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Anthony N. Kounadis

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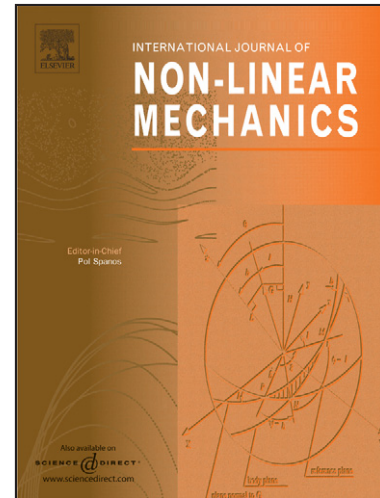
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Anthony N. Kounadis

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FLUTTER INSTABILITY AND OTHER SINGULARITY PHENOMENA IN SYMMETRIC  
SYSTEMS VIA COMBINATION OF MASS DISTRIBUTION AND WEAK DAMPING

by  
Anthony N. Kounadis  
Academy of Athens

**Abstract**

*The local dynamic instability of autonomous conservative, lumped-mass (discrete) systems, is thoroughly discussed when negligibly small dissipative forces are included. It is shown that such small forces may change drastically the response of these systems. Hence, existing, widely accepted, findings based on the omission of damping could not be valid if damping, being always present in actual systems, is included. More specifically the conditions under which the above systems may experience dynamic bifurcations associated either with a degenerate or a generic Hopf bifurcation are examined in detail by studying the effect of the structure of the damping matrix on the Jacobian eigenvalues. The case whereby this phenomenon may occur before divergence is discussed in connection with the individual or coupling effect of non-uniform mass and stiffness distribution. Jump phenomena in the critical dynamic loading at a certain mass distribution are also assessed. Numerical results verified by a nonlinear dynamic analysis using 2-DOF and 3-DOF models confirm the validity of the theoretical findings as well as the efficiency of the technique proposed herein.*

**1. Introduction**

The effect of *damping* on the elastic stability of flexurally vibrating *nonconservative* systems was recognized long time ago as a factor of decisive importance [Ziegler (1952), Nemat-Nasser and Hermann (1966), Crandall (1970)]. In such mechanical systems although *dissipative* forces are often very small their effect may be great. However, this effect was, in general, ignored in the case of elastic dynamic stability of *conservative* systems. Indeed, it was widely accepted that the presence of damping in *undamped* conservative systems which are *stable* does **not** change their stability [Gantmacher (1970), Huseyin (1986)]. The energy loss due to damping is either dissipated within the system or transmitted away by radiation. A classification of various sorts of damping is reported by Gaul (1999). The behavior of conservative discrete systems when damping is included can be described using a *local* (linear) analysis by the matrix-vector differential equation [Gantmacher (1970), Huseyin (1978), Kounadis (2006)].

$$M\ddot{q} + C\dot{q} + Vq = 0, \quad (1)$$

where the dot denotes a derivative with respect to time  $t$ ;  $q(t)$  is an  $n$ -dimensional *state* vector with coordinates  $q_i(t)$  ( $i=1, \dots, n$ );  $M, C$  and  $V$  are  $n \times n$  real symmetric matrices. More specifically, matrix  $M$  associated with the total kinetic energy of the system is a function of the concentrated masses  $m_i$  ( $i=1, \dots, n$ ) being always *positive definite*; matrix  $C$  with elements the damping coefficients  $c_{ij}$  ( $i, j=1, \dots, n$ ) may be *positive definite*, *positive semi-definite* as in the case of pervasive damping [Zajac (1964, 1965), Huseyin (1978)] or *indefinite* [Sygulski (1996), Laneville and Mazouzi (1996), Misra, Wong and Paidoussis (2001)];  $V$  is a generalized *stiffness* matrix whose elements  $V_{ij}$  are also linear functions of a *suddenly* applied external load  $\lambda$  with constant direction and infinite duration [Kounadis (1999)], i.e.  $V_{ij} = V_{ij}(\lambda)$ . Apparently due to this type of loading the system under discussion is *autonomous*. When the external loading  $\lambda$  is applied *statically*, one can obtain the static (divergence) instability or buckling loads  $\lambda_j^c$  ( $j=1, \dots, n$ ) by imposing the condition of vanishing the determinant of the stiffness matrix  $V(\lambda)$ , i.e.

$$|V(\lambda)| = 0. \quad (2)$$

Clearly, eq. (2) yields an  $n^{\text{th}}$  degree algebraic equation with respect to  $\lambda$ . Assuming *distinct* critical points  $V(\lambda)$  is *positive definite* for  $\lambda < \lambda_{(1)}^c$ , *positive semi-definite* for  $\lambda = \lambda_1^c$  and *indefinite* for  $\lambda_1^c < \lambda < \lambda_2^c$ .

This study was motivated by the fact that previous analyses of the author (1994, 1997<sub>1</sub>) using 2-DOF and 3-DOF Ziegler's models under partial *follower loading* have shown that in a small region of divergence instability, *flutter* (dynamic) instability may occur before divergence (e.g. for  $\lambda < \lambda_1^c$ ), if *infinitesimal* damping is included. A similar finding in an aeroelastic model was also reported by Bolotin, Grishko and Petrovsky (1996). It was also reported by Paidoussis et al (1990) that flutter can arise in an inherently conservative system but for *large* damping.

The *objective* of this paper is to present a thorough discussion of the conditions under which the above autonomous *potential* systems may exhibit dynamic bifurcational modes of instability **before** divergence (i.e. for  $\lambda < \lambda_1^c$ ) when *negligibly small damping* is included. Indeed, as will be proven in what follows, such an infinitesimal damping may change

drastically the behavior even of a conservative system. This shows the importance of inclusion of damping for the **precise** modeling of a system. Thus, previous widely accepted results referring to undamped systems (e.g. jump phenomena) should eventually be reconsidered. This work, being an extension of a very recent study of the author (2006), presents a *new* very simple and efficient approach for establishing *degenerate* and *generic Hopf bifurcations* based on *necessary* and *sufficient* conditions. To this end, the *effect of damping* on the *Jacobian* eigenvalues in connection with the influence of the *loading*  $\lambda$ , concentrated mass  $m_j$  and stiffness  $k_j$  ( $j=1,\dots,n$ ) parameters (multi-parameter system) is thoroughly discussed using 2-DOF and 3-DOF models. The case of a *positive semi-definite* or *indefinite* damping matrix  $C$  will be studied in connection with a positive definite, positive semi-definite or indefinite stiffness matrix  $V(\lambda)$ . In this respect, the *individual* and *coupling* effect of the non-uniform mass and stiffness distribution is fully assessed. Discontinuity (jump) phenomena in the critical dynamic loading may occur at a certain mass distribution.

Attention is focused on seeking *steady-state* solutions of autonomous *conservative weakly damped* mechanical systems governed by eq. (1) associated with *periodic* motions either *around centers* or due to *Hopf* bifurcations (limit cycles). The impetus of the present study was that such *local dynamic bifurcations* which could be explored via a *classical* (linear) analysis escaped the attention of eminent researchers in the past. Certainly, the *global* stability of these *dynamic* bifurcations can be established only by using a *nonlinear* dynamic analysis.

## 2. Basic equations

According to the classical analysis solutions of eq. (1) can be sought in the form

$$q = r e^{\rho t}, \quad (3)$$

where  $\rho$  is, in general, a *complex* number and  $r$  a *complex* vector independent of time  $t$ .

Inserting  $q$  from eq. (3) into eq. (1) we obtain

$$L(\rho) = (\rho^2 M + \rho C + V)r = 0, \quad (4)$$

where  $L(\rho) = \rho^2 M + \rho C + V$  is a *matrix-valued* function. Since  $V = V(\lambda)$ , matrix  $L(\rho)$  is a function of the external loading  $\lambda$  for given matrices  $M$  and  $C$ . Solutions of eq. (1) are intimately related to the algebraic properties of the matrix  $L(\rho)$ , and more specifically to the *Jacobian* eigenvalues  $\rho = \rho(\lambda)$  obtained through the *characteristic* (secular) equation [Gantmacher (1959, 1970), Huseyin (1978)].

$$\det L(\rho) = |\rho^2 M + \rho C + V| = 0, \quad (5)$$

which guarantees the *existence* of non-trivial solutions of eq. (1) or (4). Expansion of the determinantal eq. (5) leads to the *characteristic* equation [Kounadis (1994)]

$$\rho^{2n} + \alpha_1 \rho^{2n-1} + \dots + \alpha_{2n-1} \rho + \alpha_{2n} = 0, \quad (6)$$

where the real coefficients  $\alpha_i = (i = 1, \dots, 2n)$  are determined by means of Bôcher formula [Pipes & Harvill (1970)]. The eigenvalues (roots) of eq. (6)  $\rho_j = (j = 1, \dots, 2n)$  are, in general, *complex conjugate* pairs  $\rho_j = v_j \pm \mu_j i$  (where  $i = \sqrt{-1}$ ,  $v_j$  and  $\mu_j$  real numbers) with corresponding *complex conjugate* eigenvectors  $r_j$  and  $\bar{r}_j$  ( $j=1, \dots, n$ ). Since  $\rho_j = \rho_j(\lambda)$ , clearly  $v_j = v_j(\lambda)$ ,  $\mu_j = \mu_j(\lambda)$ ,  $r_j = r_j(\lambda)$  and  $\bar{r}_j = \bar{r}_j(\lambda)$ . Thus, the solutions of eq. (1) are of the form

$$Ae^{v_j t} \cos \mu_j t, \quad Be^{v_j t} \sin \mu_j t, \quad (7)$$

where  $A$  and  $B$  constants which are determined from the *initial* conditions. Solutions (7) are *bounded*, tending to zero as  $t \rightarrow \infty$ , if all eigenvalues of eq. (6) have *negative* real parts, i.e. when  $v_j < 0$  for all  $j$  [Gantmacher (1959)]. In this case the algebraic polynomial (6) is called a *Hurwitz polynomial* (since all its roots have negative real parts) and the *origin* ( $q = \dot{q} = 0$ ) is *asymptotically stable*. According to Routh-Hurwitz criterion, a *necessary* and *sufficient* condition for the polynomial to be a *Hurwitz* polynomial is that

$$\Delta_1 = \alpha_1 > 0, \Delta_2 = \begin{vmatrix} \alpha_1 & 1 \\ \alpha_3 & \alpha_2 \end{vmatrix} > 0, \Delta_3 = \begin{vmatrix} \alpha_1 & 1 & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 \\ \alpha_5 & \alpha_4 & \alpha_3 \end{vmatrix} > 0, \dots, \Delta_{2n} = \alpha_{2n} \Delta_{2n-1} > 0. \quad (8)$$

A *necessary* but **not** sufficient condition for all roots of eq. (6) to have negative real parts is  $\alpha_j > 0$  for all  $j$  ( $= 1, \dots, 2n$ ). More efficient than Routh-Hurwitz conditions is the *stability criterion* of Liénart and Chipart [Gantmacher (1970)]. According to this criterion a *necessary* and *sufficient* condition for all roots of eq. (6) to have negative real parts is  $\alpha_j > 0$  for all  $j$  ( $= 1, \dots, 2n$ ), and simultaneously all determinantal inequalities either of *odd* order  $\Delta_{2n-1}, \Delta_{2n-3}, \dots$  or *even* order  $\Delta_{2n}, \Delta_{2n-2}, \dots$  to be *positive*. In this case, the number of Hurwitz determinants is about half that of conditions (8). If some of the Hurwitz determinants are

zero one has to use Orlando's formula which expresses the determinant  $\Delta_{2n-1}$  in terms of the roots  $\rho_1, \rho_2, \dots, \rho_{2n}$ . This is given by

$$\Delta_{2n-1} = (-1)^{\frac{2n(2n-1)}{2}} \prod_{j < k}^{1, \dots, 2n} (\rho_j + \rho_k). \quad (9)$$

Clearly  $\Delta_{2n-1} = 0$ , if and only if, the sum of two roots of eq. (6) is zero. This may occur in three cases corresponding to *critical states*: if eq. (6) has at least **one** pair of conjugate **pure imaginary** roots or two **real** opposite roots or a **double** zero root (while the remaining roots have negative real parts).

In the following, attention is focused on *critical states*. More specifically, if the **real** part of at least **one** pair of eigenvalues (roots of eq. (6)) becomes (at a certain value of the slowly varied  $\lambda$ ) *positive*, say  $v_\kappa(\lambda) > 0$ , while  $\mu_\kappa(\lambda) = 0$ , the origin is *locally unstable* related to *divergence* (static) instability (Fig. 1a). If the **real** part of at least **one** pair of eigenvalues becomes zero, say  $v_\kappa(\lambda) = 0$  (with  $\mu_\kappa(\lambda) \neq 0$ ), while the remaining  $v_j$ 's are *negative*, the equilibrium state  $q = \dot{q} = 0$  is in general *critical*. In this case of existence of **one** pair of purely imaginary eigenvalues,  $\pm i\mu(\lambda) \neq 0$ , the origin becomes *locally unstable* and the system exhibits a *dynamic bifurcation*. This is related to **two** distinct cases. If this eigenvalue crosses the imaginary axis with **zero** slope, i.e.  $dv_\kappa(\lambda)/d\lambda = 0$ , the corresponding dynamic bifurcation is a *degenerate Hopf bifurcation* (Fig. 1b) associated with *periodic motions* around *centers*; otherwise, i.e. if  $dv_\kappa(\lambda)/d\lambda \neq 0$ , the system exhibits *flutter* instability (Fig. 1c) associated with *limit cycles* due to a *generic Hopf bifurcation*. Another, mixed (hybrid) type of *coupled divergence-flutter* instability occurs when the *real* and *imaginary* part of at least **one** pair of eigenvalues becomes zero, occurring at  $\lambda = \lambda_1^c$ , i.e.  $v_\kappa(\lambda_1^c) = \mu_\kappa(\lambda_1^c) = 0$  with  $dv_\kappa(\lambda_1^c)/d\lambda = 0$ , while for  $\lambda > \lambda_1^c$  the two real eigenvalues are moving in opposite directions in the real axis (Fig. 1d). This dynamic bifurcation corresponds to a *double zero* eigenvalue.

Another type of *flutter* instability may occur in *undamped circulatory* (nonconservative) systems, when **two** consecutive conjugate **purely imaginary** roots coincide at a certain value of the loading  $\lambda$  (Fig. 1e), whereas for a slight increase in it, eventually leave the imaginary axis moving in opposite directions, thus producing an eigenvalue with a *positive* real part (self-excited oscillations). However, the coincidence of

two *purely imaginary* eigenvalues is a *necessary* but **not** sufficient condition for *flutter*, since the corresponding eigenvectors must also coincide [Huseyin (1978) & (1986), Kounadis (2006)]. *Coincident* eigenvalues which may be associated with solutions composed from powers of time  $t$  multiplied with the exponential functions of relation (7) are not considered herein.

### Multi-parameter system

So far we have considered only **one** control parameter, the external loading  $\lambda$  (main control parameter). However, the response of the system is influenced by several independently varying parameters. For a given damping matrix  $C$ , the matrix-valued function  $L(\rho)$  is (in addition to  $\lambda$ ) a function of the concentrated masses  $m_j$  and stiffnesses  $\kappa_j$  ( $j=1, \dots, n$ ) which can be represented by the external parameters  $\eta_\kappa$  ( $\kappa=1, 2, \dots, 2n$ ). Hence,  $\rho = \rho(\lambda; \eta_\kappa)$  and in general  $v = v(\lambda; \eta_\kappa)$ ,  $\mu = \mu(\lambda; \eta_\kappa)$  and  $v = (\lambda; \eta_\kappa)$ .

In the following, the critical condition under which the above multi-parameter autonomous conservative system may exhibit a *dynamic mode* of instability **before** divergence if negligibly small dissipative forces are included will be discussed in detail.

### 3. Solution technique

Premultiplying eq. (4) by  $\bar{r}^T$ , the conjugate transpose of  $r$ , we obtain

$$\bar{r}^T(\rho^2 M + \rho C + V)r = 0, \quad (10)$$

Since all *quadratic* forms are real (scalar) quantities, eq. (10) is a 2<sup>nd</sup> degree algebraic polynomial with respect to  $\rho$ , from which we obtain

$$\rho = \frac{1}{2\bar{r}^T M r} \left[ -\bar{r}^T C r \pm \sqrt{(\bar{r}^T C r)^2 - 4(\bar{r}^T M r)(\bar{r}^T V r)} \right] \quad (11)$$

or 
$$\rho = v + i\mu \quad (i = \sqrt{-1}),$$

where 
$$v = -\frac{\bar{r}^T C r}{2\bar{r}^T M r}, \quad \mu^2 = \frac{\bar{r}^T V r}{\bar{r}^T M r} - v^2. \quad (12)$$

Let the corresponding to  $r$  complex conjugate eigenvectors be

$$r = x + iy, \quad \bar{r} = x - iy, \quad (13)$$

where  $x = (x_1, \dots, x_n)^T$  and  $y = (y_1, \dots, y_n)^T$ . Introducing expressions (12) into eq. (10) and setting *real* and *imaginary* parts equal to zero we get



$$\left. \begin{aligned} [(v^2 - \mu^2)M + vC + V]x &= \mu(C + 2vM)y \\ [(v^2 - \mu^2)M + vC + V]y &= -\mu(C + 2vM)x \end{aligned} \right\} \quad (14)$$

For a *non trivial* solution the determinant of the homogeneous system (14) must be zero. For  $\lambda < \lambda_1^c$  (as stated in the Introduction) matrix  $V$  for given values of the stiffness parameters  $\kappa_j$  ( $j=1, \dots, n$ ) is positive definite. If in addition matrix  $C$  is positive definite and given that matrix  $M$  is always positive definite, according to the Parodi theorem [Bellman (1970)] all eigenvalues of eq. (5) or all roots of eq. (6) have *negative* real parts. Hence, the system is *asymptotically stable*. Indeed, if  $C$  is *positive definite*, as  $\lambda$  increases gradually from zero, at least a pair of complex conjugate eigenvalues follows in the  $\rho$ -complex plane the path shown in Fig. 1a becoming a *double* negative eigenvalue at a certain  $\lambda = \lambda_0$  slightly smaller than  $\lambda_1^c$  due to the vanishing of the discriminant of eq. (11). For  $\lambda > \lambda_0$  but less than  $\lambda_1^c$  the discriminant becomes positive related to two unequal negative eigenvalues moving in opposite directions in the real axis. At  $\lambda = \lambda_1^c$  one of these eigenvalues vanishes, becoming positive and increasing for  $\lambda > \lambda_1^c$ , yielding *static* (divergence) *instability*, while the other (negative) eigenvalue decreases algebraically.

### 3.1. Conditions for dynamic bifurcation

Attention is mainly focused on dynamic bifurcations associated either with *degenerate* or *generic Hopf bifurcation* (Fig. 1b and Fig. 1c, respectively) which may occur before divergence (i.e. for  $\lambda < \lambda_1^c$ ). This will be discussed in connection with the *sign* of the *quadratic* form  $\bar{r}^T C r$  which may be *positive semi-definite* or *indefinite*.

As mentioned above, the *necessary condition* for the existence of a degenerate or a Hopf bifurcation is the existence of **one** at least pair of *conjugate pure imaginary* eigenvalues  $\pm i\mu$  (i.e.  $v = 0$ ), while the remaining eigenvalues are complex conjugate with negative real parts. Since  $v = 0$  from eq. (12), it follows that

$$\bar{r}^T C r = 0 \quad (\text{for } r \neq 0). \quad (15)$$

Eq. (15) is satisfied when the damping matrix  $C$  is either *positive semi-definite* (since  $|C| = 0$ ) or *indefinite*. Since  $C$  is a given matrix, the *indefinite* quadratic form  $\bar{r}^T C r$  may become zero for a suitable value of  $r$ , depending on the loading  $\lambda$  and the parameters  $\eta_\kappa$  ( $\kappa = 1, \dots, 2n$ ), since  $r = r(\lambda; \eta_\kappa)$ .

The *sufficient* condition for a *generic* Hopf bifurcation is the fulfillment of the *transversality condition* [Huseyin (1986)]

$$\left. \frac{dv}{d\lambda} \right|_{\lambda=\lambda_H} \neq 0, \quad (16)$$

where  $\lambda = \lambda_H (< \lambda_1^c)$  is the load for which the real part of  $\rho$  becomes zero, i.e.  $v(\lambda_H; \eta_\kappa) = 0$ . Clearly, if condition (16) is violated, namely if

$$\left. \frac{dv}{d\lambda} \right|_{\lambda=\lambda_H} = 0, \quad (17)$$

we have a *degenerate* Hopf bifurcation.

Assuming that the *necessary condition* for a Hopf (degenerate or generic) bifurcation is satisfied, we can introduce into eq. (4) the pair of conjugate pure imaginary eigenvalues  $\pm i\mu$ . This leads to

$$(A \pm i\mu C)r = 0, \quad (18)$$

where  $A = V - \mu^2 M$ . Clearly, for a non-trivial solution (i.e. for  $r \neq 0$ ) the corresponding determinant must be zero, namely

$$|A \pm i\mu C| = 0 \quad (19)$$

According to the proof by Peremans-Duparc-Lekkerkerker [Bellman (1970), p. 67] if  $A$  and  $C$  are *real symmetric* matrices such that  $A$  is non-negative definite, then eq. (19) implies that there exists a non-trivial **real** vector  $r$  satisfying eq. (18) which yields

$$Ar = 0, \quad Cr = 0 \quad (r \neq 0). \quad (20)$$

Eqs. (20) are simultaneously satisfied if the *determinants* of both matrices  $A$  and  $C$  are zero, i.e.

$$|A| = 0, \quad |C| = 0. \quad (21)$$

The second of eqs. (21) is fulfilled if matrix  $C$  is positive *semi-definite*. In this case for given  $C$  one can determine by means of the 2<sup>nd</sup> of eqs (20) the components  $r_j$  ( $j=1, \dots, n-1$ ) of  $r$  as function of the component  $r_n$ . Introducing  $r_j(r_n)$  into the 2<sup>nd</sup> of eqs (12), after setting  $v=0$ , leads to

$$\mu^2 = \frac{\mathbf{r}^T \mathbf{V} \mathbf{r}}{\mathbf{r}^T \mathbf{M} \mathbf{r}} = \mu^2(\lambda; \eta_\kappa), \quad (22)$$

from which  $r_n$  is eliminated. Clearly,  $\eta_\kappa$  ( $\kappa=1, \dots, 2n$ ) includes the *stiffness* components  $\kappa_j$  ( $j=1, \dots, n$ ) and the concentrated masses  $m_j$  ( $j=1, \dots, n$ ). Note also that  $\bar{\mathbf{r}}^T = \mathbf{r}^T$  since  $\mathbf{r}$  is a real vector.

By virtue of the 1<sup>st</sup> of eqs (21) one can obtain the determinantal equation

$$|\mathbf{V} - \mu^2 \mathbf{M}| = 0, \quad (23)$$

whose expansion is an  $n^{\text{th}}$  algebraic polynomial with respect to  $\mu^2$ . From eqs (22) and (23) we can determine  $\mu^2$  and  $\lambda$  for given stiffness components  $\kappa_j$  and masses  $m_j$ , where  $j=1, \dots, n$ . The *smallest positive* root which satisfies eqs (22) and (23) corresponds to the *critical dynamic bifurcational load*  $\lambda = \lambda_H$  associated either with a *degenerate* or a *generic* Hopf bifurcation.

#### a: C is positive semi-definite

It can readily be shown that if  $C$  is a *positive semi-definite* matrix the *dynamic* bifurcation is a *degenerate* Hopf bifurcation since the **transversality condition** is violated (Fig. 1b). Indeed, using the expression of  $v$  given in the 1<sup>st</sup> of eqs (12), one can show that condition (17) is satisfied. Clearly, since  $\mathbf{r}^T \mathbf{C} \mathbf{r} = 0$  while  $\mathbf{r}^T \mathbf{M} \mathbf{r} \neq 0$ , condition (17) is fulfilled provided that

$$\frac{d}{d\lambda}(\mathbf{r}^T \mathbf{C} \mathbf{r}) = 2 \frac{d\mathbf{r}^T}{d\lambda} \mathbf{C} \mathbf{r} = 0, \quad (24)$$

which is true since due to (20)  $\mathbf{C} \mathbf{r} = 0$ . Such a result was anticipated since  $\mathbf{r}$  evaluated through the 2<sup>nd</sup> of homogeneous eqs. (20) is independent of  $\lambda$ . Note also that a *real* eigenvector  $\mathbf{r}$  corresponds to both pure imaginary eigenvalues  $+i\mu$  and  $-i\mu$  [Kounadis (2006)]. It is worth noticing that the above finding is also valid if  $C$  is a symmetric *singular* matrix (i.e.  $C$  is not necessarily positive semi-definite).

One can now obtain a **new** finding of paramount importance for a *multi-parameter* system. By means of the 2<sup>nd</sup> of eqs (20) we can determine  $\mathbf{r} = \mathbf{r}_j(r_n)$ , where  $j=1, \dots, n-1$ . Inserting these values into the 1<sup>st</sup> of eqs (20), after setting  $\mathbf{A} = \mathbf{V} - \mu^2 \mathbf{M}$ , we obtain

$$(\mathbf{V} - \mu^2 \mathbf{M})\mathbf{r} = 0. \quad (25)$$

Writing eq. (25) *analytically* and solving each of the resulting equations with respect to  $\mu^2$  we find

$$\mu^2 = \frac{V_{11}r_1 + \dots + V_{1n}r_n}{M_{11}r_1 + \dots + M_{1n}r_n} = \frac{V_{21}r_1 + \dots + V_{2n}r_n}{M_{21}r_1 + \dots + M_{2n}r_n} = \dots = \frac{V_{n1}r_1 + \dots + V_{nn}r_n}{M_{n1}r_1 + \dots + M_{nn}r_n}. \quad (26)$$

Relations (26) and (22) furnish  $n$  equations from which we can determine  $\lambda$  and  $(n-1)$  from the  $\eta_\kappa$  ( $\kappa=1, \dots, 2n$ ) parameters. In this solution we are looking for the *minimum positive* value of  $\lambda$  which must be less than  $\lambda_1^c$ . For instance, if  $n=2$  we can determine in addition to  $\lambda = \lambda_H$  one parameter.

### b: C is indefinite

If the matrix  $C$  is *indefinite* (since  $|C| < 0$ ) the eigenvectors satisfying eq. (18) must be complex. Multiplication of eq. (18) by  $\bar{r}^T$  (complex conjugate transpose of  $r$ ) leads to

$$\bar{r}^T (A \pm i\mu C)r = 0. \quad (27)$$

Since  $\bar{r}^T A r$  and  $\bar{r}^T C r$  are *real* (scalar) quantities, eq. (27) yields

$$\bar{r}^T A r = 0, \quad \bar{r}^T C r = 0. \quad (28)$$

Eqs (28) are also obtained from relation (12), after setting  $v=0$  and  $V = A + \mu^2 M$ , since matrix  $M$  is always positive definite.

If  $C$  is an *indefinite* matrix the quadratic form  $\bar{r}^T C r$  depending on the values of  $v \neq 0$  may be *negative* or *positive*. Hence, it may also vanish for a certain  $v(\lambda; \eta_\kappa) \neq 0$  (where  $\kappa=1, 2, \dots, 2n$ ) depending on the value of  $\lambda$  for given values of the parameters  $\eta_\kappa$ .

Given that conditions (20) are **not** valid (since  $|C| \neq 0, |A| \neq 0$ ), the procedure established previously based on  $\bar{r}^T C r = 0$  **cannot** be adopted in connection with conditions (28). Instead of this, one can apply eq. (19), the expansion of which, after setting *real* and *imaginary* parts equal to zero furnishes two equations in  $\lambda$  and  $\mu^2$  that can be determined provided that all  $\eta_\kappa$  ( $\kappa=1, \dots, 2n$ ) and matrix  $C$  are known. Clearly, the determinant in eq. (19) can be established for  $n > 2$  by using *symbolic* algebra. For  $n=2$ , eq. (19), after setting  $A = V - \mu^2 M$ , yields

$$\left. \begin{aligned} (V_{11} - \mu^2 M_{11})(V_{22} - \mu^2 M_{22}) - (V_{12} - \mu^2 M_{12})^2 - \mu^2 (c_{11}c_{22} - c_{12}^2) &= 0 \\ c_{11}(V_{22} - \mu^2 M_{22}) + c_{22}(V_{11} - \mu^2 M_{11}) - 2c_{12}(V_{12} - \mu^2 M_{12}) &= 0 \end{aligned} \right\}. \quad (29)$$

Eqs. (29) can also be written as follows:

$$\left. \begin{aligned} |V - \mu^2 M| = |A| = \mu^2 |C| \\ \mu^2 = \frac{c_{11} V_{22} + c_{22} V_{11} - 2c_{12} V_{12}}{c_{11} M_{22} + c_{22} M_{11} - 2c_{12} M_{12}} \end{aligned} \right\}, \quad (30)$$

Eliminating  $\mu^2$  from eqs (30) we can find  $\lambda$  for given matrix C and values of  $\eta_\kappa$ . The *lowest positive* value of  $\lambda (< \lambda_1^c)$  is the *dynamic instability critical (flutter) load*  $\lambda = \lambda_H$  which implies a *dynamic bifurcation* occurring **before** divergence.

One may also apply **another** procedure for establishing  $\lambda_H$ . Using the expressions of  $r$  and  $\bar{r}$  given in eq. (13), after setting  $v=0$ , eqs (14) and (28) become

$$\left. \begin{aligned} Ax &= \mu Cy \\ Ay &= -\mu Cx \end{aligned} \right\} \quad \text{with } A = V - \mu^2 M \quad (31)$$

and

$$x^T Ax + y^T Ay = 0, \quad x^T Cx + y^T Cy = 0. \quad (32)$$

From eqs (31) we obtain

$$\left. \begin{aligned} y &= \frac{1}{\mu} C^{-1} Ax \\ x &= -\frac{1}{\mu} C^{-1} Ay \end{aligned} \right\} \quad (33)$$

Substituting the expression of the vector  $y$  into the 2<sup>nd</sup> of eqs (33) we get

$$(B + \mu^2 C)x = 0, \quad (34)$$

where  $B = AC^{-1}A$ . For a *non-trivial* solution the determinant of the above homogeneous system must be zero, i.e.

$$|B + \mu^2 C| = 0, \quad (35)$$

which is a function of  $\mu^2$  and  $\lambda$  for given matrix C and parameters  $\eta_\kappa$  ( $\kappa=1, \dots, 2n$ ).

Eqs (32) using relations (33) become

$$\mu^2 = \frac{x^T Vx + y^T Vy}{x^T Mx + y^T My}, \quad x^T (B + \mu^2 C)x = 0. \quad (36)$$

Introducing the expression of  $y$  from relation (33) into the 1<sup>st</sup> of eqs (36) we obtain

$$\mu^2 = \frac{\mathbf{x}^T (\mu^2 \mathbf{V} + \mathbf{A} \mathbf{C}^{-1} \mathbf{V} \mathbf{C}^{-1} \mathbf{A}) \mathbf{x}}{\mathbf{x}^T (\mu^2 \mathbf{M} + \mathbf{A} \mathbf{C}^{-1} \mathbf{M} \mathbf{C}^{-1} \mathbf{A}) \mathbf{x}}, \quad (37)$$

where  $\mathbf{A} = \mathbf{V} - \mu^2 \mathbf{M}$ .

From the homogeneous system (34) using symbolic algebra one can express via the homogeneous system 34 the  $x_1, \dots, x_{n-1}$  components of the vector  $\mathbf{x}$  as function of the component  $x_n$  as well as of  $\mu^2$  and  $\lambda$  for given matrix  $\mathbf{C}$  and parameters  $\eta_\kappa$  ( $\kappa=1, \dots, 2n$ ). Introducing  $x_1, \dots, x_{n-1}$  into the 1<sup>st</sup> of eqs. (36) we obtain a relation between  $\mu^2$  and  $\lambda$  which together with the 2<sup>nd</sup> of eqs. (36) constitute a *nonlinear algebraic system* with respect to  $\mu^2$  and  $\lambda$  which can be solved numerically. The *lowest* positive value of  $\lambda$  ( $< \lambda_1^c$ ) is the *dynamic instability critical load*  $\lambda = \lambda_H$  at which a *dynamic bifurcation* occurs **before** divergence.

This *dynamic bifurcation* is a *generic Hopf bifurcation* since the *sufficient* condition related to the *transversality condition* is satisfied, namely  $d\nu(\lambda; \eta_\kappa)/d\lambda \neq 0$  (Fig. 1c). Indeed, due to the 1<sup>st</sup> of eqs (12) it follows that  $\nu=0$  which yields  $\bar{\mathbf{r}}^T \mathbf{C} \mathbf{r} = 0$ . Thus, one has to show

$$\left. \frac{d}{d\lambda} (\bar{\mathbf{r}}^T \mathbf{C} \mathbf{r}) \right|_{\lambda=\lambda_H} \neq 0, \quad (38)$$

where  $\bar{\mathbf{r}} = \bar{\mathbf{r}}(\lambda; \eta_\kappa)$  and  $\mathbf{r} = \mathbf{r}(\lambda; \eta_\kappa)$ . For a given matrix  $\mathbf{C}$  the quadratic form  $\bar{\mathbf{r}}^T \mathbf{C} \mathbf{r}$ , corresponding to a *critical condition* (since  $\rho = \pm i\mu$ ), is a real **polynomial** of 2<sup>nd</sup> degree with respect to the loading  $\lambda$  for known values of the parameters  $\eta_\kappa$  ( $\kappa=1, \dots, 2n$ ). If the *derivative* of this polynomial with respect to  $\lambda$  is zero, then at  $\lambda = \lambda_H$  corresponds to a *twofold* root of eq. (6); a case which is excluded since only *distinct* critical points are considered.

Clearly, the proof for *transversality condition* presented here for both the *degenerate* and *generic Hopf bifurcation* is simpler than that recently reported [Kounadis (2006)].

The above, rather cumbersome, analysis for establishing the load  $\lambda = \lambda_H$  associated with a *generic Hopf bifurcation* can be drastically simplified in the case of an *indefinite* matrix  $\mathbf{C}$  when  $|\mathbf{C}| < -\varepsilon^2$  for  $\varepsilon \rightarrow 0$ . In the following an *approximate* technique for a simple, rapid and reliable evaluation of  $\lambda = \lambda_H$  and  $\mu^2$  will be presented. The accuracy of the results obtained by using this technique increases substantially as  $\varepsilon$  approaches zero.

### 3.2. Approximate technique: $|C| < -\varepsilon^2$ for $\varepsilon \rightarrow 0$

If this case in which the determinant of the damping matrix  $C$  is *negative* but *negligibly small* and  $Cr \neq 0$ , we may consider that  $\tilde{r}^T Cr = 0$  in eq. (28) can be approximately satisfied by a **real** vector,  $r(\lambda; \eta_k) \neq 0$ , as in the previous case of a positive semi-definite matrix  $C$ , where  $|C| = 0$ . Namely, we may assume that at a certain  $\lambda = \lambda_H$  there exists a vector  $r(\lambda_H; \eta_k) \neq 0$  for given  $\eta_k$  for which

$$\tilde{r}^T Cr = 0 \quad (39)$$

and due to eqs (28)  $\tilde{r}^T Ar = 0$ . (40)

We may also assume that the latter case is satisfied if  $|A| = 0$  which implies

$$Ar = 0 \quad (r \neq 0). \quad (41)$$

Analytically eq. (41) is written as follows:

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1,n-1} & \cdot & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2,n-1} & \cdot & \alpha_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{n,n-1} & \cdot & \alpha_{nn} \end{bmatrix} \begin{bmatrix} r_1 \\ \cdot \\ \cdot \\ \cdot \\ r_{n-1} \\ r_n \end{bmatrix} = 0 \quad (42)$$

or in a partitioned form

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & \alpha_{nn} \end{bmatrix} \begin{bmatrix} \tilde{r} \\ r_n \end{bmatrix} = 0, \quad (43)$$

where  $A_{11}$  is an  $(n-1) \times (n-1)$  *non-singular* symmetric matrix,  $A_{12}$  an  $n \times 1$  (column) matrix such that  $A_{12}^T = A_{21}$ ;  $\alpha_{nn}$  is a *real* (scalar) quantity,  $\tilde{r} = (r_1, \dots, r_{n-1})^T$  is a *vector* and  $r_n$  the  $n^{\text{th}}$  component of the vector  $r$ . From eq. (43) one can find

$$\tilde{r} = -r_n A_{11}^{-1} A_{12}, \quad (r_n \neq 0). \quad (44)$$

Introducing this expression of  $\tilde{r}$  into eq. (39) we obtain

$$(\tilde{r}^T \ r_n) \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{nn} \end{bmatrix} \begin{bmatrix} \tilde{r} \\ r_n \end{bmatrix} = 0, \quad (45)$$

where

$$C_{11} = \begin{bmatrix} c_{11} & \cdots & c_{1,n-1} \\ \vdots & \ddots & \vdots \\ c_{n-1,1} & \cdots & c_{n-1,n-1} \end{bmatrix}, C_{12} = C_{21}^T = \begin{bmatrix} c_{1n} \\ \vdots \\ c_{n-1,n} \end{bmatrix}. \quad (46)$$

From eq. (45) it follows that

$$\tilde{r}^T C_{11} \tilde{r} + 2r_n C_{21} \tilde{r} + r_n^2 c_{nn} = 0 \quad (47)$$

or due to eq. (44)

$$A_{12}^T A_{11}^{-1} C_{11} A_{11}^{-1} A_{12} - 2C_{21} A_{11}^{-1} A_{12} + c_{nn} = 0. \quad (48)$$

Eq. (48) along with the determinantal equation

$$|A| = 0 \quad (49)$$

furnish a *nonlinear system* of two equations with unknowns  $\lambda$  and  $\mu^2$  which can be solved *numerically* provided that the matrix  $C$  and all parameters  $\eta_\kappa$  ( $\kappa = 1, \dots, 2n$ ) are known. The *lowest positive*  $\lambda$  ( $< \lambda_1^c$ ) is the *dynamic instability load*  $\lambda = \lambda_H$  which corresponds to a *generic Hopf bifurcation*. This is so because the *transversality condition* is also satisfied according to the proof given above.

#### 4. Numerical examples

A nonlinear dynamic analysis using 2-DOF and 3-DOF cantilever models [Fig. 2 a,b] confirms the validity of the theoretical findings as well as the efficiency and simplicity of the proposed technique. The nonlinear equations of motion for the perfect 2-DOF model shown in Fig. 2a are obtained from Kounadis (1994) after setting  $\gamma = 0$  and  $\eta=1$ , as follows

$$\begin{aligned} (1+m)\ddot{\theta}_1 + \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + c_{11}\dot{\theta}_1 + c_{12}\dot{\theta}_2 + k\theta_1 - \theta_2 + \theta_1 - \lambda \sin \theta_1 &= 0, \\ \ddot{\theta}_2 + \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + c_{22}\dot{\theta}_2 + c_{12}\dot{\theta}_1 - \theta_1 + \theta_2 - \lambda \sin \theta_2 &= 0, \end{aligned} \quad (50)$$

where  $m = m_1/m_2, k = k_1/k_2, \lambda = Pl/k_2$ .

Linearization of eqs. (50) after setting

$$\Theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = e^{et} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = e^{et} \varphi$$

gives

$$(\rho^2 M + \rho C + V)\varphi = 0 \quad (51)$$



where

$$M = \begin{bmatrix} 1+m & 1 \\ 1 & 1 \end{bmatrix}, C = \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix}, V = \begin{bmatrix} k+1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix}, \quad (52)$$

The *static* buckling (divergence) equation is given by

$$\lambda^2 - (2+k)\lambda + k = 0, \quad (53)$$

whose lowest root is the *first* buckling load  $\lambda_1^c$  equal to

$$\lambda_1^c = 0.5 (k+2 - \sqrt{k^2+4}). \quad (54)$$

Clearly, for the entire interval of values  $k>0$ , eq. (54) yields  $0 \leq \lambda_1^c < 1$ . The case of equal roots is excluded. The variation of the *static buckling load*  $\lambda_1^c$  versus  $k$  is shown in Fig. 3.

The equations of motion for the perfect 3-DOF model, shown in Fig. 2b, are obtained from Kounadis (1997<sub>2</sub>) after setting  $\delta = \gamma = 0$  and  $\eta=1$ , i.e.

$$(1 + \bar{m}_1 + \bar{m}_2)\ddot{\theta}_1 + (1 + \bar{m}_2)\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \ddot{\theta}_3 \cos(\theta_1 - \theta_3) + \dot{\theta}_2^2(1 + \bar{m}_2)\sin(\theta_1 - \theta_2) + \dot{\theta}_3^2 \sin(\theta_1 - \theta_3) + c_{11}\dot{\theta}_1 + c_{12}\dot{\theta}_2 + \bar{k}_1\theta_1 - \bar{k}_2(\theta_2 - \theta_1) - \lambda \sin \theta_1 = 0,$$

$$(1 + \bar{m}_2)\ddot{\theta}_2 + (1 + \bar{m}_2)\ddot{\theta}_1 \cos(\theta_1 - \theta_2) + \ddot{\theta}_3 \cos(\theta_2 - \theta_3) - (1 + \bar{m}_2)\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + \dot{\theta}_3^2 \sin(\theta_2 - \theta_3) + c_{22}\dot{\theta}_2 + c_{12}\dot{\theta}_1 + c_{23}\dot{\theta}_3 + \bar{k}_2(\theta_2 - \theta_1) - \theta_3 + \theta_2 - \lambda \sin \theta_2 = 0.$$

$$\ddot{\theta}_3 + \ddot{\theta}_2 \cos(\theta_2 - \theta_3) + \ddot{\theta}_1 \cos(\theta_1 - \theta_3) - \dot{\theta}_2^2 \sin(\theta_2 - \theta_3) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_3) + c_{33}\dot{\theta}_3 + c_{32}\dot{\theta}_2 + \theta_3 - \theta_2 - \lambda \sin \theta_3 = 0,$$

where  $\bar{m}_1 = m_1/m_3$ ,  $\bar{m}_2 = m_2/m_3$ ,  $\bar{k}_1 = k_1/k_3$  and  $\bar{k}_2 = k_2/k_3$ ,  $\lambda = Pl/k_3$ . (55)

Linearizing eqs. (55) and then setting

$$\Theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = e^{\rho t} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = e^{\rho t} \varphi,$$

we obtain

$$(\rho^2 M + \rho C + V)\varphi = 0, \quad (56)$$

where

$$M = \begin{bmatrix} 1 + \bar{m}_1 + \bar{m}_2 & 1 + \bar{m}_2 & 1 \\ 1 + \bar{m}_2 & 1 + \bar{m}_2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & c_{23} \\ 0 & c_{23} & c_{33} \end{bmatrix}, \quad V = \begin{bmatrix} \bar{k}_1 + \bar{k}_2 - \lambda & -\bar{k}_2 & 0 \\ -\bar{k}_2 & 1 + \bar{k}_2 - \lambda & -1 \\ 0 & -1 & 1 - \lambda \end{bmatrix}. \quad (57)$$

The *static buckling* (divergence) equation is equal to

$$\lambda^3 - (2 + \bar{k}_1 + 2\bar{k}_2)\lambda^2 + (2\bar{k}_1 + 3\bar{k}_2 + \bar{k}_1\bar{k}_2)\lambda - \bar{k}_1\bar{k}_2 = 0, \quad (58)$$

whose roots are *real* and *positive*, since  $\lambda^s$  ( $i=1,2,3$ ) are the eigenvalues of the positive definite matrix  $V(\lambda=0)$  (i.e. obtained from eq. (57) for  $\lambda=0$ ). Moreover, one can show that the case of a *double* root of eq. (58) is excluded.

## 2-DOF model

For a *degenerate* Hopf bifurcation, associated with a *positive semi-definite* matrix  $C$ , using eqs (20) and (21) for  $n=2$ , we get

$$r = \tau_1 / \tau_2 = -c_{22} / c_{11}. \quad (59)$$

Application of eq. (36) due to eqs (52) gives

$$\frac{(k+1-\lambda)r-1}{(1+m)r+1} = \frac{-r+1-\lambda}{r+1}, \quad (60)$$

from which we obtain the degenerate Hopf bifurcation load  $\lambda = \lambda_H$ , i.e.

$$(r^2 - mr - 1)\lambda_H = r^2(m + k + 2) + r(k - m) - 2, \quad (61)$$

where  $m$  and  $k$  are positive quantities, while  $r$  may be positive or negative. Note that if  $r^2 - mr - 1 = 0$ , the critical load  $\lambda_H$  exhibits a *discontinuity* (varying from  $+\infty$  to  $-\infty$ ). For  $r^2(m + k + 2) + r(k - m) - 2 = 0$  it follows that  $\lambda = 0$ . Clearly, for given  $r$  the above extreme values of  $\lambda_H$  may occur for various combinations of values of the parameters  $m$  and  $k$ .

It is worth noticing that the *discontinuity* in the flutter load  $\lambda_H$  (being independent of the stiffness ratio  $k$ ) occurs at  $m = (r^2 - 1)/r$ . Namely, for a given damping ratio  $r$ , one can find a critical value of  $m$ , i.e.  $m = m_{cr}$ , which corresponds to a discontinuity in the load  $\lambda_H$ . As stated above, since  $r (= -c_{22}/c_{11})$  does **not** depend on  $\lambda$ , eq. (12) yields  $dv/d\lambda = 0$  (violation of the transversality sufficient condition for a *degenerate* Hopf bifurcation).

From eq. (26) we obtain

$$\mu^2 = -\rho^2 = \frac{-r+1-\lambda}{1+r} \quad (r \neq -1). \quad (62)$$

For the above 2-DOF model with  $k=1$ ,  $m=10$  and a *positive semi-definite matrix*  $C$  with  $c_{11} = 0.01$ ,  $c_{12} = c_{21} = 0.002$  and  $c_{22} = 0.0004$  (i.e.  $|C| = 0$ ) we find  $\lambda_1^c = 0.381966011$ ,  $r = -0.20$ ,  $\lambda_H = 0.32/1.04 = 0.307692307$ , and  $\rho^2 = -\mu^2 = -1.115384615$  [Kounadis (2006)]. The critical *mass* ratio for which a *discontinuity* in the flutter load  $\lambda_H$  for this degenerate Hopf bifurcation occurs is:  $m_{cr} = 4.80$ . The variation of  $\lambda_H$  with respect to  $m > 0$  for  $k=1$  is shown in Fig. 4a.

For a *generic* Hopf bifurcation, associated with a given *indefinite* matrix  $C$ , according to the *exact* analysis, one can obtain  $\lambda_H$  and  $\mu^2$  by solving the system of eqs. (30).

In case of a matrix  $C$  for which  $|C| < -\varepsilon^2$  with  $\varepsilon \rightarrow 0$ , the determination of the *flutter* load  $\lambda_H$  (and then  $\mu^2$ ) is appreciably simplified without diminishing its accuracy. Thus, application of eq. (26) gives

$$\mu^2 = \frac{V_{11}r + V_{12}}{M_{11}r + M_{12}} = \frac{V_{12}r + V_{22}}{M_{12}r + M_{22}}, \quad (63)$$

which leads to eq. (60) and then to eq. (61). The ratio  $r$  is obtained from eq. (39), i.e.

$$c_{11}r^2 + 2c_{12}r + c_{22} = 0,$$

or

$$r = \frac{1}{c_{11}} \left( -c_{12} \pm \sqrt{c_{12}^2 - c_{11}c_{22}} \right). \quad (64)$$

Clearly, the equation yielding a *discontinuity* in the flutter load  $\lambda_H$ , i.e.  $r^2 = mr - 1 = 0$ , is still valid. For the above 2-DOF model with  $k=1$  and  $m=10$  related to an indefinite matrix  $C$  with  $c_{11} = 0.01$ ,  $c_{12} = c_{21} = 0.0325$  and  $c_{22} = 0.012$  (i.e.  $|C| = -9.3625 \times 10^{-4} < 0$ ) we find, according to the *exact* analysis,  $\lambda_H = 0.193698381 < \lambda_1^c$ , and  $\mu = 1.109221303$  [Kounadis (2006)]. On the basis of the *approximate* analysis we obtain using eq. (64)  $r = -0.190179743$  and thereafter through eq. (63),  $\lambda_H = 0.193830151$  and  $\mu = 1.109204333$ , respectively, which practically coincide with the previously found values of  $\lambda_H$  and  $\mu$ . The plot  $\lambda_H$  versus  $m > 0$  with the corresponding asymptotes  $m = 5.068$  and  $\lambda = 1.19018$  is shown in Fig. 4b. Fig. 5a shows the Hopf bifurcation load  $\lambda_H$  versus mass ratio  $m (= m_1/m_2) > 5.068$  for various values of stiffness ratio  $k (= k_1/k_2)$ . The small circle in each  $k$ -curve corresponds to the

critical *divergence* (buckling) load  $\lambda_{(1)}^c$  ( $\equiv \lambda_H$ ). This corresponds to a dynamic bifurcation associated with a coupled *divergence-flutter* instability. This *double zero* eigenvalue dynamic bifurcation is called an Arnold-Bogdanof dynamic bifurcation [Kounadis (2006)]. Below the small circle we have  $\lambda_H < \lambda_{(1)}^c$  for each value of  $k$ . Clearly, both  $\lambda_{(1)}^c$  and  $\lambda_H$  increase with the increase of  $k$ . Fig. 5b shows Hopf bifurcations loads  $\lambda_H$  versus mass ratio  $m$  ( $= m_1/m_2$ )  $< 5.068$  for various stiffness ratios  $k$  ( $= k_1/k_2$ ). The values of  $\lambda_H$  are higher than  $\lambda = 1.19018$  (asymptote). It is worth noticing that a discontinuity in the flutter load versus mass was first observed in a continuous damped cantilever model carrying concentrated (attached) masses [Kounadis (1977), Kounadis and Katsikadelis (1980)].

**Table 1** gives values of  $\lambda_{(1)}^c \equiv \lambda_H$ , of  $\lambda_H < \lambda_{(1)}^c$  and of  $\lambda_H$  slightly smaller than  $\lambda_{(1)}^c$  for various values of the *stiffness* and *mass* ratio  $k$  and  $m$  respectively, together with the corresponding amplitudes  $\theta_1(\tau)$  of the stable limit cycles. Note that  $\theta_1(\tau)$  increases with the increase of  $k$  and  $m$ .

Fig. 6 shows the phase-plane portrait  $\theta_2$  vs  $\dot{\theta}_2$  corresponding to the above *generic* bifurcation with  $k=10$ ,  $m=46.838455646$  and  $\lambda_H = 0.898 < \lambda^c = 0.9009805$ . Note that the (absolutely) maximum amplitude of  $\theta_2(\tau)$  corresponding to this generic Hopf bifurcation, is nearly 0.008 rad. The maximum (absolutely) amplitudes of  $\theta_1(\tau)$  are, as expected, lower than the corresponding  $|\max \theta_2(\tau)|$ .

## 5. Conclusions

The conditions under which *symmetric weakly damped systems* under *potential* loading may exhibit Hopf bifurcations, discontinuities of flutter loads and other phenomena, are thoroughly discussed. The most important findings are the following:

1. A solution methodology for establishing *necessary* and *sufficient* conditions for *degenerate* and *generic* Hopf bifurcations as well as *double zero* eigenvalue bifurcations is presented.
2. Besides this *exact* analysis an efficient and readily employable technique suitable for multi-degree of freedom systems with *negligibly small* damping, is comprehensively established.
3. It was, strangely enough, found that the *coupling effect* of negligibly small *damping* with *non-uniform mass* distribution may lead to *flutter load discontinuities*. This may

have important applications in the dynamic response of multi-story buildings (simulated by cantilever models).

4. The individual and coupling effect of the *mass* and *stiffness* ratios on the *buckling* and *flutter* load,  $\lambda_{(1)}^c$  and  $\lambda_H$ , respectively, are assessed.
5. Combinations of values of  $m$  and  $k$  for which the flutter load  $\lambda_H$  of generic Hopf bifurcations, associated with *stable* limit cycles, occurs *prior* to divergence, are also assessed.
6. The effect of the above parameters  $m$  and  $k$  on the *maximum* (final) amplitudes of the angles  $\theta_1(\tau)$  and  $\theta_2(\tau)$ , being independent of the initial conditions, are also presented.

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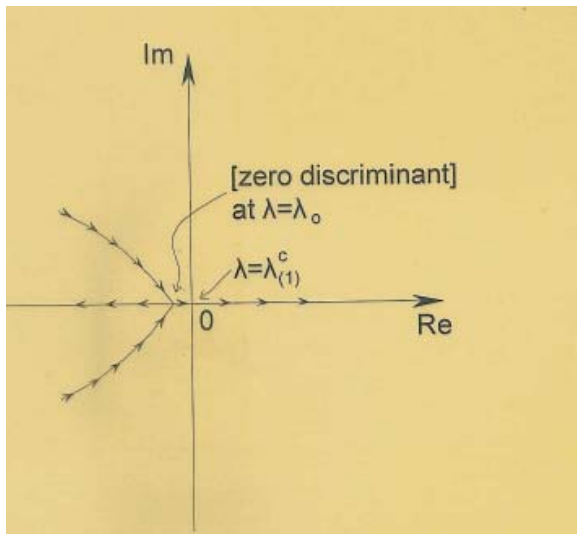


Fig. 1a

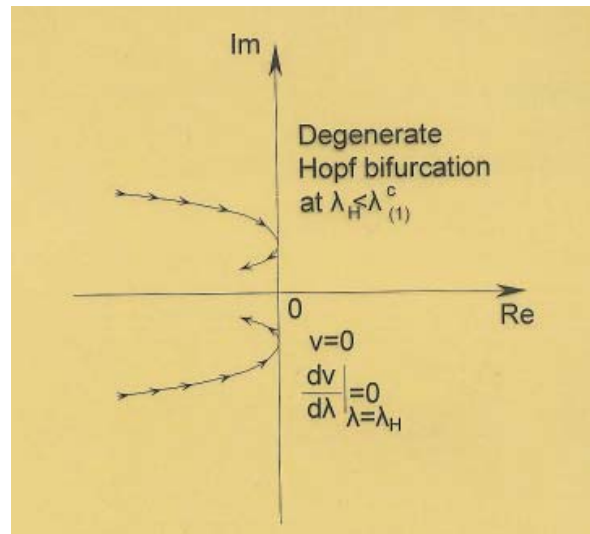


Fig. 1b

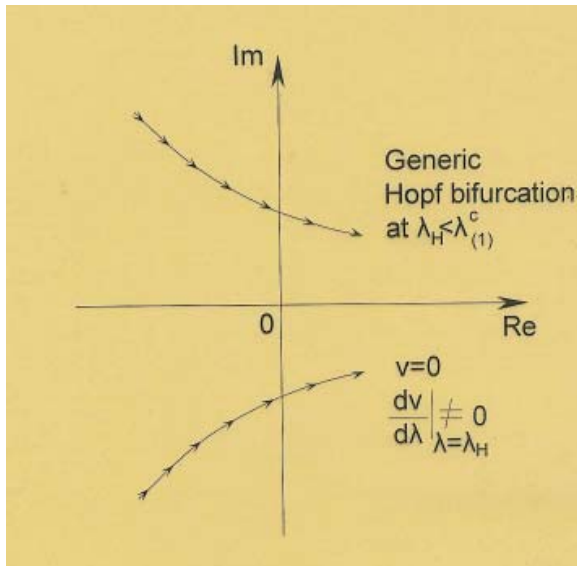


Fig. 1c

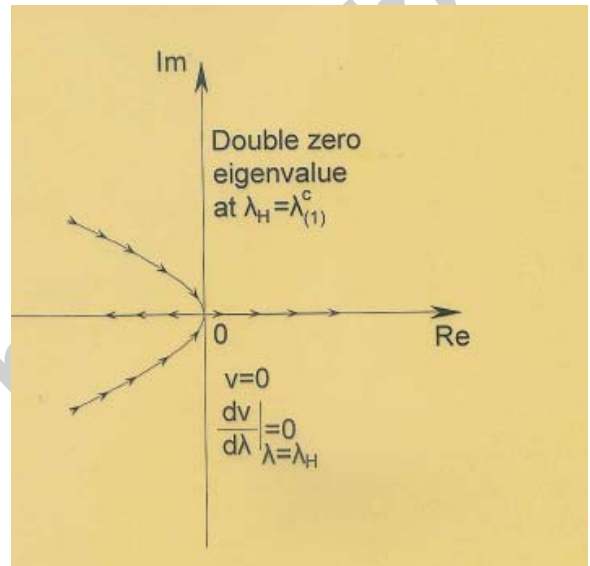


Fig. 1d

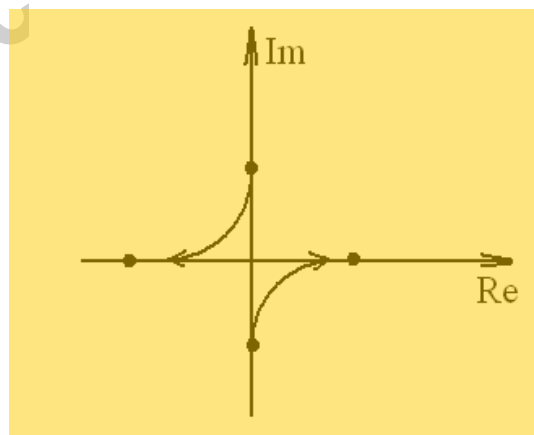


Fig. 1e

Fig.1 a,b,c,d,e. Types of dynamic bifurcations.

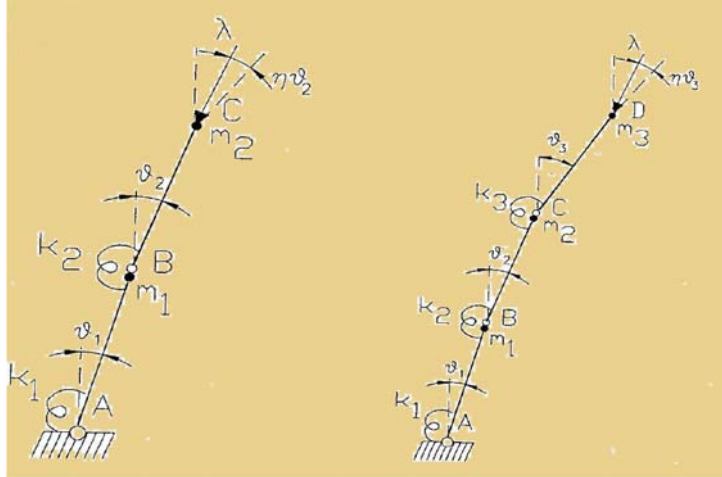


Fig. 2. 2-DOF and 3-DOF perfect models under partial follower loading

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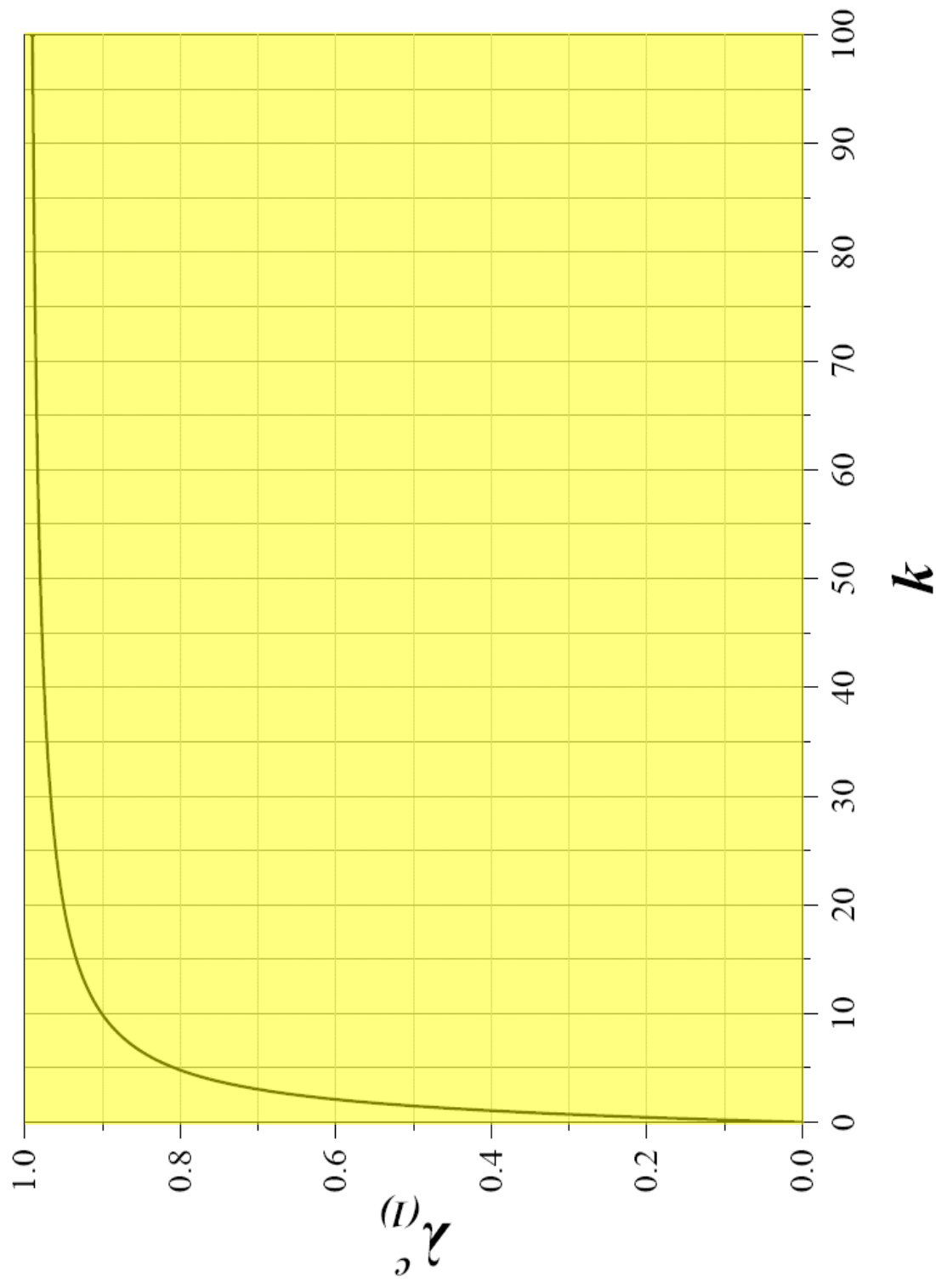


Fig.3. Variation of the critical static buckling load  $\lambda_c^{(1)}$  versus the stiffness ratio  $k$ .

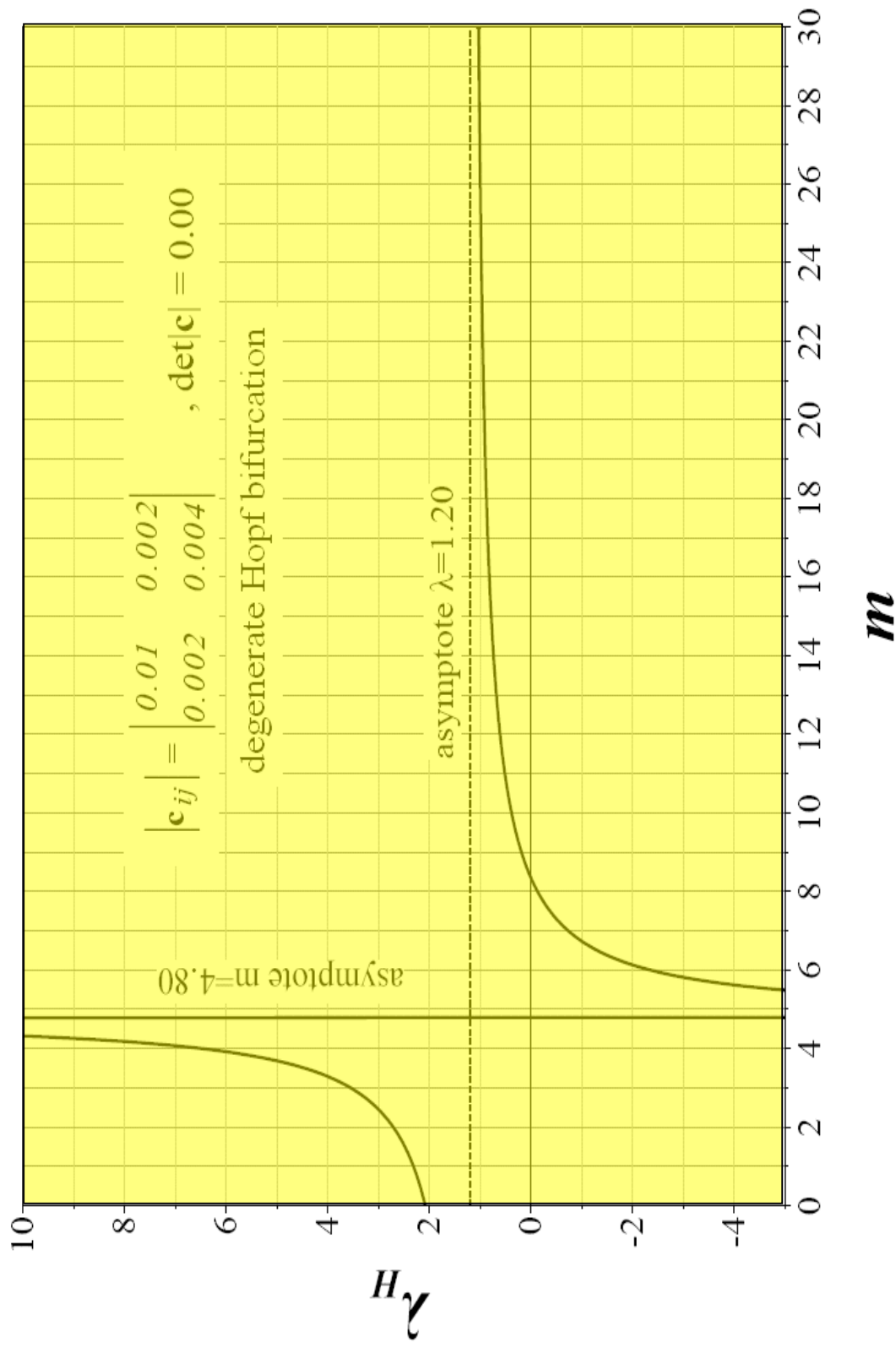


Fig.4a. Variation of  $\lambda_H$  versus mass ratio  $m$  for stiffness ratio  $k=1$ . Note that at  $m = 4.80$  there is a discontinuity of the flutter load  $\lambda_H$ , which depends on the damping ratio  $\gamma$ .

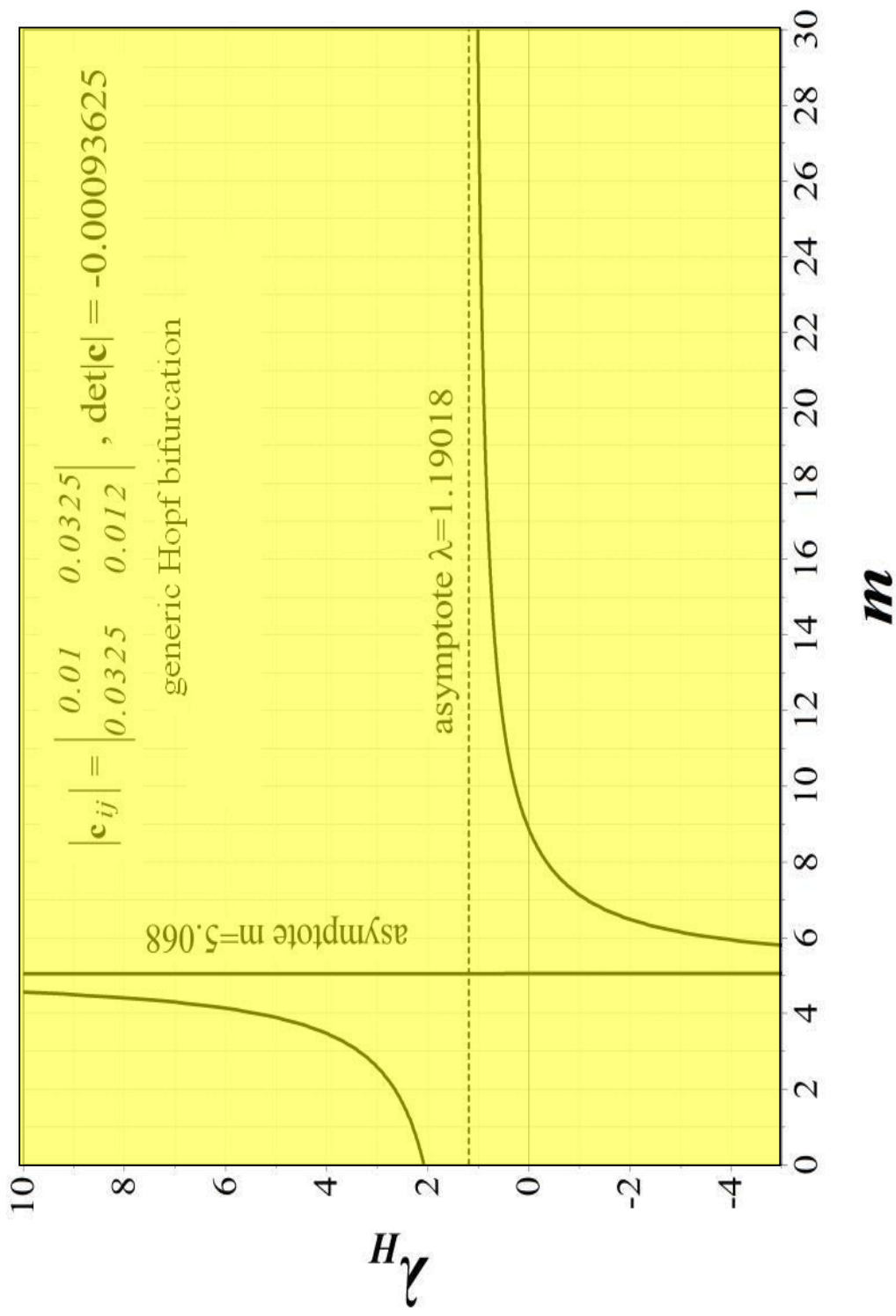


Fig.4b. Variation of  $\lambda_H$  versus mass ratio  $m$  for stiffness ratio  $k=1$ . Note that at  $m=5.068$  there is a discontinuity of the flutter load  $\lambda_H$ , which depends on the damping ratio  $\tau$ .

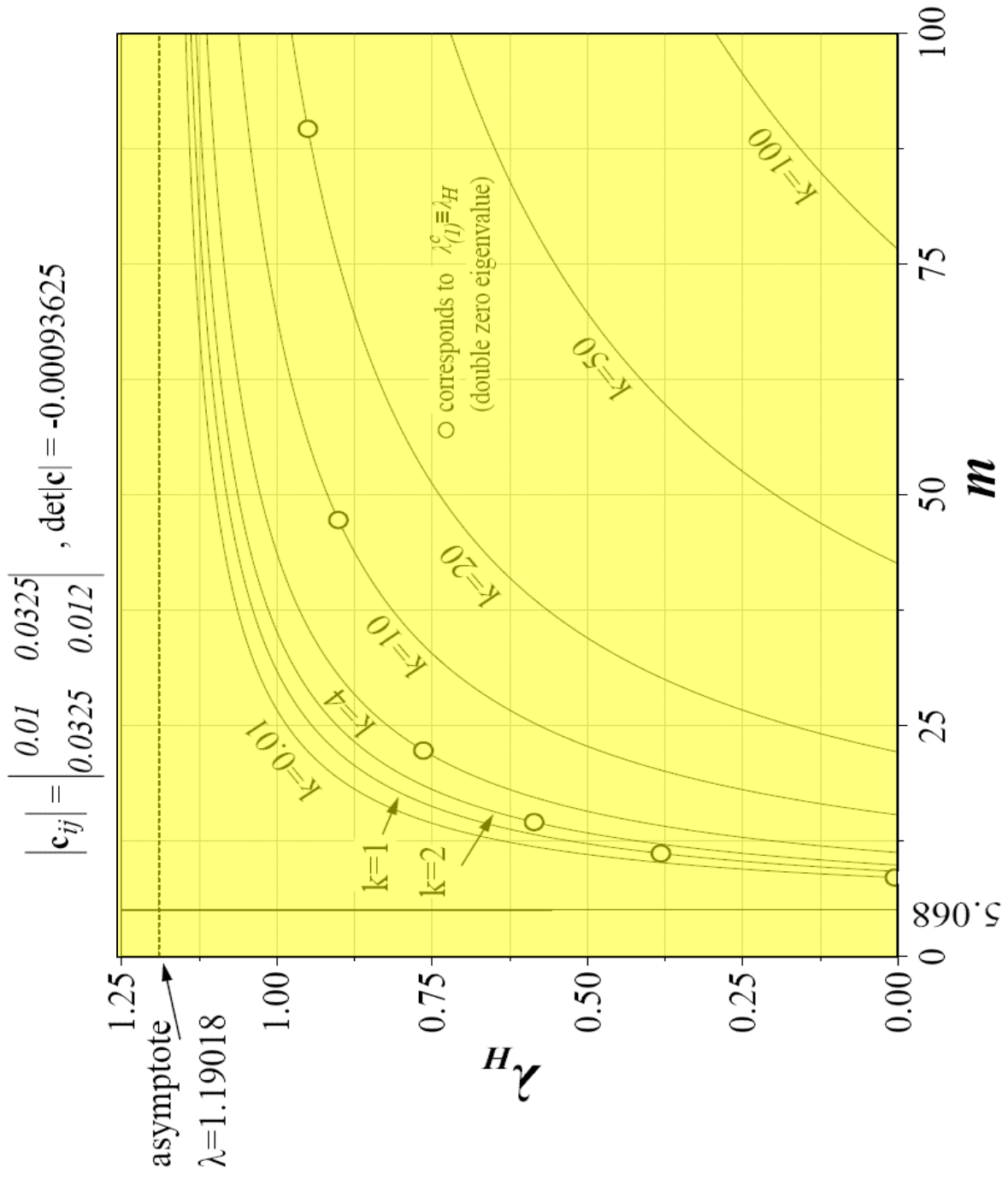


Fig.5a. Hopf bifurcation load  $\lambda_H$  versus mass ratio  $m$  ( $= m_1/m_2$ )  $> 5.068$  for various values of stiffness ratio  $k$  ( $= k_1/k_2$ ).

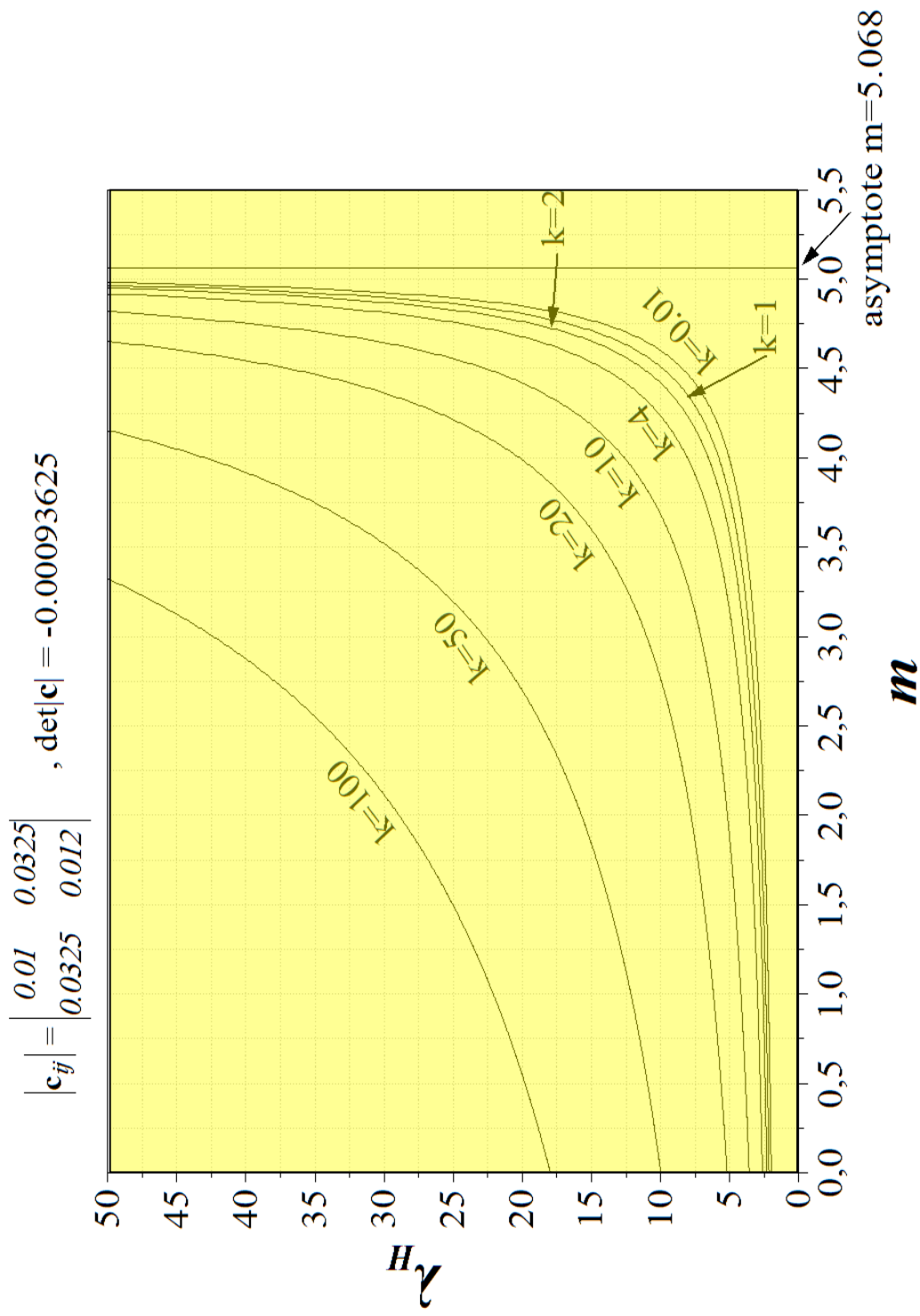


Fig.5b. Hopf bifurcation load  $\lambda_H$  versus mass ratio  $m$  ( $= m_1/m_2$ )  $< 5.068$  for various values of stiffness ratio  $k$  ( $= k_1/k_2$ ).

Table 1. Buckling and flutter loads  $\lambda_1^c$  and  $\lambda_{H-}$  as well as  $|\max\theta(\tau)|$  for various values of stiffness and mass ratios  $k$  and  $m$ .

$k$	$\lambda_1^c$	$m$ $(\lambda_H \equiv \lambda_{(1)})$	$m$		Max Absolute Amplitude of Limit Cycle Response $ \max\theta_1(\tau) $	$m$		Max Absolute Amplitude of Limit Cycle Response $ \max\theta_1(\tau) $
			$\lambda_H$	$\lambda_H$		$\lambda_H$	$\lambda_H$	
0.01	0.004987500078124096	8.5381671820524	8.530898922049763 0.0025	8.537910996697269 0.0049	1.266 × 10 <sup>-4</sup>	8.537910996697269 0.0049	1.443 × 10 <sup>-4</sup>	
0.05	0.024687548812872162	8.624620682147096	8.58721493692014 0.0123	8.624048427854003 0.0245	1.345 × 10 <sup>-4</sup>	8.624048427854003 0.0245	1.512 × 10 <sup>-4</sup>	
0.10	0.048750780274960714	8.735080373142841	8.66032855429743 0.025	8.732669827235092 0.048	1.361 × 10 <sup>-4</sup>	8.732669827235092 0.048	1.405 × 10 <sup>-4</sup>	
0.20	0.09501243788791092	8.963941923121673	8.800309959352578 0.047	8.96034346927522 0.094	1.423 × 10 <sup>-4</sup>	8.96034346927522 0.094	1.511 × 10 <sup>-4</sup>	
0.50	0.21922359359558485	9.712596429979081	9.242934965700924 0.11	9.706750465869224 0.218	1.607 × 10 <sup>-4</sup>	9.706750465869224 0.218	1.812 × 10 <sup>-4</sup>	
1.00	0.3819660112501051	11.148927626097844	10 0.1936983908834291	11.145423271996046 0.3815	1.852 × 10 <sup>-4</sup>	11.145423271996046 0.3815	2.353 × 10 <sup>-4</sup>	
2.00	0.5857864376269049	14.539770234116409	11.498810153696272 0.30	14.51185531175303 0.584	2.372 × 10 <sup>-4</sup>	14.51185531175303 0.584	3.624 × 10 <sup>-4</sup>	
4.00	0.7639320225002102	22.29891798820193	14.133179512599185 0.38	22.221165740849916 0.762	2.983 × 10 <sup>-4</sup>	22.221165740849916 0.762	6.161 × 10 <sup>-4</sup>	
10.0	0.9009804864072155	47.268958942233375	21.55582361428882 0.45	46.83845564604463 0.898	4.645 × 10 <sup>-4</sup>	46.83845564604463 0.898	0.001075	
20.0	0.95012437887911	89.649282067276	33.4568798886462 0.475	88.90731487411975 0.948	6.221 × 10 <sup>-4</sup>	88.90731487411975 0.948	0.00145	
50	0.9800079936063923	217.29095024741068	68.76698107585851 0.49	215.28247546849224 0.978	7.963 × 10 <sup>-4</sup>	215.28247546849224 0.978	0.00177	
100	0.9900009998000527	430.1966503540317	128.36449052490056 0.50	425.98894651830915 0.988	9.365 × 10 <sup>-4</sup>	425.98894651830915 0.988	0.00196	

$$k = 10, \quad m = 46.838455646$$

$$\lambda = \lambda_H = 0.898 < \lambda^c_{(1)} = 0.9009805$$

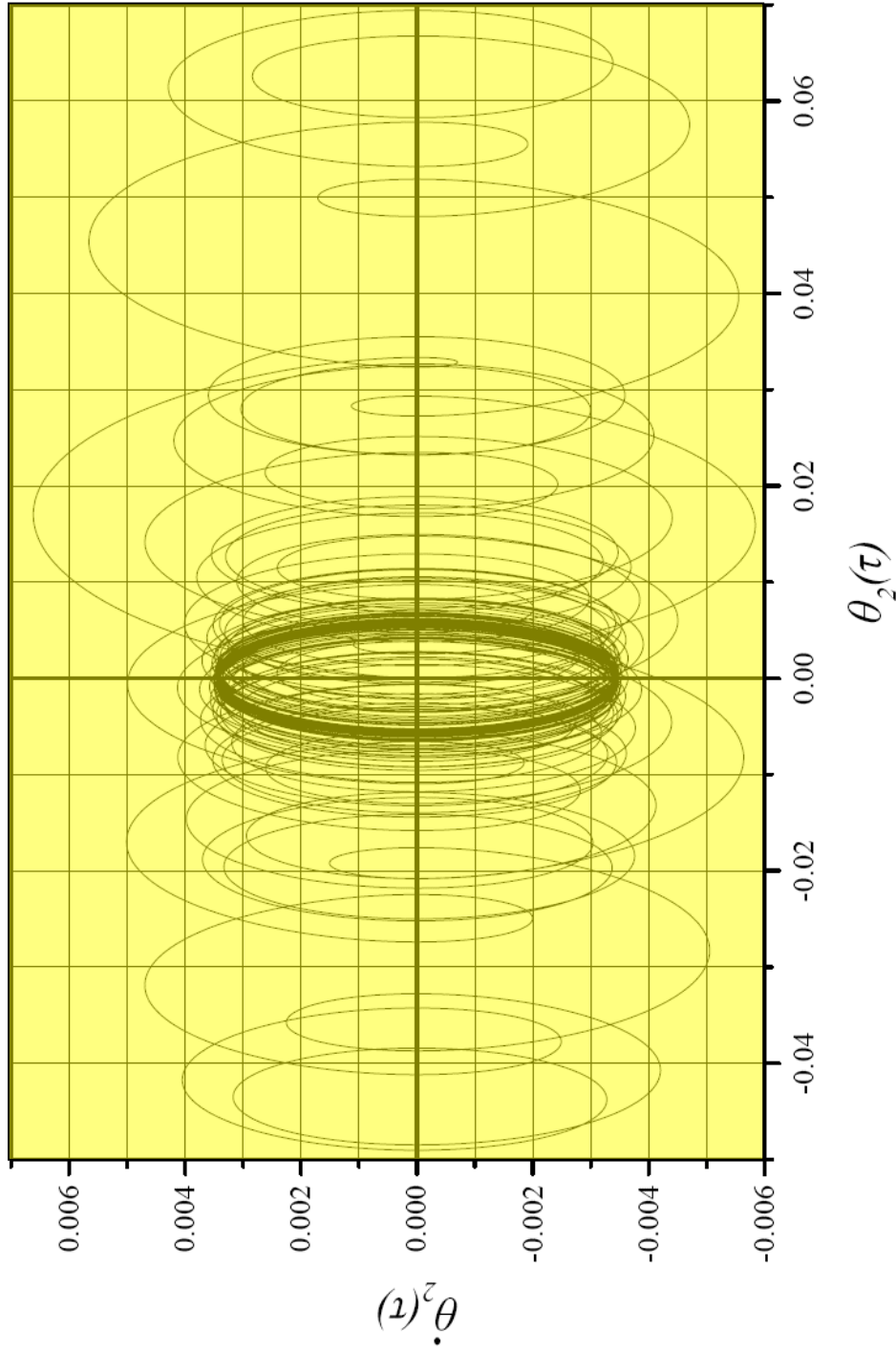


Fig.6. Phase - plane portrait of the Hopf bifurcation of a 2-D.O.F. model with  $k = 10$  and  $m = 46.838455646$ .