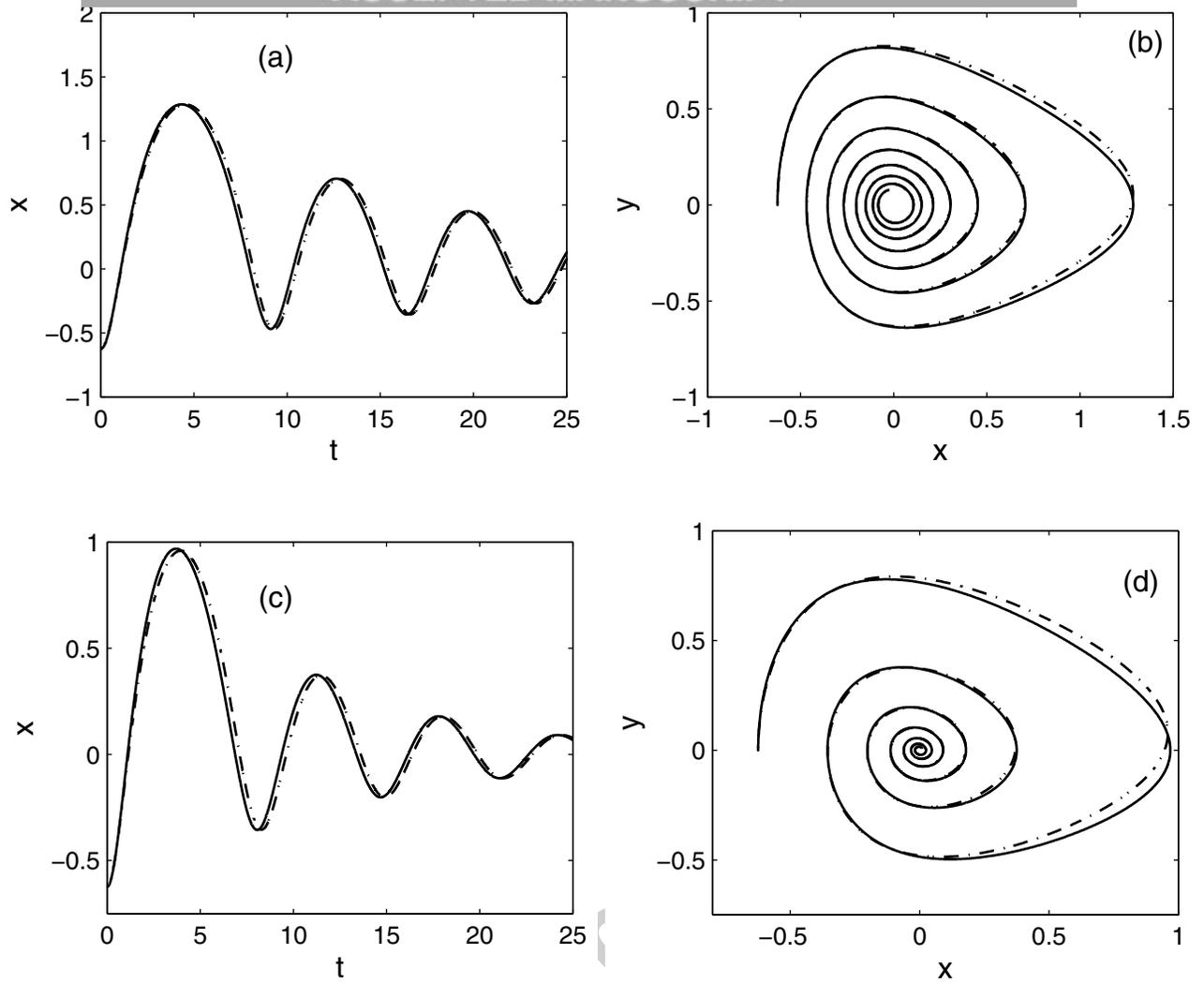


Figure 2: S. B. Yamgoué and T. C. Kofané