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Normalized causal and well-balanced multivariate fractional Brownian motion

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Abstract

This paper is devoted to study some properties of an extension of the well-known fractional Brownian motion to the multivariate case. Following recent works from Lavancier *et. al.*, we study the covariance structure of the multivariate fractional Gaussian noise. We evaluate several parameters of the model that allow to control the correlation structure at lag zero between all the components of the multivariate process. We particularly focus on two cases for which we can relate characteristic parameters of the covariance function to parameters of the stochastic representation of the processes. These cases are the causal case, a direct multivariate generalization of Mandelbrot&Van Ness representation, and the well-balanced case which adds to the previous case an anti-causal filtering of a Brownian motion. The characterization of the covariance function is then used to study the multivariate fractional Gaussian noise, defined as the increment process of the multivariate fractional Brownian motion. We study the covariance structure as well as the spectral structure of this multivariate stationary process. We exhibit the intriguing facts that two fractional Gaussian noise may be long-range interdependent when only one is long-range dependent. We then perform a wavelet analysis of the multivariate fractional Brownian motion, and show that the wavelet analysis may destroy the long-range interdependence if the wavelet is properly chosen.

1 Introduction

Long-range dependence or memory is the accepted term to design long-range correlations in time series. It is defined as the non integrability of the correlation function due to a very slow decay at infinite lags. This slow decay is usually modeled as a power law $\tau^{\alpha-1}$ with an exponent α lower than one for long-range dependence. In the frequency domain, this corresponds to the divergence of the power spectrum at small frequencies, again with a power law $1/f^\alpha$. This type

of behavior may lead to dramatic difficulties when it comes to estimate some parameters from the long-range correlated data. Indeed, rates of convergence of usual estimates are much slower than the usual $1/\sqrt{N}$ rate found for classical mixing processes. Furthermore, this property is not rare at all and is found in many different fields.

In functional Magnetic Resonance Imaging (fMRI), measurements taken from one area of the brain are well modeled by a long-range dependent Gaussian process [3]. In order to study the flow of information using fMRI, neuroscientists have access to multiple correlated measurements. In network traffic monitoring, several measurements can be performed such as IP packet or bytes, and it is now well established that corresponding times series are long-range dependent [1]. Other examples may be found in economy, in biology, in physics, ... Thus, there is a need to develop models of multivariate long-range dependent processes. Models of discrete time series have already been studied recently as generalization of FARIMA models [2, 20].

Here, we will concentrate on a continuous time model recently introduced in [10, 16] as a generalization of fractional Brownian motion (fBm) to multivariate fractional Brownian motion (mfBm). The definition given in [10] concerns a wide scope generalization of the fBm where self-similarity becomes an operator self-similarity for the multivariate case. The authors establish stochastic integral representations of operator self-similar multivariate Gaussian processes with stationary increments, and study some of their properties. Lavancier *et al.* in [16] concentrate on the covariance structure of multivariate processes that are jointly self-similar, and possess stationary increments. Joint self-similarity can be viewed as a particular case of operator self-similarity when the operator is diagonal. In the particular case of Gaussian processes, Lavancier and co-workers then link their general findings with the representations established by Didier&Pipiras. Note that the work in these two papers have close connections with the work of Stoev&Taquq [18] which concerns stochastic integral representations of the fBm with a time varying Hurst index.

The aim of the paper is to study further the multivariate fractional Brownian motion in the case of joint self-similarity based on the moving-average stochastic integral representation of [10]. In this case, a mfBm is a zero mean Gaussian process, with stationary increments, almost surely zero at time zero, that satisfies $x(\lambda t) = \lambda^H x(t)$ for all $t \in \mathbb{R}$, all $\lambda > 0$, where H is a diagonal matrix. In section 2 we will review some of its known properties. The process is evidently parameterized by the matrix parameter H but also by two matrices A_- and A_+ that control the correlation structure between the components of the process. The covariance structure is known since the work of Lavancier *et al.* However, the covariance depends on the parameters in such a way that it is difficult to generate sample paths with this covariance function. We thus propose another parameterization of the covariance function in order to ease the synthesis step of mfBm. This is done by linking the parameters (matrices A_+ , A_- , H) to the correlation between the components at times 1. Even if the choice is arbitrary, it allows to control directly the correlation. Furthermore, we will not solve the problem for any matrices A_+ , A_- but we will study two cases called the causal case for which $A_- = 0$ and the well-balanced case for which $A_+ = A_-$. A discussion concerning a more general case will be provided in section 5. The

parameterization we adopt is through the definition of a matrix A which needs to be positive-semidefinite. We give a necessary condition for semidefinite-positivity which can be used to invalidate the model. In section 3 we will concentrate on the increments of the mfBm. We will evaluate the correlation structure as well as the spectral density matrix of the process. In particular, we will exhibit long-range properties in the cross-correlation of different components or equivalently divergence at the zero frequency in the cross-spectral densities. We will also evaluate the coherence function between two components and relate it to the condition of non negativity mentioned above. We will continue the analysis of the process by analysing it through the lens of the wavelet transform. It is well-known now that fractality and long-range dependence may be adequately taken into account by wavelets. Section 4 is devoted to this analysis where we exhibit the ability of the wavelet analysis to reveal fractality (constant relative bandwidth filter bank) and to destroy long-range dependence if the wavelet is correctly chosen (nullity of the first moments). To conclude the paper, we will illustrate the process by depicting some sample paths, by discussing some points concerning more general models and by giving some ideas for further research.

2 Moving-average multivariate fractional Brownian motion

2.1 Model and properties

The fractional Brownian motion, as defined by Mandelbrot&Van Ness [17] is a causal linear transform of a Wiener process, with a kernel that respects self-similarity and which is parameterized by the self-similarity index $H \in (0, 1)$. This transform can be generalized in several ways, including time-varying index and non causal integration [18], or operator self-similarity [10]. Here, we concentrate on particular cases of the latter, and study the multivariate fractional Brownian motion (mfBm) defined *via* an integration of the mixing of independent Wiener processes. This comes after the work of Didier&Pipiras in [10] when we restrict the operators involved to be diagonal matrices. Let $x(t)$ of dimension p be defined as

$$x(t) = \int (k_H(u, t)A_+ + l_H(u, t)A_-)dW(u) \quad (1)$$

where W is a vector of p independent standardized Wiener processes or Brownian motions, A_+ and A_- are $p \times p$ matrices of reals, H is a diagonal matrix of parameters $H_j \in (0, 1), \forall j = 1, \dots, p$, $k_H(u, t)$ is a matrix of kernels that reads $(t - u)_+^{H-1/2} - (-u)_+^{H-1/2}$ and $l_H(u, t)$ reads $(u - t)_+^{H-1/2} - (u)_+^{H-1/2}$. In this notation, $(a)_+ = \max(a, 0)$ and t^H is understood as the exponential of a matrix $\exp(H \log(t))$. The terms $-(\pm u)_+^{H-1/2}$ insure that the mfBm is almost surely zero at time zero. As seen in the stochastic integral representation (1) of the mfBm, $x(t)$ is a multivariate non-stationary Gaussian process with stationary increments. Moreover, the components of $x(t)$ are correlated, and the structure of the correlation is inherited from the

presence of the mixing matrices A_+ and A_- . The correlation structure is sufficient to completely determine the process since it is Gaussian with zero mean (as a linear transform of a zero mean Gaussian process).

2.2 Covariances and cross-covariances

The following analysis relies heavily on the paper of Lavancier *et al.*, [16]. In this paper, the authors exhibit the general structure of the covariance of a zero mean multivariate self-similar process, that is a process that satisfies $x(\lambda t) = \lambda^H x(t)$ (in the sense of finite-dimensional distributions), where H is a diagonal matrix. As a particular case, the covariance of the mfBm is evaluated directly from the integral representation of Didier&Pipiras [10].

Let $r_{jk}(s, t) = E[x_j(s)x_k(t)]$ denote the cross-covariance of the components j and k of x . For the sake of simplicity, let $B_{jk} = B(H_j + .5, H_k + .5)$ where $B(x, y)$ is the beta function. Let $\sigma_j, j = 1, \dots, p$ be positive numbers, and $\rho_{jk}, j = 1, \dots, p, k > j$ be real numbers in $[-1, 1]$. Among all possible models based on (1), our objective is to concentrate ourselves on those allowing us to parameterize the matrices A_+ and A_- only in terms of $\sigma_j = E[x_j(1)^2]$ and $\sigma_j \sigma_k \rho_{jk} = E[x_j(1)x_k(1)]$ (for $j, k = 1, \dots, p$). From proposition 3.1 of Lavancier *et al.*, [16], it consists in finding A_+ and A_- such that

$$\begin{aligned} \sigma_j \sigma_k \rho_{jk} &= \frac{B_{jk}}{\sin(\pi(H_j + H_k))} (\mathcal{A}_{jk} + \mathcal{A}_{kj}) \\ \mathcal{A} &= \cos(\pi H) A_+ A_+^t + A_- A_-^t \cos(\pi H) - \sin(\pi H) A_+ A_-^t \cos(\pi H) - \cos(\pi H) A_+ A_-^t \sin(\pi H) \end{aligned} \quad (2)$$

where $\cos(\pi H)$ and $\sin(\pi H)$ are diagonal matrices, with j th diagonal term defined as $\cos(\pi H_j)$ and $\sin(\pi H_j)$, and where this equation is valid only if $H_j + H_k \neq 1$. In general, equation (3) cannot be solved to determine explicitly A_+ and A_- . In this paper, we mainly focus on two particular cases: the causal case where $A_- = 0$ and the well-balanced case where $A_+ = A_-$. In the causal case, the integral representation is a direct generalization of the integral representation of Mandelbrot&Van Ness to the multivariate case. The well-balanced case by Stoev&Taquu in one dimension [18], corresponds to $A_+ = A_-$. More general cases will be discussed in section 5. Note that in the well-balanced case, the existence of the integral representation is subjected to the restriction $H_i \neq 1/2, \forall i = 1, \dots, p$. This point is further discussed in section 2.3 below.

Theorem 2.1 of [16] states that two different cases have to be considered when evaluating the covariance, namely $H_j + H_k \neq 1$ and $H_j + H_k = 1$. We will show here that these two cases can be merged for the causal and the well-balanced mfBm. Before stating our first result let us define the matrix A :

Causal case $A_- = 0$:

$$A_{jj} = \frac{\sigma_j^2 \sin(\pi H_j)}{B_{jj}} \quad (4)$$

$$A_{jk} = \begin{cases} \frac{\sigma_j \sigma_k \rho_{jk} \sin(\pi(H_j + H_k))}{(\cos(\pi H_j) + \cos(\pi H_k)) B_{jk}} & \text{if } H_j + H_k \neq 1 \\ \frac{2\sigma_j \sigma_k \rho_{jk}}{(\sin(\pi H_j) + \sin(\pi H_k)) B_{jk}} & \text{if } H_j + H_k = 1 \end{cases} \quad (5)$$

Well-balanced case $A_- = A_+$:

$$A_{jj} = \frac{\sigma_j^2 \sin(\pi H_j)}{2(1 - \sin(\pi H_j)) B_{jj}} \quad (6)$$

$$A_{jk} = \begin{cases} \frac{\sigma_j \sigma_k \rho_{jk} \sin(\pi(H_j + H_k))}{2(\cos(\pi H_j) + \cos(\pi H_k) - \sin(\pi(H_j + H_k))) B_{jk}} & \text{if } H_j + H_k \neq 1 \\ \frac{\sigma_j \sigma_k \rho_{jk}}{(\sin(\pi H_j) + \sin(\pi H_k) - 2) B_{jk}} & \text{if } H_j + H_k = 1 \end{cases} \quad (7)$$

Note that the restriction that none Hurst parameters should be equal to $1/2$ in the well-balanced-case appears clearly in the matrix definition, since this case would lead to undefined entries. We can then state the following proposition:

Proposition 1 *if A is positive-semidefinite, the process $x(t)$ defined by*

$$x(t) = \int k_H(u, t) A_+ dW(u) \text{ in the causal case} \quad (8)$$

$$x(t) = \int (k_H(u, t) + l_H(u, t)) A_+ dW(u) \text{ in the well-balanced case} \quad (9)$$

where A_+ is a square root of A , i.e. $A = A_+ A_+^t$, is a vector of p correlated fBm of parameters $H_j, j = 1 \dots, p$ and the parameterization of the matrix A is such that $r_{jj}(1, 1) = E[x_j(1)^2] = \sigma_j^2$ and $r_{jk}(1, 1) = E[x_j(1)x_k(1)] = \sigma_j \sigma_k \rho_{jk}$.

The proof of this proposition is immediate and is a direct use of theorem 2.1 and proposition 3.1 of [16] in the restricted cases considered here. Basically, the covariance is evaluated directly from the integral representation. The diagonal form of the kernel allows an easy evaluation. Note that the matrix A needs to be positive-semidefinite. This will be discussed later.

The covariance matrix of the process can then be parameterized as follows. We introduce the function $w_{jk}(t)$ for $t \in \mathbb{R}$, which is defined as

$$w_{jk}(t) = \begin{cases} c_{kj}(t) |t|^{H_j + H_k} & \text{if } j = k \text{ or } j \neq k \text{ and } H_j + H_k \neq 1 \\ |t| + f_{jk} t \log |t| & \text{if } j \neq k \text{ and } H_j + H_k = 1. \end{cases} \quad (10)$$

where $c_{jk}(t) = c_{jk}\mathbf{1}_{\mathbb{R}^+}(t) + c_{kj}\mathbf{1}_{\mathbb{R}^-}(t)$ and where

Causal case $A_- = 0$:

$$c_{jk} = \frac{2 \cos(\pi H_j)}{\cos(\pi H_j) + \cos(\pi H_k)} \quad (11)$$

$$\begin{aligned} f_{jk} &= \frac{2(H_k - H_j)}{B_{jk}(\sin(\pi H_j) + \sin(\pi H_k))} \\ &= \frac{2}{\pi \tan(\pi H_j)} = -\frac{2}{\pi \tan(\pi H_k)} = -f_{kj}. \end{aligned} \quad (12)$$

Well-balanced case $A_- = A_+$:

$$c_{jk} = 1 \quad (13)$$

$$f_{jk} = 0. \quad (14)$$

Equipped with these definitions, we state the following:

Proposition 2 *For $(j, k) \in \{1, \dots, p\}^2$ and $(H_j, H_k) \in (0, 1)^2$, the covariance between the j th and the k th component of a mfBm reads*

$$r_{jk}(s, t) = \frac{\sigma_j \sigma_k \rho_{jk}}{2} \left\{ w_{kj}(s) + w_{jk}(t) - w_{jk}(t - s) \right\}. \quad (15)$$

Once again, the proof of this result is a direct application of proposition 1 above and of theorem 2.1 in [16]. Several comments can now be made from this result.

- For $j = 1, \dots, p$, the j -th component $x_j(t)$ of $x(t)$ is a fractional Brownian motion, and we recover from (15) the well-known form of the covariance of a scalar fBm

$$r_{jj}(s, t) = \frac{\sigma_j^2}{2} \left\{ |s|^{2H_j} + |t|^{2H_j} - |t - s|^{2H_j} \right\}.$$

where we have set $\rho_{jj} = 1$ of course.

- Then, note that A_{jj} can be obtained from A_{jk} when $j = k$ and $\rho_{jj} = 1$. Note also that when $H_j = H_k$, $c_{jk}(t) = c_{jk} = 1$. Thus, in this particular case, the cross-covariance function is proportional to the cross-covariance function of a fBm with Hurst parameter H_j .
- The limit of A_{jk} when $H_j + H_k \rightarrow 1$ is equal to the definition of A_{jk} when $H_j + H_k = 1$. This can be easily verified using elementary trigonometric identities. For example in the causal case, omitting σ 's and ρ 's, A_{jk} can be written as $\sin(\pi(H_j + H_k)/2) / \cos(\pi(H_j - H_k)/2)$ whereas A_{jj} writes $1 / (\sin(\pi(H_j + H_k)/2) \cdot \cos(\pi(H_j - H_k)/2))$. Thus A_{jk} for $H_j + H_k = 1$ could have been defined by continuity.

- In the same spirit, the form of the covariance for $H_j + H_k \neq 1$ converges as $H_j + H_k \rightarrow 1$ to the form of the covariance obtained for $H_j + H_k = 1$. This is evident in the well-balanced case since for $H_j + H_k \neq 1$, $w_{jk}(t) = |t|^{H_j+H_k}$ and for $H_j + H_k = 1$, $w_{jk}(t) = |t|$. For the causal case, the proof of the assertion needs some more care. First it is easy to prove that $(H_j + H_k - 1)c_{kj}(t) \rightarrow f_{jk}\text{Sign}(t)$ when $H_j + H_k \rightarrow 1$. For this expand $\cos(\pi H_k) = -\cos(\pi(H_j + H_k - 1) - \pi H_j)$ as $\cos(\pi H_j) + \pi \sin(\pi H_j)(H_j + H_k - 1) + o(H_j + H_k - 1)$ and remember that $f_{jk} = -f_{kj}$. Then, let $\alpha = H_j + H_k$. Note that from eq. (11) we have $c_{jk}(t) + c_{kj}(t) = 2$. Then,

$$\begin{aligned} \frac{2r_{ij}(s, t)}{\sigma_j \sigma_k \rho_{jk}} &= c_{jk}(s)|s|^{\alpha-1} + c_{kj}(t)|t|^{\alpha-1} - c_{kj}(t-s)|t-s|^{\alpha-1} \\ &= (2 - c_{kj}(s))|s| + c_{kj}(t)|t| - c_{kj}(t-s)|t-s| \\ &\quad + (2 - c_{kj}(s))|s|^{\alpha-1} - |s| + c_{kj}(t)|t|^{\alpha-1} - |t| - c_{kj}(t-s)(|t-s|^{\alpha-1} - |t-s|) \end{aligned}$$

It is easy to show that $(2 - c_{kj}(s))|s| + c_{kj}(t)|t| - c_{kj}(t-s)|t-s| = |s| + |t| - |t-s|$. Moreover, $c_{kj}(t)(|t|^{\alpha-1} - |t|) \rightarrow f_{jk}\text{Sign}(t)|t| \log |t| = f_{jk}t \log |t|$ which concludes the proof of the assertion. Thus, the case $H_j + H_k = 1$ can be defined by continuity from the case $H_j + H_k \neq 1$.

- In the well-balanced case $A_+ = A_-$, the covariance function takes the simple expression

$$r_{jk}(s, t) = \frac{\sigma_j \sigma_k \rho_{jk}}{2} (|s|^{H_j+H_k} + |t|^{H_j+H_k} - |t-s|^{H_j+H_k}).$$

This result is due to the time reversibility of the mfBm when $A_+ = A_-$, as observed by Didier&Pipiras in [10]. Time reversibility is clearly observed in the integral representation (1) when $A_+ = A_-$, but this condition is absolutely not necessary to insure time reversibility (see [10] for more details and a necessary and sufficient condition on the mixing matrices). Note finally that in the case $A_- = 0$, the process is not time reversible, and this is reflected in the more complicated structure of the covariance function.

2.3 On the validity of the stochastic representation

Didier&Pipiras give conditions for the existence of the representation. For the diagonal operator self-similarity considered here, the condition of existence of the time representation we use is that $H_i \neq 1/2, \forall i = 1, \dots, p$.

However, in the causal case, this condition can be relaxed since representation given in eq. (8) is valid. Consider $B_{c,il}(t) = \int k_{H_i}(u, t) dW_l(u)$. Since H is diagonal, the kernel matrix k_H is also diagonal and the process may be written $x(t) = \sum_{l=1}^p A_{+,il} B_{c,il}(t)$. What happens to $B_{c,il}(t)$ when $H_i \rightarrow 1/2$? The kernel $k_{H_i}(t, u)$ converges to $\mathbf{1}_{\mathbb{R}^+}(t-u) - \mathbf{1}_{\mathbb{R}^+}(-u) = \mathbf{1}_{[0,t]}(u)$, the indicator function of the interval $[0, t]$. Thus, since $W_l(0) = 0$ almost surely, $B_{il}(t) = W_l(t)$ is a standard Wiener process.

In the well-balanced case, the previous analysis leads to a problem. If we introduce $B_{ac,il}(t) = \int l_{H_i}(u, t) dW_l(u)$, then $x(t) = \sum_{l=1}^p A_{+,il}(B_{c,il}(t) + B_{ac,il}(t))$. The kernel $l_{H_i}(t, u)$ converges to $\mathbf{1}_{\mathbb{R}^+}(u - t) - \mathbf{1}_{\mathbb{R}^+}(u) = -\mathbf{1}_{[0,t]}(u)$. Thus, $B_{ac,il}(t)$ converges to $-W_l(t)$, and thus $x(t) = 0$ almost surely. We recover this fact by evaluating the variance of the well-balanced process (involved in eq. (6)) which is equal to zero.

In all other cases, $A_{+,il}B_{c,il}(t) + A_{-,il}B_{ac,il}(t)$ converges to $(A_{+,il} - A_{-,il})W_l(t)$ as $H_i \rightarrow 1/2$, and if the i th line of A_+ is not equal to the i th line of A_- then $x(t)$ is well-defined.

However, it may be shown (see [10]) that for $H_i = 1/2$ the following stochastic representation holds for $x_i(t)$

$$x_i(t) = \sum_{l=1}^p \int \left((\mathbf{1}_{\mathbb{R}^+}(t - u) - \mathbf{1}_{\mathbb{R}^+}(-u)) A_{+,il} + \log \left(\frac{|t - u|}{|u|} \right) A_{-,il} \right) dW_l(u)$$

This however introduces more special cases and we prefer to assume that $H_i \neq 1/2, \forall i = 1, \dots, p$.

2.4 Semidefinite-positivity of A

The aim of this section is to examine the semidefinite-positivity condition of the matrix A defined by equation (5) in the causal case and by equation (7) in the well-balanced case. This condition is the main limitation of this model. Indeed, if the matrix A is not positive-semidefinite, it cannot be factorized into $A_+ A_+^t$.

The first comment to be made is the fact that if $H_1 = \dots = H_p = \tilde{H}$, then A is positive-semidefinite. Indeed, it is easily verified that $A = c(\tilde{H}) \times R(1, 1)$ where $c(\tilde{H}) = \sin(\pi \tilde{H}) / B(\tilde{H} + .5, \tilde{H} + .5)$ in the causal case, and $c(\tilde{H}) = \sin(\pi \tilde{H}) / (2(1 - \sin(\pi \tilde{H})) B(\tilde{H} + .5, \tilde{H} + .5))$ in the well-balanced case. $R(1, 1)$ is the covariance matrix of the mBm at times (1,1). Hence, as the product between a positive constant and a positive-semidefinite matrix, A is positive-semidefinite. In this particular case, there is no limitation in the model: we can choose $\tilde{H} \in (0, 1)$ whatever $\rho_{jk} \in [-1, 1]$ for all j, k .

In the general case, we could not find necessary and sufficient conditions to insure that A is positive-semidefinite. However, we establish the following necessary condition. Let $g(H_j, H_k)$ be defined as

Causal case $A_- = 0$:

$$g(H_j, H_k) = \Gamma(H_j + H_k + 1) \frac{\sin\left(\frac{\pi}{2}(H_j + H_k)\right)}{\cos\left(\frac{\pi}{2}(H_j - H_k)\right)}. \quad (16)$$

Well-balanced case $A_- = A_+$:

$$g(H_j, H_k) = \Gamma(H_j + H_k + 1) \sin\left(\frac{\pi}{2}(H_j + H_k)\right) \quad (17)$$

Proposition 3 *If A defined by equations (5) or (7) is positive-semidefinite, then*

$$\rho_{jk}^2 \frac{g(H_j, H_k)^2}{g(H_j, H_j)g(H_k, H_k)} \leq 1, \forall j \neq k \quad (18)$$

Proof. Let $z_{jk}^t = (0, 0, \dots, z_j, 0, \dots, 0, z_k, 0, \dots, 0)$ be a vector whose all elements are zero except the j th and the k th. Since A is nonnegative, $z_{jk}^t A z_{jk}^t \geq 0$. Let B the 2×2 submatrix of A corresponding to the elements at the intersection of the j th and k th lines with the j th and k th columns. Then $z_{jk}^t A z_{jk}^t = (z_j z_k) B (z_j z_k)^t \geq 0$. Thus B is non negative and its determinant is positive. In the causal case, this determinant is given by

$$\sigma_j^2 \sigma_k^2 \frac{\sin(\pi H_j) \sin(\pi H_k)}{B_{jj} B_{kk}} - \sigma_j^2 \sigma_k^2 \rho_{jk}^2 \frac{\sin(\pi(H_j + H_k))^2}{(\cos(\pi H_j) + \cos(\pi H_k)) B_{jk}^2}$$

Factorize the positive quantity $\sigma_j^2 \sigma_k^2 \frac{\sin(\pi H_j) \sin(\pi H_k)}{B_{jj} B_{kk}}$, use $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ and elementary trigonometric identities to get the result. The same of kind of simple calculations and noting the identity $(1 - \sin x)(1 - \sin y) = (\cos((x-y)/2) - \sin((x+y)/2))^2$ give the result in the well-balanced case. Since j, k, z_j, z_k are arbitrary, this ends the proof. ■

Even if this condition is only a necessary one, it gives a useful condition to be fulfilled by the parameters. Indeed, if the condition is violated we are ensured that the model is not defined.

For the 2 dimensional case, the condition is obviously necessary and sufficient. From (18), the condition depends only on H_j, H_k and $\rho = \rho_{jk}$. A plot is feasible to determine the range of possible parameters, see Fig. 1. For the causal and the well-balanced cases, we observe that the higher $|H_j - H_k|$ (resp. the lower), the lower the maximal possible correlation ρ (resp. the higher).

3 Increments of the multivariate fractional Brownian motion

This section aims at exploring the covariance structure and the spectral density matrix of the increments of size δ of the multivariate fractional Brownian motion. Let $\Delta_\delta x(t) = x(t+\delta) - x(t)$ denote the increments of the multivariate fractional Brownian motion of size δ and let $\Delta_\delta x_j(t)$ be its j th component.

3.1 Covariances and cross-covariances

Let $\gamma_{jk}(h, \delta) = E[\Delta_\delta x_j(t) \Delta_\delta x_k(t+h)]$ denote the cross-covariance of the increments of size δ of the components j and k . Expanding the expectation and using the covariance (15), we deduce

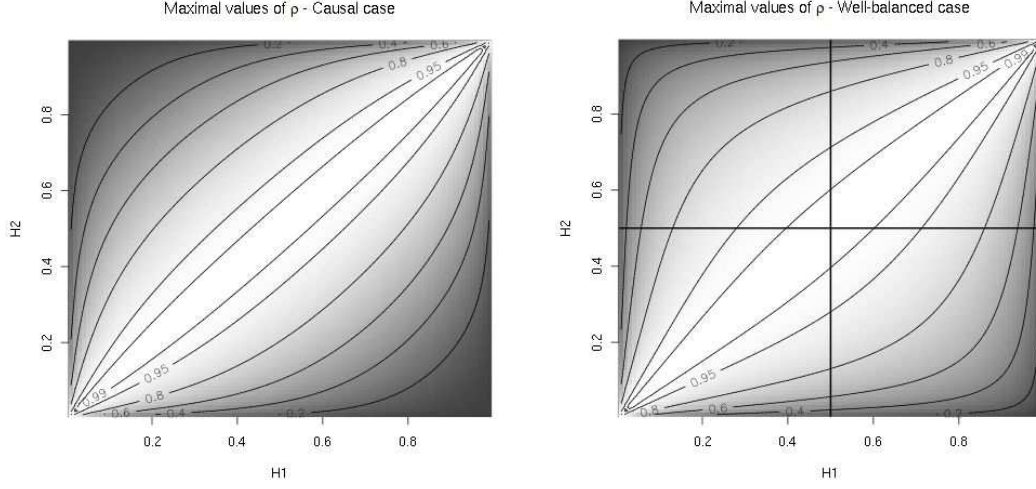


Figure 1: Maximal values of the absolute possible correlation parameter $|\rho_{12}|$ ensuring that the matrix A is positive-semidefinite, in terms of H_1 and H_2 .

that $\gamma_{jk}(h, \delta)$ is given by

$$\gamma_{jk}(h, \delta) = \frac{\sigma_j \sigma_k \rho_{jk}}{2} \left(w_{jk}(h - \delta) - 2w_{jk}(h) + w_{jk}(h + \delta) \right). \quad (19)$$

We reproduce the definition, eq. (10), of w_{jk} for convenience

$$w_{jk}(t) = \begin{cases} c_{kj}(t)|t|^{H_j+H_k} & \text{if } j = k \text{ or } j \neq k \text{ and } H_j + H_k \neq 1 \\ |t| + f_{jk}t \log |t| & \text{if } j \neq k \text{ and } H_j + H_k = 1. \end{cases}$$

The first comment is that the result confirms that the increment process is a multivariate stationary random process. Stationarity is in the strict sense since the process is Gaussian.

In the well-balanced case ($c_{jk} = 1$ and $f_{jk} = 0$), we observe that for all $H_j, H_k \neq 1/2$,

$$\gamma_{jk}(h, \delta) = \frac{\sigma_j \sigma_k \rho_{jk}}{2} \left(|h - \delta|^{H_j+H_k} - 2|h|^{H_j+H_k} + |h + \delta|^{H_j+H_k} \right). \quad (20)$$

Therefore, in this case, $\gamma_{jk}(h, \delta)$ is proportional to the covariance of a fractional gaussian noise with Hurst parameter $(H_j + H_k)/2$. In particular, it is a symmetric function with respect to h .

The causal case is different from the well-balanced case since, when $H_j + H_k \neq 1$, $c_{kj} = 2 \cos(\pi H_k) / (\cos(\pi H_j) + \cos(\pi H_k)) \neq c_{jk}$. Note that when $H_k = 1/2$, we observe (since $c_{kj} = 0$) that $\gamma_{jk}(h, \delta) = 0$ for $h \geq \delta > 0$. Let us also observe that, the case $H_j = H_k = 1/2$ leading to $f_{jk} = 0$ makes $\gamma_{jk}(h, \delta)$ proportional to the covariance of the increments of size δ of a Brownian motion. We now turn to the analysis of some of the properties of the covariance of the increments.

3.1.1 Long-memory type properties of the cross-covariance

For two functions f and g , we denote by $f \sim g$ when $\lim f(h)/g(h) = 1$, as $|h| \rightarrow +\infty$.

Proposition 4 *As $|h| \rightarrow +\infty$, we have for any $\delta > 0$*

$$\gamma_{jk}(h, \delta) \sim \frac{\sigma_j \sigma_k \rho_{jk}}{2} \delta^2 |h|^{H_j + H_k - 2} \times \tau_{jk}(h), \quad (21)$$

where

$$\tau_{jk}(h) = \begin{cases} c_{kj}(h)(H_j + H_k)(H_j + H_k - 1) & \text{if } j = k \text{ and } H_j \neq 1/2 \\ & \text{or } j \neq k \text{ and } H_j + H_k \neq 1 \\ f_{jk} \times \text{Sign}(h) & \text{if } H_j + H_k = 1 \text{ and } H_j \neq 1/2. \end{cases}$$

The cases $j = k$ and $H_j = 1/2$ and $j \neq k$ and $H_j = H_k = 1/2$ are omitted since they correspond to the covariance of the increments of a Brownian motion and therefore in these cases, $\gamma_{jk}(h, \delta) = 0$ for $|h| \geq \delta$.

In Section 2.2, we mentioned that $c_{kj}(h)(H_j + H_k - 1) \sim f_{jk} \text{Sign}(h)$ as $H_j + H_k \rightarrow 1$, which makes the second case the limit of the first one.

Proof. Define $B(h) := w_{jk}(h - \delta) - 2w_{jk}(h) + w_{jk}(h + \delta)$. Without loss of generality, let us choose h such that $|h| \geq \delta$. For the first case, let $\alpha = H_j + H_k$. In this case, it is sufficient to note that $c_{kj}(h) = c_{kj}(h + \delta) = c_{kj}(h - \delta)$ and

$$B(h) = c_{kj}(h)|h|^\alpha \left(\left(1 - \frac{\delta}{h}\right)^\alpha - 2 + \left(1 + \frac{\delta}{h}\right)^\alpha \right) \sim c_{kj}(h)|h|^\alpha \alpha(\alpha - 1) \delta^2 h^{-2}.$$

For the second case, for $|h| \geq \delta$, the expression of $B(h)$ reduces to

$$B(h) = f_{jk} \left((h + \delta) \log \left(1 + \frac{\delta}{h} \right) + (h - \delta) \log \left(1 - \frac{\delta}{h} \right) \right).$$

Using the expansion of $\log(1 \pm x)$ as $x \rightarrow 0$, we obtain $B(h) \sim f_{jk} \delta^2 h^{-1} = f_{jk} \delta^2 |h|^{-1} \times \text{Sign}(h)$, which is the expected result. ■

At this point several interesting remarks may be done. First, setting $j = k$ and $\rho_{jj} = 1$ allows us to recover the well-known asymptotic behavior for the covariance of a monovariate fGn $\sigma_j^2 H_j (2H_j - 1) \delta^2 |h|^{2H_j - 2}$ (see [17]). When $H_j + H_k = 1$ but $H_j \neq 1/2$, the increments of size δ are long-range interdependent since their cross-covariance is not summable. Note that in this case one fGn is long-range dependent and the other is necessarily not. When $H_j + H_k \neq 1$, the same conclusion may be drawn. If the two fGn are long-range dependent ($H_j > 1/2$ and $H_k > 1/2$), then necessarily they are long-range interdependent. Interestingly, two fGn can be long-range independent when only one is long-range dependent.

3.1.2 Behavior of $\gamma_{jk}(\cdot, \delta)$ for large h

Let $h \geq \delta$ and $\rho_{jk} \geq 0$. When $H_j + H_k \neq 1$

$$\gamma_{jk}(h, \delta) = \sigma_j \sigma_k \rho_{jk} c_{kj} \times \tilde{\gamma}_{\frac{H_j + H_k}{2}}(h, \delta),$$

where $\tilde{\gamma}_H(h, \delta)$ is the covariance function of a fGn (with size δ) with Hurst parameter H and with variance 1. Recall that for $h \geq \delta$, $\tilde{\gamma}_H(\cdot, \delta)$ is a negative and increasing (resp. positive and decreasing) function when $H < 1/2$ (resp. $H > 1/2$). This corresponds to the behavior of $\gamma_{jk}(\cdot, \delta)$ in the well-balanced case since $c_{kj} = 1$. In the causal case, we may derive the following statement (by studying the sign of c_{kj}) illustrated by Fig. 2:

$$\text{For } h \geq \delta, \quad \gamma_{jk}(h, \delta) \begin{cases} \text{is negative and increasing} & \text{when } H_k < 1/2 \\ \text{is positive and decreasing} & \text{when } H_k > 1/2 \\ \text{equals zero} & \text{when } H_k = 1/2. \end{cases}$$

Let us underline that the study of the function $(h - \delta) \log(h - \delta) - 2h \log(h) + (h + \delta) \log(h + \delta)$ leads to the same conclusion when $H_j + H_k = 1$.

3.2 Spectral density and cross-spectral density

In all the following, the convention adopted for the Fourier transform $FT(f(t))$ of a function f is $F(\omega) = \int f(t) \exp(-i\omega t) dt$. Depending on the context, and this will be detailed, the transform will be understood in the L^1 , L^2 or even in the generalized function sense. The inverse transform reads $f(t) = 1/(2\pi) \int F(\omega) \exp(i\omega t) d\omega$.

Even if for some values of $H_j + H_k$, the covariance $\gamma_{jk}(h, \delta)$ may be in L^1 or L^2 , it is not the case for all values, and thus we evaluate the spectral density matrix in the generalized function (distribution) sense (see *e.g.* [14]).

Proposition 5 (i) *The Fourier transform of $\gamma_{jk}(h, \delta)$, denoted by $S_{jk}(\omega, \delta)$ is given for all j, k and for all H_j, H_k by*

$$\begin{aligned} S_{jk}(\omega, \delta) &= \sigma_j \sigma_k \rho_{jk} \frac{1 - \cos(\omega\delta)}{|\omega|^{H_j + H_k + 1}} \Gamma(H_j + H_k + 1) \times \zeta_{jk}(\omega) \\ \zeta_{jk}(\omega) &= \begin{cases} -c_{kj} e^{-i \text{Sign}(\omega) \frac{\pi}{2} (H_j + H_k + 1)} - c_{jk} e^{i \text{Sign}(\omega) \frac{\pi}{2} (H_j + H_k + 1)} & \text{if } j = k \text{ or } j \neq k, H_j + H_k \neq 1 \\ 2 - i\pi f_{jk} \text{Sign}(\omega) & \text{if } j \neq k \text{ and } H_j + H_k = 1. \end{cases} \end{aligned} \quad (22)$$

In the causal and well-balanced cases, this reduces for all H_j, H_k to:

Causal case $A_- = 0$:

$$S_{jk}(\omega, \delta) = 2\sigma_j \sigma_k \rho_{jk} g(H_j, H_k) \frac{1 - \cos(\omega\delta)}{|\omega|^{H_j + H_k + 1}} e^{-i \text{Sign}(\omega) \frac{\pi}{2} (H_k - H_j)}. \quad (23)$$

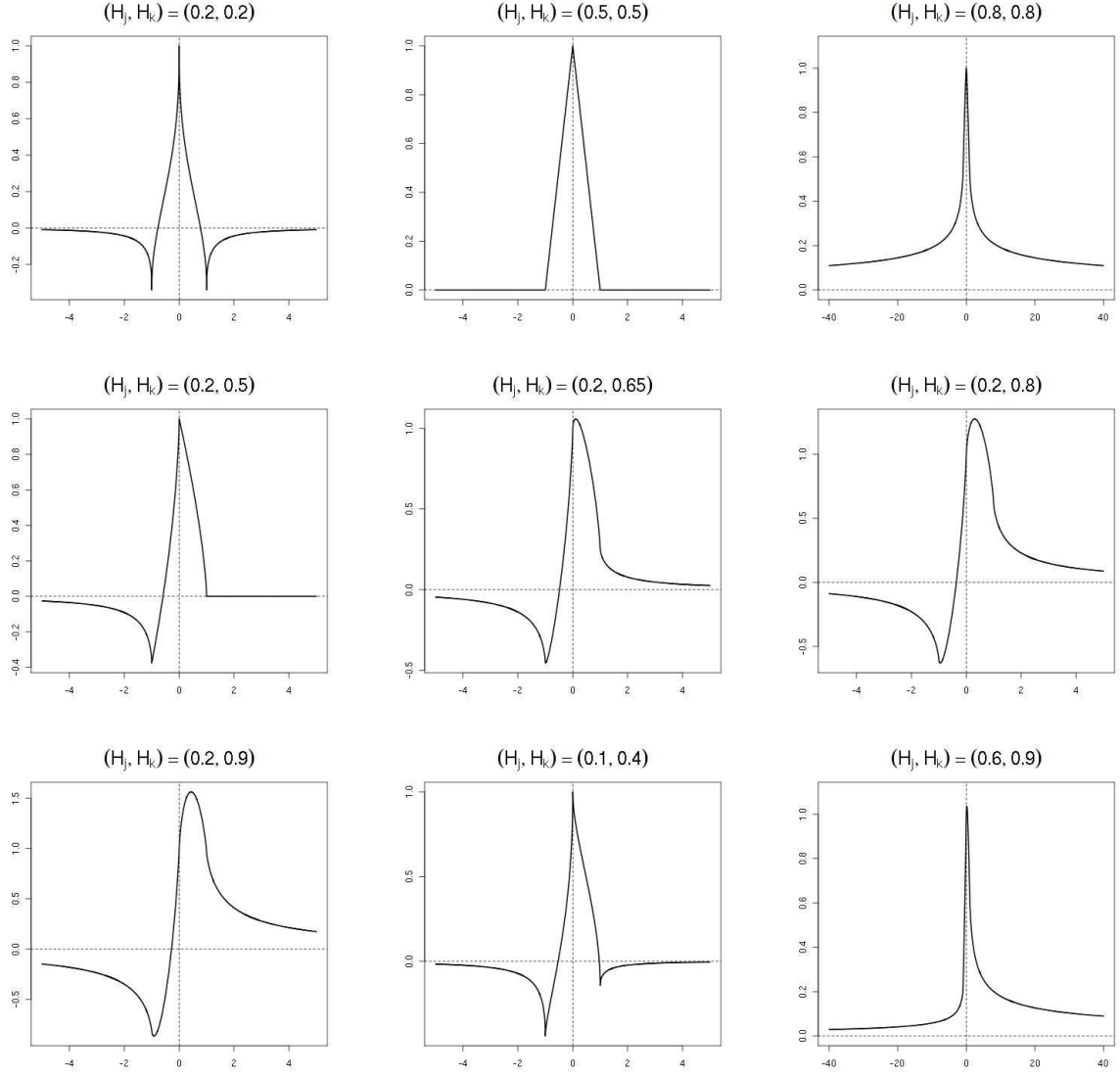


Figure 2: Examples of cross-covariance functions for the causal case for different parameters H_j, H_k . Without loss of generality, the parameters σ_j, σ_k and ρ_{jk} are fixed to 1. In the well-balanced, from (20), the top left (resp. right) plot corresponds to the covariance for all the values of H_j, H_k such that $H_j + H_k = 0.2$ (resp. 0.8). The top middle one has no sense in the well-balanced case since H_j and H_k must be different of $1/2$.

Well-balanced case $A_- = A_+$:

$$S_{jk}(\omega, \delta) = 2\sigma_j\sigma_k\rho_{jk} g(H_j, H_k) \frac{1 - \cos(\omega\delta)}{|\omega|^{H_j+H_k+1}}, \quad (24)$$

where the function g is defined by (16) and (17).

(ii) For any fixed δ , we have for both cases, as $\omega \rightarrow 0$

$$|S_{jk}(\omega, \delta)| \sim \sigma_j\sigma_k\rho_{jk} g(H_j, H_k) \delta^2 |\omega|^{1-H_j-H_k}. \quad (25)$$

(iii) Moreover, the coherence function between the two components j and k satisfies:

$$C_{jk}(\omega, \delta) := \frac{|S_{jk}(\omega, \delta)|^2}{S_{jj}(\omega, \delta)S_{kk}(\omega, \delta)} = \rho_{jk}^2 \times \frac{g(H_j, H_k)^2}{g(H_j, H_j)g(H_k, H_k)}. \quad (26)$$

Equations (23) and (24) are easily derived from (22) using elementary algebra and the definitions of c_{jk} and f_{jk} . Before turning to the proof, let us give some remarks concerning these results:

- Unlike the covariance $\gamma_{jk}(h, \delta)$, note that the expression of $S_{jk}(\omega, \delta)$ is unchanged when $j \neq k$ and $H_j + H_k = 1$. For example in the causal case, it reduces to

$$\frac{2\sigma_j\sigma_k\rho_{jk}}{\sin(\pi H_j)} \frac{1 - \cos(\omega\delta)}{|\omega|^2} e^{i\text{Sign}(\omega)\pi(H_j - \frac{1}{2})}.$$

- When $j = k$ or when $H_j = H_k$, we recover the standard real spectral density function of a fGn. Actually, in the well-balanced case, $\forall j, k$, $S_{jk}(\omega, \delta)$ corresponds to the spectral density function of a fGn with Hurst parameter $\frac{H_j+H_k}{2}$.
- The analysis of the local behavior in a neighborhood of zero of the cross-spectral density (25) leads to the same remarks as the ones done in Section 3.1.1 concerning long-memory type properties.
- Finally, let us underline the fact that the coherence function is independent of the frequency. Furthermore, we recover the necessary condition of proposition 3. Indeed, the coherence must be lower than one, a condition satisfied if

$$\rho_{jk}^2 \times \frac{g(H_j, H_k)^2}{g(H_j, H_j)g(H_k, H_k)} \leq 1$$

Proof. We only concentrate on the cross-spectra since (25) and (26) are then easily derived. We denote by $FT(\cdot)$ the Fourier transform in the generalized function sense (see [14]). From (10),

$$\begin{aligned} S_{jk}(\omega, \delta) &:= FT(\gamma_{jk}(h, \delta)) \\ &= \frac{\sigma_j \sigma_j \rho_{jk}}{2} (FT(w_{jk}(h - \delta)) - 2FT(w_{jk}(h)) + FT(w_{jk}(h + \delta))) \\ &= \frac{\sigma_j \sigma_j \rho_{jk}}{2} (2 \cos(\omega \delta) - 2) FT(w_{jk}(h)). \end{aligned} \quad (27)$$

Let us now split the proof into the two different cases.

Case 1. $j = k$ or $j \neq k$ and $H_j + H_k \neq 1$.

Let $\alpha = H_j + H_k$. For $x \in \mathbb{R}$, consider the following generalized functions

$$x_+^\alpha = x^\alpha \mathbf{1}_{\mathbb{R}^+}(x) \quad \text{and} \quad x_-^\alpha = (-x)^\alpha \mathbf{1}_{\mathbb{R}^-}(x).$$

From the definition of $c_{kj}(h)$, the Fourier transform of $w_{jk}(h) = c_{kj}(h)|h|^\alpha$ writes $FT(c_{kj}(h)|h|^\alpha) = c_{kj}FT(h_+^\alpha) + c_{jk}FT(h_-^\alpha)$. The Fourier transforms of h_\pm^α exist in the generalized function sense (see [14]) and read

$$FT(h_+^\alpha) = \Gamma(\alpha + 1) \left\{ \omega_+^{-\alpha-1} e^{-i\frac{\pi}{2}(\alpha+1)} + \omega_-^{-\alpha-1} e^{i\frac{\pi}{2}(\alpha+1)} \right\} \quad (28)$$

$$FT(h_-^\alpha) = \Gamma(\alpha + 1) \left\{ \omega_+^{-\alpha-1} e^{i\frac{\pi}{2}(\alpha+1)} + \omega_-^{-\alpha-1} e^{-i\frac{\pi}{2}(\alpha+1)} \right\} \quad (29)$$

or alternatively

$$\begin{aligned} FT(h_+^\alpha) &= \Gamma(\alpha + 1) e^{-i\text{Sign}(\omega)\frac{\pi}{2}(\alpha+1)} |\omega|^{-\alpha-1} \\ FT(h_-^\alpha) &= \Gamma(\alpha + 1) e^{i\text{Sign}(\omega)\frac{\pi}{2}(\alpha+1)} |\omega|^{-\alpha-1}. \end{aligned}$$

Therefore, we obtain

$$FT(c_{kj}(h)|h|^\alpha) = \Gamma(\alpha + 1) |\omega|^{-\alpha-1} \left(c_{kj} e^{-i\text{Sign}(\omega)\frac{\pi}{2}(\alpha+1)} + c_{jk} e^{i\text{Sign}(\omega)\frac{\pi}{2}(\alpha+1)} \right),$$

which, combined with (27), leads to the result.

Case 2. $j \neq k$ and $H_j + H_k = 1$.

We have to concentrate on the Fourier transform of $|h| + f_{jk}h \log |h|$. For $\alpha > -1$, the Fourier transform of $|h|^\alpha$ equals $FT(|h|^\alpha) = -2\Gamma(\alpha + 1) \sin(\pi\alpha/2) |\omega|^{-\alpha-1}$. Now, let us notice that $FT(|h|^\alpha \log |h|) = FT(\frac{d}{d\alpha} |h|^\alpha) = \frac{d}{d\alpha} FT(|h|^\alpha)$. Setting $\alpha = 0$ in this last equation and the explicit derivation of $FT(|h|^\alpha)$ (with respect to α) leads to $FT(\log |h|) = -\pi |\omega|^{-1}$. In order to get $FT(h \log |h|)$, we just have to note that $FT(h \log |h|) = -\frac{1}{i} \times \frac{d}{d\omega} FT(\log |h|)$, leading to $FT(h \log |h|) = i\pi \text{Sign}(\omega) |\omega|^{-2}$. We finally obtain

$$FT(|h| + f_{jk}h \log |h|) = |\omega|^{-2} (-2 + i\pi f_{jk} \text{Sign}(\omega))$$

Combined with (27), the last equation leads to (22). ■

4 Wavelet Analysis

The use of wavelet analysis in the understanding of the monovariate fractional Brownian motion, and more generally for the study of fractal signals, goes back to the early works of Flandrin [12, 13], Tewfik [19], Wornell [22] to cite some but a few. It is now well accepted that wavelet analysis is the adequate analysis to extract information properly from fractal or multifractal signals. Several causes for this fact can be put forward, such as the adequation between $1/f$ -like spectral densities of fractal signals and the constant relative bandwidth filter bank underlying the wavelet analysis, ability to “kill” long-range dependence if the wavelet is correctly chosen.

The aim of this section is thus to analyse the multivariate fractional Brownian motion through the lens of the wavelet transform. We use the continuous wavelet transform here, but a similar analysis could be performed in the multiresolution framework using orthonormal wavelet bases. We will consider complex valued wavelets, not necessarily in the Hardy class, not necessarily with compact support. The hypothesis we impose on the wavelets will be detailed when needed.

4.1 Definition and stationarity

Let ψ be a complex wavelet function, let $a > 0$ and $b \in \mathbb{R}$ and consider $\psi_{ab}(\cdot) = a^{-1/2}\psi((\cdot - b)/a)$. Let

$$d_{a,b}^j := \left\langle x_j \middle| \psi_{ab} \right\rangle_{L^2} = \int_{\mathbb{R}} x_j(t) \overline{\psi_{ab}}(t) dt \quad (30)$$

the wavelet transform of the j th component of a multivariate fractional Brownian motion. $\overline{\psi}$ denotes the complex conjugate of ψ . In this section, we assume that conditions [C1] and [C2(2)] are satisfied, where:

[C1] Admissibility condition: $\psi(t) \in L^2$ and $|\widehat{\psi}(\omega)|^2/|\omega| \in L^1$, where $\widehat{\psi}$ is the Fourier transform of ψ .

[C2(K)] $t^m \psi(t) \in L^1$ for $m = 0, 1, \dots, K$.

Condition [C1] ensures that $\widehat{\psi}(0) = 0$ and that $\int_{\mathbb{R}} \psi(t) dt = 0$. We note, as [15], that under condition [C2(1)], the integral (30) is well-defined as a sample path integral and is a second-order random variable. This follows, since under [C2(1)] we have $\int_{\mathbb{R}} |s|^H |\psi_{ab}(s)| ds < +\infty, \forall H \in (0, 1)$.

The aim of this section is to focus on the correlation between the wavelet transforms (at different scales and different times) of two components j and k of the multivariate fractional Brownian motion. The wavelet transform is a random field. It is clearly zero mean and Gaussian. We have for $a_1, a_2 > 0$ and $b_1, b_2 \in \mathbb{R}$

$$E[d_{a_1, b_1}^j \overline{d_{a_2, b_2}^k}] = \int_{\mathbb{R}^2} r_{jk}(t_1, t_2) \overline{\psi_{a_1 b_1}}(t_1) \psi_{a_2 b_2}(t_2) dt_1 dt_2.$$

Under [C1], and from (10) the last expression reduces to

$$E[d_{a_1, b_1}^j \overline{d_{a_2, b_2}^k}] = -\frac{\sigma_j \sigma_k \rho_{jk}}{2} \int_{\mathbb{R}^2} w_{jk}(t_2 - t_1) \overline{\psi_{a_1 b_1}}(t_1) \psi_{a_2 b_2}(t_2) dt_1 dt_2.$$

Let $\Gamma_\psi(v) := \int_{\mathbb{R}} \psi_{a_1 b_1}(u) \overline{\psi_{a_2 b_2}}(u + v) du$ be the correlation function between the two wavelets ψ_{a_1, b_1} and ψ_{a_2, b_2} . Then we have

$$E[d_{a_1, b_1}^j \overline{d_{a_2, b_2}^k}] = -\frac{\sigma_j \sigma_k \rho_{jk}}{2} \int_{\mathbb{R}} w_{jk}(v) \overline{\Gamma_\psi}(v) dv. \quad (31)$$

Note that [C2(2)] implies that for all the values of H_j and H_k , $\int_{\mathbb{R}} |w_{jk}(v)| |\Gamma_\psi(v)| dv < +\infty$. With two changes of variables, this may also be rewritten as

$$\begin{aligned} E[d_{a_1, b_1}^j \overline{d_{a_2, b_2}^k}] &= -\frac{\sigma_j \sigma_k \rho_{jk}}{2} \sqrt{a_1 a_2} \times \\ &\int_{\mathbb{R}^2} w_{jk}(a_2 t_2 - a_1 t_1 + b_2 - b_1) \overline{\psi}(t_1) \psi(t_2) dt_1 dt_2. \end{aligned} \quad (32)$$

If we interpret for fixed parameters a_1 and a_2 , the quantity $E[d_{a_2, b_2}^j \overline{d_{a_2, b_2}^k}]$ as the cross-correlation between two signals, we observe that it depends only on the difference between the times at which it is evaluated (i.e. $b_2 - b_1$). With the fact that the wavelet transform is a zero mean and Gaussian field, we conclude that $d_{a_1, \cdot}^j$ and $d_{a_2, \cdot}^k$ are jointly stationary signals. This is of course because the wavelet transform reveals the stationary increments property hidden in the fractional Brownian motion. The wavelet transform can be seen as a generalized derivative.

4.2 Self-similarity type property of the cross-wavelet transform

The variance of the wavelet transforms at similar scales for the fractional Brownian motion with Hurst parameter H exhibits some self-similarity. Indeed, it is proved in [12] for example that for all b

$$\text{Var}(d_{a, b}^j) = a^{2H+1} \times \left(-\frac{\sigma^2}{2} \int_{\mathbb{R}^2} |t_2 - t_1|^{2H} \overline{\psi}(t_1) \psi(t_2) dt_1 dt_2 \right).$$

We note here that the same behavior holds for the cross-wavelet variance.

Proposition 6 *Under the assumptions [C1] and [C2(2)], let $b_1 = b_2 = b$ and fix $a_1 = a_2 = a > 0$. Then,*

$$\begin{aligned} E[d_{a, b}^j \overline{d_{a, b}^k}] &= \frac{\sigma_j \sigma_k \rho_{jk}}{2} z_{jk} a^{H_j + H_k + 1} \\ \text{Corr}[d_{a, b}^j, d_{a, b}^k] &= \rho_{jk} \times \frac{z_{jk}}{\sqrt{z_{jj} z_{kk}}}, \end{aligned} \quad (33)$$

where $z_{jk} := -\int_{\mathbb{R}^2} w_{jk}(t_2 - t_1) \overline{\psi}(t_1) \psi(t_2) dt_1 dt_2$.

Proof. Consider Equation (32). When $H_j + H_k \neq 1$, it suffices to note that for $a > 0$, $c_{kj}(av) = c_{kj}(v)$ and therefore $w_{jk}(av) = a^{H_j+H_k}w_{jk}(v)$. Now, when $H_j + H_k = 1$, the result comes from Condition [C1] ensuring that $\int_{\mathbb{R}^2} f_{jk} \times (t_2 - t_1) \log(a) \bar{\psi}(t_1) \psi(t_2) dt_1 dt_2 = 0$. ■

Let us observe that the instantaneous cross-wavelet correlation is independent of the scale. This is the generalization of the fact that the coherence does not depend on the frequency.

4.3 Cross-correlation structure of the wavelet transform of the mfBm

For fixed scales, a_1, a_2 , we now specify the behavior of the cross-wavelet covariance (or correlation) as $|b_2 - b_1| \rightarrow +\infty$. In particular, our aim is to exhibit the influence of the number of vanishing moments of the wavelet function on the asymptotic cross-wavelet covariance. Such a result needs the following assumption:

[C3] The wavelet function has $M \geq 1$ vanishing moments that is

$$\int_{\mathbb{R}} t^m \psi(t) dt = 0 \text{ for } m = 0, \dots, M-1 \quad \text{and} \quad \int_{\mathbb{R}} t^M \psi(t) dt \neq 0.$$

We may now derive our result obtained as $|h| = |b_2 - b_1| \rightarrow +\infty$.

Proposition 7 *Under the assumptions [C1], [C2(2M+1)] and [C3], then as $|h| \rightarrow +\infty$, we have*

$$E[d_{a_1,b}^j \overline{d_{a_2,b+h}^k}] \sim -\frac{\sigma_j \sigma_k \rho_{jk}}{2} \kappa(\psi, M) |h|^{H_j+H_k-2M} \tilde{\tau}_{jk}(h)$$

where $\kappa(\psi, M) := \binom{2M}{M} (a_1 a_2)^M \left| \int t^M \psi(t) dt \right|^2$ and

$$\tilde{\tau}_{jk}(h) = \begin{cases} c_{kj}(h) \binom{H_j+H_k}{2M} & \text{if } j = k \text{ and } H_j \neq 1/2 \\ & \text{or } j \neq k \text{ and } H_j + H_k \neq 1 \\ \frac{f_{jk} \times \text{Sign}(h)}{2M(2M-1)} & \text{if } H_j + H_k = 1 \text{ and } H_j \neq 1/2. \end{cases}$$

Remark 1 *As for Proposition 4, we notice that the second case is the limit of the first one as $H_j + H_k \rightarrow 1$. Moreover, let us underline the importance of the number of vanishing moments for the wavelet. Similarly to the fractional Brownian motion, Proposition 7 asserts that the higher M , the less correlated the wavelet transforms of the components j and k of the multivariate fractional Brownian motion. This has many implications. In particular, this suggests that estimating the instantaneous cross-wavelet correlation at a scale a may be efficiently done by using the empirical correlation since at scale a , $d_{a,b}^j$ and $d_{a,b+h}^k$ are not too much correlated if M is large.*

The proof of this result is postponed until Appendix 6.

4.4 Cross-spectral density of the wavelet transform of the mfBm

In the case of fBm, the expression of the spectral density of the wavelet transform was provided by [12] and [13]. A rigorous proof of the existence of this spectral density in the L^1 sense was obtained by [15]. On the basis of this work, our ambition is, to provide the cross-spectral density between wavelet transforms (at different scales) of components j and k of the multivariate fractional Brownian motion.

Proposition 8 *Under Assumptions [C1], [C2(M)] and [C3] (with $M \geq 2$), we derive the following assertions.*

(i) *The cross-spectral density of the wavelet transforms of two components j and k , that is the Fourier transform of the function $E[d_{a_1, b_1}^j \overline{d_{a_2, b_2}^k}]$ (in terms of $b_2 - b_1$) exists and is given by*

$$\tilde{S}_{jk}(\omega) = \frac{\sigma_j \sigma_k \rho_{jk}}{2} \frac{\Gamma(H_j + H_k + 1)}{|\omega|^{H_j + H_k + 1}} \sqrt{a_1 a_2} \overline{\widehat{\psi}(-a_1 \omega)} \widehat{\psi}(-a_2 \omega) \times \zeta_{jk}(\omega) \quad (34)$$

where $\zeta_{jk}(\omega)$ is defined by equation (22). In the causal and well-balanced cases, this reduces for all H_j, H_k to:

Causal case $A_- = 0$:

$$\tilde{S}_{jk}(\omega) := \frac{\sigma_j \sigma_k \rho_{jk} g(H_j, H_k)}{|\omega|^{H_j + H_k + 1}} \sqrt{a_1 a_2} e^{-i \text{Sign}(\omega) \frac{\pi}{2} (H_k - H_j)} \overline{\widehat{\psi}(-a_1 \omega)} \widehat{\psi}(-a_2 \omega) \quad (35)$$

Well-balanced case $A_- = A_+$:

$$\tilde{S}_{jk}(\omega) := \frac{\sigma_j \sigma_k \rho_{jk} g(H_j, H_k)}{|\omega|^{H_j + H_k + 1}} \sqrt{a_1 a_2} \overline{\widehat{\psi}(-a_1 \omega)} \widehat{\psi}(-a_2 \omega) \quad (36)$$

where the function g is defined by (16) and (17).

(ii) We have for both cases, as $\omega \rightarrow 0$

$$\left| \tilde{S}_{jk}(\omega) \right| \sim \sigma_j \sigma_k \rho_{jk} g(H_j, H_k) (a_1 a_2)^{M+1/2} |\widehat{\psi}^{(M)}(0)|^2 |\omega|^{2M-1-\alpha}.$$

(iii) Moreover, the coherence function between the two components j and k satisfies:

$$\tilde{C}_{jk}(\omega) := \frac{\left| \tilde{S}_{jk}(\omega) \right|^2}{\tilde{S}_{jj}(\omega) \tilde{S}_{kk}(\omega)} = \rho_{jk}^2 \times \frac{g(H_j, H_k)^2}{g(H_j, H_j) g(H_k, H_k)} \frac{\overline{\widehat{\psi}(-a_1 \omega)} \widehat{\psi}(-a_2 \omega)}{\widehat{\psi}(-a_1 \omega) \widehat{\psi}(-a_2 \omega)}. \quad (37)$$

Similarly as proposition 5, equations (35) and (36) are obtained from (34) and using elementary algebra. The proof of this proposition is rejected in the appendices, see section 7. Note that the interpretation of the coherence in this case is difficult. It is complex valued, a property which is

not natural for a coherence. This comes from the fact that the quantities $\tilde{S}_{jj}(\omega)$ are not power spectral densities but cross-spectral densities (cross-spectral density between two different scales of the wavelet transform of one signal). Thus, to interpret correctly the coherence, we should look at one scale only, in which case we recover the coherence evaluated in the usual spectral domain. And this result is logical since the usual coherence is independent of the frequency.

5 Discussion

To conclude the paper, we make some comments on synthesis, on a more general case and we give some ideas for future works.

5.0.1 On synthesis

The synthesis of monovariate fractional Brownian motion has found an elegant and efficient solution through the use of Wood&Chan method of simulation of Gaussian processes [21]. The method relies on the embedding of the correlation matrix of N regularly spaced samples of a fractional Gaussian noise into a larger circulant matrix. As a circulant matrix, the diagonalization is easy since it relies on the discrete Fourier transform. Furthermore, using the fast Fourier transform, it can be implemented in a very efficient way. Wood&Chan have generalized their technique for synthesizing multivariate Gaussian homogeneous random fields [5]. As a particular case, the simulation of multivariate Gaussian stationary signals can be performed. We have implemented their algorithm to generate the fractional Brownian motion, either in the well-balanced case and in the causal case. We show in figure (3) some examples of sample paths that may be generated using this algorithm. The parameters have been chosen not only to insure that proposition 3 is satisfied, but also to insure that the simulation provided by Chan&Wood is exact (see [5] and [8] for more details on the exactness of the simulation). The complexity of the algorithm dramatically increases with the number of samples needed and with the number of dimension required. This drawback may lead to the use of simpler but non exact simulation techniques, relying for example on the spectral matrix (see *e.g.* [6]).

5.1 A more general in-between case

When examining equations (2) and (3) we observe that the two cases we have studied leads to an easy solution. A more general case also leads to an easy solution. This case generalizes the causal and well-balanced cases and is obtained when matrices A_+ and A_- are proportional. If we write $A_- = \kappa A_+$, we recover evidently the causal case when $\kappa = 0$ and the well-balanced case when $\kappa = 1$. This case can be treated in the same spirit as the two others, and if we define

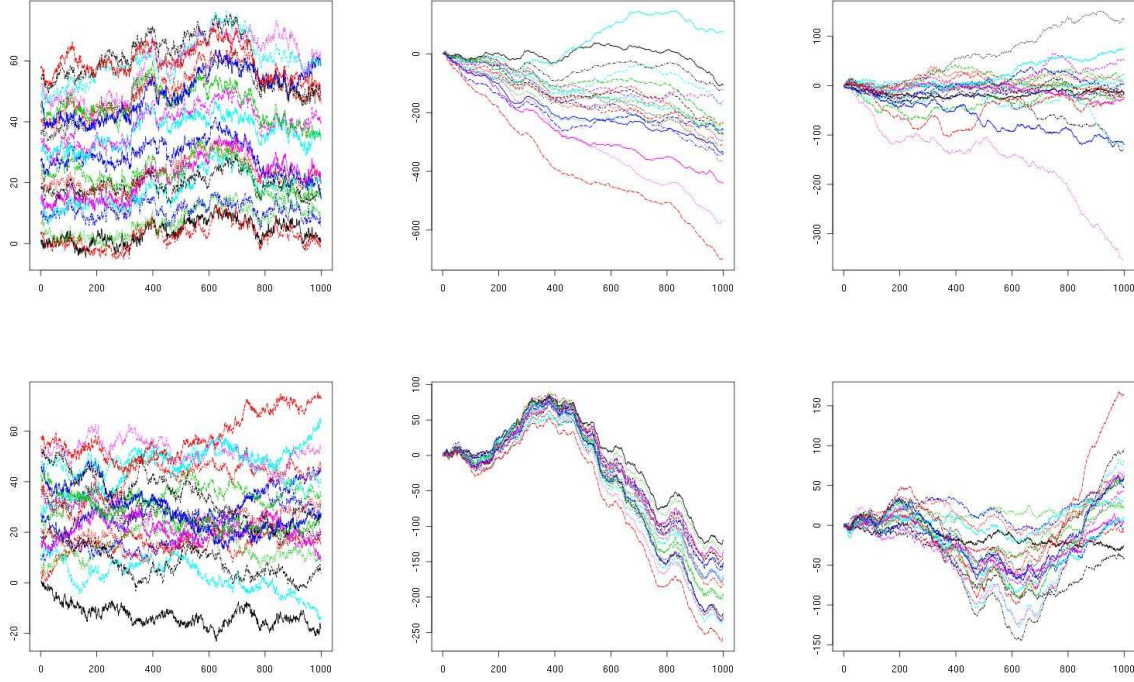


Figure 3: Examples of sample paths of length $n = 1000$ normalized causal (top) and well-balanced (bottom) multivariate fractional Brownian motion with $p = 20$ components. The Hurst parameters are equally spaced in $[0.3, 0.4]$ (left), $[0.8, 0.9]$ (middle) and $[0.4, 0.8]$ (right). The correlation parameters are all set to 0.7 (left, middle) and 0.3 (right). Note that the existence condition discussed in Proposition 3 is satisfied for these different choices of parameters. For convenience, the sample paths of the left column have been decentered.

matrix $A = (A_{jk})$ for $j, k = 1 \dots, p$ by

$$A_{jk} = \begin{cases} \frac{\sigma_j \sigma_k \rho_{jk} \sin(\pi(H_j + H_k))}{B_{jk}((\cos(\pi H_j) + \cos(\pi H_k))(1 + \kappa^2) - 2\kappa \sin(\pi(H_j + H_k)))} & \text{if } H_j + H_k \neq 1 \\ \frac{2\sigma_j \sigma_k \rho_{jk}}{B_{jk}((\sin(\pi H_j) + \sin(\pi H_k))(1 + \kappa^2) - 4\kappa)} & \text{if } H_j + H_k = 1 \end{cases}$$

$$A_{jj} = \frac{\sigma_j^2 \sin(\pi H_j)}{B_{jj}(1 + \kappa^2 - 2\kappa \sin(\pi H_j))}$$

we will end up with the same parameterization of the covariance. From the expression of A_{jj} we can show that the model leads to $\sigma_{jj} = 0$ if and only if $H_j = 1/2$ and $\kappa = 1$. We thus recover the condition of existence in the well-balanced case. For this general case, there is another problem which appears for example if $H_j + H_k \neq 1$. We observe that the denominator of A_{jk} may cancel whenever

$$\kappa = \frac{\sin(\frac{\pi}{2}(H_j + H_k)) \pm \sqrt{-\cos(\pi H_j) \cos(\pi H_k)}}{\cos(\frac{\pi}{2}(H_j - H_k))}$$

Of course this occurs if and only if the two cosine in the square root are of opposite sign, implying that H_j and H_k are not simultaneously in the same intervals $[0, 1/2)$, $(1/2, 1]$. If there is a value $\kappa(H_j, H_k)$ that cancelled the denominator of A_{jk} , then $\rho_{jk} = 0$, and we cannot use the correlation at times 1 to normalize correctly the process. Another choice has to be done, such as $E[x_j(1)x_k(-1)]$.

5.2 Some future works

A first step in the future work is to adopt another more natural parameterization of the covariance function, in which function $w_{jk}(t)$ writes $\rho_{jk} + \text{Sign}(t)\eta_{jk}$, where ρ_{jk} has the same meaning of the correlation at times 1, and where η_{jk} is an antisymmetric parameter, linked to ρ_{jk} . This parameterization should lead to a more easy study of the condition of existence of the covariance function.

A next step in our work will be to tackle the problem of inferring the Hurst parameters from the observation of a sample path of either the multivariate fractional Brownian motion or the fractional Gaussian noise. The main question to answer concerns the comparison between p estimators designed for monovariate fBm (or fGn) and a multivariate estimator to be defined.

6 Appendix: Proof of Proposition 7

Proof.

Case 1. $j = k$ and $H_j \neq 1/2$ or $j \neq k$ and $H_j + H_k \neq 1$.

Let $D_h := \{(t_1, t_2) \in \mathbb{R}^2 : |a_2 t_2 - a_1 t_1| < \frac{|h|}{2}\}$, $\alpha := H_j + H_k$ and let us write $E[d_{a,b}^j \overline{d_{a,b+h}^k}] = -\frac{\sigma_j \sigma_k \rho_{jk}}{2} \sqrt{a_1 a_2} \times T$ with

$$T := \int_{\mathbb{R}^2} c_{kj}(a_2 t_2 - a_1 t_1 + h) |a_2 t_2 - a_1 t_1 + h|^\alpha \overline{\psi}(t_1) \psi(t_2) dt_1 dt_2 = T_1 + T_2,$$

and where T_1 (resp. T_2) corresponds to the integral on D_h (resp. $\mathbb{R}^2 \setminus D_h$). Let us first prove that $|h|^{2M-\alpha} T_2 \rightarrow 0$ as $|h| \rightarrow +\infty$. We have (since $2M - \alpha > 0$ and since $|c_{kj}(h)| \leq c^\vee := \max(|c_{jk}|, |c_{kj}|)$ for all h)

$$\begin{aligned} |h|^{2M-\alpha} |T_2| &\leq 2^{2M-\alpha} c^\vee \int_{\mathbb{R}^2 \setminus D_h} (a_2 t_2 - a_1 t_1)^{2M} \left| 1 + \frac{h}{a_2 t_2 - a_1 t_1} \right|^\alpha |\psi(t_1)| |\psi(t_2)| dt_1 dt_2 \\ &\leq 2^{2M-\alpha} 3^\alpha c^\vee \int_{\mathbb{R}^2 \setminus D_h} (a_2 t_2 - a_1 t_1)^{2M} |\psi(t_1)| |\psi(t_2)| dt_1 dt_2. \end{aligned}$$

The result is then obtained by using assumption [C2(2M)] and the dominated convergence theorem. Now, within the domain D_h , one may use the series expansion of $(1+x)^\alpha$ (for $|x| < 1$).

$$\begin{aligned} T_1 &= |h|^\alpha \int_{D_h} c_{kj}(a_2 t_2 - a_1 t_1 + h) \left(1 + \frac{a_2 t_2 - a_1 t_1}{h} \right)^\alpha \overline{\psi}(t_1) \psi(t_2) dt_1 dt_2 \\ &= |h|^\alpha c_{kj}(h) \int_{D_h} \left(\sum_{\ell=0}^{\infty} \binom{\alpha}{\ell} \left(\frac{a_2 t_2 - a_1 t_1}{h} \right)^\ell \right) \overline{\psi}(t_1) \psi(t_2) dt_1 dt_2, \end{aligned}$$

where $\binom{\alpha}{\ell}$ denotes the binomial coefficient $(\alpha)(\alpha-1)\dots(\alpha-\ell+1)/\ell!$. Decompose T_1 into three terms (denoted by T'_1, T'_2 and T'_3) corresponding to the $2M$ first terms of the series, the $(2M+1)$ th ($\ell = 2M$) and the remainder terms. Then,

$$T'_1 = |h|^\alpha c_{kj}(h) \sum_{\ell=0}^{2M-1} h^{-\ell} \binom{\alpha}{\ell} \int_{D_h} (a_2 t_2 - a_1 t_1)^\ell \overline{\psi}(t_1) \psi(t_2) dt_1 dt_2.$$

Under Assumption [C3], ψ has M vanishing moments and therefore the previous expression reduces to

$$T'_1 = -|h|^\alpha c_{kj}(h) \sum_{\ell=0}^{2M-1} h^{-\ell} \binom{\alpha}{\ell} \int_{\mathbb{R}^2 \setminus D_h} (a_2 t_2 - a_1 t_1)^\ell \overline{\psi}(t_1) \psi(t_2) dt_1 dt_2.$$

Now,

$$|h|^{2M-\alpha} |T'_1| \leq c^\vee \sum_{\ell=0}^{2M-1} \left| \binom{\alpha}{\ell} \right| \int_{\mathbb{R}^2 \setminus D_h} 2^{2M-\ell} (a_2 t_2 - a_1 t_1)^{2M} |\psi(t_1)| |\psi(t_2)| dt_1 dt_2.$$

Assumption [C2(2M)] and the dominated convergence theorem may be combined to prove that $|h|^{2M-\alpha}T'_1 \rightarrow 0$ as $|h| \rightarrow +\infty$. The term T'_2 is defined as

$$T'_2 := |h|^{\alpha-2M}c_{kj}(h) \binom{\alpha}{2M} \int_{D_h} (a_2t_2 - a_1t_1)^{2M} \bar{\psi}(t_1)\psi(t_2)dt_1dt_2.$$

As previously, as $|h| \rightarrow +\infty$ we obtain

$$\begin{aligned} \frac{|h|^{2M-\alpha}T'_2}{c_{kj}(h)} &\rightarrow \binom{\alpha}{2M} \int_{\mathbb{R}^2} (a_2t_2 - a_1t_1)^{2M} \bar{\psi}(t_1)\psi(t_2)dt_1dt_2 \\ &= \binom{\alpha}{2M} \binom{2M}{M} (a_1a_2)^M \left| \int t^M \psi(t)dt \right|^2 = \binom{\alpha}{2M} \kappa(\psi, M). \end{aligned}$$

Since $T = T_1 + T_2 = T'_1 + T'_2 + T'_3 + T_2$, the proof will be completed if we manage to prove that $|h|^{2M-\alpha}T'_3 \rightarrow 0$. Let us write

$$\begin{aligned} |h|^{2M-\alpha}T'_3 &= h^{2M}c_{kj}(h) \int_{D_h} \sum_{\ell \geq 2M+1} \binom{\alpha}{\ell} \left(\frac{a_2t_2 - a_1t_1}{h} \right)^\ell \bar{\psi}(t_1)\psi(t_2)dt_1dt_2 \\ &= \frac{c_{kj}(h)}{h} \int_{D_h} (a_2t_2 - a_1t_1)^{2M+1} \left(\sum_{\ell \geq 0} \binom{\alpha}{\ell + 2M+1} \left(\frac{a_2t_2 - a_1t_1}{h} \right)^\ell \right) \bar{\psi}(t_1)\psi(t_2)dt_1dt_2. \end{aligned}$$

The binomial coefficient appearing in the last equation satisfies, with $\ell' = \ell + 2M + 1$

$$\begin{aligned} \left| \binom{\alpha}{\ell'} \right| &= \frac{|\alpha(\alpha-1)\cdots(\alpha-\ell'+1)|}{\ell'!} \\ &\leq \frac{2(2-\alpha)\cdots(\ell'-1-\alpha)}{\ell'!} \text{ since } \alpha \leq 2 \\ &\leq \frac{2(\ell'-1)!}{\ell'!} = \frac{2}{\ell'} \leq \frac{2}{\ell} \end{aligned}$$

Recall that in D_h we have $|a_2t_2 - a_1t_1|/|h| \leq 1/2$. The series in the previous integral then satisfies

$$\begin{aligned} \left| \sum_{\ell \geq 0} \binom{\alpha}{\ell + 2M+1} \left(\frac{a_2t_2 - a_1t_1}{h} \right)^\ell \right| &\leq \left| \binom{\alpha}{2M+1} \right| + \sum_{\ell \geq 1} \left| \binom{\alpha}{\ell + 2M+1} \right| \left| \frac{a_2t_2 - a_1t_1}{h} \right|^\ell \\ &\leq \frac{2}{2M+1} + \sum_{\ell \geq 1} \frac{2}{\ell} 2^{-\ell} \\ &= \frac{2}{2M+1} + 2\log(2) =: C_M \end{aligned}$$

Thus we obtain

$$|h|^{2M-\alpha}|T'_3| \leq \frac{C^M c^\vee}{|h|} \int_{\mathbb{R}^2} |a_2 t_2 - a_1 t_1|^{2M+1} |\psi(t_1)| |\psi(t_2)| dt_1 dt_2$$

Since by Assumption [C2(2M+1)], $t^{2M+1}\psi(t) \in L^1$, we have $|h|^{2M-\alpha}|T'_3| = O(h^{-1})$, whence the result.

Case 2. $H_j + H_k = 1$ and $H_j \neq 1/2$.

We take the same notation as previously. We first note that, under [C1], the term T can be rewritten as

$$T = \int_{\mathbb{R}^2} |a_2 t_2 - a_1 t_1 + h| + f_{jk}(a_2 t_2 - a_1 t_1 + h) \log \left| 1 + \frac{a_2 t_2 - a_1 t_1}{h} \right| \bar{\psi}(t_1) \psi(t_2) dt_1 dt_2.$$

We decompose T in $T_1 + T_2$ (as done in case 1). The proof that $|h|^{2M-1}T_2 \rightarrow 0$ as $|h| \rightarrow +\infty$ follows similar arguments as in the case 1 and is therefore omitted. Now, the term T_1 can be rewritten as

$$\begin{aligned} T_1 &= |h| \int_{D_h} \left(1 + \frac{a_2 t_2 - a_1 t_1}{h} \right) \bar{\psi}(t_1) \psi(t_2) dt_1 dt_2 \\ &\quad + f_{jk} h \int_{D_h} \left(1 + \frac{a_2 t_2 - a_1 t_1}{h} \right) \log \left(1 + \frac{a_2 t_2 - a_1 t_1}{h} \right) \bar{\psi}(t_1) \psi(t_2) dt_1 dt_2. \end{aligned}$$

Denote by \tilde{T}_1 and \tilde{T}_2 these two integrals. Assumption [C1] leads to

$$\tilde{T}_1 = -|h| \int_{\mathbb{R}^2 \setminus D_h} \left(1 + \frac{a_2 t_2 - a_1 t_1}{h} \right) \bar{\psi}(t_1) \psi(t_2) dt_1 dt_2.$$

Then, we assert that

$$\begin{aligned} |h|^{2M-1}|\tilde{T}_1| &\leq |h|^{2M-1} \int_{\mathbb{R}^2 \setminus D_h} \left| a_2 t_2 - a_1 t_1 \right| \left| 1 + \frac{a_2 t_2 - a_1 t_1}{h} \right| |\psi(t_1)| |\psi(t_2)| dt_1 dt_2 \\ &\leq 2^{2M-1} 3 \int_{\mathbb{R}^2 \setminus D_h} \left(a_2 t_2 - a_1 t_1 \right)^{2M} |\psi(t_1)| |\psi(t_2)| dt_1 dt_2 \rightarrow 0 \end{aligned}$$

as $|h| \rightarrow +\infty$. For the term \tilde{T}_2 , we may use the series expansion of $\log(1+x)$ (for $|x| < 1$). We omit the details and leave the reader to prove that as $|h| \rightarrow +\infty$

$$\begin{aligned} \tilde{T}_2 &\sim f_{jk} h \int_{\mathbb{R}^2} \left(1 + \frac{a_2 t_2 - a_1 t_1}{h} \right) \left(\frac{(-1)^{2M}}{2M-1} \left(\frac{a_2 t_2 - a_1 t_1}{h} \right)^{2M-1} \right. \\ &\quad \left. \frac{(-1)^{2M+1}}{2M} \left(\frac{a_2 t_2 - a_1 t_1}{h} \right)^{2M} \right) \bar{\psi}(t_1) \psi(t_2) dt_1 dt_2 \\ &\sim h^{1-2M} \times \frac{f_{jk}}{2M(2M-1)} \binom{2M}{M} (a_1 a_2)^M \left| \int t^M \psi(t) dt \right|^2. \end{aligned}$$

Hence, $T \sim |h|^{1-2M} \times \frac{f_{jk} \times \text{Sign}(h)}{2M(2M-1)} \kappa(\psi, M)$.

In this proof, Fubini's theorem and interchanges of integrals and (in)finite sums are widely used. All of these are justified by the absolute convergence of the different series related to the expansions of $(1+x)^{-\alpha}$ or $\log(1+x)$ for $|x| < 1$ and Assumption [C2(2M+1)]. ■

7 Proof of Proposition 8

Before proving the proposition itself, we need to generalize a formula due to Von Bahr and Essen [4]. In 1965, they have obtained the following representation theorem for $|v|^\alpha$ for $\alpha \in (0, 2)$:

$$|v|^\alpha = \frac{\Gamma(\alpha+1) \sin(\pi\alpha/2)}{\pi} \int_{\mathbb{R}} \frac{1 - \cos(\omega v)}{|\omega|^{\alpha+1}} d\omega. \quad (38)$$

The following lemma provides a similar representation for v_+^α and v_-^α .

Lemma 9 *For any $\alpha \in (0, 2)$, $\alpha \neq 1$, we have*

$$\begin{aligned} v_+^\alpha &= \frac{\Gamma(\alpha+1)}{2\pi} \int_{\mathbb{R}} \frac{\sin\left(\frac{\pi}{2}\right) (1 - \cos(\omega v)) + \cos\left(\frac{\pi}{2}\right) \text{Sign}(\omega) (\sin(\omega v) - g(\omega v))}{|\omega|^{\alpha+1}} d\omega \\ &= \frac{\Gamma(\alpha+1)}{2\pi} \int_{\mathbb{R}} \frac{\text{Re}\left(e^{i\text{Sign}(\omega)\frac{\pi}{2}(\alpha+1)} (e^{-i\omega v} - 1 + ig(\omega v))\right)}{|\omega|^{\alpha+1}} d\omega \end{aligned}$$

and

$$\begin{aligned} v_-^\alpha &= \frac{\Gamma(\alpha+1)}{2\pi} \int_{\mathbb{R}} \frac{\sin\left(\frac{\pi}{2}\right) (1 - \cos(\omega v)) - \cos\left(\frac{\pi}{2}\right) \text{Sign}(\omega) (\sin(\omega v) - g(\omega v))}{|\omega|^{\alpha+1}} d\omega \\ &= \frac{\Gamma(\alpha+1)}{2\pi} \int_{\mathbb{R}} \frac{\text{Re}\left(e^{-i\text{Sign}(\omega)\frac{\pi}{2}(\alpha+1)} (e^{-i\omega v} - 1 + ig(\omega v))\right)}{|\omega|^{\alpha+1}} d\omega \end{aligned}$$

where the function g equals zero when $\alpha \in (0, 1)$ and is the identity function when $\alpha \in (1, 2)$.

The second expressions for v_+^α and v_-^α are derived from the first one with elementary algebra.

Proof. Let us first observe that

$$v_+^\alpha = \frac{1}{2} (|v|^\alpha + \text{Sign}(v)|v|^\alpha) \quad \text{and} \quad v_-^\alpha = \frac{1}{2} (|v|^\alpha - \text{Sign}(v)|v|^\alpha).$$

Therefore, the proof consists in giving a representation of $\text{Sign}(v)|v|^\alpha$. Let $\alpha \in (0, 1)$, then from (38) and properties of the function Γ

$$\frac{1}{\alpha+1} |v|^{\alpha+1} = \frac{\Gamma(\alpha+1)}{\pi} \cos(\pi\alpha/2) \int_{\mathbb{R}} \frac{1 - \cos(\omega v)}{|\omega|^{\alpha+2}} d\omega.$$

Since $\int_{\mathbb{R}} |\omega|^{-\alpha-1} |\sin(\omega v)| < +\infty$ for $\alpha \in (0, 1)$, we can differentiate this integral with respect to v to obtain

$$\text{Sign}(v)|v|^\alpha = \frac{\Gamma(\alpha+1)}{\pi} \cos(\pi\alpha/2) \int_{\mathbb{R}} \frac{\text{Sign}(\omega) \sin(\omega v)}{|\omega|^{\alpha+1}} d\omega. \quad (39)$$

When $\alpha \in (1, 2)$, then from (38) and properties of the function Γ

$$\alpha|v|^{\alpha-1} = \frac{\Gamma(\alpha+1)}{\pi} (-\cos(\pi\alpha/2)) \int_{\mathbb{R}} \frac{1 - \cos(\omega v)}{|\omega|^\alpha} d\omega.$$

Since $\int_{\mathbb{R}} |\omega|^{-\alpha-1} |\sin(\omega v) - \omega v| d\omega < +\infty$ for $\alpha \in (1, 2)$, we can take the primitive of the last equation to get

$$\begin{aligned} \text{Sign}(v)|v|^\alpha &= \frac{\Gamma(\alpha+1)}{\pi} \cos(\pi\alpha/2) \int_{\mathbb{R}} \frac{\sin(\omega v)/\omega - v}{|\omega|^\alpha} d\omega \\ &= \frac{\Gamma(\alpha+1)}{\pi} \cos(\pi\alpha/2) \int_{\mathbb{R}} \frac{\text{Sign}(\omega)(\sin(\omega v) - \omega v)}{|\omega|^{\alpha+1}} d\omega, \end{aligned}$$

which ends the proof. \blacksquare

Let $\alpha \in (0, 1)$, then by differentiating (39) with respect to α and taking the limit as $\alpha \rightarrow 1^-$, we may obtain

$$\text{Sign}(h)|h| \log |h| = h \log |h| = \lim_{\alpha \rightarrow 1^-} -\frac{1}{2} \int_{\mathbb{R}} \frac{\text{Sign}(\omega) \sin(\omega v)}{|\omega|^{\alpha+1}} d\omega. \quad (40)$$

We now turn to the proof of proposition 8.

Proof. (i) We recall that under [C1] and [C2(2)], Equation (31) holds, that is $E[d_{a_1, b_1}^j \overline{d_{a_2, b_2}^k}] = -\frac{\sigma_j \sigma_k \rho_{jk}}{2} T$ with $T := \int_{\mathbb{R}} w_{jk}(v) \overline{\Gamma_\psi(v)} dv$. Furthermore, note that the Fourier transforms of $\psi_{a, b}$ and $\Gamma_\psi(v)$ exist and are equal respectively to $\sqrt{a} \hat{\psi}(a\omega) e^{-i\omega b}$ and to $\sqrt{a_1 a_2} \hat{\psi}(-a_1 \omega) \overline{\hat{\psi}(-a_2 \omega)} e^{-i\omega(b_2 - b_1)}$ which leads to

$$q(\omega) := \int_{\mathbb{R}} \overline{\Gamma_\psi(v)} e^{-i\omega v} dv = \sqrt{a_1 a_2} \overline{\hat{\psi}(a_1 \omega)} \hat{\psi}(a_2 \omega) e^{-i\omega(b_2 - b_1)}. \quad (41)$$

Now, let us split the proof into two cases.

Case 1. $j = k$ or $j \neq k$ and $H_j + H_k \neq 1$.

When $j = k$, at this step, the authors of [15] have used the representation of $|v|^\alpha$ obtained by [4]. We obtained a similar representation for the function v_+^α and v_-^α for $\alpha \in (0, 2)$ and $\alpha \neq 1$ in lemma 9. For any α , let us set $S(\omega) := \text{Sign}(\omega) \frac{\pi}{2} (\alpha + 1)$. We have by Fubini's theorem and under Assumption [C2(2M)] (with $M \geq 2$).

$$T = \frac{\Gamma(\alpha+1)}{2\pi} \int_{\mathbb{R}} |\omega|^{-\alpha-1} \int_{\mathbb{R}} \left(\frac{c_{kj}}{2} \left(e^{iS(\omega)} e^{-i\omega v} + e^{-iS(\omega)} e^{i\omega v} \right) \overline{\Gamma_\psi(v)} \right) d\omega dv$$

$$\begin{aligned}
& + \frac{c_{jk}}{2} \left(e^{-iS(\omega)} e^{-i\omega v} + e^{iS(\omega)} e^{i\omega v} \right) \overline{\Gamma_\psi(v)} \Big) dv d\omega \\
& = \frac{\Gamma(\alpha+1)}{2\pi} \int_{\mathbb{R}} |\omega|^{-\alpha-1} \left(\frac{c_{kj}}{2} \left(e^{iS(\omega)} q(\omega) + e^{-iS(\omega)} q(-\omega) \right) \right. \\
& \quad \left. + \frac{c_{jk}}{2} \left(e^{-iS(\omega)} q(\omega) + e^{iS(\omega)} q(-\omega) \right) \right) d\omega \\
& = \frac{\Gamma(\alpha+1)}{2\pi} \int_{\mathbb{R}} |\omega|^{-\alpha-1} \left(c_{kj} e^{iS(\omega)} + c_{jk} e^{-iS(\omega)} \right) q(\omega) d\omega. \tag{42}
\end{aligned}$$

Note that the condition $M \geq 2$ is required for $\alpha > 1$. For $\alpha < 1$, $M \geq 1$ is a sufficient condition. These conditions allow us to show that the contributions $\int (1 - ig(\omega v)) \exp(\pm iS(\omega)) \overline{\Gamma_\psi(v)} dv$ in this calculation are equal to zero.

Making the change of variable $\omega \leftrightarrow -\omega$ in the integral in (42) and using (41), we obtain

$$T = \Gamma(\alpha+1) \sqrt{a_1 a_2} \frac{1}{2\pi} \int_{\mathbb{R}} |\omega|^{-\alpha-1} \left(c_{kj} e^{-iS(\omega)} + c_{jk} e^{iS(\omega)} \right) \overline{\widehat{\psi}(-a_1 \omega)} \widehat{\psi}(-a_2 \omega) e^{i\omega(b_2 - b_1)} d\omega.$$

and therefore, reminding that $\zeta_{jk}(\omega) := -c_{kj} e^{-iS(\omega)} - c_{jk} e^{iS(\omega)}$, we have

$$E[d_{a_1, b_1}^j \overline{d_{a_2, b_2}^k}] = \frac{\sigma_j \sigma_k \rho_{jk}}{2} \sqrt{a_1 a_2} \frac{1}{2\pi} \int_{\mathbb{R}} \underbrace{|\omega|^{-\alpha-1} \zeta_{jk}(\omega) \overline{\widehat{\psi}(-a_1 \omega)} \widehat{\psi}(-a_2 \omega)}_{=: P(\omega)} e^{i\omega(b_2 - b_1)} d\omega.$$

By using Bochner's Theorem, the proof will be done, if one proves that the function $P(\omega)$ is integrable. Let us prove this last assertion. Under [C2(M)], $t^k \psi(t) \in L^1$ for $k = 0, \dots, M$. Therefore, $\widehat{\psi}$ is a M times continuous and differentiable function. Using a Taylor expansion

$$\widehat{\psi}(\omega) = \sum_{k=0}^{M-1} \omega^k \widehat{\psi}^{(k)}(\omega) + \omega^M \widehat{\psi}^{(M)}(\tilde{\omega}) = \omega^M \widehat{\psi}^{(M)}(\tilde{\omega}), \text{ with } \tilde{\omega} \in [0 \wedge \omega, 0 \vee \omega],$$

under [C2(M)]. And since $\psi^{(M)}$ is continuous at zero, $\widehat{\psi}(\omega) \sim \omega^M \widehat{\psi}^{(M)}(0)$ as $\omega \rightarrow 0$. Then as $\omega \rightarrow 0$:

$$P(\omega) \sim \zeta_{jk}(\omega) |\omega|^{2M-1-\alpha} (a_1 a_2)^M |\widehat{\psi}^{(M)}(0)|^2. \tag{43}$$

As a consequence, for $M \geq 2$, P is continuous at zero and $\lim_{\omega \rightarrow 0^\pm} P(\omega) = 0$. Therefore for $\varepsilon > 0$, P is integrable on the interval $[-\varepsilon, \varepsilon]$ as a continuous function on this interval. Finally (recall that $c^\vee := \max(|c_{jk}|, |c_{kj}|)$),

$$\begin{aligned}
\int_{|\omega| \geq \varepsilon} |P(\omega)| & \leq 2c^\vee (a_1 a_2)^{\alpha+1} \left(\int_{|\omega| \geq a_1 \varepsilon} \frac{|\widehat{\psi}(\omega)|^2}{|\omega|^{\alpha+1}} d\omega \right)^{1/2} \left(\int_{|\omega| \geq a_2 \varepsilon} \frac{|\widehat{\psi}(\omega)|^2}{|\omega|^{\alpha+1}} d\omega \right)^{1/2} \\
& \leq \frac{(a_1 a_2)}{\varepsilon^{2\alpha}} \int_{\mathbb{R}} \frac{|\widehat{\psi}(\omega)|^2}{|\omega|} d\omega < +\infty
\end{aligned}$$

under [C1]. Hence, $P(\omega) \in L^1$ and Bochner's Theorem may be applied.

Case 2. $j \neq k$ and $H_j + H_k = 1$.

We start with the representation of $v \log |v|$ given by (40).

$$\begin{aligned} w_{jk}(v) &= |v| + f_{jk}v \log |v| = \lim_{\alpha \rightarrow 1^-} |v|^\alpha + f_{jk}v \log |v| \\ &= \lim_{\alpha \rightarrow 1^-} \frac{1}{2\pi} \int_{\mathbb{R}} \frac{2(1 - \cos(\omega v)) - \pi f_{jk} \text{Sign}(\omega) \sin(\omega v)}{|\omega|^{\alpha+1}} d\omega. \end{aligned}$$

Now, we derive the computation of the term $T := \int_{\mathbb{R}} w_{jk}(v) \bar{\Gamma}_\psi(-v) dv$, similarly as the previous case. Using dominated convergence theorem and Fubini's Theorem,

$$\begin{aligned} T &= \frac{1}{2\pi} \int_{\mathbb{R}} \left(\lim_{\alpha \rightarrow 1^-} \int_{\mathbb{R}} \frac{2(1 - \cos(\omega v)) - \pi f_{jk} \text{Sign}(\omega) \sin(\omega v)}{|\omega|^{\alpha+1}} d\omega \right) \bar{\Gamma}_\psi(v) dv \\ &= \frac{1}{2\pi} \lim_{\alpha \rightarrow 1^-} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{2(1 - \cos(\omega v)) - \pi f_{jk} \text{Sign}(\omega) \sin(\omega v)}{|\omega|^{\alpha+1}} \bar{\Gamma}_\psi(v) dv \right) d\omega \\ &= \frac{1}{2\pi} \lim_{\alpha \rightarrow 1^-} \int_{\mathbb{R}} \frac{2 - i\pi f_{jk} \text{Sign}(\omega)}{|\omega|^{\alpha+1}} q(\omega) d\omega \\ &= \frac{1}{2\pi} \lim_{\alpha \rightarrow 1^-} \int_{\mathbb{R}} |\omega|^{-\alpha-1} (2 - i\pi f_{jk} \text{Sign}(\omega)) \overline{\widehat{\psi}(a_1\omega)} \widehat{\psi}(a_2\omega) e^{-i\omega(b_2-b_1)} d\omega. \end{aligned}$$

From (43), $|\omega|^{-\alpha-1} \overline{\widehat{\psi}(a_1\omega)} \widehat{\psi}(a_2\omega)$ is an integrable function for all $\alpha \in (0, 2)$. Therefore, the integral and the limit may be interchanged. Making the change of variable $\omega \leftrightarrow -\omega$ we obtain

$$T = \sqrt{a_1 a_2} \frac{1}{2\pi} \int_{\mathbb{R}} \frac{2 - i\pi f_{jk} \text{Sign}(\omega)}{\omega^2} \overline{\widehat{\psi}(-a_1\omega)} \widehat{\psi}(-a_2\omega) e^{i\omega(b_2-b_1)} d\omega.$$

and Bochner's Theorem can be applied.

(ii) is derived from (43). ■

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