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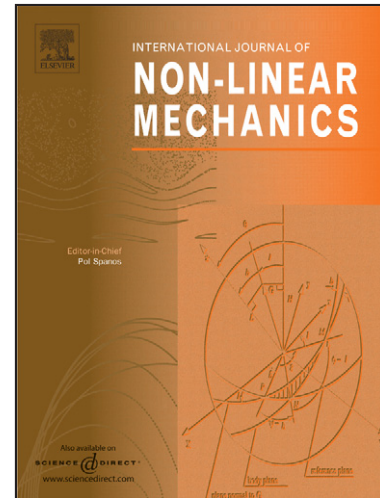
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Nonlinear Output Frequency Response Functions of MDOF Systems with Multiple Nonlinear Components

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Abstract: In engineering practice, most mechanical and structural systems are modeled as Multi-Degree-of-Freedom (MDOF) systems such as, e.g., the periodic structures. When some components within the systems have nonlinear characteristics, the whole system will behave nonlinearly. The concept of Nonlinear Output Frequency Response Functions (NOFRFs) was proposed by the authors recently and provides a simple way to investigate nonlinear systems in the frequency domain. The present study is concerned with investigating the inherent relationships between the NOFRFs for any two masses of nonlinear MDOF systems with multiple nonlinear components. The results reveal very important properties of the nonlinear systems. These properties clearly indicate how the system linear characteristic parameters govern the propagation of the nonlinear effect induced by nonlinear components in the system. One potential application of the results is to detect and locate faults in engineering structures which make the structures behave nonlinearly.

Nomenclature

$x(t), u(t)$	the output and input of the nonlinear system
$X(j\omega), U(j\omega)$	the spectrum of the system output and input
$h_n(\tau_1, \dots, \tau_n)$	the n^{th} order Volterra kernel
$H_n(j\omega_1, \dots, j\omega_n)$	the n^{th} order GFRF
$G_n(j\omega)$	the n^{th} order NOFRF
M, C, K	the system mass, damping and stiffness matrices
m_i, c_i, k_i	the i^{th} mass, damping and stiffness parameter
$FS_{L(i)}$ and $FD_{L(i)}$	the restoring forces of $L(i)^{\text{th}}$ nonlinear damper and stiffness
$r_{(L(i),l)}$ and $w_{(L(i),l)}$	the nonlinearity related parameters of the $L(i)^{\text{th}}$ nonlinear component
$(i=1, \dots, P)$	

$NonF_{L(i)}$	the nonlinear force of the $L(i)^{th}$ nonlinear component
$x_i(t), x_i(j\omega)$	the displacement and the output frequency response of the i^{th} mass
$\Theta(j\omega)$	the transform function of the linear MDOF oscillator
$Q_{(i,l)}(j\omega)$	the element of the i^{th} row and the l^{th} column of the inverse matrix of $\Theta(j\omega)$
$h_{(i,j)}(\tau_1, \dots, \tau_j)$	the j^{th} order Volterra kernel associated to the i^{th} mass
$G_{(i,l)}(j\omega)$	the l^{th} order NOFRF associated to the i^{th} mass
$\lambda_n^{i,i+1}(j\omega_1, \dots, j\omega_n)$	the ratio between the n^{th} GFRFs of the i^{th} and $(i+1)^{th}$ masses
$\lambda_n^{i+1}(j\omega)$	the ratio between the n^{th} NOFRFs of the i^{th} and $(i+1)^{th}$ masses
$\Lambda_{\frac{L(i)-1, L(i)}{N}}(j\omega_1, \dots, j\omega_N)$	the term introduced by the nonlinear force $NonF_{L(i)}$ for the n^{th} order GFRF.
$\Gamma_{(L(i),n)}(j\omega)$	the term introduced by the nonlinear force $NonF_{L(i)}$ for the n^{th} order NOFRF.

1 Introduction

In engineering practice, for many mechanical and structural systems, more than one set of coordinates are needed to describe the system behaviour. This implies a MDOF model is needed to represent the system. In addition, these systems may also behave nonlinearly due to nonlinear characteristics of some components within the systems. For example, a beam with breathing cracks behaves nonlinearly because of the cracked elements inside the beam [1]. For nonlinear systems, the classical Frequency Response Function (FRF) cannot achieve a comprehensive description for the system dynamical characteristics, which, however, can be fulfilled using the Generalised Frequency Response Functions (GFRFs) [2]. The GFRFs, which are extension of the FRFs to the nonlinear case, are defined as the Fourier transforms of the kernels of the Volterra series [3]. The Volterra series and its derivative GFRFs are powerful tools for the analysis of nonlinear systems and have been widely studied in the past two decades [4][5][6]. The applications of the Volterra series range from the electrical engineering [7]~[9], communications[10]~[12], network theory [13][14] to structure dynamics [15]~[17].

If a differential equation or discrete-time model is available for a nonlinear system, the GFRFs can be determined using the algorithm in [18]~[20]. However, the GFRFs are much more complicated than the FRF. GFRFs are multidimensional functions [21][22], which can be difficult to measure, display and interpret in practice. Recently, the novel concept known as Nonlinear Output Frequency Response Functions (NOFRFs) was proposed by the authors [23]. The concept can be considered to be an alternative

extension of the FRF to the nonlinear case. NOFRFs are one dimensional functions of frequency, which allow the analysis of nonlinear systems in the frequency domain to be implemented in a manner similar to the frequency domain analysis of linear systems and which provide great insight into the mechanisms which dominate important nonlinear behaviours. Using the NOFRF, the authors have investigated the resonance phenomena for a class of nonlinear systems [24]. Most recently, the concept of the NOFRF has been extended from the SISO case to the MIMO case by the authors [25].

The present study is concerned with the analysis of the inherent relationships between the NOFRFs for any two masses of MDOF systems with multiple nonlinear components. The results reveal, for the first time, very important properties of the nonlinear systems. These properties clearly indicate how the system linear characteristic parameters govern the propagation of the nonlinear effect induced by nonlinear components in the system. One potential application of the results is to detect and locate faults in engineering structures which make the structures behave nonlinearly. This study will focus on the derivation and verification of nine important properties of nonlinear MDOF systems using a NOFRF based analysis.

2. MDOF Systems with Multiple Nonlinear Components

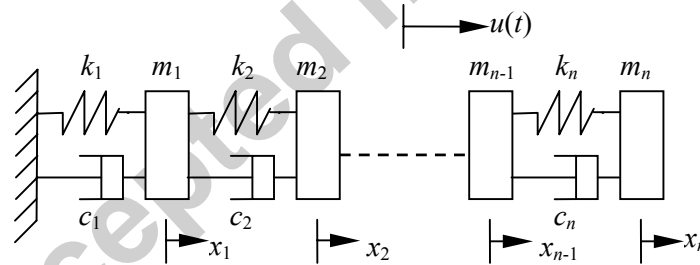


Figure 1, a multi-degree freedom oscillator

A typical multi-degree-of-freedom oscillator is shown as Figure 1, the input force is added on the J th mass.

If all springs and damping have linear characteristics, then this oscillator is a MDOF linear system, and the governing motion equation can be written as

$$M\ddot{x} + C\dot{x} + Kx = F(t) \quad (1)$$

where

$$M = \begin{bmatrix} m_1 & 0 & \dots & 0 \\ 0 & m_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & m_n \end{bmatrix}$$

is the system mass matrix, and

$$C = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 & \cdots & 0 \\ -c_2 & c_2 + c_3 & -c_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -c_{n-1} & c_{n-1} + c_n & -c_n \\ 0 & \cdots & 0 & -c_n & c_n \end{bmatrix} \quad K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & \cdots & 0 \\ -k_2 & k_2 + k_3 & -k_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -k_{n-1} & k_{n-1} + k_n & -k_n \\ 0 & \cdots & 0 & -k_n & k_n \end{bmatrix}$$

are the system damping and stiffness matrix respectively. $x = (x_1, \dots, x_n)'$ is the displacement vector, and

$$F(t) = (\overbrace{0, \dots, 0}^{J-1}, u(t), \overbrace{0, \dots, 0}^{n-J})'$$

is the external force vector acting on the oscillator.

Equation (2) is the basis of the modal analysis method, which is a well-established approach for determining dynamic characteristics of engineering structures [26]. In the linear case, the displacements $x_i(t)$ ($i = 1, \dots, n$) can be expressed as

$$x_i(t) = \int_{-\infty}^{+\infty} h_{(i)}(t - \tau) u(\tau) d\tau \quad (2)$$

where $h_{(i)}(t)$ ($i = 1, \dots, n$) are the impulse response functions that are determined by equation (1), and the Fourier transform of $h_{(i)}(t)$ is the well-known FRF.

Assume there are \bar{L} nonlinear components, which have nonlinear stiffness and damping, in the MDOF system, and they are the $L(i)$ th ($i = 1, \dots, \bar{L}$) components respectively, and the corresponding restoring forces $FS_{L(i)}(\Delta)$ and $FD_{L(i)}(\dot{\Delta})$ are the polynomial functions of the deformation Δ and $\dot{\Delta}$, i.e.,

$$FS_{L(i)}(\Delta) = \sum_{l=1}^P r_{(L(i),l)} \Delta^l, \quad FD_{L(i)}(\dot{\Delta}) = \sum_{l=1}^P w_{(L(i),l)} \dot{\Delta}^l$$

where P is the degree of the polynomial. Without loss of generality, assume $L(i) - 1$ and $L(i) \neq 1, J, n$ ($1 \leq i \leq \bar{L}$) and $k_{L(i)} = r_{(L(i),1)}$ and $c_{L(i)} = w_{(L(i),1)}$.

Denote

$$NF = (nf(1) \cdots nf(n))' \quad (3)$$

where

$$nf(l) = \begin{cases} 0 & \text{if } l \neq L(i) - 1, L(i), 1 \leq i \leq \bar{L} \\ -NonF_{L(i)} & \text{if } l = L(i) - 1, 1 \leq i \leq \bar{L} \\ NonF_{L(i)} & \text{if } l = L(i), 1 \leq i \leq \bar{L} \end{cases} \quad (1 \leq l \leq n) \quad (4)$$

and

$$NonF_{L(i)} = \sum_{l=2}^P r_{(L(i),l)} (x_{L(i)-1} - x_{L(i)})^l + \sum_{l=2}^P w_{(L(i),l)} (\dot{x}_{L(i)-1} - \dot{x}_{L(i)})^l \quad (5)$$

Then the motion of the MDOF oscillator in Figure 1 can be determined by below equation

$$M\ddot{x} + C\dot{x} + Kx = NF + F(t) \quad (6)$$

Equations (3)~(6) are the motion governing equations of nonlinear MDOF systems with multiple nonlinear components. The \bar{L} nonlinear components can lead the whole system to behave nonlinearly. In this case, the Volterra series [2] can be used to describe the relationships between the displacements $x_i(t)$ ($i=1, \dots, n$) and the input force $u(t)$ as below

$$x_i(t) = \sum_{j=1}^N \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_{(i,j)}(\tau_1, \dots, \tau_j) \prod_{i=1}^j u(t - \tau_i) d\tau_i \quad (7)$$

under quite general conditions [2]. In equation (7), $h_{(i,j)}(\tau_1, \dots, \tau_j)$, ($i=1, \dots, n$, $j=1, \dots, N$), represents the j th order Volterra kernel for the relationship between $u(t)$ and the displacement of m_i .

When a system is linear, its dynamical properties can be easily analyzed using the FRFs defined as the Fourier transform of $h_{(i)}(t)$ ($i=1, \dots, n$) in equation (2). However, as equation (7) shows, the dynamical properties of a nonlinear system are determined by a series of Volterra kernels, such as $h_{(i,j)}(\tau_1, \dots, \tau_j)$, ($i=1, \dots, n$, $j=1, \dots, N$) for the MDOF nonlinear systems considered in the present study. The objective of this paper is to study the nonlinear MDOF systems using the concept of Nonlinear Output Frequency Response Functions (NOFRFs), which is an alternative extension of the FRF to the nonlinear case and is derived based on the Volterra series approach of nonlinear systems.

3. Nonlinear Output Frequency Response Functions

The definition of NOFRFs is based on the Volterra series theory of nonlinear systems. The Volterra series extends the well-known convolution integral description for linear systems to a series of multi-dimensional convolution integrals, which can be used to represent a wide class of nonlinear systems [2].

Consider the class of nonlinear systems which are stable at zero equilibrium and which can be described in the neighbourhood of the equilibrium by the Volterra series

$$x(t) = \sum_{n=1}^N \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \dots, \tau_n) \prod_{i=1}^n u(t - \tau_i) d\tau_i \quad (8)$$

where $x(t)$ and $u(t)$ are the output and input of the system, $h_n(\tau_1, \dots, \tau_n)$ is the n th order Volterra kernel, and N denotes the maximum order of the system nonlinearity. Lang and Billings [2] derived an expression for the output frequency response of this class of nonlinear systems to a general input. The result is

$$\begin{cases} X(j\omega) = \sum_{n=1}^N X_n(j\omega) & \text{for } \forall \omega \\ X_n(j\omega) = \frac{1/\sqrt{n}}{(2\pi)^{n-1}} \int_{\omega_1 + \dots + \omega_n = \omega} H_n(j\omega_1, \dots, j\omega_n) \prod_{i=1}^n U(j\omega_i) d\sigma_{n\omega} \end{cases} \quad (9)$$

This expression reveals how nonlinear mechanisms operate on the input spectra to produce the system output frequency response. In (9), $X(j\omega)$ is the spectrum of the system output, $X_n(j\omega)$ represents the n th order output frequency response of the system,

$$H_n(j\omega_1, \dots, j\omega_n) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_n(\tau_1, \dots, \tau_n) e^{-(\omega_1\tau_1 + \dots + \omega_n\tau_n)j} d\tau_1 \dots d\tau_n \quad (10)$$

is the n th order Generalised Frequency Response Function (GFRF) [2], and

$$\int_{\omega_1 + \dots + \omega_n = \omega} H_n(j\omega_1, \dots, j\omega_n) \prod_{i=1}^n U(j\omega_i) d\sigma_{n\omega}$$

denotes the integration of $H_n(j\omega_1, \dots, j\omega_n) \prod_{i=1}^n U(j\omega_i)$ over the n -dimensional hyper-plane $\omega_1 + \dots + \omega_n = \omega$. Equation (9) is a natural extension of the well-known linear relationship $X(j\omega) = H(j\omega)U(j\omega)$, where $H(j\omega)$ is the frequency response function, to the nonlinear case.

For linear systems, the possible output frequencies are the same as the frequencies in the input. For nonlinear systems described by equation (9), however, the relationship between the input and output frequencies is more complicated. Given the frequency range of an input, the output frequencies of system (9) can be determined using the explicit expression derived by Lang and Billings in [2].

Based on the above results for the output frequency response of nonlinear systems, a new concept known as the Nonlinear Output Frequency Response Function (NOFRF) was recently introduced by Lang and Billings [24]. The NOFRF is defined as

$$G_n(j\omega) = \frac{\int_{\omega_1 + \dots + \omega_n = \omega} H_n(j\omega_1, \dots, j\omega_n) \prod_{i=1}^n U(j\omega_i) d\sigma_{n\omega}}{\int_{\omega_1 + \dots + \omega_n = \omega} \prod_{i=1}^n U(j\omega_i) d\sigma_{n\omega}} \quad (11)$$

under the condition that

$$U_n(j\omega) = \int_{\omega_1 + \dots + \omega_n = \omega} \prod_{i=1}^n U(j\omega_i) d\sigma_{n\omega} \neq 0 \quad (12)$$

Notice that $G_n(j\omega)$ is valid over the frequency range of $U_n(j\omega)$, which can be determined using the algorithm in [2].

By introducing the NOFRFs $G_n(j\omega)$, $n = 1, \dots, N$, equation (9) can be written as

$$X(j\omega) = \sum_{n=1}^N X_n(j\omega) = \sum_{n=1}^N G_n(j\omega) U_n(j\omega) \quad (13)$$

which is similar to the description of the output frequency response for linear systems. The NOFRFs reflect a combined contribution of the system and input to the system output frequency response behaviour. It can be seen from equation (11) that $G_n(j\omega)$

depends not only on H_n ($n=1, \dots, N$) but also on the input $U(j\omega)$. For a nonlinear system, the dynamical properties are determined by the GFRFs H_n ($n=1, \dots, N$). However, from equation (10) it can be seen that the GFRF is multidimensional [21][22] which makes the GFRFs difficult to measure, display and interpret in practice. According to equation (11), the NOFRF $G_n(j\omega)$ is a weighted sum of $H_n(j\omega_1, \dots, j\omega_n)$ over $\omega_1 + \dots + \omega_n = \omega$ with the weights depending on the test input. Therefore $G_n(j\omega)$ can be used as an alternative representation of the dynamical properties described by H_n . The most important property of the NOFRF $G_n(j\omega)$ is that it is one dimensional, and thus allows the analysis of nonlinear systems to be implemented in a convenient manner similar to the analysis of linear systems. Moreover, there is an effective algorithm [23] available which allows the estimation of the NOFRFs to be implemented directly using system input output data.

4. Analysis of MDOF Systems with Multiple Nonlinear Components Using NOFRFS

4.1 GFRFs of MDOF Systems with Multiple Nonlinear Components

From equation (6), the GFRFs $H_{(i,j)}(j\omega_1, \dots, j\omega_j)$, ($i=1, \dots, n$, $j=1, \dots, N$) can be determined using the harmonic probing method [18][19].

First consider the input $u(t)$ is of a single harmonic

$$u(t) = e^{j\omega t} \quad (14)$$

Substituting (14) and

$$x_i(t) = H_{(i,1)}(j\omega)e^{j\omega t} \quad (i=1, \dots, n) \quad (15)$$

into equation (6) and extracting the coefficients of $e^{j\omega t}$ yields,

$$(-M\omega^2 + jC\omega + K)H_1(j\omega) = \begin{pmatrix} \overbrace{0 \dots 0}^{J-1} & 1 & \overbrace{0 \dots 0}^{n-J} \end{pmatrix}^T \quad (16)$$

where

$$H_1(j\omega) = \begin{pmatrix} H_{(1,1)}(j\omega) & \dots & H_{(n,1)}(j\omega) \end{pmatrix}^T \quad (17)$$

From equation (16), it is known that

$$H_1(j\omega) = \begin{pmatrix} -M\omega^2 + jC\omega + K \end{pmatrix}^{-1} \begin{pmatrix} \overbrace{0 \dots 0}^{J-1} & 1 & \overbrace{0 \dots 0}^{n-J} \end{pmatrix}^T \quad (18)$$

Denote

$$\Theta(j\omega) = -M\omega^2 + jC\omega + K \quad (19)$$

and

$$\Theta^{-1}(j\omega) = \begin{pmatrix} Q_{(1,1)}(j\omega) & \dots & Q_{(1,n)}(j\omega) \\ \vdots & \ddots & \vdots \\ Q_{(n,1)}(j\omega) & \dots & Q_{(n,n)}(j\omega) \end{pmatrix} \quad (20)$$

where $Q_{(i,l)}(j\omega)$ ($i=1,\dots,n; l=1,\dots,n$) is the element of the i th row and the l th column of the inverse matrix of $\Theta(j\omega)$.

It is obtained from equations (18)~(20) that

$$H_{(i,1)}(j\omega) = Q_{(i,J)}(j\omega) \quad (i=1,\dots,n) \quad (21)$$

Thus, for any two consecutive masses, the relationship between the first order GFRFs can be expressed as

$$\frac{H_{(i,1)}(j\omega)}{H_{(i+1,1)}(j\omega)} = \frac{Q_{(i,J)}(j\omega)}{Q_{(i+1,J)}(j\omega)} = \lambda_1^{i,i+1}(\omega) \quad (i=1,\dots,n-1) \quad (22)$$

The above procedure used to analyze the relationships between the first order GFRFs can be extended to investigate the relationship between the \bar{N} th order GFRFs with $\bar{N} \geq 2$. To achieve this, consider the input

$$u(t) = \sum_{k=1}^{\bar{N}} e^{j\omega_k t}$$

Substituting this input and

$$\begin{aligned} x_i(t) = & H_{(i,1)}(j\omega_1)e^{j\omega_1 t} + \dots + H_{(i,1)}(j\omega_{\bar{N}})e^{j\omega_{\bar{N}} t} + \dots \\ & + \bar{N}! H_{(i,\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}})e^{j(\omega_1 + \dots + \omega_{\bar{N}})t} + \dots \end{aligned} \quad (i=1,\dots,n) \quad (23)$$

into equation (6), and, for the first row of equation (6), extracting the coefficients of $e^{j(\omega_1 + \dots + \omega_{\bar{N}})t}$ yields

$$\begin{aligned} & (-m_1(\omega_1 + \dots + \omega_{\bar{N}})^2 + j(c_1 + c_2)(\omega_1 + \dots + \omega_{\bar{N}}) + (k_1 + k_2))H_{(1,\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}}) \\ & - (jc_2(\omega_1 + \dots + \omega_{\bar{N}}) + k_2)H_{(2,\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}}) = 0 \end{aligned} \quad (24)$$

Similarly, it can be easily deduced that, for the masses that are not connected to the $L(i)$ th ($i=1,\dots,\bar{L}$) spring, the GFRFs satisfy the following relationships

$$\begin{aligned} & (-m_n(\omega_1 + \dots + \omega_{\bar{N}})^2 + jc_n(\omega_1 + \dots + \omega_{\bar{N}}) + k_n)H_{(n,\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}}) \\ & - (jc_n(\omega_1 + \dots + \omega_{\bar{N}}) + k_n)H_{(n-1,\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}}) = 0 \end{aligned} \quad (25)$$

$$\begin{aligned} & (-m_i(\omega_1 + \dots + \omega_{\bar{N}})^2 + j(c_i + c_{i+1})(\omega_1 + \dots + \omega_{\bar{N}}) + k_i + k_{i+1})H_{(i,\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}}) \\ & - (jc_i(\omega_1 + \dots + \omega_{\bar{N}}) + k_i)H_{(i-1,\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}}) \\ & - (jc_{i+1}(\omega_1 + \dots + \omega_{\bar{N}}) + k_{i+1})H_{(i+1,\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}}) = 0 \end{aligned} \quad (i \neq 1, L(i)-1, L(i), n, 1 \leq l \leq \bar{L}) \quad (26)$$

For the mass that is connected to the left of the $L(i)$ th spring, the GFRFs satisfy the following relationships

$$\begin{aligned} & \left(-m_{L(i)-1}(\omega_1 + \dots + \omega_{\bar{N}})^2 + j(c_{L(i)-1} + c_{L(i)})(\omega_1 + \dots + \omega_{\bar{N}}) \right. \\ & \left. + k_{L(i)-1} + k_{L(i)} \right) H_{(L(i)-1,\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}}) \\ & - (jc_{L(i)-1}(\omega_1 + \dots + \omega_{\bar{N}}) + k_{L(i)-1})H_{(L(i)-2,\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}}) \\ & - (jc_{L(i)}(\omega_1 + \dots + \omega_{\bar{N}}) + k_{L(i)})H_{(L(i),\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}}) + \Lambda_{\bar{N}}^{L(i)-1,L(i)}(j\omega_1, \dots, j\omega_{\bar{N}}) = 0 \end{aligned}$$

$$(1 \leq i \leq \bar{L}) \quad (27)$$

For the mass that is connected to the right of the $L(i)$ th spring, the GFRFs satisfy the following relationships

$$\begin{aligned} & \left(\begin{array}{l} -m_{L(i)}(\omega_1 + \dots + \omega_{\bar{N}})^2 + j(c_{L(i)} + c_{L(i)+1})(\omega_1 + \dots + \omega_{\bar{N}}) \\ + k_{L(i)} + k_{L(i)+1} \end{array} \right) H_{(L(i), \bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}}) \\ & - (jc_{L(i)}(\omega_1 + \dots + \omega_{\bar{N}}) + k_{L(i)}) H_{(L(i)-1, \bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}}) \\ & - (jc_{L(i)+1}(\omega_1 + \dots + \omega_{\bar{N}}) + k_{L(i)+1}) H_{(L(i)+1, \bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}}) - \Lambda_{\bar{N}}^{L(i)-1, L(i)}(j\omega_1, \dots, j\omega_{\bar{N}}) = 0 \end{aligned} \quad (1 \leq i \leq \bar{L}) \quad (28)$$

In equations (27) and (28), $\Lambda_{\bar{N}}^{L(i)-1, L(i)}(j\omega_1, \dots, j\omega_{\bar{N}})$ represents the extra terms introduced by $NonF_{L(i)}$ for the \bar{N} th order GFRFs, for example, for the second order GFRFs,

$$\begin{aligned} \Lambda_2^{L(i)-1, L(i)}(j\omega_1, j\omega_2) &= (-w_{(L(i), 2)}\omega_1\omega_2 + r_{(L(i), 2)}) (H_{(L(i)-1, 1)}(j\omega_1)H_{(L(i)-1, 1)}(j\omega_2) \\ &+ H_{(L(i), 1)}(j\omega_1)H_{(L(i), 1)}(j\omega_2) - H_{(L(i)-1, 1)}(j\omega_1)H_{(L(i), 1)}(j\omega_2) - H_{(L(i)-1, 1)}(j\omega_2)H_{(L(i), 1)}(j\omega_1)) \end{aligned} \quad (1 \leq i \leq \bar{L}) \quad (29)$$

Denote

$$H_{\bar{N}}(j\omega_1, \dots, j\omega_{\bar{N}}) = \left(H_{(1, \bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}}) \ \dots \ H_{(n, \bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}}) \right)^T \quad (30)$$

and

$$\bar{A}_{\bar{N}}(j\omega_1, \dots, j\omega_{\bar{N}}) = \left[\bar{A}_{\bar{N}}(1) \ \dots \ \bar{A}_{\bar{N}}(n) \right]^T \quad (31)$$

where

$$\bar{A}_{\bar{N}}(l) = \begin{cases} 0 & \text{if } l \neq L(i)-1, L(i), \ 1 \leq i \leq \bar{L} \\ -\Lambda_{\bar{N}}^{L(i)-1, L(i)}(j\omega_1, \dots, j\omega_{\bar{N}}) & \text{if } l = L(i)-1, \ 1 \leq i \leq \bar{L} \\ \Lambda_{\bar{N}}^{L(i)-1, L(i)}(j\omega_1, \dots, j\omega_{\bar{N}}) & \text{if } l = L(i), \ 1 \leq i \leq \bar{L} \end{cases} \quad (1 \leq l \leq n) \quad (32)$$

then equations (25)~(28) can be written in a matrix form as

$$\Theta(j(\omega_1 + \dots + \omega_{\bar{N}})) H_{\bar{N}}(j\omega_1, \dots, j\omega_{\bar{N}}) = \bar{A}_{\bar{N}}(j\omega_1, \dots, j\omega_{\bar{N}}) \quad (33)$$

so that

$$H_{\bar{N}}(j\omega_1, \dots, j\omega_{\bar{N}}) = \Theta^{-1}(j(\omega_1 + \dots + \omega_{\bar{N}})) \bar{A}_{\bar{N}}(j\omega_1, \dots, j\omega_{\bar{N}}) \quad (34)$$

Therefore, for each mass, the \bar{N} th order GFRF can be calculated as

$$H_{(i, \bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}}) = \sum_{l=1}^{\bar{L}} \left(\begin{array}{l} Q_{i, L(l)-1}(j(\omega_1 + \dots + \omega_{\bar{N}})) \\ Q_{i, L(l)}(j(\omega_1 + \dots + \omega_{\bar{N}})) \end{array} \right)^T \left(\begin{array}{l} -\Lambda_{\bar{N}}^{L(l)-1, L(l)}(j\omega_1, \dots, j\omega_{\bar{N}}) \\ \Lambda_{\bar{N}}^{L(l)-1, L(l)}(j\omega_1, \dots, j\omega_{\bar{N}}) \end{array} \right) \quad (i=1, \dots, n) \quad (35)$$

Define

$$\lambda_{\bar{N}}^{i+1}(\omega_1, \dots, \omega_{\bar{N}}) = \frac{H_{(i, \bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}})}{H_{(i+1, \bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}})} \quad (i=1, \dots, n-1) \quad (36)$$

then from equation (36), it can be known that, for two consecutive masses, the \bar{N} th order GFRFs have the following relationships

$$\lambda_{\bar{N}}^{i,i+1}(\omega_1, \dots, \omega_{\bar{N}}) = \frac{\sum_{l=1}^{\bar{L}} \begin{pmatrix} \mathcal{Q}_{i,L(l)-1}(j(\omega_1 + \dots + \omega_{\bar{N}})) \\ \mathcal{Q}_{i,L(l)}(j(\omega_1 + \dots + \omega_{\bar{N}})) \end{pmatrix}^T \begin{pmatrix} -\Lambda_{\bar{N}}^{L(l)-1,L(l)}(j\omega_1, \dots, j\omega_{\bar{N}}) \\ \Lambda_{\bar{N}}^{L(l)-1,L(l)}(j\omega_1, \dots, j\omega_{\bar{N}}) \end{pmatrix}}{\sum_{l=1}^{\bar{L}} \begin{pmatrix} \mathcal{Q}_{i+1,L(l)-1}(j(\omega_1 + \dots + \omega_{\bar{N}})) \\ \mathcal{Q}_{i+1,L(l)}(j(\omega_1 + \dots + \omega_{\bar{N}})) \end{pmatrix}^T \begin{pmatrix} -\Lambda_{\bar{N}}^{L(l)-1,L(l)}(j\omega_1, \dots, j\omega_{\bar{N}}) \\ \Lambda_{\bar{N}}^{L(l)-1,L(l)}(j\omega_1, \dots, j\omega_{\bar{N}}) \end{pmatrix}} \quad (i=1, \dots, n-1) \quad (37)$$

Equations (22) and (37) give a comprehensive description for the relationships between the GFRFs of any two consecutive masses for the nonlinear MDOF system (6).

In addition, denote

$$\lambda_{\bar{N}}^{0,1}(j\omega_1, \dots, j\omega_{\bar{N}}) = 0 \quad (\bar{N} = 1, \dots, N) \quad (38)$$

$$\Lambda_{(J-1, \bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}}) = 0 \quad (\bar{N} = 1, \dots, N) \quad (39)$$

$$\Lambda_{(J, \bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}}) = \begin{cases} 1 & \text{if } \bar{N} = 1 \\ 0 & \text{if } \bar{N} = 2, \dots, N \end{cases} \quad (40)$$

$$\Lambda_{(L(i)-1, \bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}}) = -\Lambda_{\bar{N}}^{L(i)-1,L(i)}(j\omega_1, \dots, j\omega_{\bar{N}}) \quad (\bar{N} = 1, \dots, N, 1 \leq i \leq \bar{L}) \quad (41)$$

$$\Lambda_{(L(i), \bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}}) = \Lambda_{\bar{N}}^{L(i)-1,L(i)}(j\omega_1, \dots, j\omega_{\bar{N}}) \quad (\bar{N} = 1, \dots, N, 1 \leq i \leq \bar{L}) \quad (42)$$

and $L(0) = J$, then, for the first two masses, from equations (16) and (33), it can be known that

$$\lambda_{\bar{N}}^{1,2}(\omega_1, \dots, \omega_{\bar{N}}) = \frac{H_{(1, \bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}})}{H_{(2, \bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}})} = \frac{jc_2(\omega_1 + \dots + \omega_{\bar{N}}) + k_2}{\left[-m_1(\omega_1 + \dots + \omega_{\bar{N}})^2 + k_2 + jc_2(\omega_1 + \dots + \omega_{\bar{N}}) \right] + \left[(1 - \lambda_{\bar{N}}^{0,1}(\omega_1, \dots, \omega_{\bar{N}}))(jc_1(\omega_1 + \dots + \omega_{\bar{N}}) + k_1) \right]} \quad (\bar{N} = 1, \dots, N) \quad (43)$$

Starting with equation (43), and iteratively using equations (16) and (33) from the first mass, it can be deduce that, for the masses that aren't connected to nonlinear components and the J th spring, the following relationships hold for the GFRFs.

$$\begin{aligned} & \lambda_{\bar{N}}^{i,i+1}(\omega_1, \dots, \omega_{\bar{N}}) \\ &= \frac{H_{(i, \bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}})}{H_{(i+1, \bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}})} = \frac{jc_{i+1}(\omega_1 + \dots + \omega_{\bar{N}}) + k_{i+1}}{\left[j \left((1 - \lambda_{\bar{N}}^{i-1,i}(\omega_1, \dots, \omega_{\bar{N}}))c_i + c_{i+1} \right) (\omega_1 + \dots + \omega_{\bar{N}}) \right] + \left[(1 - \lambda_{\bar{N}}^{i-1,i}(\omega_1, \dots, \omega_{\bar{N}}))k_i + k_{i+1} - m_i(\omega_1 + \dots + \omega_{\bar{N}})^2 \right]} \end{aligned} \quad (1 \leq i < n, i \neq J, L(l)-1, L(l), l=0, \dots, \bar{L}, \bar{N} = 1, \dots, N) \quad (44)$$

For the masses that are connected to nonlinear components and the J th spring, from equations (16), (27) and (28), it can be known that the following relationships hold for the GFRFs.

$$\begin{aligned}
\lambda_{\bar{N}}^{i,i+1}(\omega_1, \dots, \omega_{\bar{N}}) &= \frac{H_{(i,\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}})}{H_{(i+1,\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}})} \\
&= \bar{\lambda}_{\bar{N}}^{-i,i+1}(\omega_1, \dots, \omega_{\bar{N}}) \left(1 + \frac{1}{k_i + jc_i(\omega_1 + \dots + \omega_{\bar{N}})} \frac{\Lambda_{(i,\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}})}{H_{(i+1,\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}})} \right) \\
&\quad (i = L(l) - 1, L(l), l = 0, \dots, \bar{L}, \bar{N} = 1, \dots, N) \quad (45)
\end{aligned}$$

where

$$\begin{aligned}
\bar{\lambda}_{\bar{N}}^{-i,i+1}(\omega_1, \dots, \omega_{\bar{N}}) &= \frac{jc_{i+1}(\omega_1 + \dots + \omega_{\bar{N}}) + k_{i+1}}{\left[j \left((1 - \lambda_{\bar{N}}^{i-1,i}(\omega_1, \dots, \omega_{\bar{N}}))c_i + c_{i+1} \right) (\omega_1 + \dots + \omega_{\bar{N}}) \right.} \\
&\quad \left. + (1 - \lambda_{\bar{N}}^{i-1,i}(\omega_1, \dots, \omega_{\bar{N}}))k_i + k_{i+1} - m_i(\omega_1 + \dots + \omega_{\bar{N}})^2 \right]} \\
&\quad (i = L(l) - 1, L(l), l = 0, \dots, \bar{L}, \bar{N} = 1, \dots, N) \quad (46)
\end{aligned}$$

Moreover, denote $\lambda_{\bar{N}}^{n+1,n}(\omega_1, \dots, \omega_{\bar{N}}) = 1$, ($\bar{N} = 1, \dots, N$), $c_{n+1} = 0$ and $k_{n+1} = 0$. Then, for the last two masses, from equations (16) and (33) it can be deduced that

$$\begin{aligned}
\lambda_{\bar{N}}^{n,n-1}(\omega_1, \dots, \omega_{\bar{N}}) &= \frac{1}{\lambda_{\bar{N}}^{n-1,n}(\omega_1, \dots, \omega_{\bar{N}})} = \frac{H_{(n,\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}})}{H_{(n-1,\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}})} \\
&= \frac{jc_n(\omega_1 + \dots + \omega_{\bar{N}}) + k_n}{\left[-m_n(\omega_1 + \dots + \omega_{\bar{N}})^2 + (1 - \lambda_{\bar{N}}^{n+1,n}(\omega_1, \dots, \omega_{\bar{N}}))k_{n+1} + k_n \right.} \\
&\quad \left. + j \left((1 - \lambda_{\bar{N}}^{n+1,n}(\omega_1, \dots, \omega_{\bar{N}}))c_{n+1} + c_n \right) (\omega_1 + \dots + \omega_{\bar{N}}) \right]} \quad (\bar{N} = 1, \dots, N) \quad (47)
\end{aligned}$$

Starting with equation (47), and iteratively using equations (16) and (33), it can be deduce that, for the masses that aren't connected to nonlinear components and the J th spring, the following relationships hold for the GFRFs.

$$\begin{aligned}
\lambda_{\bar{N}}^{i,i-1}(\omega_1, \dots, \omega_{\bar{N}}) &= \frac{1}{\lambda_{\bar{N}}^{i-1,i}(\omega_1, \dots, \omega_{\bar{N}})} = \frac{H_{(i,\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}})}{H_{(i-1,\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}})} \\
&= \frac{jc_i(\omega_1 + \dots + \omega_{\bar{N}}) + k_i}{\left[-m_i(\omega_1 + \dots + \omega_{\bar{N}})^2 + (1 - \lambda_{\bar{N}}^{i+1,i}(\omega_1, \dots, \omega_{\bar{N}}))k_{i+1} + k_i \right.} \\
&\quad \left. + j \left((1 - \lambda_{\bar{N}}^{i+1,i}(\omega_1, \dots, \omega_{\bar{N}}))c_{i+1} + c_i \right) (\omega_1 + \dots + \omega_{\bar{N}}) \right]} \\
&\quad (2 \leq i \leq n, i \neq L(l) - 1, L(l), l = 0, \dots, \bar{L}, \bar{N} = 1, \dots, N) \quad (48)
\end{aligned}$$

For the masses that are connected to nonlinear components and the J th spring, from equations (16), (27) and (28), it can be known that the following relationships hold for the GFRFs.

$$\begin{aligned}
\lambda_{\bar{N}}^{i,i-1}(\omega_1, \dots, \omega_{\bar{N}}) &= \frac{H_{(i,\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}})}{H_{(i-1,\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}})} \\
&= \bar{\lambda}_{\bar{N}}^{-i,i-1}(\omega_1, \dots, \omega_{\bar{N}}) \left(1 + \frac{1}{k_i + jc_i(\omega_1 + \dots + \omega_{\bar{N}})} \frac{\Lambda_{(i,\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}})}{H_{(i-1,\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}})} \right)
\end{aligned}$$

$$(i = L(l) - 1, L(l), l = 0, \dots, \bar{L}, \bar{N} = 1, \dots, N) \quad (49)$$

where

$$\bar{\lambda}_{\bar{N}}^{i,i-1}(\omega_1, \dots, \omega_{\bar{N}}) = \frac{jc_i(\omega_1 + \dots + \omega_{\bar{N}}) + k_i}{\left[-m_i(\omega_1 + \dots + \omega_{\bar{N}})^2 + (1 - \lambda_{\bar{N}}^{i+1,i}(\omega_1, \dots, \omega_{\bar{N}}))k_{i+1} + k_i \right] + j\left[(1 - \lambda_{\bar{N}}^{i+1,i}(\omega_1, \dots, \omega_{\bar{N}}))c_{i+1} + c_i \right](\omega_1 + \dots + \omega_{\bar{N}})} \quad (50)$$

From different perspectives, equations (43)~(46) and equations (47)~(50) give two alternative descriptions for the relationships between the GFRFs of any two consecutive masses for the nonlinear MDOF system (6).

4.2 NOFRFs of MDOF Systems with Multiple Nonlinear Components

According to the definition of NOFRF in equation (11), the \bar{N} th order NOFRF of the i th mass can be expressed as

$$G_{(i,\bar{N})}(j\omega) = \frac{\int_{\omega_1 + \dots + \omega_{\bar{N}} = \omega} H_{(i,\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}}) \prod_{q=1}^{\bar{N}} U(j\omega_q) d\sigma_{\bar{N}\omega}}{\int_{\omega_1 + \dots + \omega_{\bar{N}} = \omega} \prod_{q=1}^{\bar{N}} U(j\omega_q) d\sigma_{\bar{N}\omega}} \quad (1 \leq \bar{N} \leq N, 1 \leq i \leq n) \quad (51)$$

where $U(j\omega)$ is the Fourier transform of $u(t)$.

According to equation (44), for the masses that aren't connected to nonlinear components and the J th spring, equation (51) can be rewritten as

$$G_{(i,\bar{N})}(j\omega) = \frac{\int_{\omega_1 + \dots + \omega_{\bar{N}} = \omega} \lambda_{\bar{N}}^{i,i+1}(\omega_1, \dots, \omega_{\bar{N}}) H_{(i+1,\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}}) \prod_{q=1}^{\bar{N}} U(j\omega_q) d\sigma_{\bar{N}\omega}}{\int_{\omega_1 + \dots + \omega_{\bar{N}} = \omega} \prod_{q=1}^{\bar{N}} U(j\omega_q) d\sigma_{\bar{N}\omega}} = \frac{jc_{i+1}\omega + k_{i+1}}{\left[-m_i\omega^2 + (1 - \lambda_{\bar{N}}^{i-1,i}(\omega))jc_i\omega + k_i \right] + jc_{i+1}\omega + k_{i+1}} G_{(i+1,\bar{N})}(j\omega) \quad (1 \leq i < n, i \neq L(l) - 1, L(l), l = 0, \dots, \bar{L}, \bar{N} = 1, \dots, N) \quad (52)$$

Denote

$$\lambda_{\bar{N}}^{i,i+1}(\omega) = \frac{G_{(i,\bar{N})}(j\omega)}{G_{(i+1,\bar{N})}(j\omega)} = \lambda_{\bar{N}}^{i,i+1}(\omega_1, \dots, \omega_{\bar{N}}) \Big|_{(\omega_1 + \dots + \omega_{\bar{N}}) = \omega} \quad (53)$$

Therefore, for two consecutive masses that aren't connected to these nonlinear components and the J th spring, the NOFRFs have the following relationship

$$\lambda_{\bar{N}}^{i,i+1}(\omega) = \frac{jc_{i+1}\omega + k_{i+1}}{\left[-m_i\omega^2 + (1 - \lambda_{\bar{N}}^{i-1,i}(\omega))jc_i\omega + k_i \right] + jc_{i+1}\omega + k_{i+1}} \quad (1 \leq i < n, i \neq L(l) - 1, L(l), l = 0, \dots, \bar{L}, \bar{N} = 1, \dots, N) \quad (54)$$

where $\lambda_{\bar{N}}^{0,1}(\omega) = 0$.

Similarly, for the masses that are connected to nonlinear components and the J th spring, from equations (44) and (45), it can be deduced that

$$\lambda_{\bar{N}}^{i,i+1}(\omega) = \bar{\lambda}_{\bar{N}}^{i,i+1}(\omega) \left(1 + \frac{1}{k_i + jc_i \omega} \frac{\Gamma_{(i,\bar{N})}(j\omega)}{G_{(i+1,\bar{N})}(j\omega)} \right) \quad (i = L(l) - 1, L(l), l = 0, \dots, \bar{L}, \bar{N} = 1, \dots, N) \quad (55)$$

where

$$\bar{\lambda}_{\bar{N}}^{i,i+1}(\omega) = \frac{jc_{i+1}\omega + k_{i+1}}{\left[-m_i \omega^2 + (1 - \lambda_{\bar{N}}^{i-1,i}(\omega))(jc_i \omega + k_i) + jc_{i+1}\omega + k_{i+1} \right]} \quad (56)$$

and

$$\Gamma_{(i,\bar{N})}(j\omega) = \frac{\int_{\omega_1 + \dots + \omega_{\bar{N}} = \omega} \Lambda_{(i,\bar{N})}(j\omega_1, \dots, j\omega_{\bar{N}}) \prod_{q=1}^{\bar{N}} U(j\omega_q) d\sigma_{\bar{N}\omega}}{\int_{\omega_1 + \dots + \omega_{\bar{N}} = \omega} \prod_{q=1}^{\bar{N}} U(j\omega_q) d\sigma_{\bar{N}\omega}} \quad (57)$$

Equations (54)~(57) give a comprehensive description for the relationships between the NOFRFs of two consecutive masses of the nonlinear MDOF system (6).

Using the same procedure, from equations (47)~(50), an alternative description can be established for the following relationships between the NOFRFs of two consecutive masses. For the masses that aren't connected to nonlinear components and the J th spring

$$\lambda_{\bar{N}}^{i,i-1}(\omega) = \frac{1}{\lambda_{\bar{N}}^{i-1,i}(\omega)} = \frac{G_{(i,\bar{N})}(j\omega)}{G_{(i-1,\bar{N})}(j\omega)} = \frac{jc_i \omega + k_i}{\left[-m_i \omega^2 + (1 - \lambda_{\bar{N}}^{i+1,i}(\omega))(jc_{i+1}\omega + k_{i+1}) + jc_i \omega + k_i \right]} \quad (2 \leq i \leq n, i \neq L(l) - 1, L(l), l = 0, \dots, \bar{L}, \bar{N} = 1, \dots, N) \quad (58)$$

For the masses that are connected to nonlinear components and the J th spring

$$\lambda_{\bar{N}}^{i,i-1}(\omega) = \frac{1}{\lambda_{\bar{N}}^{i-1,i}(\omega)} = \frac{G_{(i,\bar{N})}(j\omega)}{G_{(i-1,\bar{N})}(j\omega)} = \bar{\lambda}_{\bar{N}}^{i,i-1}(\omega) \left(1 + \frac{1}{k_i + jc_i \omega} \frac{\Gamma_{(i,\bar{N})}(j\omega)}{G_{(i-1,\bar{N})}(j\omega)} \right) \quad (i = L(l) - 1, L(l), l = 0, \dots, \bar{L}, \bar{N} = 1, \dots, N) \quad (59)$$

where $\lambda_{\bar{N}}^{n+1,n}(\omega) = 1$ ($\bar{N} = 1, \dots, N$) and

$$\bar{\lambda}_{\bar{N}}^{i,i-1}(\omega) = \frac{jc_i \omega + k_i}{\left[-m_i \omega^2 + (1 - \lambda_{\bar{N}}^{i+1,i}(\omega))(k_{i+1} + jc_{i+1}\omega) + k_i + \omega c_i \right]} \quad (60)$$

From different perspectives, both equations (54)~(57) and equations (58)~(60) give a comprehensive description for the relationships between the NOFRFs of any two consecutive masses of the nonlinear MDOF system (6).

4.3 The Properties of NOFRFs of Locally Nonlinear Systems

Without loss of generality, assume $L(1) < \dots < L(\bar{L})$. Then, from equations (54)~(60), the following important properties of the NOFRFs of MDOF systems with multiple nonlinear components can be obtained.

i) If $J \leq L(1)$, then for the masses ($1 \leq i \leq J-1$ and $L(\bar{L}) \leq i < n$), the following relationships hold.

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)} = \dots = \frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)} \quad (1 \leq i \leq J-1 \text{ and } L(\bar{L}) \leq i < n) \quad (61)$$

for the masses ($J \leq i < L(1)-1$), the following relationships hold.

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)} \neq \frac{G_{(i,2)}(j\omega)}{G_{(i+1,2)}(j\omega)} = \dots = \frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)} \quad (J \leq i < L(1)-1) \quad (62)$$

for the masses ($L(1) \leq i < L(\bar{L})$), the following relationships hold.

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)} \neq \frac{G_{(i,2)}(j\omega)}{G_{(i+1,2)}(j\omega)} \neq \dots \neq \frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)} \quad (L(1)-1 \leq i < L(\bar{L})) \quad (63)$$

ii) If $L(1) \leq J \leq L(\bar{L})$, then for the masses ($1 \leq i < L(1)-1$ and $L(\bar{L}) \leq i < n$), the following relationships hold.

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)} = \dots = \frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)} \quad (1 \leq i < L(1)-1 \text{ and } L(\bar{L}) \leq i < n) \quad (64)$$

for the masses ($L(1) \leq i < L(\bar{L})$), the following relationships hold

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)} \neq \frac{G_{(i,2)}(j\omega)}{G_{(i+1,2)}(j\omega)} \neq \dots \neq \frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)} \quad (L(1)-1 \leq i < L(\bar{L})) \quad (65)$$

iii) If $J \geq L(\bar{L})$, then for the masses ($1 \leq i < L(1)-1$ and $J \leq i < n$), the following relationships hold.

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)} = \dots = \frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)} \quad (1 \leq i < L(1)-1 \text{ and } J \leq i < n) \quad (66)$$

for the masses ($L(\bar{L}) \leq i < J$), the following relationships hold.

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)} \neq \frac{G_{(i,2)}(j\omega)}{G_{(i+1,2)}(j\omega)} = \dots = \frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)} \quad (L(\bar{L}) \leq i < J) \quad (67)$$

for the masses ($L(1)-1 \leq i < L(\bar{L})$), the following relationships hold.

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)} \neq \frac{G_{(i,2)}(j\omega)}{G_{(i+1,2)}(j\omega)} \neq \dots \neq \frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)} \quad (L(1)-1 \leq i < L(\bar{L})) \quad (68)$$

iv) For the masses ($1 \leq i \leq \min(J, L(1)-1)-1$ and $\max(L(\bar{L}), J) \leq i < n$), the following relationships of the output frequency responses hold

$$x_i(j\omega) = \lambda^{i,j+1}(\omega)x_{i+1}(j\omega)$$

$$(1 \leq i \leq \min(J, L(1)) - 1 \text{ and } \max(L(\bar{L}), J) \leq i < n) \quad (69)$$

where

$$\lambda^{i,i+1}(\omega) = \frac{jc_{i+1}\omega + k_{i+1}}{[-m_i\omega^2 + (1 - \lambda^{i-1,i}(\omega))(jc_i\omega + k_i) + jc_{i+1}\omega + k_{i+1}]} \quad (70)$$

Above fourth properties can be easily extended to a more general case, as the following.

v) For any two masses whose positions are $i, k \subseteq [1, \min(J, L(1)) - 1]$, or $i, k \subseteq [\max(J, L(\bar{L})), n - 1]$, the following relationships hold.

$$\frac{G_{(i,1)}(j\omega)}{G_{(k,1)}(j\omega)} = \dots = \frac{G_{(i,N)}(j\omega)}{G_{(k,N)}(j\omega)} = \lambda^{i,k}(\omega) \quad (i, k \subseteq [1, \min(J, L(1)) - 1], \text{ or } i, k \subseteq [\max(J, L(\bar{L})), n - 1]) \quad (71)$$

and

$$\lambda^{i,k}(\omega) = \prod_{d=0}^{k-i-1} \lambda^{i+d,i+d+1}(\omega) \quad (72)$$

Moreover, the following relationships of their output frequency responses hold

$$x_i(j\omega) = \lambda^{i,k}(\omega)x_k(j\omega) \quad (73)$$

vi) If $J \leq L(1)$, for any two masses whose positions are $i, k \subseteq [J, L(1) - 1]$, the following relationships hold.

$$\frac{G_{(i,1)}(j\omega)}{G_{(k,1)}(j\omega)} \neq \frac{G_{(i,2)}(j\omega)}{G_{(k,2)}(j\omega)} = \dots = \frac{G_{(i,N)}(j\omega)}{G_{(k,N)}(j\omega)} = \lambda^{i,k}(\omega) \quad (i, k \subseteq [J, L(1) - 1]) \quad (74)$$

vii) If $J \geq L(\bar{L})$, then for any masses whose positions are $i, k \subseteq [L(\bar{L}), J]$, the following relationships hold.

$$\frac{G_{(i,1)}(j\omega)}{G_{(k,1)}(j\omega)} \neq \frac{G_{(i,2)}(j\omega)}{G_{(k,2)}(j\omega)} = \dots = \frac{G_{(i,N)}(j\omega)}{G_{(k,N)}(j\omega)} = \lambda^{i,k}(\omega) \quad (i, k \subseteq [L(\bar{L}), J]) \quad (75)$$

viii) For any two masses whose positions are $i, k \subseteq [L(1) - 1, L(\bar{L})]$, the following relationships hold.

$$\frac{G_{(i,1)}(j\omega)}{G_{(k,1)}(j\omega)} \neq \frac{G_{(i,2)}(j\omega)}{G_{(k,2)}(j\omega)} \neq \dots \neq \frac{G_{(i,N)}(j\omega)}{G_{(k,N)}(j\omega)} \quad (76)$$

ix) For any two masses whose positions are $i \subseteq [1, L(1) - 1]$ and $k \subseteq [L(1), n]$ or $i \subseteq [1, L(\bar{L}) - 1]$ and $k \subseteq [L(\bar{L}), n]$, the following relationships hold.

$$\frac{G_{(i,1)}(j\omega)}{G_{(k,1)}(j\omega)} \neq \frac{G_{(i,2)}(j\omega)}{G_{(k,2)}(j\omega)} \neq \dots \neq \frac{G_{(i,N)}(j\omega)}{G_{(k,N)}(j\omega)} \quad (77)$$

The nine relationships between NOFRFs of nonlinear MDOF systems reveal, for the first time, very important properties of the nonlinear systems. They reveal how the linear

system parameters govern the propagation of the nonlinear effect induced by the nonlinear components in the whole system. These properties can be applied to detect and locate the nonlinear components in engineering structures which often is related to faults.

5 Numerical Study and Discussions

5.1 Case Studies

To verify above analysis results, a damped 10-DOF oscillator was adopted, in which the fourth and sixth springs were nonlinear. The damping was assumed to be proportional damping, e.g., $C = \mu K$. The values of the system parameters are

$$\begin{aligned} m_1 = \dots = m_{10} = 1, \quad k_1 = \dots = k_5 = k_{10} = 3.6 \times 10^4, \quad k_6 = k_7 = k_8 = 0.8 \times k_1, \\ k_9 = 0.9 \times k_1, \quad \mu = 0.01, \quad w_{(4,2)} = w_{(4,3)} = w_{(6,2)} = w_{(6,3)} = 0 \\ r_{(4,2)} = 0.8 \times k_1^2, \quad r_{(4,3)} = 0.4 \times k_1^3, \quad r_{(6,2)} = 0.5 \times r_{(4,2)}, \quad r_{(6,3)} = 0.1 \times r_{(4,3)} \end{aligned}$$

and the input is a harmonic force, $u(t) = A \sin(2\pi \times 20t)$.

When a nonlinear system is subject to a harmonic input,

$$u(t) = A \cos(\omega_F t + \beta) \quad (78)$$

it can be derived that the output spectrum $Y(j\omega)$ of nonlinear systems can be simply expressed as [12]

$$Y(jk\omega_F) = \sum_{n=1}^{[(N-k+1)/2]} G_{k+2(n-1)}(jk\omega_F) A_{k+2(n-1)}(jk\omega_F) \quad (k = 0, 1, \dots, N) \quad (79)$$

where $[\cdot]$ means to take the integer part, and

$$A_n(j(-n+2k)\omega_F) = \frac{1}{2^n} \frac{n!}{k!(n-k)!} |A|^n e^{j(-n+2k)\beta} \quad (80)$$

$$G_n(j(-n+2k)\omega_F) = H_n(\overbrace{j\omega_F, \dots, j\omega_F}^k, \overbrace{-j\omega_F, \dots, -j\omega_F}^{n-k}) \quad (81)$$

According to equations (79)~(81), if only the NOFRFs up to the 4th order is considered, the frequency components of the outputs of the 10 masses

$$\begin{aligned} x_i(j\omega_F) &= G_{(i,1)}(j\omega_F)A_1(j\omega_F) + G_{(i,3)}(j\omega_F)A_3(j\omega_F) \\ x_i(j2\omega_F) &= G_{(i,2)}(j2\omega_F)A_2(j2\omega_F) + G_{(i,4)}(j2\omega_F)A_4(j2\omega_F) \\ x_i(j3\omega_F) &= G_{(i,3)}(j3\omega_F)A_3(j3\omega_F) \\ x_i(j4\omega_F) &= G_{(i,4)}(j4\omega_F)A_4(j4\omega_F) \end{aligned} \quad (i = 1, \dots, 10) \quad (82)$$

From equation (82), it can be seen that, using the method in [23], two different inputs with the same waveform but different strengths are sufficient to estimate the NOFRFs up to 4th order. Therefore, in the numerical studies, two different inputs were $A=0.8$ and $A=1.0$ respectively. The simulation studies were conducted using a fourth-order *Runge-Kutta* method to obtain the forced response of the system.

Case 1. Input Force Acting on the Eighth Mass ($J = 8$)

In this case, the position of the input force is on the right of the two nonlinear components. The evaluated results of $G_1(j\omega_F)$, $G_3(j\omega_F)$, $G_2(j2\omega_F)$ and $G_4(j2\omega_F)$ for all masses are given in Table 1. According to Property iii) in the previous section, it can be known that the following relationships should be tenable.

$$\begin{aligned}
 \lambda_1^{i,i+1}(j\omega_F) &= \frac{G_{(i,1)}(j\omega_F)}{G_{(i+1,1)}^H(j\omega_F)} = \frac{G_{(i,3)}(j\omega_F)}{G_{(i+1,3)}^H(j\omega_F)} = \lambda_3^{i,i+1}(j\omega_F) && \text{for } i = 1,2,8,9 \\
 \lambda_1^{i,i+1}(j\omega_F) &= \frac{G_{(i,1)}(j\omega_F)}{G_{(i+1,1)}(j\omega_F)} \neq \frac{G_{(i,3)}(j\omega_F)}{G_{(i+1,3)}(j\omega_F)} = \lambda_3^{i,i+1}(j\omega_F) && \text{for } i = 3,4,5,6,7 \\
 \lambda_2^{i,i+1}(j2\omega_F) &= \frac{G_{(i,2)}(j2\omega_F)}{G_{(i+1,2)}(j2\omega_F)} = \frac{G_{(i,4)}(j2\omega_F)}{G_{(i+1,4)}(j2\omega_F)} = \lambda_4^{i,i+1}(j2\omega_F) && \text{for } i = 1,2,6,7,8,9 \\
 \lambda_2^{i,i+1}(j2\omega_F) &= \frac{G_{(i,2)}(j2\omega_F)}{G_{(i+1,2)}(j2\omega_F)} \neq \frac{G_{(i,4)}(j2\omega_F)}{G_{(i+1,4)}(j2\omega_F)} = \lambda_4^{i,i+1}(j2\omega_F) && \text{for } i = 3,4,5
 \end{aligned} \tag{83}$$

Table 1, the evaluated results of $G_1(j\omega_F)$, $G_3(j\omega_F)$, $G_2(j2\omega_F)$ and $G_4(j2\omega_F)$

	$G_1(j\omega_F)$ ($\times 10^{-6}$)	$G_3(j\omega_F)$ ($\times 10^{-9}$)	$G_2(j2\omega_F)$ ($\times 10^{-8}$)	$G_4(j2\omega_F)$ ($\times 10^{-10}$)
Mass 1	0.7415+1.81675i	-1.4839 -3.1863i	1.2028+0.4681i	0.0118-0.6146i
Mass 2	0.9687+3.4830i	-2.0346-6.1478i	1.8350+1.5489i	0.3497-1.0888i
Mass 3	0.2866+4.76390i	-0.9248-8.4985i	1.0931+3.3649i	1.2839-1.0913i
Mass 4	-1.4623+ 5.2958i	2.2399-7.6670i	-2.6863+1.5531i	2.6082+1.7940i
Mass 5	-4.0944+4.6145i	6.5809-6.9866i	-6.0115+1.2279i	2.1843+3.9698i
Mass 6	-7.7469+1.6880i	-9.3252+ 6.1777i	5.1062-1.7667i	-1.8932-4.1325i
Mass 7	-10.2035-3.6670i	-3.0998+6.4668i	1.6279-2.6939i	-2.3878-1.1449i
Mass 8	-9.5104-10.9684i	2.0570+4.5521i	-0.3077-1.5854i	-1.3026+0.3861i
Mass 9	-16.4606-2.6060i	5.1710+2.4786i	-0.8845-0.3838i	-0.2497+0.7718i
Mass 10	-19.3586+1.8453i	6.5645+1.2960i	-0.9602+0.2564i	0.2926+0.7833i

From the NOFRFs in Table 1, $\lambda_1^{i,i+1}(j\omega_F)$, $\lambda_3^{i,i+1}(j\omega_F)$, $\lambda_2^{i,i+1}(j2\omega_F)$ and $\lambda_4^{i,i+1}(j2\omega_F)$ ($i = 1, \dots, 9$) can be calculated. The results are given in Table 2. It can be seen that the results shown in Table 2 have a strict accordance with the relationships in (83).

Table 2, the evaluated values of $\lambda_1^{i,i+1}(j\omega_F)$, $\lambda_3^{i,i+1}(j\omega_F)$, $\lambda_2^{i,i+1}(j2\omega_F)$ and $\lambda_4^{i,i+1}(j2\omega_F)$

	$\lambda_1^{i,i+1}(j\omega_F)$	$\lambda_3^{i,i+1}(j\omega_F)$	$\lambda_2^{i,i+1}(j2\omega_F)$	$\lambda_4^{i,i+1}(j2\omega_F)$
$i=1$	0.5391-0.0630i	0.5391-0.0630i	0.5085-0.1741i	0.5085-0.1741i
$i=2$	0.7407-0.1588i	0.7407-0.1588i	0.5766-0.3580i	0.5766-0.3580i
$i=3$	0.8219-0.2811i	0.9888-0.4095i	0.2378-1.1151i	0.1388-0.5139i
$i=4$	0.7994-0.3924i	0.7415-0.3778i	0.4796-0.1604i	0.6244-0.3135i
$i=5$	0.6285-0.4587i	-0.8354+0.1958i	-1.1257-0.1490i	-0.9941+0.0731i
$i=6$	0.6197-0.3882i	1.3389+0.8002i	1.3194+1.0981i	1.3194+1.0981i
$i=7$	0.6513-0.3656i	0.9242+1.0986i	1.4455+1.3074i	1.4455+1.3074i
$i=8$	0.6666+0.5608i	0.6666+0.5608i	0.9473+1.3813i	0.9473+1.3813i
$i=9$	0.8299+0.2137i	0.8299+0.2137i	0.7602+0.6028i	0.7602+0.6028i

Case 2. Input Force Acting on the Fifth Mass ($J = 5$)

In this case, the input force is located between the two nonlinear components. The evaluated results of $G_1(j\omega_F)$, $G_3(j\omega_F)$, $G_2(j2\omega_F)$ and $G_4(j2\omega_F)$ for all masses are given in Table 3. According to Property ii) in the previous section, it can be shown that the following relationships should be tenable.

$$\begin{aligned}
 \lambda_1^{i,i+1}(j\omega_F) &= \frac{G_{(i,1)}(j\omega_F)}{G_{(i+1,1)}(j\omega_F)} = \frac{G_{(i,3)}(j\omega_F)}{G_{(i+1,3)}(j\omega_F)} = \lambda_3^{i,i+1}(j\omega_F) \quad \text{for } i = 1,2,6,7,8,9 \\
 \lambda_1^{i,i+1}(j\omega_F) &= \frac{G_{(i,1)}(j\omega_F)}{G_{(i+1,1)}(j\omega_F)} \neq \frac{G_{(i,3)}(j\omega_F)}{G_{(i+1,3)}(j\omega_F)} = \lambda_3^{i,i+1}(j\omega_F) \quad \text{for } i = 3,4,5 \\
 \lambda_2^{i,i+1}(j2\omega_F) &= \frac{G_{(i,2)}(j2\omega_F)}{G_{(i+1,2)}(j2\omega_F)} = \frac{G_{(i,4)}(j2\omega_F)}{G_{(i+1,4)}(j2\omega_F)} = \lambda_4^{i,i+1}(j2\omega_F) \quad \text{for } i = 1,2,6,7,8,9 \\
 \lambda_2^{i,i+1}(j2\omega_F) &= \frac{G_{(i,2)}(j2\omega_F)}{G_{(i+1,2)}(j2\omega_F)} \neq \frac{G_{(i,4)}(j2\omega_F)}{G_{(i+1,4)}(j2\omega_F)} = \lambda_4^{i,i+1}(j2\omega_F) \quad \text{for } i = 3,4,5
 \end{aligned} \tag{84}$$

Table 3, the evaluated results of $G_1(j\omega_F)$, $G_3(j\omega_F)$, $G_2(j2\omega_F)$ and $G_4(j2\omega_F)$

	$G_1(j\omega_F)$ ($\times 10^{-6}$)	$G_3(j\omega_F)$ ($\times 10^{-8}$)	$G_2(j2\omega_F)$ ($\times 10^{-8}$)	$G_4(j2\omega_F)$ ($\times 10^{-10}$)
Node1	-6.0043+0.5981i	0.1295+1.5561i	-6.9092-1.9495i	6.1399+20.0025i
Node2	-11.1153-0.1888i	-0.0956+2.8751i	-10.9876-7.5957i	-1.2465+38.9120i
Node3	-14.2954-3.3193i	-0.9191+3.6854i	-7.8461-18.0461i	-31.8055+47.7222i
Node4	-14.3350-8.9404i	-8.1475+0.9043i	-3.9589+2.5420i	-6.8951-9.7451i

Node5	-10.0249-16.1049i	-8.4096-1.0040i	11.3654-11.3058i	-51.0635-33.3886i
Node6	-11.6678-7.3967i	5.4052+5.0502i	-0.0705+21.5252i	51.8267-1.4614i
Node7	-8.8533-0.2333i	4.6355+1.0013i	7.9904+9.6650i	22.6618-19.9710i
Node8	-4.0944+4.6145i	2.6124-2.0219i	6.3668+0.9278i	1.7503-15.3984i
Node9	-0.1864+7.0794i	0.8006-3.7067i	2.6067-2.8217i	-6.9911-6.0614i
Node10	1.8495+8.0539i	-0.1740-4.4215i	0.2983-3.9483i	-9.5281-0.4184i

From the NOFRFs in Table 3, $\lambda_1^{i,i+1}(j\omega_F)$, $\lambda_3^{i,i+1}(j\omega_F)$, $\lambda_2^{i,i+1}(j2\omega_F)$ and $\lambda_4^{i,i+1}(j2\omega_F)$ ($i=1, \dots, 9$) can be calculated. The results are given in Table 4. It can be seen that the results shown in Tables 4 have a strict accordance with the relationships in (84).

Table 4, the evaluated values of $\lambda_1^{i,i+1}(j\omega_F)$, $\lambda_3^{i,i+1}(j\omega_F)$, $\lambda_2^{i,i+1}(j2\omega_F)$ and $\lambda_4^{i,i+1}(j2\omega_F)$

	$\lambda_1^{i,i+1}(j\omega_F)$	$\lambda_3^{i,i+1}(j\omega_F)$	$\lambda_2^{i,i+1}(j2\omega_F)$	$\lambda_4^{i,i+1}(j2\omega_F)$
$i=1$	0.5391-0.0630i	0.5391-0.0630i	0.5085-0.1741i	0.5085-0.1741i
$i=2$	0.7405-0.1588i	0.7405-0.1588i	0.5766-0.3582i	0.5767-0.3582i
$i=3$	0.8219-0.2811i	0.1610-0.4345i	-0.6692+4.1287i	-1.7245-4.4839i
$i=4$	0.7994-0.3925i	0.9426-0.2201i	-0.2869-0.0617i	0.1820+0.0718i
$i=5$	1.2371+0.5961i	-0.9233+0.6770i	-0.5270-0.5263i	-0.9663-0.6715i
$i=6$	1.3390+0.8002i	1.3389+0.8003i	1.3194+1.0980i	1.3192+1.0981i
$i=7$	0.9242+1.0986i	0.9242+1.0986i	1.4455+1.3074i	1.4456+1.3074i
$i=8$	0.6666+0.5608i	0.6666+0.5608i	0.9473+1.3813i	0.9473+1.3813i
$i=9$	0.8299+0.2137i	0.8299+0.2137i	0.7602+0.6028i	0.7602+0.6028i

The two numerical case studies verify the properties of the NOFRFs of MDOF systems with multiple nonlinear components derived in the present study. These properties can provide a convenient method to detect the positions of the nonlinear components in a MDOF system by analyzing the relationships between the NOFRFs.

5.2 Discussion

In engineering practice, a wide class of real life structures can be modeled as periodic-structures-like MDOF systems, which are defined as structures consisting of identical substructures connected to each other in identical manner, such as periodically supported beams [28]~[33] and plates[32][33]. If one or more components are of nonlinear properties, then the systems can behave nonlinearly. Efforts have been made to study the

dynamics of the nonlinear structures [34]-[38] using one-dimensional and multi-dimensional MDOF models. In addition, the detection and location of faults and defects in periodic structures and machines are also interesting to many researchers. Zhu and Wu [39] have studied the detection of damages in large periodic structures. In their studies, the periodic structure with damage is still considered to be a linear system, and the location and magnitude of damage in large mono-coupled periodic systems were estimated using measured changes in the natural frequencies. However, in engineering practices, the local faults and defects can often make the structures and machines behave nonlinearly. In the latter cases, obviously the properties discovered in this study can provide a convenient way to detect the positions of the faults. For example, consider the 10-element periodic mass-spring system shown in Figure 2, which is a specific form of the mass-spring system investigated in [39], and assume the sixth spring is damaged and of nonlinear property. To detect the position of the damaged spring, two excitations can be used to excite the system, and then the NOFRFs up to the fourth order of all masses can be estimated from the responses. Obviously, if the excitation force is acting on the 10th mass, then according to the property iii), there exist the following relationships:

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)} = \frac{G_{(i,3)}(j\omega)}{G_{(i+1,3)}(j\omega)} \quad (i=1, 2, 3, 4) \quad (85)$$

$$\frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)} \neq \frac{G_{(i,3)}(j\omega)}{G_{(i+1,3)}(j\omega)} \quad (i=5, 6, 7, 8, 9) \quad (86)$$

Clearly, the relationship (86) provides a direct way to detect the position of the damaged spring.

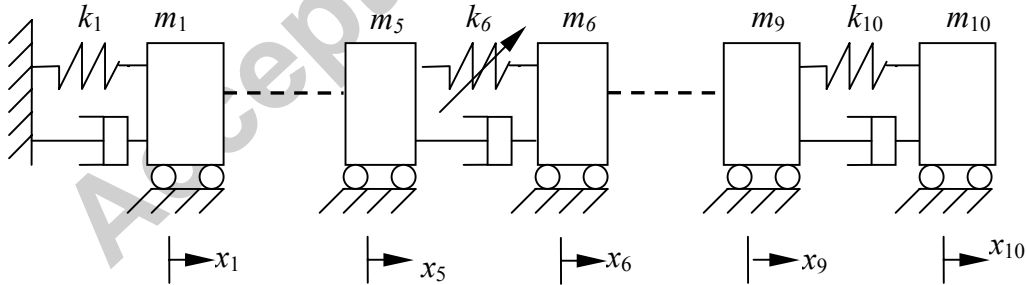


Figure 2, The 10-element periodic mass-spring system whose 6th spring is fault

The locally nonlinear MDOF (6) can also be used to describe the transversal motion of the tall apartment block with damages shown in Figure 3. Based the same model, Sakellariou and Fassois [40][41] have used a stochastic output error vibration-based methodology to detect the damage in structures where the damage elements were modeled as components of cubic stiffness. No doubt the properties obtained in this study can also provide a convenient way to detect the position of the damage in this kind of structures.

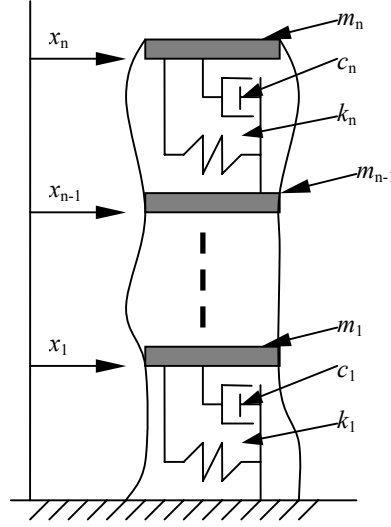


Figure 3, Schematic diagram of the n -storey building block

In addition, for simplicity, in this study the one-dimensional nonlinear MDOF system, each mass of which has only one freedom degree, is adopted. Nevertheless, following the same procedure used in this study, the nine properties can be extended to the multi-dimensional nonlinear MDOF systems by simply replacing the scalar forms of m_i , c_i and k_i as matrix forms, consequently, the $\lambda_N^{i,i+1}(j\omega)$ ($i = 0, \dots, n-1$; $\bar{N} = 1, \dots, N$) also have a matrix form. The multi-dimensional cases could be more complicated than the one-dimensional cases, but the NOFRF properties of multi-dimensional nonlinear MDOF systems can be used to analyze a more wide class of structures. For example, the beam shown in Figure 4(a) can be represented as a series of rigid blocks connected by rotational and transverse springs shown in Figure 4(b). The rotational spring approximates the bending of the beam and the transverse spring approximates the shear. Neild, Mcfadden and Williams [42] have used this model to analyze the beam with one breathing crack. When cracks present in this beam, the whole beam can behave nonlinearly. Clearly, this beam is a two-dimensional nonlinear MDOF system, and the detection of the crack position in this beam could be easily achieved using the properties of the NOFRFs for multi-dimensional MDOF systems.

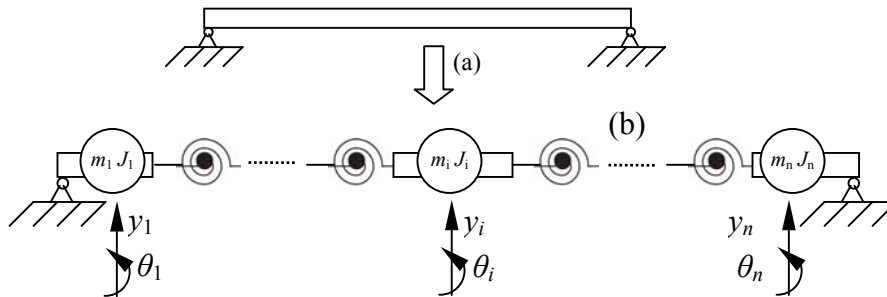


Figure 4, Modelling of a beam as discrete blocks

Moreover, as the response spectrum of each mass of nonlinear MDOF systems is totally determined by the associated NOFRFs and the input, based on the relationships between the NOFRFs, new spectra analysis methods can be developed to detect the position of nonlinear elements in the systems, which may only involve a very simple procedure. We are currently working on this and the results will be presented in a future publication.

From equation (70) it can be seen that the ratios between the NOFRFs of two consecutive masses are mainly determined by the linear parameters. Therefore equation (70) reveals how the linear system parameters govern the relationships between the NOFRFs of two consecutive masses. This fact provides a convenient way to estimate these linear parameters, which has been elaborated in [43].

6 Conclusions

In this paper, significant relationships between the NOFRFs of MDOF systems with multiple nonlinear components have been derived and verified by numerical studies. The results reveal, for the first time, important properties of this general class of MDOF nonlinear systems. The potential of using these properties to detect and locate faults in engineering structures is also discussed.

It is worth noting here that, theoretically, the obtained relationships about NOFRFs are valid for nonlinear systems whose responses can be described using the Volterra series, which covers a considerably wide range of operating conditions of nonlinear systems [4]~[6][44][45]. In practice, because the validity of the Volterra series is dependent on the amplitude of the external force input which is normally controllable during fault detection oriented structural tests, provided that the amplitude of testing input is properly selected, the important relationships between the NOFRFs will hold and can therefore be used for structural fault diagnosis. Moreover, for convenience, in this study the one-dimensional MDOF system is adopted, that is each mass has only one degree of freedom, but the obtained results can be easily extended to the multi-dimensional MDOF cases where each mass in the MDOF system has more than one degree of freedom. In the analysis, if m_i , c_i and k_i are taken as matrix forms and $x_i, (i = 1, \dots, n)$ are taken as vectors, the same results can be achieved.

Appendix 1:

Proof of Property (i)

The first property is straightforward. For the masses on the left of the J th mass, substituting $\lambda_{\bar{N}}^{0,1}(\omega) = 0$ ($\bar{N} = 1, \dots, N$) into equation (54), it is obtained that

$$\lambda_1^{1,2}(\omega) = \dots = \lambda_{\bar{N}}^{1,2}(\omega) = \frac{k_2 + jc_2\omega}{(-m_1\omega^2 + j\omega(c_1 + c_2) + k_1 + k_2)} = \lambda^{1,2}(\omega) \quad (\text{A-1})$$

Subsequently, substituting (A-1) into equation (55) yields

$$\lambda_1^{2,3}(\omega) = \dots = \lambda_{\bar{N}}^{2,3}(\omega) = \frac{jc_3\omega + k_3}{[-m_2\omega^2 + (1 - \lambda^{1,2}(j\omega))(jc_2\omega + k_2) + jc_3\omega + k_3]} = \lambda^{2,3}(\omega) \quad (\text{A-2})$$

Iteratively using above procedure until $i=(J-1)$, for the masses ($1 \leq i \leq J-1$), property (61) can be proved.

Similarly, substituting $\lambda_{\bar{N}}^{n+1,n}(\omega) = 1$ ($\bar{N} = 1, \dots, N$) into equation (58), it is known that

$$\lambda_1^{n,n-1}(\omega) = \frac{1}{\lambda_1^{n-1,n}(\omega)} = \dots = \lambda_{\bar{N}}^{n,n-1}(\omega) = \frac{1}{\lambda_{\bar{N}}^{n-1,n}(\omega)} = \frac{jc_n\omega + k_n}{[-m_n\omega^2 + jc_n\omega + k_n]} = \lambda^{n,n-1}(\omega) \quad (\text{A-3})$$

Subsequently, substituting (A-3) into equation (58), it can be deduced that

$$\lambda_1^{n-1,n-2}(\omega) = \frac{1}{\lambda_1^{n-2,n-1}(\omega)} = \dots = \lambda_{\bar{N}}^{n-1,n-2}(\omega) = \frac{1}{\lambda_{\bar{N}}^{n-2,n-1}(\omega)} = \frac{jc_{n-1}\omega + k_{n-1}}{[-m_{n-1}\omega^2 + (1 - \lambda^{n,n-1}(\omega))(jc_n\omega + k_n) + jc_{n-1}\omega + k_{n-1}]} = \lambda^{n-1,n-2}(\omega) \quad (\text{A-4})$$

Iteratively using above procedure until $i = L(\bar{L})$, for the masses ($L(\bar{L}) \leq i < n$), property (61) can be proved.

Obviously, from equations (40) and (57), it is known that

$$\Gamma_{(J,\bar{N})}(j\omega) = \begin{cases} 1 & \text{if } \bar{N} = 1 \\ 0 & \text{if } \bar{N} = 2, \dots, N \end{cases} \quad (\text{A-5})$$

Substituting $\lambda_1^{J-1,J}(\omega) = \dots = \lambda_{\bar{N}}^{J-1,J}(\omega)$ and equation (A-5) into (54), it can be deduced that

$$\lambda_2^{J,J+1}(\omega) = \dots = \lambda_{\bar{N}}^{J,J+1}(\omega) = \frac{(k_J + jc_J\omega)G_{(J+1,\bar{N})}(j\omega)}{1 + (k_J + jc_J\omega)G_{(J+1,\bar{N})}(j\omega)} \lambda_1^{J,J+1}(\omega) \quad (\text{A-6})$$

Obviously,

$$\lambda_1^{J,J+1}(\omega) \neq \lambda_2^{J,J+1}(\omega) = \dots = \lambda_{\bar{N}}^{J,J+1}(\omega) \quad (\text{A-7})$$

Substituting $\lambda_1^{J,J+1}(\omega) \neq \lambda_2^{J,J+1}(\omega) = \dots = \lambda_{\bar{N}}^{J,J+1}(\omega)$ into equation (54), it can be proved that

$$\lambda_1^{J+1,J+2}(\omega) \neq \lambda_2^{J+1,J+2}(\omega) = \dots = \lambda_{\bar{N}}^{J+1,J+2}(\omega) \quad (\text{A-8})$$

Iteratively using this procedure until $i = L(1) - 2$, for the masses ($J \leq i < L(1) - 1$), property (62) can be proved.

Then, substituting $\lambda_1^{L(1)-2, L(1)-1}(\omega) \neq \lambda_2^{L(1)-2, L(1)-1}(\omega) = \dots = \lambda_N^{L(1)-2, L(1)-1}(\omega)$ into equation (56), it can be known that

$$\bar{\lambda}_1^{L(1)-1, L(1)}(\omega) \neq \bar{\lambda}_2^{L(1)-1, L(1)}(\omega) = \dots = \bar{\lambda}_N^{L(1)-1, L(1)}(\omega) \quad (\text{A-9})$$

Moreover, generally,

$$\frac{\Gamma_{(L(1)-1, 1)}(j\omega)}{G_{(L(1), 1)}(j\omega)} \neq \dots \neq \frac{\Gamma_{(L(1)-1, \bar{N})}(j\omega)}{G_{(L(1), \bar{N})}(j\omega)} \quad (\text{A-10})$$

Substituting (A-9) and (A-10) into equation (56), it can be deduced that

$$\lambda_1^{L(1)-1, L(1)}(\omega) \neq \lambda_2^{L(1)-1, L(1)}(\omega) \neq \dots \neq \lambda_N^{L(1)-1, L(1)}(\omega) \quad (\text{A-11})$$

Iteratively using the procedure until $i = L(\bar{L}) - 1$, for the masses ($L(1) - 1 \leq i < L(\bar{L})$), property (63) can be proved.

Proof of Property (ii) and (iii)

Following the same procedure used for proof of Property (i), the second and third properties can be proved. The details are omitted here.

Proof of Property (iv)

The fourth property is also straightforward since, according to equation (13), the output frequency response of the i th mass can be expressed as

$$x_{i+1}(j\omega) = \sum_{k=1}^N G_{(i+1, k)}(j\omega) U_k(j\omega) \quad (\text{A-12})$$

Equation (A-12) can be rewritten as

$$x_{i+1}(j\omega) = \sum_{k=1}^N \lambda_k^{i, i+1}(\omega) G_{(i, k)}(j\omega) U_k(j\omega) \quad (\text{A-13})$$

Using the first three properties, it can be known that,

$$\lambda_1^{i, i+1}(\omega) = \dots = \lambda_N^{i, i+1}(\omega) = \lambda^{i, i+1}(\omega) \quad (1 \leq i \leq \min(J, L(1)) - 1 \text{ and } \max(L(\bar{L}), J) \leq i < n) \quad (\text{A-14})$$

Substituting (A-13) into (A-14) yields

$$x_{i+1}(j\omega) = \sum_{k=1}^N \lambda^{i, i+1}(\omega) G_{(i, k)}(j\omega) U_k(j\omega) \quad (\text{A-15})$$

Obviously, $x_{i+1}(j\omega) = \lambda^{i, i+1}(\omega) x_i(j\omega)$, then the fourth property is proved.

Proof of Properties (v)~(ix)

The proof of the properties (v)~(ix) only needs some simple calculations. The details are therefore omitted here.

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