



HAL
open science

Nonlinear dynamic stability of damped Beck's column with variable cross-section

J.T. Katsikadelis, G.C. Tsiatas

► **To cite this version:**

J.T. Katsikadelis, G.C. Tsiatas. Nonlinear dynamic stability of damped Beck's column with variable cross-section. *International Journal of Non-Linear Mechanics*, 2007, 42 (1), pp.164. 10.1016/j.ijnonlinmec.2006.10.019 . hal-00501737

HAL Id: hal-00501737

<https://hal.science/hal-00501737>

Submitted on 12 Jul 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Author's Accepted Manuscript

Nonlinear dynamic stability of damped Beck's column with variable cross-section

J.T. Katsikadelis, G.C. Tsiatas

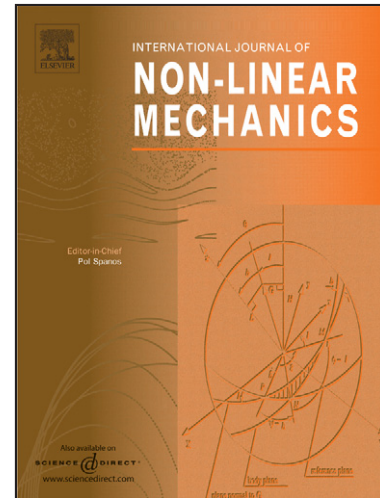
PII: S0020-7462(07)00032-7
DOI: doi:10.1016/j.ijnonlinmec.2006.10.019
Reference: NLM 1317

To appear in: *International Journal of Non-Linear Mechanics*

Received date: 14 July 2006
Revised date: 6 September 2006
Accepted date: 2 October 2006

Cite this article as: J.T. Katsikadelis and G.C. Tsiatas, Nonlinear dynamic stability of damped Beck's column with variable cross-section, *International Journal of Non-Linear Mechanics* (2007), doi:10.1016/j.ijnonlinmec.2006.10.019

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting galley proof before it is published in its final citable form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.



www.elsevier.com/locate/nlm

Nonlinear dynamic stability of damped Beck's column with variable cross-section

J.T. Katsikadelis^{*}, G.C. Tsiatas

School of Civil Engineering, National Technical University of Athens, Zografou Campus GR-15773 Athens, Greece

Abstract

In this paper the nonlinear dynamic stability of Beck's column with variable mass and stiffness properties in the presence of damping (both internal and external) is investigated using a complete nonlinear dynamic analysis. This approach permits the examination of the global stability of the system in contrast to the static nonlinear one, which, though more economical in computational cost, is associated only with the loss of local stability via flutter or divergence. The governing equations describing the dynamic response are derived in terms of the displacements taking also into account the axial deformation, which has a striking influence on the critical load. Since the cross-sectional properties of the beam vary along its axis, the resulting coupled nonlinear differential equations have variable coefficients. Their solution is achieved using the analog equation method (AEM) of Katsikadelis. Besides its accuracy and effectiveness, this method overcomes the shortcoming of a FEM solution, which may experience lack of convergence. Interesting conclusions are drawn. The important, however, finding is that the inclusion of the axial deformation affects highly the critical load of Beck's column with varying cross sectional properties, while it leaves it unaltered for Beck's column with uniform cross section.

Keywords: Beck's column; nonlinear dynamic stability; nonlinear dynamic analysis; analog equation method; variable cross-section

^{*} Corresponding author. Tel.: +30-2107721654; fax: +30-2107721655.

E-mail addresses: jkats@central.ntua.gr (J.T. Katsikadelis), gtsiatas@central.ntua.gr (G.C. Tsiatas).

1. Introduction

The influence of damping on the stability of linear elastic nonconservative uniform columns has been extensively investigated. The related main conclusions are the stabilizing effect of the external damping [1] and to the paradox of the destabilizing effect of the internal damping [2, 3]. However, when nonuniform columns are examined [4, 5] the influence of internal damping can have a stabilizing character under certain conditions, while the external damping has always a stabilizing effect [4]. Some recent results for linear viscoelastic models, in which creep, relaxation and hysteresis effects are taken into account, can be found in a survey paper by Gaul [6]. A special case of the influence of damping on Hamiltonian autonomous systems is very recently reported by Kounadis [7].

The stability behavior of nonconservative columns can be examined only via nonlinear analysis, static or dynamic [8-11]. The latter one permits the examination of the global stability of the system instead of the static nonlinear analysis, which is associated with the loss of local stability, via flutter or divergence [12, 13]. The work that has been done on the nonlinear dynamic analysis is limited only to uniform damped Beck's columns. The bifurcation may be subcritical or supercritical. The subcritical can exhibit instability below the critical load (from the linear theory), while the supercritical is the stable solution to which all solutions tend independently of the initial conditions. The initial studies of Kolkka [14] and Chen [14] have shown that the bifurcation is supercritical for the uniform Beck's column. Recently, Andersen and Thomsen [16] studying a uniform Beck's column with a tip mass at its free end observed that the rotary inertia of the mass can change the type of bifurcation from supercritical to subcritical. To the authors knowledge publications on the solution of the problem of Beck's column with variable mass and stiffness properties are not available in literature.

In this paper the nonlinear dynamic stability of Beck's column with variable mass and stiffness properties in the presence of damping (both internal and external) using a complete nonlinear dynamic analysis is investigated. For homogeneous material variable mass and stiffness is due to the variation of the beam cross-section. The nonlinearity results from retaining the square of the slope in the strain-displacement relations (intermediate nonlinear theory). In this case the transverse deflection affects the axial force and the resulting equations, in terms of the displacements, are coupled nonlinear with variable coefficients. The analysis is performed with and without considering the axial deformation. This requires the solution of two different initial boundary value problems. The deviations of the two approaches are studied and compared with the linear theory. The solution of the problems was achieved using the analog equation method (AEM) of Katsikadelis [17] as it was developed for the nonlinear dynamic analysis of beams [18]. According to this method, the two coupled nonlinear hyperbolic partial differential equations with variable coefficients are replaced by two uncoupled linear ones pertaining to the axial and transverse deformation of a substitute beam with unit axial and bending stiffness, respectively, under fictitious time dependent load distributions. Besides its accuracy, this method overcomes the shortcoming of a FEM solution, which experiences lack of convergence [16] and depends on discretization. Example problems of uniform Beck' column and Beck' column with linearly varying height are presented, which illustrate the effectiveness of the employed method to handle this problem. Moreover, useful conclusions are drawn concerning the influence of the axial deformation. Thus, in case of uniform Beck' column this influence is negligible, while in case of Beck's column with linearly varying height is dominant and should be always included in the analysis.

2. Governing equations

Consider an initially straight beam of length l of viscoelastic material vibrating in a viscous medium. The beam has variable cross-section $A = A(x)$ and moment of inertia $I = I(x)$. The x axis coincides with the neutral axis of the beam, which is bent in its plane of symmetry xz under the action of a tangential compressive follower tip load P (see Fig. 1a). We assume that there is no abrupt variation in cross-section of the beam so that the *Euler-Bernoulli* theory is valid [19]. In the following the equations are derived (a) for nonlinear analysis including the axial deformation, (b) for nonlinear analysis excluding the axial deformation and (c) for linear analysis.

2.1 Nonlinear theory including axial deformation

Moderate large deflections are considered. In this case the nonlinear kinematic relation retains the square of the slope of the deflection, while the strain component remains still small compared with the unity. Thus we have

$$\varepsilon_x(x, z) = u_{,x} + \frac{1}{2} w_{,x}^2 + z\kappa \quad (1)$$

where $u = u(x, t)$ and $w = w(x, t)$ are displacements along the x and z axis, respectively, and κ is the curvature of the deflected axis given as

$$\kappa = -\frac{w_{,xx}}{(1 + w_{,x}^2)^{3/2}} \quad (2)$$

The equations of motion are derived by considering the equilibrium of the deformed element. Thus, referring to Fig. 1b and taking into account the inertia and external damping forces we obtain

$$-m\ddot{u} - c\dot{u} + (N \cos \theta - Q \sin \theta)_{,x} = 0 \quad (3)$$

$$-m\ddot{w} - c\dot{w} + (N \sin \theta + Q \cos \theta)_{,x} = 0 \quad (4)$$

$$M_{,x} = Q \quad (5)$$

where $m = m(x) = \rho A(x)$ being the mass density per unit length and c is the coefficient of the external viscous damping. For the case of moderate large deflections the following relations are valid [18]

$$ds = dx \quad (6)$$

$$\cos \theta \simeq 1, \quad \sin \theta \simeq w_{,x} \simeq \theta \quad (7a,b)$$

$$\kappa = -w_{,xx} \quad (8)$$

The stress resultants are evaluated by integrating appropriately the normal stress

$$\sigma = E\varepsilon + E^* \frac{\partial \varepsilon}{\partial t} = \left(E + E^* \frac{\partial}{\partial t} \right) \varepsilon \quad (9)$$

Thus, the axial force and the bending moment are obtained as

$$\begin{aligned} N &= \left(EA + E^* A \frac{\partial}{\partial t} \right) \left(u_{,x} + \frac{1}{2} w_{,x}^2 \right) \\ &= EA \left(u_{,x} + \frac{1}{2} w_{,x}^2 \right) + E^* A \left(\dot{u}_{,x} + 2w_{,x} \dot{w}_{,x} \right) \end{aligned} \quad (10)$$

$$\begin{aligned} M &= - \left(EI + E^* I \frac{\partial}{\partial t} \right) w_{,xx} \\ &= -EI w_{,xx} - E^* I \dot{w}_{,xx} \end{aligned} \quad (11)$$

where E^* is the coefficient of dynamic visco-elastic resistance [20].

Substituting Eqs. (7) into Eqs. (3) and (4) and using Eq. (5) to eliminate Q , we obtain equations of motion in the form

$$-m\ddot{u} - c\dot{u} + N_{,x} - (M_{,x} w_{,x})_{,x} = 0 \quad (12)$$

$$-m\ddot{w} - c\dot{w} + M_{,xx} + (Nw_{,x})_{,x} = 0 \quad (13)$$

which by virtue of Eqs. (10) and (11) become

$$\begin{aligned} -m\ddot{u} - c\dot{u} + \left[EA \left(u_{,x} + \frac{1}{2} w_{,x}^2 \right) + E^* A \left(\dot{u}_{,x} + 2w_{,x} \dot{w}_{,x} \right) \right]_{,x} \\ + \left[(EI w_{,xx} + E^* I \dot{w}_{,xx})_{,x} w_{,x} \right]_{,x} = 0 \end{aligned} \quad (14)$$

$$\begin{aligned}
& -m\ddot{w} - c\dot{w} - (EIw_{,xx} + E^*I\dot{w}_{,xx})_{,xx} + \\
& \left\{ \left[EA \left(u_{,x} + \frac{1}{2} w_{,x}^2 \right) + E^*A \left(\dot{u}_{,x} + 2w_{,x} \dot{w}_{,x} \right) \right] w_{,x} \right\}_{,x} = 0
\end{aligned} \tag{15}$$

The pertinent boundary conditions are

$$u(0) = 0 \quad \text{and} \quad N(l) = -P = \left(EA + E^*A \frac{\partial}{\partial t} \right) \left[u_{,x} + \frac{1}{2} w_{,x}^2 \right] \Big|_{x=l} \tag{16a,b}$$

$$w(0) = 0 \quad \text{and} \quad Q(l) = 0 = - \left[\left(EI + E^*I \frac{\partial}{\partial t} \right) w_{,xx} \right]_{,x} \Big|_{x=l} \tag{17a,b}$$

$$w_{,x}(0) = 0 \quad \text{and} \quad M(l) = 0 = - \left(EI + E^*I \frac{\partial}{\partial t} \right) w_{,xx} \Big|_{x=l} \tag{18a,b}$$

and the initial conditions are

$$u(x,0) = \tilde{u}(x), \quad \dot{u}(x,0) = \dot{\tilde{u}}(x) \tag{19a,b}$$

$$w(x,0) = \tilde{w}(x), \quad \dot{w}(x,0) = \dot{\tilde{w}}(x) \tag{20a,b}$$

where $\tilde{u}(x), \dot{\tilde{u}}(x), \tilde{w}(x), \dot{\tilde{w}}(x)$ are prescribed spatial functions.

Without restricting the generality in our analysis we neglect the axial inertia and damping forces, whose influence will be the subject of further investigation. Thus the equations of motion are reduced to

$$\left[EA \left(u_{,x} + \frac{1}{2} w_{,x}^2 \right) \right]_{,x} + \left[(EIw_{,xx})_{,x} w_{,x} \right]_{,x} = 0 \tag{21}$$

$$-m\ddot{w} - c\dot{w} - (EIw_{,xx} + E^*I\dot{w}_{,xx})_{,xx} + \left\{ \left[EA \left(u_{,x} + \frac{1}{2} w_{,x}^2 \right) \right] w_{,x} \right\}_{,x} = 0 \tag{22}$$

Moreover, the boundary conditions after dropping the time dependent terms [4, 5, 21] become

$$u(0) = 0 \quad \text{and} \quad EA \left(u_{,x} + \frac{1}{2} w_{,x}^2 \right) \Big|_{x=l} = -P \tag{23a,b}$$

$$w(0) = 0 \quad \text{and} \quad - (EIw_{,xx})_{,x} \Big|_{x=l} = 0 \tag{24a,b}$$

$$w_{,x}(0) = 0 \quad \text{and} \quad -EIw_{,xx} \Big|_{x=l} = 0 \tag{25a,b}$$

while the initial conditions are limited only to Eqs. (20).

2.2 Nonlinear theory excluding axial deformation

We start with Eq. (12), in which we drop the axial inertia and damping forces. This results in equation

$$N_{,x} - (M_{,x} w_{,x})_{,x} = 0 \quad (26)$$

which can be readily integrated independently to yield

$$N = M_{,x} w_{,x} + c \quad (27)$$

Then using the boundary conditions (16b) and (18b) for the beam end $x = l$, we obtain

$$c = -P \quad (28)$$

Thus, the axial force is given as

$$N(x) = -P - M_{,x} w_{,x} \quad (29)$$

Introducing the bending moment M from Eq. (11) without the contribution of the time dependent term into Eq. (29), we obtain

$$N(x) = -P - (EIw_{,xx})_{,x} w_{,x} \quad (30)$$

which is substituted into Eq. (13) to yield the counterpart of Eq. (15), when the axial deformation is neglected. Thus we obtain the nonlinear equation of motion excluding the axial deformation as

$$m\ddot{w} + c\dot{w} + (EIw_{,xx} + E^* I\dot{w}_{,xx})_{,xx} + Pw_{,xx} + [(EIw_{,xx})_{,x} w_{,x}^2]_{,x} = 0 \quad (31)$$

The boundary conditions (24) and (25) hold also in this case together with the initial conditions (20).

2.3 Linear theory

In this case Eqs. (30) and (31) are simplified to [22]

$$N(x) = -P \quad (32)$$

$$m\ddot{w} + c\dot{w} + (EIw_{,xx} + E^*I\dot{w}_{,xx})_{,xx} + Pw_{,xx} = 0 \quad (33)$$

under the boundary conditions (24) and (25) and the initial conditions (20).

3. The AEM solution for the nonlinear dynamic analysis of Beck's column

Eqs. (21) and (22) are solved using the AEM, which for the problem at hand is applied as follows. Let $u = u(x, t)$ and $w = w(x, t)$ be the sought solutions, which are two and four times differentiable in $(0, l)$, respectively. Noting that Eqs. (21) and (22) are of the second order with respect to u and of fourth order with respect to w , respectively, we obtain by differentiating

$$u_{,xx} = b_1(x, t) \quad (34)$$

$$w_{,xxxx} = b_2(x, t) \quad (35)$$

Eqs. (34) and (35) describe the axial and bending linear response of a beam with constant unit axial and flexural stiffness subjected to the fictitious time dependent axial b_1 , and transverse b_2 , respectively. They indicate that the solution of Eqs. (21) and (22) can be established by solving Eqs. (34) and (35) under the boundary conditions (23)-(25), provided that the fictitious load distributions b_1 , b_2 are first determined. Eqs. (34) and (35) are quasi-static, that is the time is considered as a parameter. Note that Eqs. (34) and (35) are referred to as the analog equations to Eqs. (21) and (22). The fictitious loads are established by developing a procedure based on the integral equation method for one-dimensional problems. Thus, the integral representations of the solutions of Eqs. (34) and (35) are written as

$$u(x, t) = c_1x + c_2 + \int_0^l G_1(x, \xi) b_1(\xi, t) d\xi \quad (36)$$

$$w(x, t) = c_3x^3 + c_4x^2 + c_5x + c_6 + \int_0^l G_2(x, \xi) b_2(\xi, t) d\xi \quad (37)$$

where $c_i = c_i(t)$ ($i = 1, 2, \dots, 6$) are arbitrary time dependent integration constants to be determined from the boundary conditions and

$$G_1 = \frac{1}{2}|x - \xi| \quad (38)$$

$$G_2 = \frac{1}{12}|x - \xi|(x - \xi)^2 \quad (39)$$

are the fundamental solutions (free space Green's functions) of Eqs. (34) and (35), respectively.

The derivatives of u and w are obtained by direct differentiation of Eqs. (36) and (37).

This yields

$$u_{,x}(x, t) = c_1 + \int_0^l G_{1,x}(x, \xi) b_1(\xi, t) d\xi, \quad u_{,xx}(x, t) = b_1(x, t) \quad (40a,b)$$

$$w_{,x}(x, t) = 3c_3x^2 + 2c_4x + c_5 + \int_0^l G_{2,x}(x, \xi) b_2(\xi, t) d\xi \quad (41a)$$

$$w_{,xx}(x, t) = 6c_3x + 2c_4 + \int_0^l G_{2,xx}(x, \xi) b_2(\xi, t) d\xi \quad (41b)$$

$$w_{,xxx}(x, t) = 6c_3 + \int_0^l G_{2,xxx}(x, \xi) b_2(\xi, t) d\xi, \quad w_{,xxxx}(x, t) = b_2(x, t) \quad (41c,d)$$

Substituting the above derivatives into Eqs. (21) and (22) yields the equations, from which the fictitious sources b_1 and b_2 can be determined. This can be implemented only numerically as follows.

The interval $(0, l)$ is divided into N equal elements (see Fig. 2) on which b_1 and b_2 are assumed to vary according to a certain law (constant, linear, parabolic etc). The constant element assumption is employed here, because the numerical implementation becomes very simple and the obtained numerical results are very good.

After discretization of Eqs. (36) and (37) we obtain

$$u(x, t) = \sum_{j=1}^2 x^{2-j} c_j(t) + b_1(\xi_j) \sum_{j=1}^N \int_j G_1(x, \xi) d\xi \quad (42)$$

$$w(x, t) = \sum_{j=1}^4 x^{4-j} c_{j+2}(t) + b_2(\xi_j) \sum_{j=1}^N \int_j G_2(x, \xi) d\xi \quad (43)$$

or

$$u(x, t) = \mathbf{H}_1(x)\mathbf{c}_1 + \mathbf{G}_1(x)\mathbf{b}_1 \quad (44)$$

$$w(x, t) = \mathbf{H}_2(x)\mathbf{c}_2 + \mathbf{G}_2(x)\mathbf{b}_2 \quad (45)$$

where $\mathbf{G}_1(x)$ and $\mathbf{G}_2(x)$ are $1 \times N$ known matrices originating from the integration of the kernels $G_1(x, \xi)$ and $G_2(x, \xi)$ on the elements, respectively; $\mathbf{H}_1(x) = \begin{bmatrix} x & 1 \end{bmatrix}$ and $\mathbf{H}_2(x) = \begin{bmatrix} x^3 & x^2 & x & 1 \end{bmatrix}$; $\mathbf{c}_1 = \{c_1, c_2\}^T$; $\mathbf{c}_2 = \{c_3, c_4, c_5, c_6\}^T$; \mathbf{b}_1 , \mathbf{b}_2 are the vectors containing the values of the fictitious loads at the nodal points, respectively. Similarly, we obtain for Eqs. (40) and (41)

$$u_{,x}(x, t) = \mathbf{H}_{1x}(x)\mathbf{c}_1 + \mathbf{G}_{1x}(x)\mathbf{b}_1, \quad u_{,xx}(x, t) = \mathbf{b}_1 \quad (46a,b)$$

$$w_{,x}(x, t) = \mathbf{H}_{2x}(x)\mathbf{c}_2 + \mathbf{G}_{2x}(x)\mathbf{b}_2, \quad w_{,xx}(x, t) = \mathbf{H}_{2xx}(x)\mathbf{c}_2 + \mathbf{G}_{2xx}(x)\mathbf{b}_2 \quad (47a,b)$$

$$w_{,xxx}(x, t) = \mathbf{H}_{2xxx}(x)\mathbf{c}_2 + \mathbf{G}_{2xxx}(x)\mathbf{b}_2, \quad w_{,xxxx}(x, t) = \mathbf{b}_2 \quad (47c,d)$$

where $\mathbf{G}_{1x}(x)$, $\mathbf{G}_{2xx}(x)$, ..., $\mathbf{G}_{2xxx}(x)$ are $1 \times N$ known matrices, originating from the integration of the derivatives of the kernels $G_1(x, \xi)$, $G_2(x, \xi)$ on the elements; $\mathbf{H}_{1x}(x)$ is a 1×2 known matrix resulting from the differentiation of $\mathbf{H}_1(x)$, whereas $\mathbf{H}_{2x}(x)$, $\mathbf{H}_{2xx}(x)$, $\mathbf{H}_{2xxx}(x)$ are 1×4 known matrices resulting from the differentiation of $\mathbf{H}_2(x)$.

Finally, collocating Eqs. (21) and (22) at the N nodal points and substituting the relevant derivatives from Eqs. (46) and (47) yields the following equations of motion

$$\mathbf{K}_1(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}_1, \mathbf{c}_2) = \mathbf{0}, \quad (48)$$

$$\mathbf{M}\ddot{\mathbf{b}}_2 + \mathbf{C}\dot{\mathbf{b}}_2 - \mathbf{K}_2(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}_1, \mathbf{c}_2) = \mathbf{0}, \quad (49)$$

where \mathbf{M} and \mathbf{C} are known $N \times N$ generalized mass and damping matrices and $\mathbf{K}_i(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}_1, \mathbf{c}_2)$ ($i = 1, 2$) generalized stiffness vectors. The boundary conditions (23)-(25) are in general nonlinear and can be written as

$$\mathbf{f}(\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}_1, \mathbf{c}_2) = \mathbf{0} \quad (50)$$

Eq. (49) is the semi-discretized equation of motion of the Beck's column. The associated initial conditions result from Eq. (45) when applied to all nodal points and combined with Eqs. (20). Thus, we have

$$\mathbf{b}_2(0) = \mathbf{G}_2^{-1}(\tilde{\mathbf{w}} - \mathbf{H}_2\mathbf{c}_2) \quad (51)$$

$$\dot{\mathbf{b}}_2(0) = \mathbf{G}_2^{-1}\dot{\tilde{\mathbf{w}}} \quad (52)$$

The time step integration method for nonlinear equations of motion can be employed to solve Eq. (49). In each iteration for \mathbf{b}_2 within a time step, the current value of \mathbf{b}_2 is utilized to update the vectors \mathbf{b}_1 and $\mathbf{c}_1, \mathbf{c}_2$ on the basis of Eqs. (48) and (50). This demands the solution of a nonlinear system of algebraic equations, which is performed using the modified Newton-Raphson method. In this paper, the average acceleration time step integration method was employed to solve Eq. (49) and the results were cross-checked by a time step integration method based on the analog equation method [23]. Once the vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}_1, \mathbf{c}_2$ are computed the displacements $u = u(x, t)$ and $w = w(x, t)$ and their derivatives at any instant t are evaluated from Eqs. (44) through (47).

4. Numerical examples

On the base of the procedure described in previous section a FORTRAN program has been written for establishing the dynamic nonlinear response of damped, both externally and internally, beam columns with variable mass and stiffness properties subjected to follower

forces. The uniform and nonuniform Beck's column has been studied as an illustrative example of the developed analysis and solution method.

Example 1: Uniform Beck's column

The nonlinear dynamic stability of Beck's column with uniform rectangular cross-section $b \times h_0$ and length $l = 1.0$ m has been studied. The employed data are: $E = 3.556 \times 10^5$ kN/m², $b = 0.01$ m, $h_0 = 0.15$ m, $\rho = 666.667$ kNm⁻¹ sec², $c = 0.1$ and $E^* = 0.01E$. The employed initial conditions are $\tilde{w}(x) = 8w_l(\xi^4 - 4\xi^3 + 6\xi^2)/24$ and $\dot{\tilde{w}}(x) = 0$ with $\xi = x/l$ and w_l is the initial tip deflection of the beam. In Table 1 the computed critical loads are presented from (i) linear theory, (ii) nonlinear theory excluding the axial deformation and (iii) nonlinear theory including the axial deformation. The convergence of the method is shown by increasing the number of elements N . In all three theories, the computed values of the critical load are identical and coincide with those obtained by Andersen and Thomsen [16], who used a perturbation analysis of the nonlinear equations of motion ignoring, however, the axial deformation of the beam. The FEM solution using geometric nonlinear dynamic finite element model for case (ii) gives results which are qualitative and quantitative the same, which, however, exhibit lack of convergence in some cases [16].

In Table 2 the finite tip amplitude is presented from the nonlinear theory (i) excluding and (ii) including the axial deformation for various values of the axial load P . The two nonlinear theories give the same results. However, as compared with those obtained in [16] only the case for $P = 13.53$ are found in good agreement. From the obtained results, it becomes apparent that the influence of the axial deformation on the critical load is negligible for uniform Beck's column. This is an interesting finding.

Moreover, the time histories of the tip deflection for two different initial tip deflections w_t are shown in Fig. 3 validating the supercritical bifurcation character of uniform Beck's column, while in Fig. 4 and Fig. 5 the variation of the critical load in regard to the external and internal damping is depicted, respectively. From Fig. 4, it can be pointed out that the critical load increases monotonically with the coefficient of the external viscous damping. On the contrary, the curve of Fig. 5 shows that the value of the critical load decreases to a minimum value ($P_{cr} = 12.03$ at $E^* = 0.03E$) and thereafter it increases with increasing value of the coefficient of dynamic visco-elastic resistance (internal damping).

Example 2: Beck's column with variable mass and stiffness properties

The nonlinear dynamic stability of Beck's column with variable cross section has been studied. The employed data are the same with those in previous example. The height of the beam varies according to the linear law $h(x) = h_0 + a(x - l/2)$ (see Fig. 6) with $a = 2 \tan \phi$ being the taper ratio and h_0 the height at the half length. In order to compare the results with those of the previous example, the volume of the beam i.e. $V = bh_0l$, was kept constant. The resulting beam should have no abrupt change of the cross-section so that the *Euler-Bernoulli* theory remains valid [19]. Boley [24] has shown that, for a beam with unit constant width, a rate of change of the cross-section $|a| \simeq 0.35$ yields an error of 7.5%, while for $|a| \simeq 0.17$ the error is 1.8%. This was also verified by the authors who treated the beam as a 2D elasticity problem and used the BEM to obtain the solution [25].

The computed critical loads from (i) linear theory, (ii) nonlinear theory excluding and (iii) including the axial deformation for various values of the ratio a are shown in Table 3. The results obtained on the basis of the first two theories are coincident but differ considerably from those of the third one. This significant finding demands that the axial deformation on the nonlinear dynamic stability of Beck's column should be always included

in the analysis when the beam has variable mass and stiffness properties. It should be also observed in the nonlinear theory (iii) that the critical load increases monotonically with the taper ratio, which means that the material must be shifted towards the free end in order to obtain the maximum critical load.

Moreover, in Fig. 7 are shown the bifurcation diagrams from nonlinear theory (iii) for various values of the taper ratio. The shape of the bifurcation branches becomes steeper as the taper ratio decreases. For this reason the case of taper ratio $a = -0.15$ is investigated for possible subcritical bifurcation. Fig. 8 and Fig. 9 show the variation of the critical load in regard to the external and internal damping, respectively. It is apparent from these figures that the same qualitatively conclusions with those of uniform Beck's column can be drawn. Namely, the critical load increases monotonically with the coefficient of the external viscous damping, while it is decreasing to a minimum value ($P_{cr} = 5.16$ at $E^* = 0.05E$) and thereafter increases for further increase of the internal damping. Finally, in order to establish the bifurcation character, two nonlinear dynamic analysis are performed, one below ($P = 5.00$) and another above ($P = 5.20$) the critical load. The time histories of the tip deflection are shown in Fig. 10 validating the supercritical bifurcation character of nonuniform Beck' column.

5. Conclusions

In this paper the nonlinear dynamic stability of Beck's column with variable mass and stiffness properties in the presence of damping (both internal and external) has been investigated using a complete nonlinear dynamic analysis which includes the axial deformation. The solution of the derived coupled nonlinear equations of motion was achieved effectively using the analog equation method. This investigation has reached to certain striking effects concerning the influence of the axial deformation on the critical load of Beck's column with variable cross section. The main conclusions can be summarized as

- The bifurcation is always supercritical in uniform and nonuniform Beck's column.
- The axial deformation affects considerably the critical load, when the beam has variable cross section. It may give remarkably lower or larger critical loads. Therefore, it should be always included in the analysis. On the contrary, it has negligible influence for uniform cross section.
- The exclusion of the axial deformation in the nonlinear dynamic analysis of Beck's column with variable mass and stiffness properties may yield larger loads, reducing thus the safety of the structure.
- The critical load increases monotonically in uniform and nonuniform Beck's column with the coefficient of the external viscous damping.
- The critical load decreases to a minimum value and thereafter increases for further increase of the value of the coefficient of dynamic visco-elastic resistance (internal damping) in uniform and nonuniform Beck's column.
- In the nonlinear theory including the axial deformation, the critical load increases monotonically with the taper ratio. This suggests shifting of the material towards the free end in order to maximize the critical load
- The shape of the bifurcation branches becomes steeper as the taper ratio decreases.

References

- [1] R.H. Plaut, E.F. Infante, The effect of external damping on the stability of Beck's column, *Int. J. Solids Struct.* 6 (1970) 491-496.
- [2] V.V. Bolotin, *Nonconservative Problems of the Theory of Elastic Stability* (Moscow, English translation), Pergamon Press Ltd., Oxford, 1963.
- [3] G. Herrmann, I.C. Jong, On the destabilizing effect of damping in nonconservative elastic systems, *J. Appl. Mech.* 32 (1965) 592-597.

- [4] R.C. Kar, Stability of a nonuniform cantilever subjected to dissipative and nonconservative force, *Comput. Struct.* 11 (1980) 175-180.
- [5] B.N. Rao, G.V. Rao, Stability of tapered cantilever columns subjected to a tip-concentrated follower force with or without damping, *Comput. Struct.* 37 (1990) 333-342.
- [6] L. Gaul, The influence of damping on waves and vibrations, *Mechanical Systems and Signal Processing*, 13 (1991) 1-30.
- [7] A.N. Kounadis, Hamiltonian weakly damped autonomous systems exhibiting periodic attractors, *Z. angew. Math. Phys.* 57 (2006) 324-350.
- [8] R.V. Vitaliani, A.M. Gasparini, A.V. Saetta, Finite element solution of the stability problem for nonlinear undamped and damped systems under nonconservative loading, *Int. J. Solids Struct.* 34 (1997) 2497-2516.
- [9] A.N. Kounadis, On the paradox of the destabilizing effect of damping in non-conservative systems, *Int. J. Non-Linear Mech.* 27 (1992) 597-609.
- [10] A.N. Kounadis, Non-potential dissipative systems exhibiting periodic attractors in region of divergence, *Chaos, Solitons & Fractals*, 8 (1997) 583-612.
- [11] V.V. Bolotin, A.A. Grishko, M.Yu. Panov, Effect of damping on the postcritical behaviour of autonomous non-conservative systems, *Int. J. Non-Linear Mech.* 37 (2002) 1163-1179.
- [12] A.N. Kounadis, Some new instability aspects for nonconservative systems under follower loads, *Int. J. Mech. Sci.* 33 (1991) 297-311.
- [13] A.N. Kounadis, On the failure of static stability analyses of nonconservative systems in regions of divergence instability, *Int. J. Solids Struct.* 31 (1994) 2099-2120.
- [14] R.W. Kolkka, On the non-linear Beck's problem with external damping, *Int. J. Non-Linear Mech.* 14 (1984) 497-505.

- [15] M. Chen, Hopf bifurcation in Beck's problem, *Nonlinear Anal. Theory, Methods Appl.* 11 (1987) 1061-1073.
- [16] S.B. Andersen, J.J. Thomsen, Post-critical behaviour of Beck's column with a tip mass, *Int. J. Non-Linear Mech.* 37 (2002) 135-151.
- [17] J.T. Katsikadelis, The analog equation method. A boundary-only integral equation method for nonlinear static and dynamic problems in general bodies, *Theor. Appl. Mech.* 27 (2002) 13-38.
- [18] J.T. Katsikadelis, G.C. Tsiatas, Non-linear dynamic analysis of beams with variable stiffness, *J. Sound Vib.* 270 (2004) 847-863.
- [19] J.T. Katsikadelis, G.C. Tsiatas, Buckling load optimization of beams, *Arch. Appl. Mech.* 74 (2005) 790-799.
- [20] J.A. Hudson, *The Excitation and Propagation of Elastic Waves*, Cambridge University Press, Cambridge, 1980.
- [21] J.L. CLaudon, Détermination et maximisation de la charge critique d' une colonne de Hauger en présence d' amortissement, *J. Appl. Math. Physic.* 29 (1978) 226-236.
- [22] M. Beck, Knicklast des einseitig eingespannten, tangential gedrückten Stabes, *Zeitschrift für Angewandte Mathematik und Physik* 3 (1952) 225-228.
- [23] J.T. Katsikadelis, A new time step integration scheme for structural dynamics based on the analog equation method, in *Collection of papers dedicated to Prof. P.S. Theocaris*, National Technical University of Athens (1994) 80-100.
- [24] B.A. Boley, On the accuracy of the Bernoulli-Euler theory for beams of variable section, *J. Appl. Mech. ASME* 30 (1963) 373-378.
- [25] J.T. Katsikadelis, *Boundary Elements: Theory and Applications*. Amsterdam-London: Elsevier 2002.

Table 1. Example 1: Critical load of Beck's column ($c = 0.1$, $E^* = 0.01E$) from (i) linear theory, (ii) nonlinear theory excluding and (iii) nonlinear theory including the axial deformation for various values of the boundary elements N .

	$N = 20$	$N = 25$	$N = 30$	$N = 35$	$N = 65$	[16]
(i)	12.96	12.95	12.94	12.94	12.93	12.93
(ii)	12.96	12.95	12.94	12.94	12.93	12.93
(iii)	12.96	12.95	12.94	12.94	12.93	-

Table 2. Example 1: Finite tip amplitude of Beck's column ($c = 0.1$, $E^* = 0.01E$) from nonlinear theory (i) excluding and (ii) including the axial deformation for various values of boundary elements N .

P		$N = 20$	$N = 25$	$N = 30$	$N = 35$	[16]
13.18	(i)	0.155	0.158	0.160	0.161	0.208
	(ii)	0.153	0.157	0.159	0.160	
13.33	(i)	0.209	0.212	0.214	0.215	0.254
	(ii)	0.208	0.211	0.213	0.214	
13.53	(i)	0.288	0.293	0.296	0.298	0.310
	(ii)	0.289	0.294	0.296	0.298	

Table 3. Example 2: Critical load of Beck's column with linearly varying height ($c = 0.1$, $E^* = 0.01E$) from (i) linear theory, (ii) nonlinear theory excluding and (iii) nonlinear theory including the axial deformation for various values of the taper ratio a ($N = 30$).

a	(i)	(ii)	(iii)
-0.15	10.11	10.10	5.96
-0.10	11.72	11.71	8.44
-0.05	12.66	12.65	10.82
-0.01	12.94	12.93	12.54
0.00	12.94	12.94	12.94
0.01	12.92	12.92	13.33
0.05	12.59	12.58	14.68
0.10	11.61	11.60	15.85
0.15	10.04	10.03	16.29

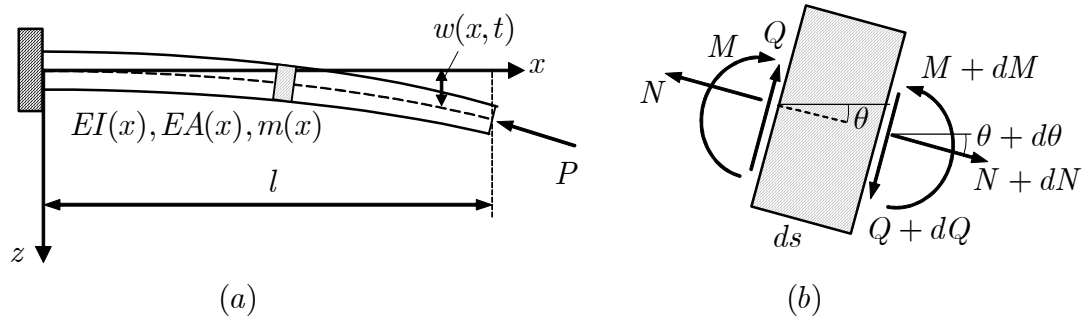


Fig. 1. (a) Beck' column with variable mass and stiffness properties and (b) forces and moments acting on the deformed element.

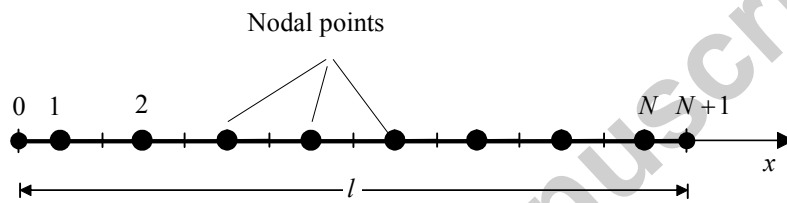


Fig. 2. Discretization of the interval and distribution of the nodal points.

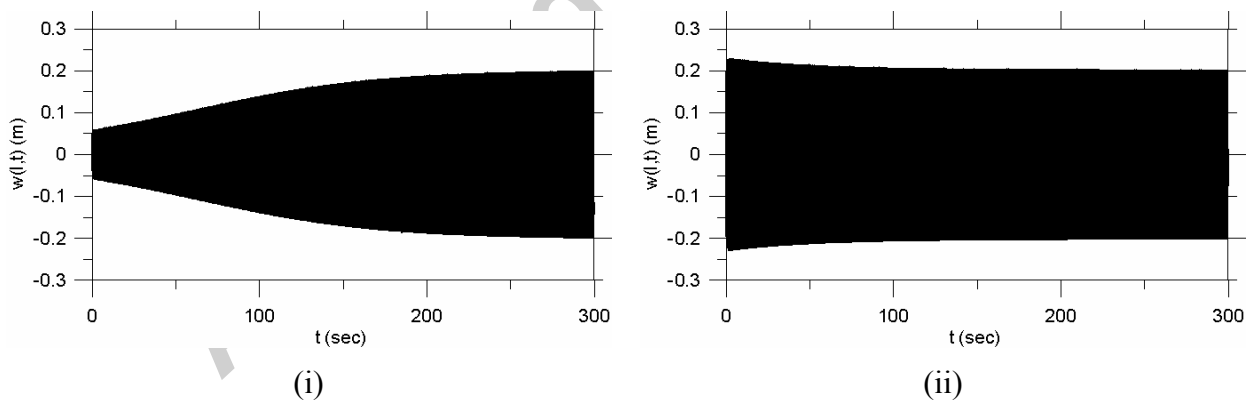


Fig. 3. Example 1: Time history of the tip deflection of uniform Beck' column ($c = 0.1$, $E^* = 0.01E$, $P_{cr} = 12.94$) for $P = 13.33$ (i) $w_l = 0.05$ and (ii) $w_l = 0.20$.

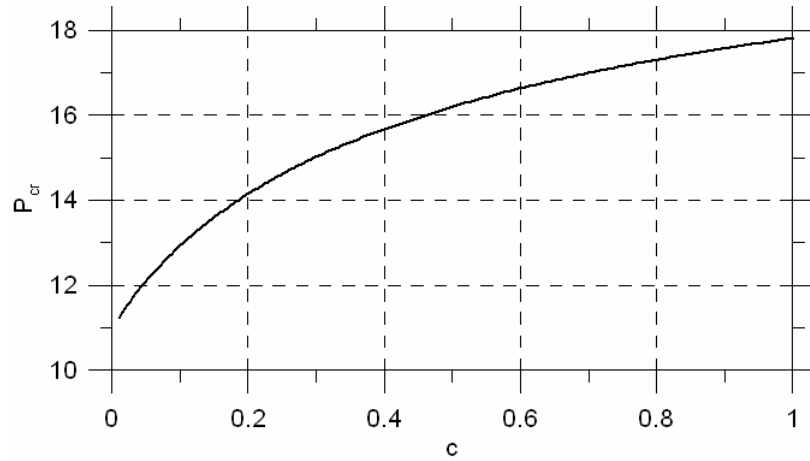


Fig. 4. Example 1: Critical load of uniform Beck' column versus external damping ($E^* = 0.01E$, $N = 30$).

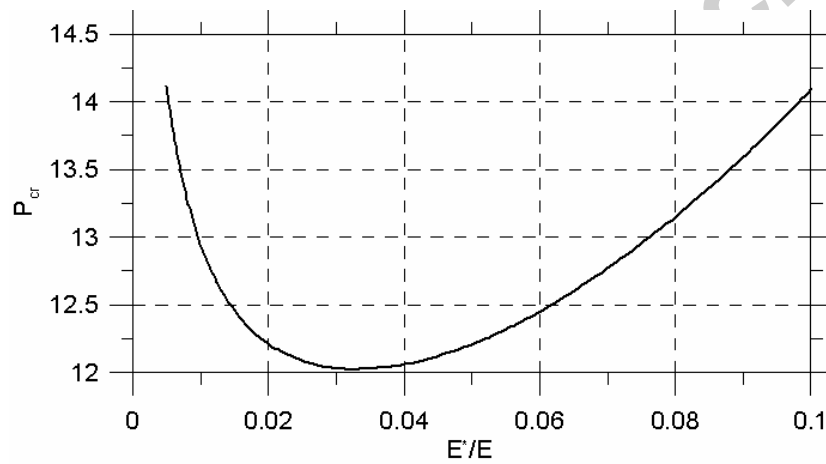


Fig. 5. Example 1: Critical load of uniform Beck' column versus internal damping ($c = 0.1$, $N = 30$).

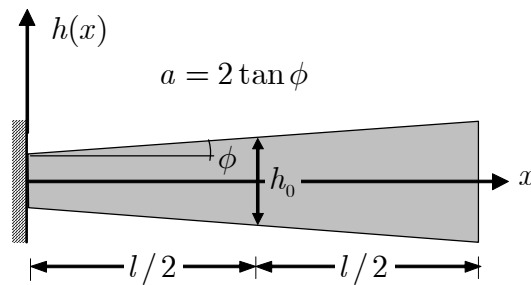


Fig. 6. Linear variation of the height of the Beck' column in Example 2.

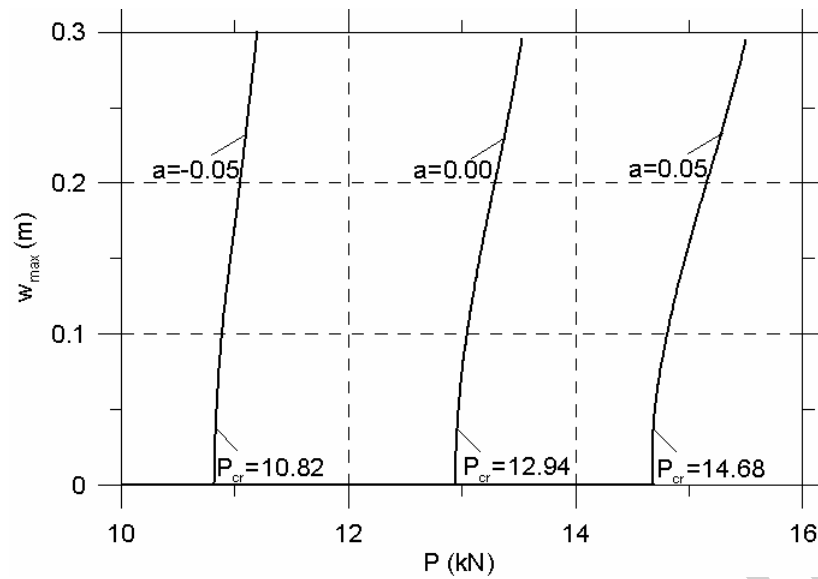


Fig. 7. Example 2: Bifurcation diagrams of Beck's column with linearly varying height from nonlinear theory including the axial deformation for various values of the taper ratio a ($c = 0.1$, $E^* = 0.01E$, $N = 30$).

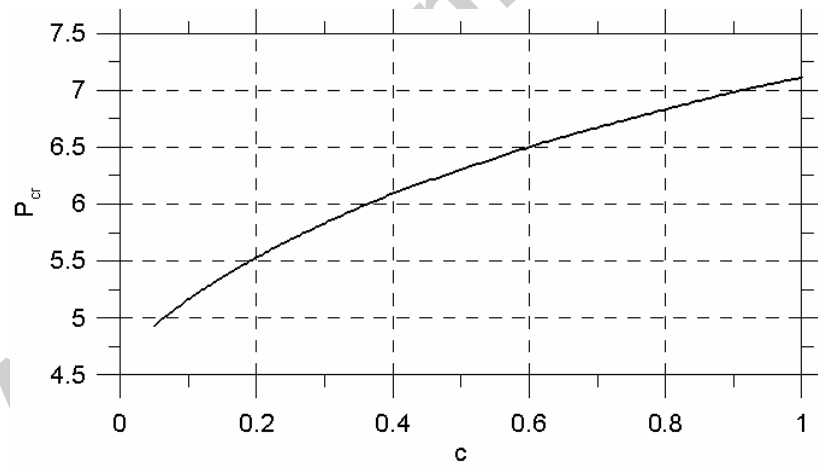


Fig. 8. Example 2: Critical load of Beck's column with linearly varying height ($a = -0.15$) versus external damping ($E^* = 0.05E$, $N = 30$).

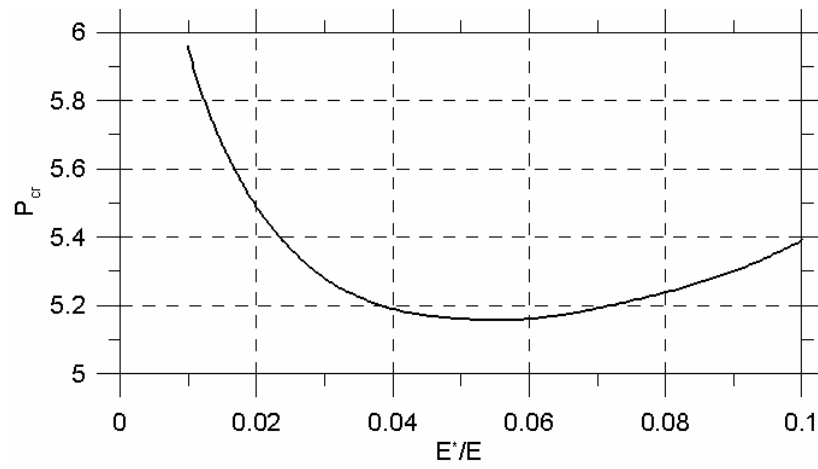


Fig. 9. Example 2: Critical load of Beck' column with linearly varying height ($a = -0.15$) versus internal damping ($c = 0.1$, $N = 30$).

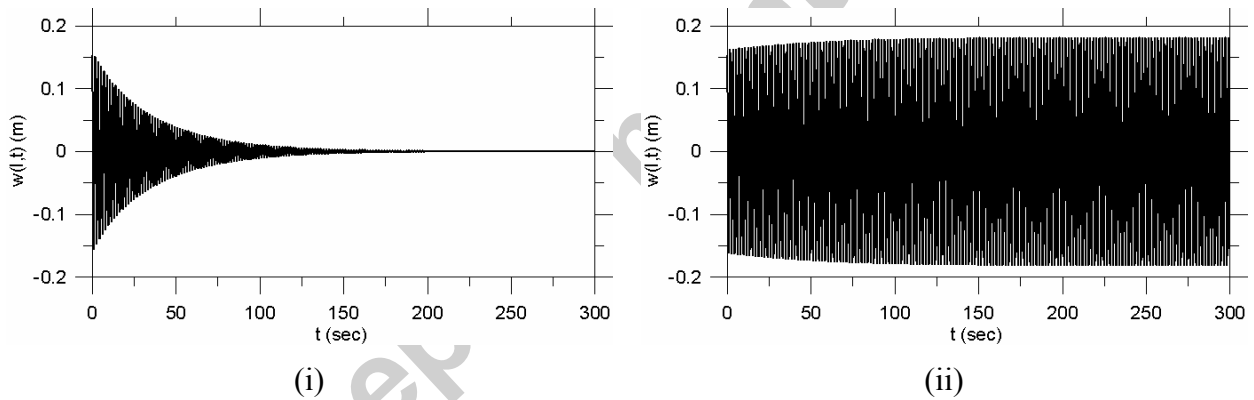


Fig. 10. Example 2: Time history of the tip deflection of Beck' column with linearly varying height ($c = 0.1$, $E^* = 0.05E$, $a = -0.15$, $P_{cr} = 5.16$, $w_l = 0.15$) for (i) $P = 5.00$ and (ii) $P = 5.20$.