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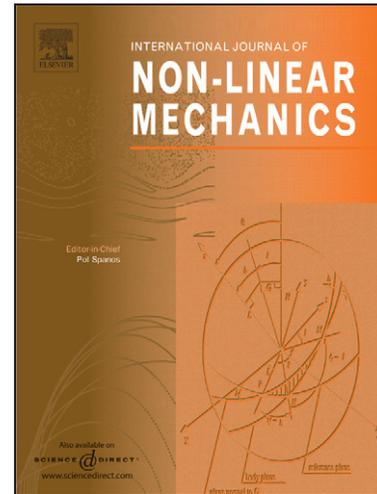
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**ITÔ CALCULUS EXTENDED TO SYSTEMS DRIVEN BY
ALPHA-STABLE LÉVY WHITE NOISES
(A NOVEL CLIP ON THE TAILS OF LÉVY MOTION)**

by

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ABSTRACT

The paper deals with probabilistic characterization of the response of non-linear systems under α -stable Lévy white noise input. It is shown that, by properly selecting a clip in the probability density function of the input, the moments of the increments of Lévy motion process remain all of the same order (dt), like the increments of the Compound Poisson process. It follows that the Itô calculus extended to Poissonian input, may also be used for α -stable Lévy white noise input processes. It is also shown that, when the clip on the tails of the probability of the increments of the Lévy motion approaches to infinity, the Einstein-Smoluchowsky equation is restored. Once these concepts are outlined extension to single oscillator is readily obtained. A discussion on the proper way to perform Monte Carlo simulation is also exploited.

Keywords: α -Stable Processes, Einstein-Smoluchowsky Equation, Itô Calculus, Truncated Lévy Motion.

1. INTRODUCTION

Normal white noises are very popular stochastic processes and they have been used to model several types of physical phenomena. Such processes may be defined as formal time derivative of Wiener processes. In this setting the powerful machinery of Itô stochastic differential calculus may be used to yield the response probabilistic characterization of systems driven by normal

white noises, in terms of moments or of conditional probability density function (PDF), that is the Fokker-Planck-Kolmogorov (FPK) equation. The Fourier transform, of the FPK equation, rules the evolution of the characteristic function (CF) of the response, often termed as *spectral* FPK equation.

However, several real phenomena observed in physics, seismology, electrical engineering, economics and in some other research fields show evident non-Gaussianity either in heavy tails distribution or in the impulsive nature of the recorded samples. According to this new need, the Itô stochastic differential calculus has been extended to Poissonian white noises too, providing the equation governing the evolution of the probability density, known as Kolmogorov-Feller equation [1-4]. The need for non-Gaussian models, to describe the fluctuations exhibited by non-Gaussian phenomena, has also raised the interest in the so-called α -stable Lévy processes [4-7]. This kind of stochastic process is characterized by the knowledge of four parameters, which are, respectively, the stability index α , the scale factor σ , the skewness β and the shift μ [8]. The choice for different values of the stability index α ($0 < \alpha \leq 2$) leads to formidable variety of stochastic processes including the Gaussian white noise obtained for $\alpha = 2$. Linear and non-linear systems driven by external α -stable Lévy white noise processes (formal time derivative of the Lévy motion processes labeled as $L_\alpha(t)$) have been treated, in the past, either in terms of PDF Einstein-Smoluchowsky (ES) equation or in terms of CF (see e.g. [9]). Moreover closed-form expressions of the probability density function of dynamical systems driven by external α -stable Lévy noises have been obtained for limited values of the stability index.

The main challenge in the analysis of dynamical systems in presence of α -stable Lévy white noise is related to the divergence of statistical moments of the α -stable random variable $L_\alpha(t)$, namely $E[L_\alpha^p(t)] = \infty$ if $p \geq \alpha$. Then unless the case of normal white noise ($\alpha = 2$) all the moments starting from the variance would diverge because of the heavy tails of the distribution of the random variable $L_\alpha(t)$. In order to overcome this drawback some truncation of the PDF of α -stable random variable $L_\alpha(t)$ have recently been performed [10] in which a power law with exponent $(5-\alpha)$ has been used to truncate the PDF of α -stable distribution. In this context a modified fractional FPK equation has been obtained and the PDF of the response converges towards a Gaussian density in the central part. Tails of solution of the response of non-linear systems are discussed in [11].

In this paper, the Itô calculus will be extended to non-linear systems under α -stable Lévy white noise. In order to aim at this, a new form of truncation is introduced, different from that proposed in [10]. We, in fact, exclude the tails on the PDF for value greater than $(\Delta t/n_\alpha)^{-1/\alpha}$, being Δt the time step and n_α a control clip parameter. It is shown that when a truncation is performed, all the moments of $dL_\alpha(t)$ exist and are all of order dt like the increments of the Compound Poisson process. It follows that the Itô rule extended to Poisson white noise process may be used. It is also shown that when $n_\alpha \rightarrow \infty$, the ES equation is fully recovered. In this setting a correct way for Monte-Carlo simulation (MCS) for the response analysis of non-linear systems under α -stable Lévy white noise is given. Based on this solid ground, the extension to single oscillator is readily found (see Appendix B). Some numerical examples have been reported to assess the validity of the proposed formulation for linear and non-linear Langevin equation under α -stable Lévy white noise.

2. INTRODUCTORY REMARKS ON ITÔ DIFFERENTIAL CALCULUS

In this section some well known concepts of Itô calculus are briefly summarized for sake of clarity as well as for introducing appropriate symbology.

Let the non-linear system under normal white noise be given in the form:

$$\dot{Z} = f(Z, t) + g(Z, t)W_2(t) \quad (1)$$

where $W_2(t)$ is the normal white noise characterized by the correlation function $R_{W_2}(t_1, t_2) = q_2 \delta(t_2 - t_1)$ being $\delta(\cdot)$ a Dirac delta function. Moreover q_2 is the strength of the white noise, $f(Z, t)$ and $g(Z, t)$ are deterministic non-linear functions of the random process $Z(t)$ and of time t . Eq. (1) is a non-linear one excited by a *parametric excitation*.

In the following we will confine ourselves to the case in which $g(Z, t) = g(t)$, that is the function g does not depend on the response $Z(t)$, Eq. (1) is referred as a non-linear one (for the presence of the non-linear term $f(Z, t)$) driven by an *external excitation*.

From the definition of $W_2(t)$, as formal time derivative of Wiener process $B(t)$ having independent increments, it follows that Eq.(1) may be rewritten in Itô form as follows:

$$dZ = f(Z, t)dt + g(t)dB(t) \quad (2)$$

Once the original equation is transformed into an Itô type stochastic differential equation, the Itô differential rule may be used for the probabilistic characterization of the response process $Z(t)$:

$$d\psi(Z, t) = \frac{\partial\psi}{\partial t}dt + \frac{\partial\psi}{\partial Z}dZ + \frac{1}{2} \frac{\partial^2\psi}{\partial Z^2}(dZ)^2 \quad (3)$$

where $\psi(Z, t)$ is any non-linear function of Z and t , continuous and differentiable on t and twice differentiable on Z . The third term in the right-hand side of eq.(3) is essential since $dB(t)$ is of order of magnitude $(dt)^{1/2}$ and then term $(dZ)^2$ is of the same order of the first term. In order to derive the equation governing the evolution of the characteristic function $\phi_Z(\theta, t) = E[\exp(i\theta Z(t))]$ we put $\psi(Z, t) = \exp(i\theta Z)$ in eq.(3), taking mathematical expectation and accounting for the non-anticipative property of Itô calculus $(E[\psi(Z, t)dB(t)^k] = E[\psi(Z, t)]E[dB(t)^k])$ the differential equation governing the evolution of the CF is readily written as:

$$\frac{\partial\phi_Z(\theta, t)}{\partial t} = i\theta E[\exp(i\theta Z)f(Z, t)] - \frac{\theta^2 q_2}{2} \phi_Z(\theta, t) g(t)^2 \quad (4)$$

An inverse Fourier transform of eq.(4) yields the so-called Fokker-Planck-Kolmogorov (FPK) equation governing the evolution of the PDF of the response:

$$\frac{\partial p_Z(z, t)}{\partial t} = -\frac{\partial}{\partial z}[p_Z(z, t)f(z, t)] + \frac{q_2}{2} g^2(t) \frac{\partial^2 p_Z(z, t)}{\partial z^2} \quad (5)$$

Itô rule extended to the case of Poissonian white noise is reported in Appendix A.

3. THE CASE OF α -STABLE WHITE NOISE EXTERNAL EXCITATION

Let us now suppose that $W_2(t)$ is substituted by $W_\alpha(t)$, being $W_\alpha(t)$ the α -stable Lévy white noise process. In analogy to the definition of the previous white noise, an α -stable Lévy white noise may be defined as the formal derivative of a corresponding α -stable Lévy motion $L_\alpha(t)$ (α is the stability index of the α -stable process) having the following properties:

i) It has independent, stationary increments following the α – stable distribution, that is

$$L_\alpha(t) - L_\alpha(s) \sim S_\alpha\left((t-s)^{1/\alpha}, 0, 0\right) \quad (6a)$$

ii) The CF of an increment of $dL_\alpha(t)$, takes the form:

$$\phi_{dL_\alpha}(\theta) = \exp\left[-dt|\theta|^\alpha\right] \quad (6b)$$

where $dL_\alpha(t)$ represents the increment of α -stable Lévy motion $L_\alpha(t)$ and $L_\alpha(t_k)$, at a selected time (t_k), is a symmetric α -stable ($S_\alpha S$) random variable.

iii) For $\alpha = 2$, $L_\alpha(t)$ reverts to $B(t)$ and then the non-normal α – stable process reverts to normal white noise process, or in other words the Wiener process is a particular case of the random process $L_\alpha(t)$.

In the case of Lévy white noise the equation governing the evolution of the PDF, for $g(Z, t) = g(t)$ (external excitation), is the so-called Einstein-Smoluchowsky (ES) equation [9] involving Riesz-Weil fractional derivative in the diffusion term, that is:

$$\frac{\partial p_Z(z, t)}{\partial t} = -\frac{\partial}{\partial z}(p_Z(z, t) f(z, t)) + |g(t)|^\alpha \frac{\partial^\alpha}{\partial |z|^\alpha} p_Z(z, t) \quad (7)$$

in which $\partial/\partial|z|^\alpha$ is the symmetric functional space derivative [12, 13] which is defined for a “sufficiently well-behaved” function through its Fourier transform $\mathfrak{S}[\cdot]$:

$$\mathfrak{S}\left[\frac{\partial^\alpha p_Z(z, t)}{\partial |z|^\alpha}\right] = -|\theta|^\alpha \phi_Z(\theta, t) \quad (8)$$

or in terms of the Riemann-Liouville derivatives as:

$$\frac{\partial^\alpha p_Z(z, t)}{\partial |z|^\alpha} = -\frac{1}{2 \cos(\pi\alpha/2)} \left[D_+^\alpha p_Z(z, t) + D_-^\alpha p_Z(z, t) \right] \quad (9)$$

where $\alpha > 0$ and if $0 < \alpha < 1$ then Riemann-Liouville derivatives reads:

$$\begin{aligned} D_+^\alpha p_Z(z, t) &= \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial z} \int_{-\infty}^z \frac{p_Z(\xi, t)}{(z-\xi)^\alpha} d\xi \\ D_-^\alpha p_Z(z, t) &= -\frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial z} \int_z^\infty \frac{p_Z(\xi, t)}{(\xi-z)^\alpha} d\xi \end{aligned} \quad (10 \text{ a, b})$$

else for $\alpha > 1$:

$$D_{\pm}^{\alpha} p_Z(z, t) = \frac{(\pm 1)^n}{\Gamma(n - \alpha)} \frac{\partial^n}{\partial z^n} \int_{-\infty}^z \xi^{n-\alpha-1} p_Z(z \mp \xi, t) d\xi \quad (11)$$

with $n-1 < \alpha \leq n \in \mathbb{N}$. In the specific case $\alpha = 1$ the Riemann-Liouville derivative coalesces with the Hilbert transform of the function $p_Z(\xi, t)$. The Dirichlet conditions about the existence of the $\mathfrak{S}[\cdot]$ are guaranteed since the area under the PDF is one. The derivatives in eqs.(10, 11) are characterized in the Fourier transform space as:

$$\mathfrak{S}[D_{\pm}^{\alpha} p_Z(z, t)] = (\mp i\theta)^{\alpha} \phi_Z(\theta, t) \quad (12)$$

with:

$$(-i\theta)^{\alpha} = |\theta|^{\alpha} \exp\left[\mp \frac{i\alpha\pi}{2} \text{sgn}(\theta)\right] \quad (13)$$

and $\text{sgn}(\cdot)$ is the well-known signum function. Combining eqs.(9, 11) with eq.(12), eq.(8) is recovered. Fourier transform of eq.(7) yields the CF equation in the form:

$$\frac{\partial \phi_Z(\theta, t)}{\partial t} = i\theta E[f(Z, t) e^{i\theta Z}] - |\theta|^{\alpha} \phi_Z(\theta, t) \quad (14)$$

in the following termed as *spectral ES*. Eq. (14) has also been obtained in [4, 14] using semimartingales.

In the next section it will be presented an alternative way for finding ES equation or the spectral counterpart as well as for properly performing Monte Carlo simulation.

4. ITÔ CALCULUS FOR EXTERNAL LÉVY WHITE NOISE

Let us consider the dynamic system reported in eq.(1) (with function $g(t) = 1$) enforced by Lévy white noise and let us cast the differential equation of motion in incremental form as:

$$Z(t_k + \Delta t) = Z(t_k) + f(Z_k, t_k) \Delta t + (\Delta t)^{1/\alpha} L_{\alpha}(t_k) \quad (15)$$

in which Δt is a small time increment and $L_{\alpha}(t_k)$ is a realization of an α -stable random variable, so that $\Delta t^{1/\alpha} L_{\alpha}(t_k)$ is the area of the impulse in the interval $t_k \div t_k + \Delta t$.

Peculiarity of α -stable random variables is long tail PDF. This feature will be reflected on experiencing eventual very high value $L_\alpha(t_k)$, when generating samples of $L_\alpha(t)$. This leads to numerical overflow in the response evaluation. To avoid the latter problem, it should be selected the highest value of $L_\alpha(t_k)$, according to a negligible committed error in the evaluation of $L_\alpha(t)$ PDF. Evaluating the probability that $L_\alpha(t_k)$ is greater than a large real value ν of an α -stable random variable $L_\alpha(t_k) \sim S_\alpha(\sigma, 0, 0)$ by:

$$\text{Pr ob} \left\{ |L_\alpha(t_k)| > \nu \right\} = C_\alpha \nu^{-\alpha} \quad ; \quad \nu \rightarrow \infty \quad (16)$$

where $C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin(x) dx \right)^{-1}$ is a real number dependent on the stability index α , (for instance, $C_\alpha = 0$ for $\alpha = 2$, $C_\alpha = 2/\pi$ for $\alpha = 1$, $C_\alpha = 1/\Gamma(2) = 1$ for $\alpha = 0$ [8]). It follows that the probability of occurrence of values of $L_\alpha(t_k)$ larger than $(\Delta t)^{-1/\alpha}$ is $C_\alpha \Delta t$, that is

$$\text{Pr ob} \left\{ |L_\alpha(t_k)| > (\Delta t)^{-1/\alpha} \right\} \simeq C_\alpha \Delta t \quad ; \quad \Delta t \rightarrow 0 \quad (17)$$

Based on this consideration in [15], realization of $L_\alpha(t_k)$ greater than $\Delta t^{-1/\alpha}$ may be neglected.

Looking at eq.(17), we may say that, introducing a real value n_α , ($n_\alpha \geq 1 \in \mathbb{R}$), in such a way:

$$\text{Pr ob} \left\{ |L_\alpha(t_k)| > \left(\frac{\Delta t}{n_\alpha} \right)^{-1/\alpha} \right\} \simeq C_\alpha \frac{\Delta t}{n_\alpha} \quad ; \quad \Delta t \rightarrow 0 \quad (18)$$

the probability remains of order Δt also for $(\Delta t/n_\alpha)^{-1/\alpha}$. This means that neglecting outcomes of $|L_\alpha(t_k)|$ larger than $(\Delta t/n_\alpha)^{-1/\alpha}$, we introduce an approximation of order $(\Delta t/n_\alpha)$. This approximation corresponds “de facto” in a proper clip of the tails of PDF, through a right choice of n_α .

Increasing values of n_α it results into smaller probability that $|L_\alpha(t_k)|$ exceeds the value $(\Delta t/n_\alpha)^{-1/\alpha}$. For instance, if values of n_α reach the order of magnitude of Δt^{-1} , then the probability that a realization of $|L_\alpha(t_k)|$ is greater than $(\Delta t/n_\alpha)^{-1/\alpha}$ is of order of magnitude

Δt^2 and in the limit as $\Delta t \rightarrow 0$ becomes an infinitesimal of higher order thus it may be really neglected. It is worth stressing that, at the limit for $n_\alpha \rightarrow \infty$ the real PDF is restored.

Previous considerations lead us to represent α -stable increments of the Lévy process $dL_\alpha(t)$ in the form:

$$dL_\alpha(t) = dt^{1/\alpha} L_\alpha(t) = \lim_{\substack{n_\alpha \rightarrow \infty \\ \Delta t \rightarrow 0}} \left(\frac{\Delta t}{n_\alpha} \right)^{\frac{1}{\alpha}} L_\alpha(t) \quad (19)$$

It is worth noting, that if $\alpha = 2$ ($L_\alpha(t) \rightarrow B(t)$), then $C_\alpha = 0$ and the area of each impulse that is $\Delta t^{1/2} B(t_k)$ remains finite setting $n_\alpha = 1$ in eq.(19).

In order to better clarify the role played by eq.(19) we may write

$$E[dL_\alpha^j(t_k)] = E[L_\alpha^j(t_k)] dt^{j/\alpha} = dt^{j/\alpha} \int_{-\infty}^{\infty} x^j p_{L_\alpha}(x) dx \quad (20)$$

Notice that in eq.(20) it is present the product of an infinitesimal quantity $dt^{j/\alpha}$ and a divergent integral term. The question is: what is the order of magnitude of $dL_\alpha^j(t_k)$? In order to check out the convergence of eq.(20) we introduce the representation of $dL_\alpha(t)$ as in eq.(19) yielding

$$E[dL_\alpha^j(t_k)] = \lim_{\Delta t \rightarrow 0} \Delta t^{j/\alpha} \int_{-(\Delta t/n_\alpha)^{-1/\alpha}}^{(\Delta t/n_\alpha)^{-1/\alpha}} x^j p_{L_\alpha}(x) dx = K_j(n_\alpha) dt \quad (21)$$

where $K_j(n_\alpha)$ are real finite numbers and then from eq.(21) we may state that the increments of the Lévy motion are of order dt . This remarkable result allows us to assess that in this perspective, once n_α has been selected, the increments of the Lévy motion behave exactly as the increments of the compound Poisson process [1-4] and then Itô rule for Poissonian white noise may be used (see eq. (A5) in the Appendix A.).

As an example we may assume that $L_\alpha(t)$ is Cauchy distributed, then PDF reads:

$$p_{L_\alpha}(x) = 1/\pi(1+x^2) \quad (22)$$

and terms $K_j(n_\alpha) = 0$ for j odd and $K_j(n_\alpha) = 2 n_\alpha/\pi(j-1)$ for j even.

Moreover if $\alpha = 2$, $n_\alpha = 1$ and then

$$E[dB^j(t)] = \lim_{\Delta t \rightarrow 0} \Delta t^{j/2} \int_{-(\Delta t)^{-1/2}}^{(\Delta t)^{-1/2}} x^j \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) dx = \Delta t^{j/2} K_j \quad (23)$$

where $K_{2j+1} = 0$, and $K_{2j} = q_2^{2j} (2j-1)!!$, (being $(2j-1)!! = (2j-1) \cdot (2j-3) \cdots 3 \cdot 1$; with $(-1)!! = 1$), that is $dB(t)$ is of order $dt^{1/2}$. It follows that the Itô rules according eq.(3) contains terms up to the order $(dZ)^2$ since the latter remains of order dt . Conversely, when the system is driven by Lévy α -stable white noise, for fixed n_α , all the increments $dL_\alpha^j(t)$ are of order dt , as shown in eq.(21). Then by selecting $\psi(Z, t) = \exp(i\theta Z(t))$ in eq.(A5) of the Appendix A, we get

$$d[\exp(i\theta Z)] = i\theta f(Z, t) \exp(i\theta Z) dt + \exp(i\theta Z) \sum_{j=1}^{\infty} \frac{(i\theta)^j}{j!} dL_\alpha^j(t) \quad (24)$$

By inserting eq.(21) into eq.(24), performing mathematical expectation, dividing by dt and using the non-anticipative property of Itô calculus, we get:

$$\frac{\partial}{\partial t} \phi_z(\theta, t) = i\theta E[f(Z, t) \exp(i\theta Z)] + \phi_z(\theta, t) \sum_{j=1}^{\infty} \frac{(i\theta)^j}{j!} K_j(n_\alpha) \quad (25)$$

Summation in eq.(25) for $\alpha = 1$ leads to

$$\sum_{j=1}^{\infty} \frac{(i\theta)^j}{j!} K_j(n_\alpha) = 2 \sum_{j=1}^{\infty} \frac{(i\theta)^{2j}}{2j!} \frac{n_\alpha^{2j-1}}{\pi(2j-1)} \quad (26)$$

And in the limit when $n_\alpha \rightarrow \infty$ converges to $-|\theta|$ as shown in fig.(1a) in which the sum in the series (26) has been reported for different values of n_α .

Summation in eq.(25) for $\alpha = 1/2$ (Lévy distribution) is given as

$$\sum_{j=1}^{\infty} \frac{(i\theta)^j}{j!} K_j(n_\alpha) = \sqrt{2} \sum_{j=1}^{\infty} \frac{(i\theta)^j}{j!} \frac{n_\alpha^{2j-1}}{(2j-1)\sqrt{\pi}} \quad (27)$$

and in the limit when $n_\alpha \rightarrow \infty$ converges to $-|\theta|^{1/2}$ as shown in fig.(2a). Many other intermediate values of $K_j(n_\alpha)$ have been tested for different values of α and always they give that at the

limit for $n_\alpha \rightarrow \infty$ summation in eq.(25) converges to $-|\theta|^\alpha$. Then we may conclude that eq.(25) when $n_\alpha \rightarrow \infty$ coincides with eq.(14), and its Fourier transform fully restores the ES equation. On the other hand once these results have been obtained we may also assert that for every value of α the following relation

$$\sum_{j=1}^{\infty} \frac{(i\theta)^j}{j!} E[dL_\alpha^j(t)] = \exp(-dt|\theta|^\alpha) - 1 = \phi_{dL_\alpha}(\theta) - 1 \simeq -dt|\theta|^\alpha \quad (28)$$

holds true.

At this stage some further observations will be exploited about the validity of eq.(28) as claimed in [4, 14]. It may be observed that formal analytical derivations led to eq.(28) but along way we introduced two fundamental assumptions:

- Setting the value of n_α , moments of increments $E[dL_\alpha^j]$ exist for every value of j and they are of order dt (eq.21).
- We consider that the series in eq.(28) converges to the CF of the increment $dL_\alpha(t)$, that coincides with eq.(25) for $n_\alpha \rightarrow \infty$.

Summing up, the realization of the α -stable random variable may attain infinite values, if we accept that for $\Delta t \rightarrow 0$ the probability that the random variable $|L_\alpha(t_k)| > (\Delta t/n_\alpha)^{-1/\alpha}$ is negligible, then $E[dL_\alpha^j]$ is of order $dt \forall j$. This means, “de facto” a proper clip of the tails of the PDF of the random variable $L_\alpha(t_k)$. If this operation is performed then Itô rule, (eq.A5) for Poissonian white noise is still valid also for α -stable Lévy white noise, and the ES equation is recovered. The latter concept has to be accounted for the MCS, in which we have to neglect realizations of values of $|L_\alpha(t_k)|$ larger than $(\Delta t/n_\alpha)^{-1/\alpha}$. In any cases, it suffices selecting at most $n_\alpha = 1/\Delta t$, for assessing that the error in clipping the tails of the PDF of $L_\alpha(t_k)$ is of order Δt^2 . Obviously if we select n_α and we compute $K_j(n_\alpha)$ and insert these values in eq.(24) we get $E[\psi(Z,t)]$. The same result is obtained by performing MCS excluding realizations of $|L_\alpha(t_k)|$ larger than $(\Delta t/n_\alpha)^{-1/\alpha}$. In this framework some numerical applications will be reported in the following sections.

5. NUMERICAL APPLICATIONS

In this section applications to linear and cubic non-linear system under α -stable Lévy white noise ($\alpha=1$) will be presented. The problem involving cubic non-linear oscillator has been selected since the solution in terms of spectral ES equation has been extensively studied (see i.e. [4],[9],[14]) and the existent solutions may be used as benchmark to highlight the role played by the truncation coefficient n_α . The analyzed system is

$$\dot{Z} = a Z + b Z^3 + W_\alpha(t) \quad (29)$$

5.1 Linear system under Cauchy white noise

The linear system, which is the hitherto studied case, is obtained setting $b=0$ in eq.(29). In this case $Z(t)$ is also an α -stable process having the same stability index as the Lévy white noise, but with different scale (or amplitude). For this case the spectral ES may be written as:

$$\frac{\partial \phi_Z(\theta, t)}{\partial t} = a \theta \frac{\partial \phi_Z(\theta, t)}{\partial \theta} - |\theta| \phi_Z(\theta, t) \quad (30 \text{ a})$$

$$\phi_Z(\theta) \Big|_{\theta=0} = 1 \quad (30 \text{ b})$$

With eq.(30 b) representing the attendant boundary condition. Steady-state solution of the differential equation described in eqs.(30) reads:

$$\phi_Z(\theta) = \exp\left[-|\theta|/a\right] \quad (31)$$

The Ornstein-Uhlenbeck CF described in eq.(31) has been reported in fig.(2) with continuous line. Such exact solution has been contrasted with results obtained by MCS for different values of n_α . The values of the time interval has been selected to value $\Delta t = 10^{-3}$ for the numerical simulation reported. Characteristic function $\phi_Z(\theta)$ in eq.(31) with the decaying factor $a = -1$, has been depicted in fig.(2a) with continuous line. The CF obtained via MCS have been contrasted in fig. (2a) for several values of the truncation parameter n_α set equal to $n_\alpha = 1$ (star), $n_\alpha = 50$ (triangle) and for the limit value $n_\alpha = 1/\Delta t = 1000$ (square).

The selected values of n_α and Δt led us to exclude values of $|L_\alpha(t_k)| > 1000$ (stars); $|L_\alpha(t_k)| > 50000$ (triangles); $|L_\alpha(t_k)| > 1000000$ (squares) according to

sec.4. It may be observed that for small values of n_α the Monte-Carlo estimate of the CF does not coincide with CF in eq.(31). A different scenario is observed increasing the value of n_α until its upper bound $1/\Delta t$ and it may be noticed that the CF obtained via numerical simulation coincides with the exact reported in eq.(31). A more precise description of the trend of the CF, for different n_α , is furnished by observation of fig.(2b). In more detail, it may be observed that, a smooth CF at the origin appears, assuming $n_\alpha = 1$. On the other hand for values of the coefficient n_α approaching the limit $1/\Delta t$ the peak of the exact CF at the origin is captured. Several other clipping values of $n_\alpha = 2/\Delta t$ or $n_\alpha = 3/\Delta t$ have been considered for numerical analysis, but the obtained results match the reported CF for the limit value $1/\Delta t$. This aspect is much more important for numerical simulation of non-linear dynamical systems as it will be shown in the following section.

5.2 Non-linear system under Cauchy white noise

Let us now consider that the non-linear coefficient $b \neq 0$ in eq.(29). The associated spectral ES equation is [4],[9],[14].

$$\begin{aligned} \frac{\partial \phi_z(\theta, t)}{\partial t} &= a \theta \frac{\partial \phi_z(\theta, t)}{\partial \theta} + b \theta \frac{\partial^3 \phi_z(\theta, t)}{\partial \theta^3} - |\theta| \phi_z(\theta, t) \\ \phi_z(\theta, t)|_{\theta=0} &= 1; \quad \frac{\partial \phi_z(\theta, t)}{\partial \theta} = 0; \quad \lim_{\theta \rightarrow \infty} \phi_z(\theta, t) = 0 \end{aligned} \quad (32)$$

The boundary conditions in eq. (32 c) has been used in [9] and it is valid only if $\phi_z(\theta)$ is differentiable for $\theta = 0$. For the steady-state case the boundary value problem described in eq.(32 a, b) may be solved for $\theta > 0$ and $\theta < 0$ assuming that $\phi_z(\theta) = \rho \exp[\lambda \theta]$ with ρ a normalization constant and λ obtained by the solution of the algebraic equation:

$$b\lambda^3 - a\lambda + 1 = 0 \quad (33)$$

which for the selected values of parameters $a = -1.0$; $b = -0.1$ as reported in [14], yields the real roots $\lambda_1 = -1.15347$, $\lambda_2 = -2.42362$, $\lambda_3 = 3.57709$. The latter positive real root must be

disregarded since it does not fulfill the boundary condition $\lim_{\theta \rightarrow \infty} \phi_Z(\theta, t) = 0$ and the stationary CF reads:

$$\begin{aligned}\phi_Z(\theta) &= \left(\frac{\lambda_1}{\lambda_1 - \lambda_2} \exp(\lambda_2 \theta) - \frac{\lambda_2}{\lambda_1 - \lambda_2} \exp(\lambda_1 \theta) \right) ; \theta \geq 0 \\ \phi_Z(\theta) &= \left(-\frac{\lambda_1}{\lambda_1 - \lambda_2} \exp(-\lambda_2 \theta) + \frac{\lambda_2}{\lambda_1 - \lambda_2} \exp(-\lambda_1 \theta) \right) ; \theta < 0\end{aligned}\quad (34)$$

In order to assess the validity of eqs.(34 a,b) MCS has been performed estimating the CF with 747.000.000 deviates and for different values of n_α . The time step of numerical integration has been set to $\Delta t = 10^{-3}$ and the results of the estimation have been reported in fig.(3a) contrasted with solution in eq.(34). Values of the truncation parameter n_α has been set to $n_\alpha = 1$ (star), $n_\alpha = 50$ (triangle) and for the limit value $n_\alpha = 1/\Delta t = 1000$ (square). The selected values of n_α and Δt lead us to exclude values of $|L_\alpha(t_k)| > 1000$ (stars) ; $|L_\alpha(t_k)| > 50000$ (triangles) ; $|L_\alpha(t_k)| > 1000000$ (squares) according to sec.4. Close observation of fig.(3a) shows that in the limiting case $n_\alpha = 1/\Delta t$ the estimated CF totally overlaps the exact CF in eqs.(34 a,b) leading to conclude that the assumption reported in [9] about the differentiability of the CF at $\theta = 0$ is correct. This consideration is still more evident in fig. (3b) where the behavior of the CF in the close vicinity of the origin has been investigated. In more detail we notice that for values of the truncation parameter too small, say $n_\alpha = 1$, the estimated CF does not coincide with the exact expression in eq.(34 a,b). However as soon as the value of n_α reaches $1/\Delta t$ the estimated CF perfectly match the exact expression and also for higher values of the truncation parameter, exceeding the limit value $n_\alpha = 1/\Delta t$, results do not change. Moreover, it must be stressed that Monte Carlo simulation of non-linear systems fails due to numerical overflows as soon as the truncation parameter exceeds the value $n_\alpha = 20/\Delta t$ and then a clipping control parameter $n_\alpha = 1/\Delta t$ is the right choice, being the committed error of order Δt^2 . Moreover at the limit when $\Delta t \rightarrow 0$ and the $n_\alpha \rightarrow \infty$ the ES equation is fully restored. Question raised in [14] about the validity of the spectral ES equation remains clarified in sense that the spectral ES and hence the ES equation are fully valid and the Itô calculus may still be applied.

6. CONCLUSIONS

In this paper the Itô stochastic differential rule holding for normal and Poissonian white noise has been extended to analysis of Lévy white noise in presence of external excitation. Extension of the differential rule to analysis of Lévy white noise has led to the formulation of the Einstein-Smoluchowsky fractional differential equation in terms of characteristic function. This remarkable result relies on the assumption to neglect increments of the Lévy white noise larger than $(\Delta t/n_\alpha)^{-1/\alpha}$ and in this context the Einstein-Smoluchowsky equation is restored. This latter consideration allows to perform Monte-Carlo simulation of dynamical systems under α – stable external excitation with an opportune clip on the increments of the α – stable Lévy flights. Some numerical analyses, contrasting the proposed formulation with a Monte-Carlo simulation obtained neglecting values of the realization of the Lévy motion $L_\alpha(t)$ higher than $(\Delta t/n_\alpha)^{-1/\alpha}$, have been reported to assess the validity of the concepts here exploited. Analyses have been conducted for linear and non-linear dynamical systems excited by a particular class of Lévy flights with stability index $\alpha = 1$, namely Cauchy white noises, which allows for an exact solution in terms of spectral ES.

In this context a simple Langevin equation has been investigated to yield the stationary characteristic function of the response contrasted with the proposed Monte-Carlo simulation method. The numerical analyses conducted have shown that the proposed clip on tails of the probability density function of the increments of the Lévy flights yields estimates which are in good agreement with the estimate via the Monte-Carlo simulation, contrasted by solving the spectral ES.

Remarkably it has been shown that the use of stochastic differential calculus yields the same differential equation for the characteristic function obtained via Fourier transform of the fractional Einstein-Smoluchowsky differential equation for the pdf of the dynamic response. This latter consideration may raise some comments about the range of validity of the Einstein-Smoluchowsky equation or of its spectral counterpart in the analysis of non-linear dynamical systems under α – stable Lévy flights. As in fact we may conclude that the ES equation may be considered as the limit case when the clip on the tails reaches infinity and the error made is of

order dt^2 . In this perspective question raised in [14] about the validity of the spectral ES equation remains fully clarified. That is ES equation and own spectral counterpart are valid as soon as differentials of order greater than dt^2 are neglected as usual done.

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Appendix A: Itô Rule for Poissonian White Noise

Let the equation of motion be given as:

$$\dot{Z}(t) = f(Z, t) + W_p(t) \quad (\text{A1})$$

where $W_p(t)$ is a Poissonian white noise, that is a stochastic process constituted by random impulses randomly distributed in time in accordance to Poisson law. If R is the random variable representing the intensities of the impulses and λ is the mean number of impulses per unit time, then $W_p(t)$ is characterized by the cumulants:

$$\begin{aligned} K_1[W_p(t)] &= \lambda E[R] \\ K_2[W_p(t_1)W_p(t_2)] &= \lambda E[R^2] \delta(t_2 - t_1) \\ &\dots \\ K_r[W_p(t_1)W_p(t_2)\dots W_p(t_r)] &= \lambda E[R^r] \delta(t_2 - t_1) \dots \delta(t_r - t_1) \end{aligned} \quad (\text{A2})$$

White noise process may be considered as the formal derivative of the Compound Poisson process $C(t)$ with increments characterized by the expressions:

$$E[dC(t)^r] = \lambda E[R^r] dt \quad (\text{A3})$$

Eq.(A1) may be reported in incremental form as:

$$dZ(t) = f(Z, t)dt + dC(t) \quad (\text{A4})$$

and since $dC^r(t)$ is of order $dt \forall r$ the Taylor expansion of $\psi(Z, t)$ will be rewritten in its complete form (omitting arguments) as:

$$\begin{aligned} d\psi &= \frac{\partial \psi}{\partial t} dt + \sum_{j=1}^{\infty} \frac{\partial^j \psi}{\partial Z^j} (dZ)^j = \\ &= \frac{\partial \psi}{\partial t} dt + \frac{\partial \psi}{\partial Z} f(Z, t)dt + \sum_{j=1}^{\infty} \frac{\partial^j \psi}{\partial Z^j} (dC)^j \end{aligned} \quad (\text{A5})$$

Since $dC(t)$ has independent increments, the non-anticipative property of Itô calculus remains still valid. By putting $\psi(Z, t) = \exp(i\theta Z(t))$ in eq.(A5) taking into account eq.(A3) making stochastic average and using the non-anticipative property of Itô calculus we get

$$\frac{\partial \phi_z(\mathcal{G}, t)}{\partial t} = i\mathcal{G} E[f(Z, t) \exp(i\mathcal{G}Z)] + \lambda \phi_z(\mathcal{G}, t) \sum_{j=1}^{\infty} \frac{(i\mathcal{G})^j}{j!} E[R^j] \quad (\text{A6})$$

Since the Taylor expansion of CF of the random variable R is given as

$$\phi_R(\vartheta) = 1 + \sum_{j=1}^{\infty} \frac{(i\vartheta)^j}{j!} E[R^j] \quad (\text{A7})$$

Then eq.(A6) may also be rewritten

$$\frac{\partial \phi_Z(\vartheta, t)}{\partial t} = i\vartheta E[f(Z, t) \exp(i\vartheta Z)] + \lambda \phi_Z(\vartheta, t) (\phi_R(\vartheta) - 1) \quad (\text{A8})$$

An inverse Fourier transform of eq. (A8) yields the Kolmogorov-Feller equation ruling the evolution of the PDF

$$\frac{\partial p_Z(z, t)}{\partial t} = -\frac{\partial}{\partial z} (f(z, t) p_Z(z)) - \lambda p_Z(z) + \lambda \int_{-\infty}^{\infty} p_Z(z - \xi) p_R(\xi) d\xi \quad (\text{A9})$$

Appendix B: Extension to a Single Oscillator System

In this appendix the analysis of a non-linear system with two degrees of freedom has been reported. Let the equation of motion of the system represented in the form:

$$\ddot{X} + f(X, \dot{X}) = W_\alpha(t) \quad (\text{B1})$$

Introducing the Lagrangian parameters $X_1(t) = X(t); X_2(t) = \dot{X}(t)$ the second-order equation of motion in eq.(B1) may be represented by a system of first-order differential equations in the form:

$$\begin{cases} \dot{X}_1(t) = -aX_1 + X_2 + W_\alpha(t) \\ \dot{X}_2(t) = -bX_2 + \varepsilon X_1^3 + W_\alpha(t) \end{cases} \quad (\text{B2})$$

with system parameters $a > 0, b > 0, \varepsilon < 0$. System of differential equations in eqs.(B2 a,b) may be cast in an alternative form by means of a linear transform which reads:

$$\begin{cases} X_1 + X_2 = Y_1 \\ X_1 - X_2 = Y_2 \end{cases} \quad (\text{B3})$$

which may be substituted into system in eqs.(B2 a,b) yielding the equivalent non-linear system:

$$\begin{cases} \dot{Y}_1(t) = Y_1(t) \left(\frac{1}{2} - \frac{a+b}{2} \right) - Y_2(t) \left(\frac{a}{2} + \left(\frac{1-b}{2} \right) \right) + \frac{\varepsilon}{8} (Y_1(t) + Y_2(t))^3 + 2W_\alpha(t) \\ \dot{Y}_2(t) = Y_1(t) \left(-\frac{a}{2} + \frac{1+b}{2} \right) - Y_2(t) \left(\frac{a}{2} + \frac{1+b}{2} \right) + \frac{\varepsilon}{8} (Y_1(t) + Y_2(t))^3 \end{cases} \quad (\text{B4})$$

Eqs.(B4 a,b) may be cast in an equivalent vector form as:

$$d\mathbf{Z}(t) = \mathbf{g}(\mathbf{Z}, t) dt + \mathbf{v} dL_\alpha(t) \quad (\text{B5})$$

where $\mathbf{Z}(t)^T = [Y_1(t) \ Y_2(t)]$, $\mathbf{g}(\mathbf{Z}, t)^T = [g_1(\mathbf{Z}, t) \ g_2(\mathbf{Z}, t)]$, $\mathbf{v}^T = [1 \ 0]$ with the non-linear functions:

$$\begin{aligned} g_1(\mathbf{Z}, t) &= Y_1 \left(\frac{1}{2} - \frac{a+b}{2} \right) - Y_2 \left(\frac{a}{2} + \left(\frac{1-b}{2} \right) \right) + \frac{\varepsilon}{8} (Y_1 + Y_2)^3 \\ g_2(\mathbf{Z}, t) &= Y_1 \left(-\frac{a}{2} + \frac{1+b}{2} \right) - Y_2 \left(\frac{a}{2} + \frac{1+b}{2} \right) + \frac{\varepsilon}{8} (Y_1 + Y_2)^3 \end{aligned} \quad (\text{B6})$$

The Itô rule for Poissonian white noise is written as:

$$d\psi(\mathbf{Z}, t) = \frac{\partial \psi}{\partial t} + \sum_{j=1}^{\infty} \frac{1}{j!} \nabla_{\mathbf{Z}}^{[j]T} \psi(\mathbf{Z}, t) (d\mathbf{Z})^{[j]} \quad (\text{B7})$$

where $\nabla_{\mathbf{Z}}^T = [\partial/\partial Y_1 \quad \partial/\partial Y_2]$ is the gradient vector, the exponent into brackets is the Kronecker power and $\psi(\mathbf{Z}, t)$ is any real-valued function continuously differentiable in t and ∞ -times differentiable in \mathbf{Z} . Particularization of eq.(B7) with $\psi(\mathbf{Z}, t) = \exp(i \boldsymbol{\theta}^T \mathbf{Z})$ results in the expression:

$$\begin{aligned} d \exp[i \boldsymbol{\theta}^T \mathbf{Z}] &= i \boldsymbol{\theta}^T \mathbf{g}(\mathbf{Z}, t) \exp[i \boldsymbol{\theta}^T \mathbf{Z}] \\ &+ \sum_{j=1}^{\infty} \frac{(i)^j}{j!} \boldsymbol{\theta}^{[j]T} \mathbf{v}^{[j]} (dL_{\alpha})^j \exp[i \boldsymbol{\theta}^T \mathbf{Z}] \end{aligned} \quad (\text{B8})$$

The application of mathematical expectation operator to both sides of eq.(B8), taking into account the non-anticipative property of the Itô calculus reads:

$$\begin{aligned} d\phi_{\mathbf{Z}}(\boldsymbol{\theta}, t) &= i \boldsymbol{\theta}^T E[\mathbf{g}(\mathbf{Z}, t) \exp[i \boldsymbol{\theta}^T \mathbf{Z}]] dt \\ &+ E\left[\sum_{j=1}^{\infty} \frac{(i)^j}{j!} \boldsymbol{\theta}^{[j]T} \mathbf{v}^{[j]} (dL_{\alpha})^j \right] \phi_{\mathbf{Z}}(\boldsymbol{\theta}, t) \end{aligned} \quad (\text{B9})$$

and resorting to eq.(21):

$$E\left[\sum_{j=1}^{\infty} \frac{(i)^j}{j!} \boldsymbol{\theta}^{[j]T} \mathbf{v}^{[j]} (dL_{\alpha})^j \right] = \lim_{n_{\alpha} \rightarrow \infty} \sum_{j=1}^{\infty} \frac{(i)^j}{j!} \theta_1^j K_j(n_{\alpha}) dt = -|\theta_1|^{\alpha} dt \quad (\text{B10})$$

yielding the spectral equation, ruling the evolution of the CF as:

$$\frac{\partial \phi_{\mathbf{Z}}(\boldsymbol{\theta}, t)}{\partial t} = i \boldsymbol{\theta}^T E[\mathbf{g}(\mathbf{Z}, t) \exp[i \boldsymbol{\theta}^T \mathbf{Z}]] - |\theta_1|^{\alpha} \phi_{\mathbf{Z}}(\boldsymbol{\theta}, t) \quad (\text{B11})$$

Fourier transform of eq.(B11) yields the Einstein-Smoluchowsky equation extended for the single non-linear oscillator reported in eq.(B1) in the form:

$$\frac{\partial p_{\mathbf{Z}}(\mathbf{Z}, t)}{\partial t} = -\nabla_{\mathbf{Z}}^T [\mathbf{g}(\mathbf{Z}, t) p_{\mathbf{Z}}(\mathbf{Z}, t)] + \frac{\partial^{\alpha}}{\partial |\mathbf{Z}_1|^{\alpha}} p_{\mathbf{Z}}(\mathbf{Z}, t) \quad (\text{B12})$$

Eq.(B12) represents the extension of the ES equation for the single degree of freedom non-linear system.

Figure Caption

Figure 1a: Asymptotic trend of the series $f(\theta, n_\alpha)$ for various n_α contrasted with $-|\theta|$.

Figure 1b: Asymptotic trend of the series $f(\theta, n_\alpha)$ for various n_α contrasted with $-|\theta|^{1/2}$.

Figure 2a: Characteristic function of the linear system with $a = -1$ contrasted with Monte Carlo estimates with different values of the truncation parameter n_α .

Figure 2b: Behavior of the Characteristic function of linear system in the neighborhood of the origin contrasted with different values of the truncation parameter n_α .

Figure 3a: Characteristic function of the non-linear system with $a = -1, b = -0.1$ contrasted with Monte Carlo estimates with different values of the truncation parameter n_α .

Figure 3b: Behavior of the Characteristic function of non-linear system in the neighborhood of the origin contrasted with different values of the truncation parameter n_α .

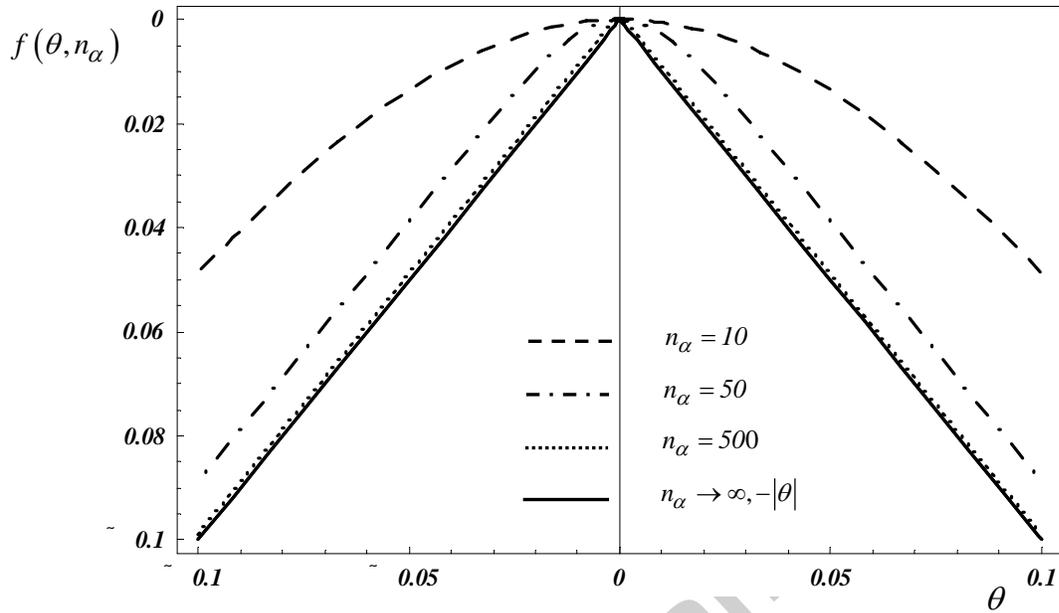


Figure 1a: Asymptotic trend of the series $f(\theta, n_\alpha)$ for various n_α contrasted with $-|\theta|$.

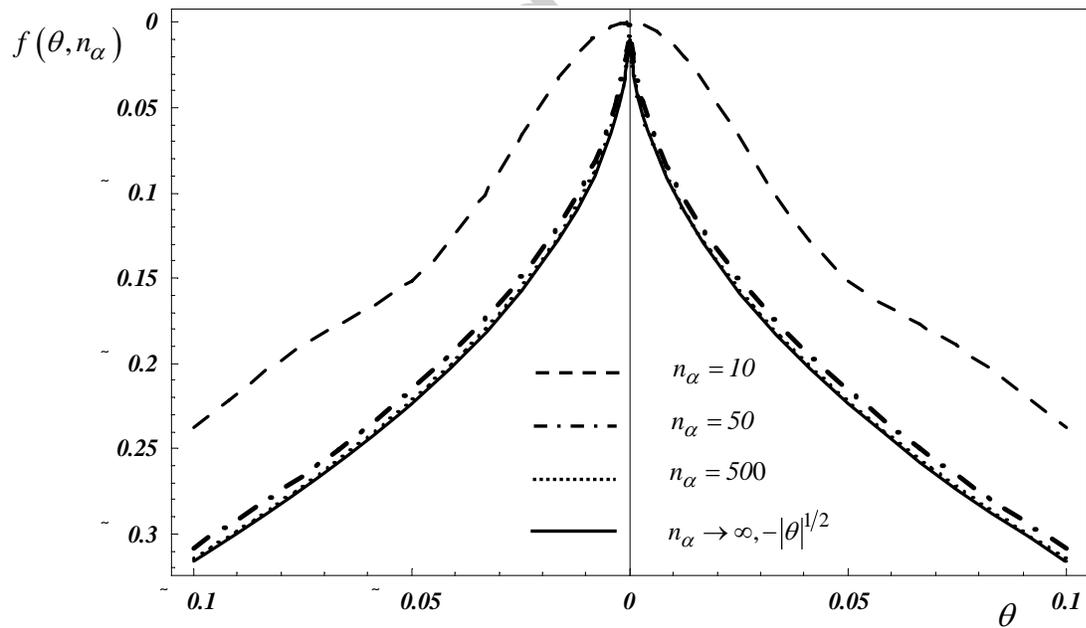


Figure 1b: Asymptotic trend of the series $f(\theta, n_\alpha)$ for various n_α contrasted with $-|\theta|^{1/2}$.

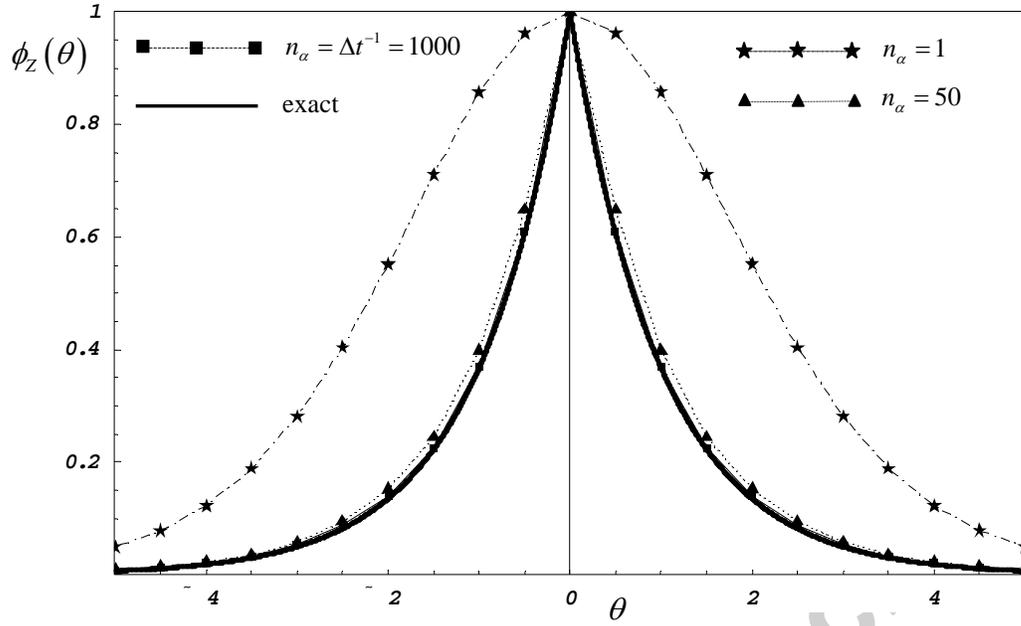


Figure 2a: Characteristic function of the linear system with $a = -1$ contrasted with Monte Carlo estimates with different values of the truncation parameter n_α

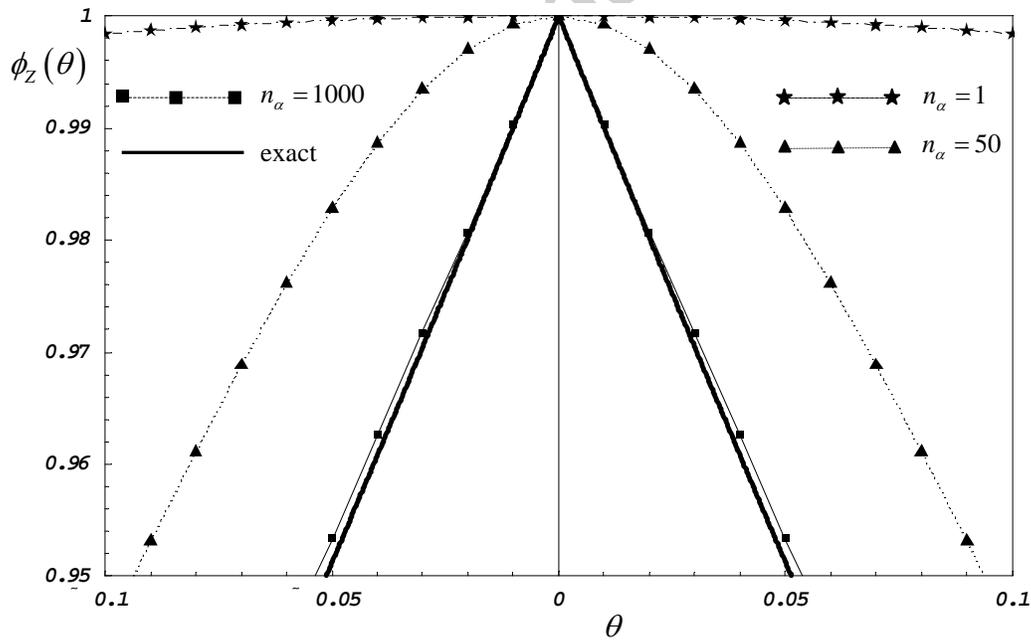


Figure 2b: Behavior of the Characteristic function of linear system in the neighborhood of the origin contrasted with different values of the truncation parameter n_α

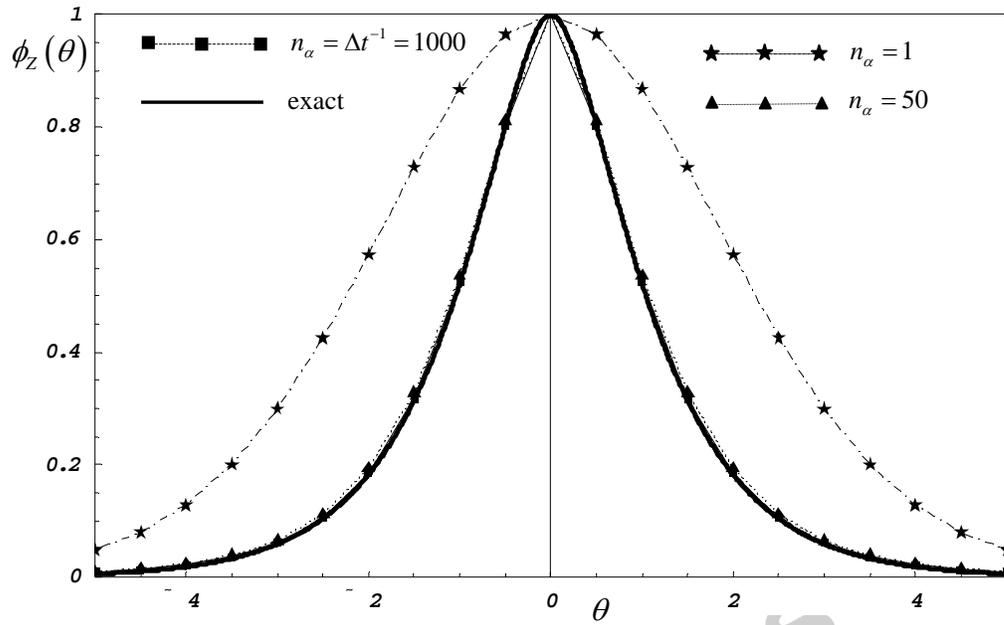


Figure 3a: Characteristic function of the non-linear system with $a = -1, b = -0.1$ contrasted with Monte Carlo estimates with different values of the truncation parameter n_α .

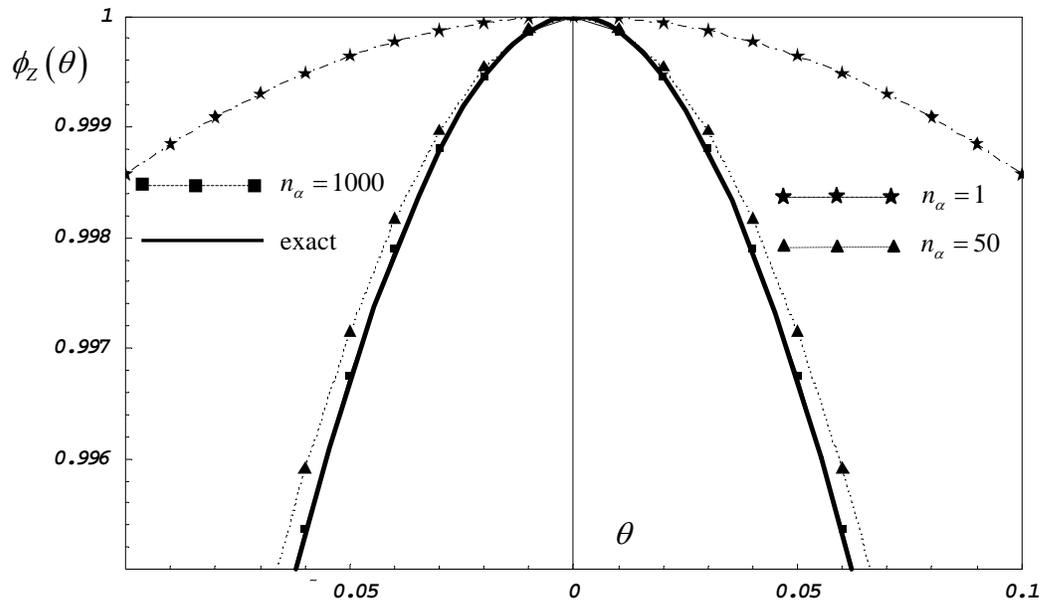


Figure 3b: Behavior of the Characteristic function of non-linear system in the neighborhood of the origin contrasted with different values of the truncation parameter n_α .